PHYSICAL REVIEW D **96,** 023014 (2017)

Expanded solutions of force-free electrodynamics on general Kerr black holes

Huiquan Li* and Jiancheng Wang

Yunnan Observatories, Chinese Academy of Sciences, 650216 Kunming, China;
Key Laboratory for the Structure and Evolution of Celestial Objects,
Chinese Academy of Sciences, 650216 Kunming, China;
and Center for Astronomical Mega-Science, Chinese Academy of Sciences, 100012 Beijing, China

(Received 19 December 2016; published 27 July 2017)

In this work, expanded solutions of force-free magnetospheres on general Kerr black holes are derived through a radial distance expansion method. From the regular conditions both at the horizon and at spatial infinity, two previously known asymptotical solutions (one of them is actually an exact solution) are identified as the only solutions that satisfy the same conditions at the two boundaries. Taking them as initial conditions at the boundaries, expanded solutions up to the first few orders are derived by solving the stream equation order by order. It is shown that our extension of the exact solution can (partially) cure the problems of the solution: it leads to magnetic domination and a mostly timelike current for restricted parameters.

DOI: 10.1103/PhysRevD.96.023014

I. INTRODUCTION

Black hole magnetospheres are believed to play essential roles in many high-energy astronomical objects. In the popular Blandford-Znajek model [1,2], energy can be extracted from a rotating black hole via a stationary and force-free magnetosphere to eject dipole relativistic jets, which may account for most of the high-energy phenomena in active galactic nuclei, gamma-ray bursts, and microquasars.

In the simplest configuration, the force-free magnetosphere is well described by the clean and precise electrodynamics on a Kerr black hole. However, the present understanding of such a system relies strongly on numerical simulations. Existing analytical approaches leave us with very few options.

One of the approaches is the perturbation method first given in the original work of Blandford and Znajek [1]. Based on the split monopole and paraboloidal solutions on a nonrotating black hole, analytical solutions on a slowly rotating black hole are derived by expanding the functions and stream equation to leading orders of the spin parameter. So the solutions apply to slowly rotating black holes. To get analytical properties of magnetospheres on rapidly rotating black holes, which may be more interesting to us, we need to calculate higher-order corrections, but this seems difficult to do [3]. Recently, the solution up to the fourth order was derived [4].

Solutions that go beyond the slow-rotation limit in the perturbation approach can be obtained in some limited regions. In the work [5,6] of Menon and Dermer (MD), asymptotic solutions (and their generalization [7]) were derived in regions far away from the horizon. The solutions can apply to black holes with general angular momentum. But these solutions are radial-distance independent. In particular, one of the solutions is the only known exact solution so far that can solve the full stream equation, which makes it very interesting. However, the current for this solution is along the infalling principle null geodesic. This means that charged particles must move at the speed of light, which is not allowed. Besides, the electromagnetic fields from the solution are also null. A method was given in Refs. [8,9] by the same authors, in which the lightlike current is artificially decomposed into a linear combination of two timelike currents with opposite charges.

On the other hand, in past years, exact solutions on extremely fast rotating black holes were obtained by focusing on the near-horizon region [10–13]. But, a smooth connection between these near- and far-region solutions is lacking.

In this work, we consider a different expansion method other than the one in the traditional perturbation approach. In terms of the boundary conditions of a magnetosphere, we expand the functions and stream equation in series of the radial distance, instead of the spin parameter. Analytical solutions that depend on both poloidal coordinates can be derived order by order following a precise procedure. So this approach hopefully can help us extend solutions in the far region to the ones in the near region. Moreover, this provides a method to generalize the MD exact solution and relax the problems of the null current and electromagnetic fields.

The paper is organized as follows. In Sec. II, the stream equation of a force-free magnetosphere is constructed and presented. In Sec. III, we show the boundary conditions at the horizons and at infinity, which can be determined from the stream equation. Two special cases of the boundary conditions lead to the previously known asymptotic solutions. In Sec. IV, the expansion forms and solving

*lhq@ynao.ac.cn

procedure of the stream equation are introduced in terms of the boundary conditions. Examples of solutions are derived and analyzed in Sec. V. Then we summarize in the last section.

II. THE STREAM EQUATION

Using the Boyer-Lindquist coordinates, a Kerr black hole is depicted by the metric

$$ds^2 = -\Lambda^2 dt^2 + \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 + \varpi^2 (d\phi - \omega dt)^2, \quad (1)$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta, \qquad a = \frac{J}{M},$$
$$\Lambda^2 = \frac{\rho^2 \triangle}{A}, \qquad \varpi^2 = \frac{A \sin^2\theta}{\rho^2}, \qquad \omega = \frac{2Mar}{A},$$
$$\triangle = (r - r_+)(r - r_-), \qquad r_\pm = M \pm \sqrt{M^2 - a^2},$$
$$A = (r^2 + a^2)^2 - a^2 \triangle \sin^2\theta = 2Mr(r^2 + a^2) + \triangle \rho^2.$$

The spin parameter a measures the angular momentum J per unit mass M of the black hole. The inner and outer horizons are located at $r = r_-$ and $r = r_+$, respectively. From the above relations, the velocities of the black hole at the horizons are

$$\omega_\pm \equiv \frac{a}{r_\pm^2 + a^2}. \quad (2)$$

In the $3 + 1$ split formulation [14], the four-dimensional spacetime (1) is replaced by a three-dimensional absolute space and a universal time coordinate. The electrodynamics on a Kerr black hole can be equivalently dealt with on the following absolute space:

$$ds_A^2 = \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 + \varpi^2 d\phi^2. \quad (3)$$

The four-dimensional quantities and equations are split accordingly. The quantities we deal with on the absolute space are measured by the so-called zero-angular momentum observers (ZAMOs). From the inverse metric, the unit basis vectors are given by

$$\mathbf{e}_{\hat{r}} = \sqrt{\frac{\triangle}{\rho^2}} \partial_r, \quad \mathbf{e}_{\hat{\theta}} = \frac{1}{\sqrt{\rho^2}} \partial_\theta, \quad \mathbf{e}_{\hat{\phi}} = \frac{\sqrt{\rho^2}}{\sqrt{A} \sin\theta} \partial_\phi. \quad (4)$$

The Kerr spacetime has Killing vectors along the time and along the toroidal directions. For simplicity, we consider the stationary and axisymmetric case of electrodynamics on the spacetime. The relevant inhomogeneous Maxwell's equations relate the electromagnetic fields to the electric charge and current densities (ρ_e, $\mathbf{j}$):

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e, \quad (5)$$

$$\nabla \times (\Lambda \mathbf{B}) = 4\pi \Lambda \mathbf{j} - \varpi (\mathbf{E} \cdot \nabla \omega) \mathbf{e}_{\hat{\phi}}. \quad (6)$$

Throughout this paper, the operator ∇ is the covariant derivative associated with the three-dimensional spatial dimensions (3). The homogeneous Maxwell's equations tell us that the electromagnetic fields can be expressed as the gauge potentials (A_0, $\mathbf{A}$):

$$\mathbf{E} = \frac{1}{\Lambda} (\nabla A_0 + \omega \nabla A_\phi), \quad (7)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (8)$$

We ignore the dynamics of plasma and impose the force-free condition

$$\rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B} = 0, \quad (9)$$

which automatically satisfies

$$\mathbf{j} \cdot \mathbf{E} = 0, \qquad \mathbf{E} \cdot \mathbf{B} = 0. \quad (10)$$

Under these conditions, the electrodynamics is described by three correlated functions: the flux $\psi = 2\pi A_\phi$ and the total electric current $I(\psi)$ flowing through the area enclosed by an axisymmetric loop, and the angular velocity of the electromagnetic field lines $\Omega(\psi) = -dA_0/dA_\phi$ on the loop.

The electromagnetic fields read

$$\mathbf{E} = -\frac{\Omega - \omega}{2\pi \Lambda \sqrt{\rho^2}} (\sqrt{\triangle} \partial_r \psi \mathbf{e}_{\hat{r}} + \partial_\theta \psi \mathbf{e}_{\hat{\theta}}), \quad (11)$$

$$\mathbf{B} = \frac{1}{2\pi \sqrt{A} \sin\theta} \left(\partial_\theta \psi \mathbf{e}_{\hat{r}} - \sqrt{\triangle} \partial_r \psi \mathbf{e}_{\hat{\theta}} + \frac{4\pi I \sqrt{\rho^2}}{\Lambda} \mathbf{e}_{\hat{\phi}} \right). \quad (12)$$

The charge and current densities are, respectively,

$$\rho_e = -\frac{1}{8\pi^2} \nabla \cdot \left(\frac{\Omega - \omega}{\Lambda} \nabla \psi \right), \quad (13)$$

$$\mathbf{j} = \frac{1}{\Lambda} [\rho_e \varpi (\Omega - \omega) \mathbf{e}_{\hat{\phi}} + I' \mathbf{B}], \quad (14)$$

where the prime stands for a derivative with respect to ψ. Note that the total current I is defined to flow upwards, with opposite sign to that defined in the original paper [14].

From the above equations and expressions, we will find that the force-free electrodynamics on a Kerr spacetime can be described by the following unique stream equation [14]:

$$\nabla \cdot \left\{ \frac{\Lambda}{\varpi^2} \left[1 - \frac{(\Omega - \omega)^2 \varpi^2}{\Lambda^2} \right] \nabla \psi \right\} + \frac{\Omega - \omega}{\Lambda} \Omega' (\nabla \psi)^2 + \frac{16\pi^2}{\Lambda \varpi^2} I I' = 0. \quad (15)$$

For the electromagnetic system, the poloidal components of the energy and angular momentum flux densities from the hole are given by

$$\mathcal{E}^r = \Omega\mathcal{L}^r = -\Omega\frac{I\partial_\theta\psi}{2\pi\sin\theta\rho^2}, \tag{16}$$

$$\mathcal{E}^\theta = \Omega\mathcal{L}^\theta = \Omega\frac{I\partial_r\psi}{2\pi\sin\theta\rho^2}. \tag{17}$$

III. BOUNDARY BEHAVIOURS

In regions that are accessible to us, the differential equation (15) has two boundaries: one at the horizon and the other at spatial infinity (if the force-free region extends far away from the outer horizon). In some sense, the two boundaries have similar behaviors and features, e.g., they both attain the radiation condition [15]

$$E^\theta = \pm B^\phi \tag{18}$$

for the electromagnetic fields (11) and (12) as they are approached. The condition can be obtained directly from the stream equation (15). To make this clear, we reexpress the stream equation in the following form:

$$\begin{aligned}
&\frac{\triangle}{A}\Bigg\{\left[\frac{A^2\sin^2\theta(\Omega-\omega)^2}{\rho^4}-\triangle\right]\partial_r^2\psi\\
&\quad+\frac{A^2\sin^2\theta(\Omega-\omega)}{\rho^4}\partial_r\Omega\partial_r\psi+\frac{2A\sin^2\theta}{\rho^2}\\
&\quad\times\left[r\Omega^2-\frac{Ma^2\sin^2\theta(r^2-a^2\cos^2\theta)(\Omega-\Omega_N)^2}{\rho^4}\right]\partial_r\psi\\
&\quad-\partial_\theta^2\psi+\frac{1}{2}\left[\frac{\triangle}{A}+\frac{A\sin^2\theta(\Omega^2-\Omega_N^2)}{\rho^4}\right]\partial_\theta\rho^2\partial_\theta\psi\Bigg\}\\
&\quad+\frac{\sqrt{A}\sin\theta(\Omega-\omega)}{\rho^2}\partial_\theta\frac{\sqrt{A}\sin\theta(\Omega-\omega)\partial_\theta\psi}{\rho^2}\\
&\quad-16\pi^2 II'=0,
\end{aligned} \tag{19}$$

where $\Omega_N = 1/(a\sin^2\theta)$.

A. Conditions at horizons

From Eq. (19), we can see that only the last line of the equation remains at the horizons $r = r_\pm$ or $\triangle = 0$:

$$\begin{aligned}
16\pi^2 I\frac{\partial}{\partial\psi}I &= \frac{\sqrt{A}\sin\theta(\Omega-\omega)}{\rho^2}\partial_\theta\frac{\sqrt{A}\sin\theta(\Omega-\omega)\partial_\theta\psi}{\rho^2}\\
&= \frac{\sqrt{A}\sin\theta(\Omega-\omega)\partial_\theta\psi}{\rho^2}\frac{\partial}{\partial\psi}\frac{\sqrt{A}\sin\theta(\Omega-\omega)\partial_\theta\psi}{\rho^2}.
\end{aligned} \tag{20}$$

Approaching the event horizon, ψ is only dependent on θ [14]. This structure in the stream equation is also found in the near-horizon treatments of magnetospheres on near-extreme Kerr black holes [12].

From the above relation at the horizons, we have

$$r = r_\pm:\ I^2 = C_\pm + \left[\frac{\sqrt{A}\sin\theta(\Omega-\omega)}{4\pi\rho^2}\partial_\theta\psi\right]^2, \tag{21}$$

where $C_\pm$ are constants. So we can conclude that any solution ψ [with any given correlated functions $\Omega_F(\psi)$ and $I(\psi)$] satisfying the stream equation (19) will always satisfy the condition (21), only if the sum of the terms within the brackets in Eq. (19) is nonsingular compared with the rest of the terms at the horizons.

It is easy to find that the Znajek boundary condition can be obtained by setting the special value

$$C_\pm = 0. \tag{22}$$

When the positive sign is chosen, the conditions (21) at the horizons read

$$I_+ = \frac{Mr_+\sin\theta(\Omega_+-\omega_+)}{2\pi\rho_+^2}\partial_\theta\psi_+, \tag{23}$$

$$I_- = \frac{Mr_-\sin\theta(\Omega_--\omega_-)}{2\pi\rho_-^2}\partial_\theta\psi_-, \tag{24}$$

where $\psi_\pm = \psi(r_\pm)$, $\Omega_\pm = \Omega(\psi(r_\pm))$, $I_\pm = I(\psi(r_\pm))$, and $\rho_\pm^2 = r_\pm^2 + a^2\cos^2\theta$. The former is exactly the Znajek regularity condition at the outer horizon [2], which corresponds to the positive-sign case of the radiation condition (18): $E^\theta = B^\phi$. The positive sign is chosen because this means current flow is directed outwards for $0 < \Omega_+ < \omega_+$ [14,16], which leads to energy and angular momentum extraction from the hole across the event horizon, as implied by Eq. (16).

B. Condition at spatial infinity

Let us now turn to the behaviors at spatial infinity. As shown in Eqs. (16) and (17), the energy and momentum extraction rates differ by the angular velocity Ω. Since the energy and momentum extracted from the hole must be finite at spatial infinity, Ω should be independent of r at infinity,

$$\Omega(r,\theta) \to \Omega_0(\theta) \quad \text{as } r\to\infty, \tag{25}$$

as noticed in Refs. [5,6]. Further, since $I(\Omega)$ and $\psi(\Omega)$ are functions of Ω, the associated functions I_0 and ψ_0 at infinity should be functions of Ω_0 as well:

$$r\to\infty:\ \psi(\Omega)\to\psi_0(\Omega_0(\theta)),\quad I(\Omega)\to I_0(\Omega_0(\theta)). \tag{26}$$

That is, all three correlated functions should be independent of r at infinity if one is. This is quite similar to the situation at the outer horizon, where the functions are also only dependent on θ.

In the present work, we consider the case in which Ω_0 is not a constant or other trivial function of θ (and similarly for ψ_0 and I_0). With the asymptotic conditions (25) and (26), we can find that the stream equation (19) takes the following simple form at infinity:

$$16\pi^2 I_0 \frac{\partial I_0}{\partial \psi_0} = \sin\theta \Omega_0 \partial_\theta (\sin\theta \Omega_0 \partial_\theta \psi_0). \tag{27}$$

Similarly, we have

$$I_0^2 = C_0 + \frac{1}{16\pi^2}(\sin\theta \Omega_0 \partial_\theta \psi_0)^2, \tag{28}$$

where C_0 is a constant. So any solutions satisfying the boundary conditions (25) and (26) must satisfy this relation. Here, we also choose the special case $C_0 = 0$, for which the above relation reads

$$I_0 = -\frac{1}{4\pi} \sin\theta \Omega_0 \partial_\theta \psi_0. \tag{29}$$

Here, the negative sign is chosen when $\Omega_+ \leq \omega_+$, which guarantees an outflow of energy by inserting the current (29) into Eq. (19). This also corresponds to the positive-sign case of the radiation condition (18). Note that, when $\Omega_+ > \omega_+$, we need to choose the positive sign in Eq. (29) (corresponding to the minus-sign case of the radiation condition: $E^\theta = -B^\phi$), which leads to an influx of energy at spatial infinity. The reason is that the direction of energy cannot reverse on a field line [1]. If the energy inflows across the event horizon for $\Omega_+ > \omega_+$, we should also have influx at infinity.

C. The cases with identical boundary conditions

As in usual second-order differential equations, a set of solutions can be defined by constraining appropriate conditions on the two boundaries. On the other hand, as we stated above, the behaviors are similar at the two boundaries: the functions are purely θ-dependent and satisfy the radiation condition (18). So it is natural to consider the special cases in which the conditions on the two boundaries are identical.

Generalizing the condition (29) to include the positive-sign case, we can express the boundary condition at infinity as

$$\pm \frac{4\pi}{\sin\theta} = \Omega_0 \frac{\partial_\theta \psi_0}{I_0}. \tag{30}$$

On the other hand, the Znajek boundary condition (23) can be reexpressed as

$$4\pi\left(\frac{1}{\sin\theta} - a \sin\theta \omega_+\right) = (\Omega_+ - \omega_+)\frac{\partial_\theta \psi_+}{I_+}. \tag{31}$$

Now we consider the special case that the functions satisfy the same boundary conditions at the horizon and at infinity[1]:

$$\Omega_0 = \Omega_+, \qquad \psi_0 = \psi_+, \qquad I_0 = I_+. \tag{32}$$

(1) If we choose the positive sign in Eq. (30), we can have $\Omega_+ \partial_\theta \psi_+ / I_+ = 4\pi/\sin\theta$ and $\partial_\theta \psi_+ / I_+ = 4\pi a \sin\theta$ by comparing Eqs. (30) and (31). From the difference between them, we have

$$\Omega_+(\theta) = \Omega_0(\theta) = \Omega_N \equiv \frac{a}{2Mr_+ - \rho_+^2} = \frac{1}{a\sin^2\theta}. \tag{33}$$

As expected, the angular velocity is larger than that of the black hole. That is why we have chosen the positive sign in the condition (30), as stated in the previous subsection.

(2) If we choose the negative sign in Eq. (30), we have $\Omega_+ \partial_\theta \psi_+ / I_+ = -4\pi/\sin\theta$ and $\partial_\theta \psi_+ / I_+ = -4\pi(2Mr_+ + \rho_+^2)/(a\sin\theta)$, which leads to

$$\Omega_+(\theta) = \Omega_0(\theta) = \Omega_P \equiv \frac{a}{2Mr_+ + \rho_+^2}. \tag{34}$$

The two solutions (33) and (34) at the boundaries are exactly the same as the asymptotical solutions found in Refs. [5,6] (the MD solutions). The first solution is the only known exact solution to date that can solve the full stream equation. In deriving the above solutions, the functions I and ψ are identical but not specified at the boundaries.

In what follows, we only take them as initial values at the two identical boundaries, instead of asymptotical solutions, to explore analytical solutions that are (r, θ)-dependent in between the boundaries.

IV. THE EXPANSION METHOD

In terms of the boundary properties, we may derive solutions to the stream equation by expanding the functions in series of the radial distance r, as done in the Appendix for the Schwarzschild black hole case. If Ω_0, ψ_0, and I_0 are all nontrivial functions of θ (i.e., not zero or constant), we can take the three correlated functions in the following general expanded forms:

$$\Omega = \sum_{n=0}^{\infty} \Omega_{-n}(\theta) r^{-n},$$
$$\psi = \sum_{n=0}^{\infty} \psi_{-n}(\theta) r^{-n},$$
$$I = \sum_{n=0}^{\infty} I_{-n}(\theta) r^{-n}. \tag{35}$$

[1]Actually, the latter two stringent conditions can be simply replaced by the unique one $\partial_\theta \psi_0 / I_0 = \partial_\theta \psi_+ / I_+$ when the relations between I, ψ, and Ω are not necessarily the same at the two boundaries.

We assume these expanded forms to be valid in all force-free regions outside the event horizon of the Kerr spacetime. These forms saturate the conditions (25) and (26) at infinity. Inserting the expanded forms into the stream equation, solutions can be derived order by order. The solving procedure is as follows.

First, we need to choose the right zeroth-order functions Ω_0, ψ_0, and I_0, i.e., the conditions at infinity. But we only need to know two of them, because the third one can be determined via Eq. (29) [or Eq. (30), more generally] when the other two are given.

Second, we need to know the functional forms $\Omega(\psi)$ and $I(\psi)$ of ψ [we can also take $\psi(\Omega)$ and $I(\Omega)$ as functions of Ω]. The functional relations can be simply determined by the zeroth-order ones,

$$\psi, \Omega(\psi), I(\psi) \Leftrightarrow \psi_0, \Omega_0(\psi_0), I_0(\psi_0), \tag{36}$$

since the former will always lead to the latter as $r \to \infty$. With the specific forms of the functions $\Omega(\psi)$ and $I(\psi)$, we can determine the values ψ_+, Ω_+, and I_+ at the horizon by inserting the functions into the Znajek regularity condition (23). This is how the conditions at the two boundaries are correlated. So the zeroth-order functions can be adjusted if the conditions at the horizon are found to be inappropriate.

Finally, with all of the functions and expanded forms inserted into the stream equation, we can solve the equation order by order. The obtained solutions should apply for rotating black holes with general a.

In summary, the derived solutions in this method completely rely on the choices of the conditions at the two boundaries. Given any two of ψ_0, Ω_0, and I_0, the general functional relations among ψ, Ω, and I can be determined by the zeroth-order ones. This further leads to the determinant of the condition at the horizon. So, with appropriate conditions at both boundaries, a set of solutions are defined.

Besides, there is an extra problem that needs to be classified: the convergency at the horizon in the extreme limit. The coefficient of the nth-order term of the derived solution should be of the order of

$$\psi_{-n} \sim \mathcal{O}(m^n) \quad (n \geq 1), \tag{37}$$

where

$$m^n = \prod_{i=1}^{n} m_i, \qquad m_i = (a, M). \tag{38}$$

Thus, in the extreme limit $r_+ = M = a$, each term of the expanded forms (35) is order $\mathcal{O}(1)$ at the coincident horizon. So every term is important close to the horizon in the extreme case. We must check this convergency of the solution, which is hard to do because we usually cannot derive the full solution of all orders. Fortunately, its convergency should be guaranteed by the Znajek regularity condition at the horizon, since it applies for arbitrary a.

V. SOLUTIONS

As examples, we shall adopt the special boundary conditions obtained in Sec. III C to make solutions in what follows.

A. $\Omega_0 = \Omega_P$

As shown in Ref. [5], this case may correspond to the split monopole because it is expanded to the leading order of a as $\Omega_P = a/(8M^2) + \cdots$ in the slow-rotating limit. Thus, we take the zeroth-order flux ψ_0 as that in the split monopole solution (on the upper half hemisphere $0 \leq \theta < \pi/2$)

$$\psi_0 = \alpha(1 - \cos\theta), \tag{39}$$

where α is a constant. Then we have from Eq. (29)

$$I_0 = -\frac{\alpha}{4\pi}\sin^2\theta\Omega_P. \tag{40}$$

In terms of the relations between ψ_0 and $I_0(\psi_0)$, $\Omega_0(\psi_0)$, we can generalize them by assuming that the relations apply for any r:

$$\Omega(\psi) = \frac{\alpha^2 a}{B(\psi)}, \qquad I(\psi) = -\frac{\alpha a\psi(2\alpha - \psi)}{4\pi B(\psi)}, \tag{41}$$

where $B(\psi) = \alpha^2(r_+^2 + 2Mr_+) + a^2(\alpha - \psi)^2$. Thus, we have

$$II' = \frac{Mr_+\alpha^4 a^2\psi(\alpha - \psi)(2\alpha - \psi)}{2\pi^2 B^3}. \tag{42}$$

Inserting the functions $\Omega(\psi)$ and $I(\psi)$ into Eq. (19), we get an equation for ψ. The resulting equation can be solved order by order by using the expanded forms (35), in analogy to the Schwarzschild case shown in the Appendix. Similarly, let us define

$$L_\theta^2 \equiv \partial_\theta^2 + (2a\Omega_P\sin 2\theta + \cot\theta)\partial_\theta + 6 - 8a\Omega_P\cos^2\theta - \frac{4}{\sin^2\theta}. \tag{43}$$

The vanishing of the coefficients of r^0 gives rise to the equation about ψ_0, which is automatically saturated because it is just the condition chosen at infinity. Comparing all the terms at order r^{-1} leads to the following equation about ψ_{-1}:

$$L_\theta^2\psi_{-1} = 0. \tag{44}$$

This equation can be solved by

$$\psi_{-1} = \beta a\sin^2\theta, \tag{45}$$

with β being an arbitrary dimensionless constant.

In order to make higher-order calculations simpler, we may set the free parameter β to be the special value 0. Then the equation about ψ_{-2} can be obtained and simplified:

$$(L_\theta^2+2)\psi_{-2} = -8Mr_+\alpha a\cos\theta\sin^2\theta\Omega_P. \tag{46}$$

A solution to the equation is

$$\psi_{-2} = \frac{1}{2}\alpha a^2\cos\theta\sin^2\theta. \tag{47}$$

Accurate to this order, the solution is somehow similar to the slow-rotating solution in the large-r limit, $A_\phi = C(1-\cos\theta) + \mathcal{O}(a^2/M)(C\cos\theta\sin^2\theta)r^{-1} + \cdots$, obtained in the perturbation approach [1]. The difference is that the next-to-leading order is at r^{-2} for the former and it is at r^{-1} for the latter.

The equation for ψ_{-3} is

$$\begin{aligned}(L_\theta^2+6)\psi_{-3} \\ = 2\alpha M\cos\theta(3a\Omega_P^{-1} + 4Mr_+ - 8Mr_+ a\sin^2\theta\Omega_P).\end{aligned} \tag{48}$$

No analytical solution is found for the equation and so the calculation procedure cannot proceed.

Inserting the relations in Eq. (41) into the Znajek condition (23), we can find that the condition of ψ at the horizon is the same as the one at infinity, $\psi_+ = \psi_0$, as we mentioned in the previous section. This means that all higher-order terms ψ_{-n} $(1\le n<\infty)$ of a legal solution ψ must cancel out on the horizon, which is a constraint of the Znajek regularity condition.

At boundaries, the solution satisfies $\Omega_0 = \Omega_+ \ge \omega_+/2$, where the equality occurs at $\theta = 0$. Generally, the solution up to the second order also satisfies

$$\Omega(r,\theta) = \frac{a}{2Mr_+ + r_+^2 + a^2\cos^2\theta(1-\frac{1}{2}a^2\sin^2\theta r^{-2})^2} > \frac{1}{2}\omega_+. \tag{49}$$

This means that the magnetosphere in the valid regions is stable [4,17] against the screw instability [18]. Since the obtained solution is quite similar to the split monopole perturbation solution at large r, other properties about the solution will not be reconsidered here.

B. $\Omega_0 = \Omega_N$

1. The expanded solution

As stated previously, $\Omega = \Omega_N$ is the MD exact solution of the force-free magnetosphere on general rotating black holes. But this r-independent solution is unrealistic. It is interesting to investigate the situation by extending the solution to the r-dependent case through the expansion approach given above.

We take Ω_N as the initial value at the boundary to derive the r-dependent solution. Obviously, $\Omega_0 = \Omega_N$ is singular at the poles $\theta = 0, \pi/2$. Thus, we demand ψ_0 to be nonsingular by taking the simple form

$$\psi_0 = c\Omega_0^{-k} + d \quad (c>0, k>0), \tag{50}$$

where c, d, and k are constants. By choosing the positive sign in Eq. (30) instead, we have

$$I_0 = \frac{kc}{2\pi}\cos\theta\Omega_0^{1-k}, \tag{51}$$

since $\Omega_0 = 1/(a\sin^2\theta)$ is already faster than ω_+.

In terms of the relations among the zeroth-order functions, we can get the functional relations at general r:

$$\psi(\Omega) = c\Omega^{-k} + d, \tag{52}$$

$$I(\Omega) = \frac{kc}{2\pi}\sqrt{1-(a\Omega)^{-1}}\Omega^{1-k}, \tag{53}$$

which lead to

$$\frac{16\pi^2}{kc}II'\Omega^{2+k} = 4(k-1)\Omega^4 - \frac{2(2k-1)}{a}\Omega^3. \tag{54}$$

It is convenient for later calculations to redefine

$$\tilde{\Omega} \equiv \frac{\Omega}{\Omega_N} \quad \text{with} \quad \tilde{\Omega}_{-n}(\theta) \equiv \frac{\Omega_{-n}(\theta)}{\Omega_N}. \tag{55}$$

With the above relations, the stream equation can be expressed as

$$\begin{aligned}&\rho^2\tilde{\Omega}[A\tilde{\Omega}(\tilde{\Omega} - 2a\sin^2\theta\omega) - a^2\sin^2\theta(\rho^2-2Mr)](\triangle\partial_r^2\tilde{\Omega} + \partial_\theta^2\tilde{\Omega}) \\ &- \rho^2[A\tilde{\Omega}(k\tilde{\Omega} - (2k+1)a\sin^2\theta\omega) - (k+1)a^2\sin^2\theta(\rho^2-2Mr)][\triangle(\partial_r\tilde{\Omega})^2 + (\partial_\theta\tilde{\Omega})^2] \\ &+ 2\triangle\tilde{\Omega}[r\rho^4\tilde{\Omega}^2 - Ma^2\sin^2\theta(r^2-a^2\cos^2\theta)(\tilde{\Omega}-1)^2]\partial_r\tilde{\Omega} \\ &+ \cot\theta\tilde{\Omega}\{(4k-1)\rho^2[A\tilde{\Omega}(\tilde{\Omega}-2a\sin^2\theta\omega) - a^2\sin^2\theta(\rho^2-2Mr)] \\ &+ 2a^2\sin^2\theta A[\tilde{\Omega}(\tilde{\Omega}-2a\sin^2\theta\omega) + a\sin^2\theta\omega] - 2\rho^2(r^2+a^2)^2\tilde{\Omega}^2\}\partial_\theta\tilde{\Omega} \\ &+ 2[2(k-1)\rho^2(A - a^2(\rho^2+2Mr)) + 2Mr(r^2+a^2)(r^2-a^2\cos^2\theta) + \rho^4\triangle]\tilde{\Omega}^4 \\ &+ 2[(2k-1)\rho^2(4Mra^2\cos^2\theta - \rho^4) - 4Mra^2\sin^2\theta(r^2-a^2\cos^2\theta)]\tilde{\Omega}^3 \\ &+ 2[2(k-1)\rho^2a^2\cos^2\theta(\rho^2-2Mr) - a^2\sin^2\theta(\rho^4 - 2Mr(r^2-a^2\cos^2\theta))]\tilde{\Omega}^2 = 0.\end{aligned} \tag{56}$$

Inserting the expanded form of $\tilde{\Omega}$ into the above equation, we can get an expanded equation. The vanishing of the coefficients of r^{6-n} gives rise to the equation about $\tilde{\Omega}_{-n}$ $(n \geq 1)$, which can be formally expressed as

$$[L_\theta^2 + n(n-1)]\tilde{\Omega}_{-n} = F_{-n}(\tilde{\Omega}_0, \tilde{\Omega}_{-1}, \ldots, \tilde{\Omega}_{-(n-1)}) \quad (n \geq 1), \tag{57}$$

where

$$L_\theta^2 = \partial_\theta^2 + (4k-3)\cot\theta\partial_\theta + 2(2k-1). \tag{58}$$

The functions F_{-n} at order $-n$ are some functions of $\tilde{\Omega}_{-i}$ with $0 \leq i \leq n-1$.

The fact that $\Omega = \Omega_N$ is an exact solution to the full stream equation means that we can always have for all $n \geq 1$

$$[L_\theta^2 + n(n-1)]\tilde{\Omega}_{-n} = 0 \quad \text{with} \quad \tilde{\Omega}_{-n} = 0. \tag{59}$$

In what follows we shall consider more general (r-dependent) solutions other than this trivial case.

With $\Omega_0 = \Omega_N$, the vanishing of the terms at order r^6 is automatically satisfied, as expected. For the order r^5, the obtained equation is

$$L_\theta^2\tilde{\Omega}_{-1} = 0. \tag{60}$$

The equation has the simple solution

$$\tilde{\Omega}_{-1} = -2\alpha a\cos\theta, \tag{61}$$

where α is an arbitrary dimensionless constant. We assume that the solution applies to the upper hemisphere $\theta \in [0, \pi/2]$ since it is asymmetric about the equatorial plane.

Equation (60) has the second kind of solution, which is symmetric. The solution generally can be expressed in terms of the hypergeometric functions. But we can have their explicit forms when $4k-3$ is an odd number. For example, the solution for $k = 3/2$ is

$$\tilde{\Omega}_{-1} = -\beta a\left[1 - \frac{1}{2}\cot^2\theta + \frac{3}{4}\cos\theta\ln\frac{1-\cos\theta}{1+\cos\theta}\right], \tag{62}$$

where β is a dimensionless constant. This solution is symmetric under $\cos\theta \to -\cos\theta$. But the solution forms closed magnetic field lines, which is excluded for a force-free magnetosphere [14,19]. So this solution is abandoned.

At order r^4, the resulting equation about Ω_{-2} can be simply reduced by inserting the solution (61):

$$(L_\theta^2 + 2)\tilde{\Omega}_{-2} = 4k\alpha^2 a^2. \tag{63}$$

A simple solution to this equation is

$$\tilde{\Omega}_{-2} = \alpha^2 a^2. \tag{64}$$

The equation for $\tilde{\Omega}_{-2}$ [obtained by inserting the second solution (62)] is difficult to solve and is not considered.

The vanishing of the coefficients of r^3 leads to the equation about Ω_{-3}, which can be reduced to

$$(L_\theta^2 + 6)\tilde{\Omega}_{-3} = 4\alpha^2 Ma^2(1 - 2k\cos^2\theta) + 4\alpha a^3\cos\theta[3 + 2(1-2k)\cos^2\theta]. \tag{65}$$

When $k \neq 3/2$, a solution to this equation is

$$\tilde{\Omega}_{-3} = 2\alpha a^3\cos^3\theta + \frac{\alpha^2 Ma^2}{2k-3}\left(4k\cos^2\theta - \frac{3}{k+1}\right). \tag{66}$$

The solution at the critical value $k = 3/2$ is not found.

The solutions at higher orders are hard to derive due to the involvement of many more terms. But we can perform a simple analysis based on Eq. (56) and the derived solutions above. For higher orders, we can find that the function F_{-n} for each n in Eq. (57) should be some polynomial of $\cos\theta$:

$$F_{-n}(\theta) = \sum_i^n f_i(\alpha, k)\cos^i\theta, \tag{67}$$

where f_i are coefficients and are of order m^n, as pointed out in Eq. (37). So Eq. (57) with the form of F_{-n} should be solvable. This implies that an exact solution may eventually be obtained or guessed by following the procedure if we could successfully handle all of the terms to higher enough orders.

At the moment, the above solution up to the first few orders should be valid for asymptotical regions far away from the horizon. The solution is consistent with our near-horizon solution for near-extreme black holes [12]. It can be checked that the solution forms open magnetic field lines, which may be separated by a current sheet on the equatorial plane, just like the split monopole solution.

2. Analysis of the solution

Our solution generalizes the MD exact solution $\Omega = \Omega_N$ [5,6] to the (r, θ)-dependent case. The MD exact solution is taken as an initial condition at both boundaries and is recovered from the generalized solution when the parameter $\alpha = 0$. The exact solution has difficulties describing a realistic magnetosphere since its four-current and the electromagnetic field are both null. Here we examine the situation for our generalized solution.

Before doing that, we first determine the conditions for which the quantities from the solution are not singular on the poles $\theta = 0$, which are summarized in Table I. For $k \geq 3/2$, all of the quantities in the table (as well as the electromagnetic fields) are nonsingular on the poles.

TABLE I. The conditions of k for the corresponding quantities to be nonsingular on the rotation axis.

ψ	$\mathcal{L}^\theta$	I, $\mathcal{L}^r$	$\mathcal{E}^\theta$	$\mathcal{E}^r$
$k \geq 0$	$k \geq \frac{3}{4}$	$k \geq 1$	$k \geq \frac{5}{4}$	$k \geq \frac{3}{2}$

The existence of a magnetosphere in all frames requires that it should be magnetically dominated, i.e., the invariant should be

$$F^2 = 2(\mathbf{B}^2 - \mathbf{E}^2) > 0. \tag{68}$$

Inserting the solution into the expressions, we can get the invariant. We find that its sign is strongly affected by the parameter k but it is not sensitive to other parameters like α and a. As shown in Fig. 1, the invariant F^2 is positive for $k < 3/2$, while it is negative for $k > 3/2$ (the case $k = 3/2$ cannot be judged since the solution is not available here). This indicates that the magnetic fields can be dominated only when (part of) the quantities are singular on the poles. As expected, the values all asymptotically approach 0 at large r as the (r, θ)-dependent solution recovers the MD exact solution at the far boundary.

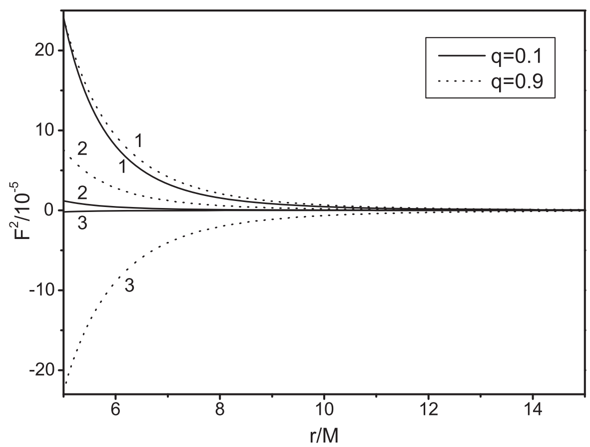

FIG. 1. Illustrations of the invariant F^2 at different radial distances r and different poloidal angles $\theta = q\pi/2$. The parameter $c = 1$ and the spin parameter $a = 0.8M$. The line groups are (1) $\alpha = 1$ and $k = 1.3$, (2) $\alpha = 0.1$ and $k = 1.49$, and (3) $\alpha = 1$ and $k = 2$.

Whether the four-current J^μ is timelike, lightlike, or spacelike can be determined by determining whether $J^2 = J_\mu J^\mu$ [contracted by the four-dimensional metric (1)] is negative, null, or positive, respectively. The contracted current is related to the charge and current densities measured in ZAMOs via

$$J^2 = -\rho_e^2 + \mathbf{j} \cdot \mathbf{j}, \tag{69}$$

with their components satisfying

$$\frac{1}{\Lambda}\rho_e = J^0, \qquad j^r = J^r, \tag{70}$$

$$j^\theta = J^\theta, \qquad j^\phi = -\omega J^0 + J^\phi. \tag{71}$$

The three-current $\mathbf{j}$ is contracted by the metric (3) of the absolute space.

The sign of J^2 is also sensitive to k, as shown in Fig. 2. For $k > 3/2$, the values of J^2 are almost all positive at all angles θ. For $k < 3/2$, they are not always positive and are negative for larger θ, i.e., near the equatorial plane. The only case in which its values are mostly negative happens when $k \to 3/2$ from the $k < 3/2$ side. The case $k = 1.49$ (to regularize $\tilde{\Omega}_3$ to be not too large, we adopt a small $\alpha = 0.1$) is shown in the right panel of Fig. 2. It can be seen that the values of J^2 grow with increasing θ from negative values at the small angle $\theta = 0.01\pi/2$, and become slightly positive at around $\theta = \pi/4$. Then they become negative again for larger angles. It can be checked that the values of

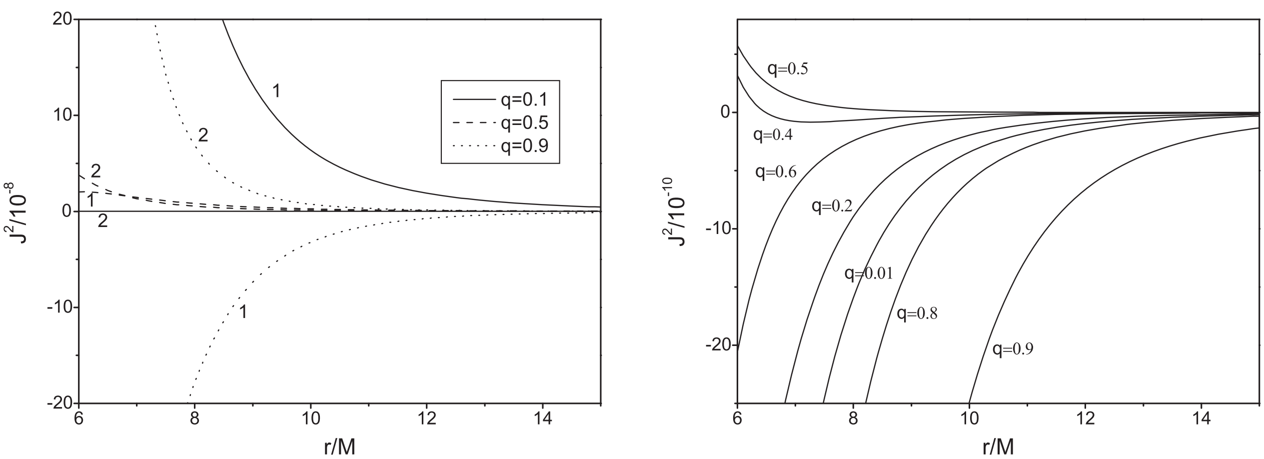

FIG. 2. Illustrations of J^2 at different distances r and different angles $\theta = q\pi/2$. The parameters are chosen to be $c = 1$ and $a = 0.8M$. Left panel: (1) $\alpha = 1$ and $k = 1.3$, and (2) $\alpha = 1$ and $k = 3$. Right panel: $\alpha = 0.1$ and $k = 1.49$.

J^2 are also negative for angles smaller than $\theta = 0.01\pi/2$. But they all will tend to be null, $J^2 = 0$, at exactly $\theta = 0$.

VI. SUMMARY

In this work, we adopted a new expansion method to explore analytical solutions of force-free magnetospheres on black holes with an arbitrary spin parameter. The functions and stream equation were expanded in series of the radial distance in terms of the boundary conditions at the event horizon and at spatial infinity. With the conditions at the two boundaries chosen, a set of solutions can be defined and solved order by order.

In terms of the regular conditions at both boundaries, the two asymptotical solutions found by MD in Refs. [5,6] were identified as the solutions that have the same conditions at the two boundaries. By taking them as initial conditions at the boundaries, we derived the corresponding expanded solutions to higher orders. The first one corresponds to the split monopole solution obtained in the perturbation approach when we take the $a \to 0$ limit. It was found to have a similar asymptotical profile to the latter at large r, though not with the same r dependence.

The second solution can be viewed as an extension of the r-independent MD exact solution to the (r, θ)-dependent case. With an appropriate choice of the relation between ψ and Ω at the far boundary, we found that the expanded stream equation should be solvable at each order. So an exact solution (probably with a closed form) can hopefully be derived or guessed if we could calculate to all or high enough orders, though we only derived the expanded solution up to the first few orders in this work.

Based on the obtained solution, we showed that the extended solution can (partially) avoid the problems of the r-independent MD solution: the four-current and the electromagnetic field are both null. When the parameter k tends to the critical value $3/2$ from the $k < 3/2$ side, our solution leads to a force-free magnetosphere which is magnetically dominated with a timelike current in most directions θ. The current becomes slightly spacelike at around $\theta = \pi/4$ and lightlike at exactly $\theta = 0$. A difficulty for the solution with k less than and close to $3/2$ is that the energy extraction (integration of $\mathcal{E}^r$) highly converges along the rotation axis in a singular way. Similar singular behaviors also exist in the relieving method [8,9]. But, in our case, the singular mode is very slight for $k \to 3/2$. Nevertheless, we may still have to exclude the $\theta = 0$ direction or assume that the force-free condition is violated by dense plasma in this region.

As we can see, the solution with $k = 3/2$ is an interesting case, but it was not found in this work and is left for a future study. We also need to derive the expanded solution to higher orders and to check whether the results from the present solution still hold (or even improve). Moreover, more varieties of the relation between ψ_0 and Ω_0 other than Eq. (50) are under consideration to find saturated results.

ACKNOWLEDGMENTS

This work is partially supported by the National Natural Science Foundation of China 11573060, 11661161010.

APPENDIX: SOLUTIONS ON A SCHWARZSCHILD BLACK HOLE

A detailed discussion of exact solutions of magnetospheres on Schwarzschild black holes can be found in Ref. [20]. Here, we use the expansion method in the text to rederive the solutions. The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad \text{(A1)}$$

where the horizon is located at $r_0 = 2M$.

In the nonrotating case, the stream equation reduces to

$$x^2 \partial_x[(1 - x^{-1})\partial_x \psi] + L_\theta^2 \psi = 0, \quad \text{(A2)}$$

where

$$x \equiv \frac{r}{r_0}, \quad \text{(A3)}$$

$$L_\theta^2 = \sin\theta \partial_\theta(\sin^{-1}\theta \partial_\theta) = \partial_\theta^2 - \cot\theta \partial_\theta. \quad \text{(A4)}$$

Here, we take ψ to be dimensionless to simplify the notation. This equation is essentially Maxwell's equation on a Schwarzschild black hole in the absence of sources, i.e., $\rho_e = \mathbf{j} = 0$.

So the force-free condition (9) is trivial here and does not really provide any extra constraint. We need to find alternative boundary conditions. Let us adopt the ansatz of a general solution:

$$\psi(x, \theta) = \psi_1(\theta)x + \psi_* \ln x + \psi_0(\theta) + \psi_{-1}(\theta)x^{-1} + \psi_{-2}(\theta)x^{-2} + \cdots. \quad \text{(A5)}$$

We choose this ansatz to guarantee that the electromagnetic fields vanish at $x \to \infty$.

Inserting the expanded form into the equation and comparing the coefficients of each order of x, we get the following equations:

$$L_\theta^2 \psi_1 = 0, \quad \text{(A6)}$$

$$L_\theta^2 \psi_* = 0, \quad \text{(A7)}$$

$$L_\theta^2 \psi_0 = \psi_* - \psi_1, \quad \text{(A8)}$$

$$(L_\theta^2 + 2)\psi_{-1} = -2\psi_*, \quad \text{(A9)}$$

$$[L_\theta^2 + n(n+1)]\psi_{-n} = (n^2 - 1)\psi_{-(n-1)} \quad (n \geq 2). \quad \text{(A10)}$$

1. Nonseparable solutions

The first four equations (A6)–(A9) are closed and complete, and thus they give exact solutions. We first set the coefficients $\psi_{-n} = 0$ $(n \geq 1)$ so that $\psi_* = 0$ since they are always solutions. Then, from Eq. (A6) a solution of ψ_1 can be written in the form

$$\psi_1 = 1 - \cos\theta. \tag{A11}$$

With this expression, Eq. (A8) can be expressed as

$$\partial_y^2 \psi_0 = -\frac{1}{y}, \tag{A12}$$

where $y = 1 + \cos\theta$. A general solution to this is

$$\psi_0 = \alpha + \beta\cos\theta - (1 + \cos\theta)\ln(1 + \cos\theta), \tag{A13}$$

where α and β are constants. When $-\alpha = \beta = 1$, the full solution is

$$\psi = (x - 1)(1 - \cos\theta) - (1 + \cos\theta)\ln(1 + \cos\theta), \tag{A14}$$

which is exactly the nonseparable solution [20].

2. Separable solutions

a. Zeroth-order solution

From Eqs. (A6)–(A8), we can impose the general solution

$$\psi_1 = \psi_* = b + c\cos\theta, \qquad \psi_0 = d\psi_1 + \alpha\cos\theta + \beta, \tag{A15}$$

where b, c, d, and e are constants. So we have

$$\psi_{-n} = -\frac{1}{n}\psi_1 \quad (n \geq 1). \tag{A16}$$

Adopting the relation $\ln(1 - z) = -\sum_{n=1}^{\infty} z^n/n$, we can express the solution as

$$\psi = \alpha\cos\theta + \beta + (b + c\cos\theta)[d + x + \ln(x - 1)]. \tag{A17}$$

This is the lowest-order separable solution with $m = 0$ given in Ref. [20]. The case $b = c = 0$ is the (split) monopole solution.

b. First-order solution

From the first three equations (A6)–(A8), we consider the case

$$\psi_1 = \psi_* = 0 \tag{A18}$$

and

$$\psi_0 = \alpha\cos\theta + \beta. \tag{A19}$$

Then, Eq. (A9) becomes $L_\theta^2 \psi_{-1} = -2\psi_{-1}$. So the general solution of ψ_{-1} can be

$$\psi_{-1} = g\sin^2\theta, \tag{A20}$$

where g is an arbitrary constant. Thus, we can have generically from Eq. (A10)

$$\psi_{-n} = \frac{3g}{n + 2}\sin^2\theta \quad (n \geq 2). \tag{A21}$$

By using the expansion expression of $\ln(1 - z)$, we can express the full solution as

$$\psi = \alpha\cos\theta + \beta - 3g\sin^2\theta\left[\frac{1}{2} + x + x^2\ln\left(1 - \frac{1}{x}\right)\right]. \tag{A22}$$

The solution with $\alpha = \beta = 0$ is clearly the separable solution at the order $m = 1$ given in Ref. [20].

c. Higher-order solutions

If we consider the case $\psi_1 = \psi_* = \psi_0 = \psi_{-1} = 0$, then Eq. (A10) becomes $L_\theta^2 \psi_{-2} = -6\psi_{-2}$. Its solution is $\psi_{-2} = 3h\cos\theta\sin^2\theta$. We can then insert the solution into the general ψ_{-n}. Following the same approach above, we can derive the separable solution at the $m = 3$ order. Similarly, we can derive all higher-order separable solutions.

[1] R. Blandford and R. Znajek, Electromagnetic extractions of energy from Kerr black holes, Mon. Not. R. Astron. Soc. **179**, 433 (1977).

[2] R. Znajek, Black hole electrodynamics and the Carter tetrad, Mon. Not. R. Astron. Soc. **179**, 457 (1977).

[3] K. Tanabe and S. Nagataki, Extended monopole solution of the Blandford-Znajek mechanism: Higher order terms for a Kerr parameter, Phys. Rev. D **78**, 024004 (2008).

[4] Z. Pan and C. Yu, Fourth-order split monopole perturbation solutions to the Blandford-Znajek mechanism, Phys. Rev. D **91**, 064067 (2015).

[5] G. Menon and C. D. Dermer, Analytic solutions to the constraint equation for a force-free magnetosphere around a kerr black hole, Astrophys. J. **635**, 1197 (2005).

[6] G. Menon and C. D. Dermer, A class of exact solution to the Blandford-Znajek process, Gen. Relativ. Gravit. **39**, 785 (2007).

[7] T. D. Brennan, S. E. Gralla, and T. Jacobson, Exact solutions to force-free electrodynamics in black hole backgrounds, Classical Quantum Gravity **30**, 195012 (2013).

[8] G. Menon and C. D. Dermer, Jet formation in the magnetospheres of supermassive black holes: analytic solutions describing energy loss through Blandford-Znajek processes, Mon. Not. R. Astron. Soc. **417**, 1098 (2011).

[9] G. Menon and C. Dermer, Local, non-geodesic, timelike currents in the force-free magnetosphere of a Kerr black hole, Gen. Relativ. Gravit. **47**, 52 (2015).

[10] A. Lupsasca, M. J. Rodriguez, and A. Strominger, Force-free electrodynamics around extreme Kerr black holes, J. High Energy Phys. 12 (2014) 185.

[11] A. Lupsasca and M. J. Rodriguez, Exact solutions for extreme black hole magnetospheres, J. High Energy Phys. 07 (2015) 090.

[12] H. Li, C. Yu, J. Wang, and Z. Xu, Force-free magnetosphere on near-horizon geometry of near-extreme Kerr black holes, Phys. Rev. D **92**, 023009 (2015).

[13] F. Zhang, H. Yang, and L. Lehner, Towards an understanding of the force-free magnetosphere of rapidly spinning black holes, Phys. Rev. D **90**, 124009 (2014).

[14] D. MacDonald and K. S. Thorne, Black-hole electrodynamics: An absolute-space/universal-time formulation, Mon. Not. R. Astron. Soc. **198**, 345 (1982).

[15] A. Nathanail and I. Contopoulos, Black hole magnetospheres, Astrophys. J. **788**, 186 (2014).

[16] M. Ruiz, C. Palenzuela, F. Galeazzi, and C. Bona, The role of the ergosphere in the Blandford-Znajek process, Mon. Not. R. Astron. Soc. **423**, 1300 (2012).

[17] A. Tomimatsu, T. Matsuoka, and M. Takahashi, Screw instability in black hole magnetospheres and a stabilizing effect of field line rotation, Phys. Rev. D **64**, 123003 (2001).

[18] L.-X. Li, Screw instability and Blandford-Znajek mechanism, Astrophys. J. **531**, L111 (2000).

[19] S. E. Gralla and T. Jacobson, Spacetime approach to force-free magnetospheres, Mon. Not. R. Astron. Soc. **445**, 2500 (2014).

[20] P. Ghosh, The structure of black hole magnetospheres. 1. Schwarzschild black holes, Mon. Not. R. Astron. Soc. **315**, 89 (2000).