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Instantons from perturbation theory

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In quantum mechanics and quantum field theory perturbation theory generically requires the inclusion of extra contributions nonperturbative in the coupling, such as instantons, to reproduce exact results. We show how full nonperturbative results can be encoded in a suitable modified perturbative series in a class of quantum mechanical problems. We illustrate this explicitly in examples which are known to contain nonperturbative effects, such as the (supersymmetric) double-well potential, the pure anharmonic oscillator, and the perturbative expansion around a false vacuum.

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The coefficients of saddle-point expansions of (path) integrals generically grow factorially and produce nonconvergent asymptotic series. In special cases exact results can be obtained from such a series by "Borel resummation," consisting in taking the Laplace transform of the function obtained by resumming the original series after dividing their terms by a factorially growing coefficient. However this procedure generically fails, because exact results take the form of trans-series, i.e. series in powers of the coupling λ , $e^{-1/\lambda}$ and $\log(-\lambda)$, which capture behaviors nonperturbative in the coupling λ .

There are several well understood tools which determine the properties of perturbative series of finite-dimensional integrals. For path integrals some results are available in specific cases (mostly within quantum mechanics or supersymmetric theories using localization techniques) but a systematic characterization of the behavior of their perturbation theories is still lacking. Quantum mechanics (QM) represents the simplest playground to test our understanding of the interplay between perturbative and nonperturbative effects in path integrals, for exact results can be extracted efficiently by other means. In this paper we present a method to recover the full nonperturbative answer from a deformation of the perturbative series in a certain class of OM problems that contain nonperturbative effects. Such a technique could be used to improve numerical computations based on perturbation theory in QM and might, in principle, be extended to higher dimensional quantum field theories (QFT).

Consider one-dimensional quantum mechanical systems described by the Hamiltonian

$$H = \frac{p^2}{2} + V(x;\lambda), \tag{1}$$

where the potential $V(x; \lambda)$ is a regular function of x describing a bounded system [i.e. $\lim_{|x|\to\infty} V(x) = \infty$]. If the dependence on the coupling λ is such that $V(x; \lambda) =$

 $V(x\sqrt{\lambda}; 1)/\lambda$ then the perturbative expansion in λ coincides with the \hbar expansion. We call such potential *classical*. If $V_0(x; \lambda)$ and $V_1(x; \lambda)$ are two such potentials then the combination $V(x; \lambda) = V_0(x; \lambda) + \lambda V_1(x; \lambda)$ is a sum of a classical contribution (V_0) and a quantum one (V_1).

Consider now the classical anharmonic oscillator

$$V^{ao}(x;\lambda) = \frac{1}{2}x^2 + \frac{\lambda}{2}x^4,$$
 (2)

whose energy eigenvalues at small λ are close to those of the harmonic oscillator $E_n^{ao} = n + \frac{1}{2} + O(\lambda)$. By studying the analytic properties of the eigenvalues, their perturbative series has been shown to be Borel resummable to the exact result [1,2]. The conditions under which Borel resummability holds can be characterized more systematically [3], extending the result to other potentials and observables. Similar results seem to hold also in higher dimensions, i.e. $\lambda \phi^4$ theories in two and three dimensions [4,5].

Note that the Borel resummability of the theory crucially depends on the presence and sign of the quadratic term x^2 . In fact, in the case of the double-well potential, where the quadratic term is negative, perturbation theory in λ around the minima is known to be non-Borel resummable. Nonperturbative effects, instantons, are needed to cure the ambiguities of the Borel transform and to reproduce the exact answer. A particularly interesting example is the supersymmetric double-well potential where perturbative corrections to the vacuum energy vanish at all orders, making the perturbative series trivially Borel resummable. Still supersymmetry is broken nonperturbatively [6] and the vacuum energy is lifted by instanton contributions.

For the special case of vanishing mass term, i.e. for the *pure* anharmonic oscillator

$$V^{pa}(x) = \frac{1}{2}x^4,$$
 (3)

the situation is more subtle. This case corresponds to the strong coupling limit, $\lambda \rightarrow \infty$, of the anharmonic

potential, obtained after the rescaling $x \to x/\lambda^{1/6}$, $p \to p\lambda^{1/6}$, and $H \to H/\lambda^{1/3}$. As such, normal perturbation theory (i.e. perturbation theory around the harmonic oscillator) cannot be used. In this special case a different semiclassical expansion in \hbar , the WKB approximation, is more useful. Not only such perturbative expansion is not Borel resummable but real instantons are not enough to reproduce the full answer [7]—complex saddles are also required to recover the full result (see [8] for a recent reanalysis).

In this work we present a different approach, which instead uses the normal perturbation theory and exploits the Borel resummability of the anharmonic potential, to show that the full result can be recovered by a single perturbative series (with no need of trans-series) in all the cases above, even at strong coupling ($\lambda \gtrsim 1$).

We will make use of the following property. If $V_0(x; \lambda)$ is a classical potential with a perturbation theory which is Borel resummable to the exact result (as in the case of the anharmonic oscillator), the perturbative series of a QM system with potential $V(x; \lambda) = V_0(x; \lambda) + \lambda V_1(x; \lambda)$ is still Borel resummable to the exact result for any V_1 if $\lim_{|x|\to\infty} V_1(x; 1)/V_0(x; 1) = 0$. We will provide a derivation of this property in [3] using the Leftschetz-thimbles techniques along the lines of [9], in the current work we limit ourselves to provide numerical evidence in a number of nontrivial examples.

The property above allows us to compute a certain class of QM problems through a Borel resummable perturbative expansion even if the initial naive \hbar expansion was not. Imagine we have a system with potential $V(x; \lambda)$ and we want to compute a quantity at some value of the coupling $\lambda = \lambda_0$. If we can find a potential $\hat{V}(x; \lambda) = V_0(x; \lambda) + \lambda V_1(x; \lambda)$ satisfying the conditions above and such that $\hat{V}(x; \lambda_0) = V(x; \lambda_0)$ we will have a way to extract the full answer by computing the perturbative expansion in λ of the modified potential $\hat{V}(x; \lambda)$ and setting $\lambda = \lambda_0$ after the Borel resummation. We name such expansion "exact perturbation theory" (EPT), while we refer to the trans-series originating from the potential $V(x; \lambda)$ as "standard perturbation theory" (SPT).

In practical computations where the perturbative series is truncated at a fixed order, the accuracy of EPT depends on the choice of the potential \hat{V} and on the value of the coupling constant.

As a first application consider the anharmonic potential in Eq. (2) perturbed at the quantum level by a negative mass term, namely

$$\hat{V}^{pa}(x;\lambda) = \left[\frac{1}{2}x^2 + \frac{\lambda}{2}x^4\right] - \lambda \left[\frac{1}{2}x^2\right].$$
(4)

At $\lambda = 1$ the potential $\hat{V}^{pa}(x; 1)$ reduces to the potential $V^{pa}(x)$ of the pure anharmonic oscillator. According to our

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criterion a Borel resummation of the perturbative series of $\hat{V}^{pa}(x;\lambda)$ computed at $\lambda = 1$ should therefore reproduce the exact results of the pure anharmonic potential. Perturbative coefficients of eigenvalues and eigenfunctions in QM can be efficiently computed using the recursion relations by Bender and Wu [10,11].

Using only the first 10 orders of perturbation theory (and performing a Padé-Borel resummation) the estimate for the ground state energy $E_0^{pa} = 0.5302$ matches the accuracy obtained by the WKB expansion of [8], using 320 orders of perturbative expansions and including the leading non-perturbative complex and real saddle contributions. The accuracy can easily be improved by including higher-order terms without the need of extra nonperturbative effects; for example with 320 orders of perturbation theory the accuracy reaches the 10^{-46} level. The result agrees with that obtained using Rayleigh-Ritz methods (see e.g. [12]), which remains however the most efficient for this type of problems.

Few remarks are in order. The pure anharmonic potential is a genuinely nonperturbative problem, in the sense that there is no small parameter to expand around. The modified potential $\hat{V}(x,\lambda)$ admits a perturbative expansion but it coincides with the pure anharmonic potential only at strong coupling $\lambda = 1$. Still the Borel resummability of the theory allows us to reconstruct the full answer from the perturbative terms alone without having to include nonperturbative effects (unlike the WKB expansion). The precision that can be achieved in this way is remarkable, especially considering that we are working at $\lambda = 1$. We should say that the WKB approach is particularly weak for the ground state while it improves considerably for higher energy states, still the expansion is not Borel resummable and the full answer requires the inclusion of nontrivial saddles. EPT instead does not require extra nonperturbative contributions and the full answer is always contained in the perturbative series. We checked numerically also the wave function as well as higher excited states, which can be computed with similar high precision (although the accuracy slowly reduces for higher energy states), confirming that indeed all observables are Borel resummable to the exact result. We also checked numerically other pure anharmonic potentials of the type $V(x) = x^{2n}$ with n > 2.

We now turn to the discussion of the ground state eigenvalue of the supersymmetric double well or Fokker-Planck potential,

$$V^{sdw}(x;\lambda) = \frac{\lambda}{2} \left[x^2 - \frac{1}{4\lambda} \right]^2 + \sqrt{\lambda}x, \qquad (5)$$

which is the sum of a classical symmetric double-well potential (first term) plus a quantum linear tilt correction (second term). The latter can be interpreted as the contribution to the classical bosonic potential from integrating

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out the fermionic variables. Because of supersymmetry the ground state energy vanishes at all orders in λ . Supersymmetry is however known to be broken nonperturbatively [6]. At weak coupling the leading contribution comes from instantons interpolating the two classical vacua at $x = \pm 1/\sqrt{4\lambda}$. This contribution has been computed at high orders in [13] using a generalized Bohr-Sommerfeld quantization formula [14] and the result agrees extremely well at weak coupling with the exact answer from nonperturbative numerical computations such as Raylegh-Ritz methods. While the instanton computation is particularly efficient at weak coupling it becomes useless at strong coupling—for $\lambda \gtrsim 1$ the perturbative expansion around the leading instanton solution is divergent and non-Borel resummable, an infinite series of multi-instanton solutions and the corresponding perturbative series have to be properly resummed altogether, making the full approach impractical. The presence of such instantons may seem puzzling since the full quantum corrected potential (5) has no nontrivial finite-action solutions. This fact led several authors [15,16] to reinterpret the nonperturbative effects in terms of complex instantons. From our point of view however there is no puzzle: The asymptotic properties of the λ expansion are determined only by the classical part of Eq. (5), which is a symmetric double-well potential known to have a non-Borel resummable perturbative expansion and requiring the inclusion of the corresponding (real) instantons. For the ground state energy the perturbative saddle-point expansion around the minimum of the full quantum corrected potential (5) is instead Borel resummable to the exact result [3].

Consider now the potential

$$\hat{V}^{sdw}(x;\lambda,\lambda_0) = \left[\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4\right] + \sqrt{\lambda}x - \lambda \left[1 + \frac{1}{2\lambda_0}\right]\frac{x^2}{2}, \quad (6)$$

which for $\lambda_0 = \lambda$ reduces to Eq. (5), i.e. $\hat{V}^{sdw}(x; \lambda, \lambda) = V^{sdw}(x; \lambda)$. Expanding in λ at fixed λ_0 the terms in the first line of Eq. (6) represent the classical part of the potential and correspond to a Borel-resummable anharmonic oscillator. Similarly to the case of the pure anharmonic oscillator we conclude that the perturbative expansion in λ (at fixed λ_0) is Borel resummable to the exact result for any λ . Setting $\lambda = \lambda_0$ we thus recover the result for the supersymmetric double well.

Note that the choice of $\hat{V}^{sdw}(x; \lambda, \lambda_0)$ above is particularly good at moderate and large values of λ_0 but is not ideal for small values of the coupling. In the latter regime, nonperturbative effects become exponentially small and anyway harder to be resolved, as discussed below.

In Fig. 1 we compare the accuracy of SPT from the leading instanton contribution of $V^{sdw}(x; \lambda)$ computed in

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FIG. 1. The ground state energy (top) and the relative error (bottom) as a function of the coupling λ for the supersymmetric double-well potential (5) computed using the SPT up to the ninth order from [13] (orange dots) and using our EPT (blue crosses). The exact result (black line) has been computed via a Rayleigh-Ritz method.

[13] against EPT using 200 orders and Padé-Borel resummation. At small coupling SPT becomes more accurate, since the leading order instanton formula provides the asymptotic $\lambda \rightarrow 0$ limit of the result. On the other hand EPT is less and less accurate at weak coupling-such expansion is not supersymmetric, at $\lambda = \lambda_0$ perturbative corrections at each order cancel up to the next order, leading to an asymptotic series whose Borel resummation encodes the full nonperturbative result. This means that at small coupling more orders of perturbation theory are required to capture the exponentially small nonperturbative effects, with a corresponding weakening of accuracy. Notice however that there is a nontrivial interval of small couplings where both SPT and EPT are under control and agree at the 10^{-4} level. At strong coupling on the other hand EPT works extremely well. Paradoxically, while the instanton calculation only works in the perturbative limit, EPT works best in the nonperturbative regime.

We can repeat a similar analysis for the nonsupersymmetric double-well potential

$$V^{dw}(x;\lambda) = \frac{\lambda}{2} \left[x^2 - \frac{1}{4\lambda} \right]^2.$$
(7)

In this case the ground state energy receives corrections also from the perturbative expansion in λ , which is not Borel resummable. The combination of the perturbative corrections and the instanton ones requires some care, see e.g. [17]. As usual this approach works better and better at weak coupling while it breaks down at strong coupling. As before we can recover the same result by considering, for example, the potential

$$\hat{V}^{dw}(x;\lambda,\lambda_0) = \left[\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4\right] - \lambda \left[1 + \frac{1}{2\lambda_0}\right]\frac{x^2}{2},\qquad(8)$$

whose classical part (in the first brackets) corresponds to a Borel resummable anharmonic oscillator and it matches the potential (7) for $\lambda = \lambda_0$. Using 300 orders of perturbation theory at $\lambda = \lambda_0 = 0.03$ we get a ground state energy $E_0^{dw} =$ 0.4546(15), which differs from the exact value (0.4531) only by 1.5×10^{-3} . On the other hand, the difference between the two lowest energies, which represents the size of the instanton contribution and the level of precision that can be reached by just truncating the perturbative asymptotic series, is 0.02, i.e. 1 order of magnitude larger. The accuracy rapidly improves increasing the coupling. Already at $\lambda = 0.04$, the difference between EPT and the exact answer drops to 3×10^{-9} , while the instanton contributions amount to 0.06. It is thus clear that EPT is able to correctly reproduce the full answer, automatically combining perturbative and multi-instanton contributions at weak coupling. Considerations similar to those of the supersymmetric double well apply regarding the complementarity of SPT versus EPT: while the former is more powerful at weak coupling the latter extends also at strong coupling. Furthermore note that observables that receive both a perturbative and a nonperturbative contribution in SPT (as the ground state energy) require a nontrivial resummation before the instanton contributions could improve the accuracy.

The final example we present, although very similar to the previous ones, is maybe the most amusing from a theoretical point of view. Consider the potential

$$V^{fv}(x;\lambda) = \frac{1}{2}x^2 - \frac{3}{2}\sqrt{\lambda}x^3 + \frac{\lambda}{2}x^4,$$
 (9)

corresponding to an asymmetric double well with a false vacuum at x = 0 and a true one at $x = 2/\sqrt{\lambda}$. Perturbation theory around the false vacuum is known to be non-Borel resummable, with the corresponding ambiguity related to the instanton describing the tunneling to the true vacuum. Information about the true vacuum is completely nonperturbative from the point of view of the perturbative expansion around the false vacuum. Consider now the potential

$$\hat{V}^{fv}(x;\lambda,\lambda_0) = \left[\frac{1}{2}x^2 + \frac{\lambda}{2}x^4\right] - \lambda \left[\frac{3}{2}\frac{\sqrt{\lambda}}{\lambda_0}x^3\right], \quad (10)$$

whose classical part is a Borel resummable anharmonic oscillator around the false vacuum. At small coupling the

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potential is an asymmetric anharmonic oscillator with a unique vacuum at x = 0. As λ increases a new vacuum develops becoming deeper than the one at the origin. At $\lambda = \lambda_0$ the potential coincides with the one in Eq. (9). Analogously to the previous cases we therefore expect that the perturbative λ expansion around the origin of $\hat{V}^{fv}(x; \lambda, \lambda_0)$ reproduces the exact results of the potential $V^{fv}(x; \lambda)$ for $\lambda = \lambda_0$ when the perturbative series is properly Borel resummed.

At $\lambda = 1$ the ground state energy is $E_0^{fv} = -0.828$, deeply below zero. Using $\hat{V}^{fv}(x;\lambda,1)$ with 280 orders of perturbation theory and performing a Padé-Borel resummation we get the estimate $\hat{E}_0^{fv} = -0.847$. This result is in quite good agreement with the exact value, especially considering that the ground state energy of the corresponding classical anharmonic potential around the false vacuum is large and positive (+0.7). Despite in this example the accuracy is not on par with the previous cases, it is notable that perturbation theory around the false vacuum can still be used to recover a completely nonperturbative result, such as the energy of the true vacuum, with a few percent accuracy.

The examples above show how, in a class of QM problems, a suitable modification of the perturbative expansion allows us to recover all nonperturbative effects from perturbation theory alone, in the spirit of the resurgence program [18], but without the need of trans-series. The high accuracy (especially at strong coupling) of the numerical checks presented above confirms the absence of extra contributions not accounted for by the EPT. The success of this expansion can be understood more easily by extending Lefschetz-thimble techniques to the QM path integral [3], suggesting that similar methods might also apply to more general theories. Complications such as ultraviolet divergences and the possible presence of phase transitions could make the extension to QFT less straightforward. However a certain class of nontrivial QFT has been proven to be Borel resummable [4,5] and may serve as a starting point (the analogue of V_0) to apply the technique to more general theories. The idea of EPT may then represent a new avenue for the study of strongly coupled theories, by allowing all nonperturbative effects to be recovered from perturbation theory alone.

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