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Quark and gluon production from a boost-invariantly expanding color electric field

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Particle production from an expanding classical color electromagnetic field is extensively studied, motivated by the early stage dynamics of ultrarelativistic heavy ion collisions. We develop a formalism at one-loop order to compute the particle spectra by canonically quantizing quark, gluon, and ghost fluctuations under the presence of such an expanding classical color background field; the canonical quantization is done in the τ-η coordinates in order to take into account manifestly the expanding geometry. As a demonstration, we model the expanding classical color background field by a boost-invariantly expanding homogeneous color electric field with lifetime T, for which we obtain analytically the quark and gluon production spectra by solving the equations of motion of QCD nonperturbatively with respect to the color electric field. In this paper we study (i) the finite lifetime effect, which is found to modify significantly the particle spectra from those expected from the Schwinger formula; (ii) the difference between the quark and gluon production; and (iii) the quark mass dependence of the production spectra. Implications of these results to ultrarelativistic heavy ion collisions are also discussed.

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I. INTRODUCTION

Early stage dynamics of ultrarelativistic heavy ion collisions (HIC) is a big missing piece in our current understanding of the spacetime evolution of HIC: Before a collision, two incident nuclei at very high energies are saturated with a huge number of gluons, which behave like coherent classical color fields [color glass condensate (CGC) picture [1–4]] rather than incoherent particles. A collision of these classical non-Abelian fields results in a formation of longitudinal color electromagnetic fields between the two nuclei receding from each other [5–7]. The strength of the longitudinal fields are very strong as $gA_\mu \sim Q_s \sim$ a few GeV, where Q_s is the so-called saturation scale of CGC. Subsequently, the color fields would decay into a huge number of particles (quarks and gluons) to form a quark-gluon plasma (QGP). However, this stage of nonequilibrium dynamics is not well understood—the questions are (a) how the huge number of quark and gluon *particles* are produced from the classical gluon *fields* (experimentally known is that about 1000 hadrons are produced per unit rapidity), and (b) how the system thermalizes to eventually form a QGP, which behaves almost like a perfect liquid as suggested by the success of hydrodynamical models (for reviews, see, for example, [8,9]). In particular, applications of hydrodynamical models assume that the formation time of QGP is extremely short $\tau_{\rm form} \lesssim 1$ fm$/c$ [10–12]. There is no satisfactory understanding of such a short formation time starting from QCD, despite numerous theoretical attempts. Thus, unveiling the early stage dynamics is not only an important piece for completing our understanding of the whole spacetime evolution of HIC but also a challenge to nonequilibrium QCD physics.

The purpose of this paper is to investigate the quark and gluon production from expanding classical color electromagnetic fields starting from QCD.

Study of the particle production from classical electromagnetic fields has a long history in quantum electrodynamics (QED). Sauter [13] was the first who claimed that spontaneous particle (electron and positron pair) production occurs when a system is exposed to strong classical electromagnetic fields. Some years later this particle production mechanism was theoretically formulated by Heisenberg and Euler [14], and by Schwinger [15] for a static and homogeneous electric field. They derived the vacuum persistency probability $\mathcal{P} = |\langle {\rm vac;in}|{\rm vac;out}\rangle|^2$, from which one can deduce the average number of particles produced at transverse and longitudinal momenta $\boldsymbol{p}_\perp$ and p_z with respect to the electric field as [16]

$$\frac{d^3N^{(e^-)}}{d^2\boldsymbol{p}_\perp dp_z} = \frac{d^3N^{(e^+)}}{d^2\boldsymbol{p}_\perp dp_z} = \frac{V}{(2\pi)^3}\exp\left[-\pi\frac{m_e^2+\boldsymbol{p}_\perp^2}{|eE|}\right], \quad (1)$$

where m_e is the electron mass, e is the coupling constant of QED, E is the electric field strength, and V is the system volume. Equation (1), often called the *Schwinger formula*, depends on eE inversely in the exponential, and hence one can understand that the particle production from a static electric field is a nonperturbative phenomenon. This is in contrast to usual perturbative phenomena, whose dependence on eE always appears with positive powers.

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The Schwinger formula was generalized to the QCD case [17–20], and then applied to the early stage phenomenology of HIC, e.g., the color flux tube model [21–26]. However, these preceding studies may be problematic because the situation in HIC is much more complicated than the static and homogeneous field that the Schwinger formula assumes. Thus, the particle production will be different from the naive estimate of the Schwinger formula, and therefore one needs to formulate the particle production in a more dynamic situation starting from the first principle, i.e., QCD.

In particular, we consider the following effects on particle production in QCD, which are missing in previous studies:

(i) *Effects of longitudinal expansion*: In HIC, two highly Lorentz-contracted nuclei pass through each other at almost the speed of light, and color electromagnetic fields are formed between the two receding nuclei with approximate boost invariance in the beam direction (Bjorken expansion [27]). Here, the longitudinal extent of the fields is finite and increases with time, which is obviously very different from what the Schwinger formula assumes. Hence, the applicability of the Schwinger formula must be reconsidered, and one has to deal with particle production from space- and time-dependent color electromagnetic fields. Recently, there is progress in a theoretical treatment of particle production from such an expanding electromagnetic field within scalar QED by Tanji [28]. We will extend this study to the case of quark and gluon production from expanding color electric fields in QCD.

(ii) *Finite lifetime effects*: The color electromagnetic fields decay in time according to the classical Yang-Mills equation. The typical scale of their lifetime is very short, where the order would be given by the inverse of the saturation scale $1/Q_{\rm s}$ [5]. Such a short lifetime of the fields should significantly affect the particle production mechanism. Indeed, for a nonexpanding electric field, Refs. [29,30] have shown that there is an interplay between perturbative particle production at shorter lifetimes and Schwinger's nonperturbative particle production at longer lifetimes. As a result, the particle spectra will heavily depend on the lifetime of the fields; in particular, production of heavy particles, such as charm quarks, from a pulse field is significantly enhanced compared to the value of the Schwinger formula [30,31]. It is thus phenomenologically important to understand finite lifetime effects on particle production. No studies have paid much attention to them so far, though there are several studies that discussed particle production from an expanding (color) electromagnetic field in QED [28,32–34] and in QCD (but quark production only) [35].

In order to examine the above-mentioned points, we study quark and gluon production from a given homogeneous classical color electric background field applied for finite duration (lifetime) from $\tau = 0$ to T with longitudinally expanding geometry. We solve mode equations for fluctuations nonperturbatively with respect to the classical field and compute the Bogoliubov coefficients among creation/annihilation operators at asymptotic times ($t \to \pm\infty$). We ignore backreaction from produced particles on the electric field, and we fix the electric field strength constant during its lifetime. For the sake of clarity, we ignore here a possible existence of color magnetic fields, which may bring interesting effects including the chiral magnetic effect. Effects of the backreaction and of scatterings between produced particles will be decisive for thermalization of the system, but we leave it for our future study.

This paper is organized as follows: In Sec. II, the general formalism for particle production from classical color electromagnetic fields employed in this work is explained. Our formalism is based on a canonical quantization under the presence of classical color background fields [17,36,37], where a nonexpanding system was treated. We extend it to quark and gluon production in an expanding system by following Ref. [28]. In Sec. III, we model the classical field by a boost-invariantly expanding homogeneous color electric field with lifetime T as a demonstration of our formalism. In such a field configuration, one can analytically obtain quark and gluon production spectra and can investigate physical consequences of the longitudinal expansion and the finite lifetime effects in detail. We also discuss some implications to the early stage dynamics of HIC of these results. Section IV is devoted to a summary and an outlook of this work. In Appendix A, details of analytical solutions of equations of motion of QCD are presented.

II. GENERAL FORMALISM

Let us explain the general formalism employed in this work for particle production from a boost-invariant classical gauge field in QCD. We consider a classical background field satisfying the classical Yang-Mills equation and quantum fluctuations of quark, gluon, and ghost around the classical field. By assuming that the Abelian dominance holds for the classical field, we linearize equations of motion for fluctuations and solve them nonperturbatively with respect to the classical field. Then, we adopt a canonical quantization procedure in the τ-η coordinates, instead of in the Cartesian coordinates, in order to treat the boost-invariant expansion of the system properly. Thereby, we directly compute expectation values of number operators of quark, gluon, and ghost.

We work in the Heisenberg picture throughout this paper. We implicitly take summation over repeated indices $m, n, \ldots$ and $\mu, \nu, \ldots$ for *spacetime only*, and not for other repeated indices, for instance, color labels $a, b, \ldots$, spin labels $s, s', \ldots$, and so on.

A. τ-η Coordinates

Let us begin with a brief review on the τ-η coordinates. It is very convenient to work in the τ-η coordinates $x^\mu = (\tau, x, y, \eta)$, instead of the usual Cartesian coordinates $\xi^m = (t, x, y, z)$, in order to treat the boost-invariant expansion of the system properly. We use Latin (Greek) indices $m, n, \ldots$ $(\mu, \nu, \ldots)$ for the Cartesian (τ-η) coordinates throughout this paper.

The τ-η coordinates are defined by the following change of variables:

$$\tau = \sqrt{t^2 - z^2}, \qquad \eta = \frac{1}{2}\ln\frac{t+z}{t-z}. \tag{2}$$

The line element ds^2 is then expressed as

$$ds^2 = \eta_{mn} d\xi^m d\xi^n = g_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

where

$$\eta_{mn} = \mathrm{diag}(1, -1, -1, -1), \tag{4}$$

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2) \tag{5}$$

are the metrics of the Cartesian coordinates and the τ-η coordinates, respectively.

For later discussions, it is convenient to introduce a *viervein* matrix $e^m{}_\mu$ [38], which relates the Cartesian coordinates ξ^m and the τ-η coordinates x^μ as

$$d\xi^m = e^m{}_\mu dx^\mu \tag{6}$$

with

$$e^m{}_\mu \equiv \frac{d\xi^m}{dx^\mu} = \begin{pmatrix} \cosh\eta & 0 & 0 & \tau\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \tau\cosh\eta \end{pmatrix}. \tag{7}$$

The inverse matrix of $e^m{}_\mu$, which we write $e^\mu{}_m$, is

$$e^\mu{}_m \equiv \frac{dx^\mu}{d\xi^m} = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sinh\eta}{\tau} & 0 & 0 & \frac{\cosh\eta}{\tau} \end{pmatrix} = \eta_{mn} g^{\mu\nu} e^n{}_\nu. \tag{8}$$

With the viervein matrix introduced above, one can define a vector X^μ in the τ-η coordinates for any vector X^m in the Cartesian coordinates as

$$X_\mu \equiv e^m{}_\mu X_m, \tag{9}$$

$$X^\mu \equiv e^\mu{}_m X^m = g^{\mu\nu} X_\nu. \tag{10}$$

From these definitions, Eqs. (9) and (10), one readily finds, for example,

$$\partial_\tau = \cosh\eta\,\partial_t + \sinh\eta\,\partial_z, \tag{11}$$

$$\partial_\eta = \tau\sinh\eta\,\partial_t + \tau\cosh\eta\,\partial_z \tag{12}$$

for derivatives ∂_μ,

$$\gamma^\tau = \gamma^t\cosh\eta - \gamma^z\sinh\eta, \tag{13}$$

$$\gamma^\eta = -\gamma^t\frac{\sinh\eta}{\tau} + \gamma^z\frac{\cosh\eta}{\tau} \tag{14}$$

for gamma matrices γ^μ,

$$A_\tau = A_t\cosh\eta + A_z\sinh\eta, \tag{15}$$

$$A_\eta = A_t\tau\sinh\eta + A_z\tau\cosh\eta \tag{16}$$

for vector fields A_μ. One can also generalize these definitions, Eqs. (9) and (10), to general tensors as $X^{\mu\cdots}{}_{\nu\cdots} = e^\mu{}_m \cdots e^n{}_\nu \cdots X^{m\cdots}{}_{n\cdots}$.

We also introduce a covariant derivative ∇_μ for curvilinear coordinates, $\nabla_\mu T^{\nu\cdots}{}_{\rho\cdots} = \partial_\mu T^{\nu\cdots}{}_{\rho\cdots} + \Gamma^\nu_{\mu\lambda} T^{\lambda\cdots}{}_{\rho\cdots} + \cdots - \Gamma^\lambda_{\mu\rho} T^{\nu\cdots}{}_{\lambda\cdots} - \cdots$. Here, $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol, whose nonzero elements in the τ-η coordinates are

$$\Gamma^\eta_{\eta\tau} = \Gamma^\eta_{\tau\eta} = 1/\tau, \qquad \Gamma^\tau_{\eta\eta} = \tau. \tag{17}$$

B. Classical background field

We consider a classical background field $\bar{A}_\mu$ satisfying the SU(N_c) classical Yang-Mills equation with an external classical source $\bar{J}^\mu$ as

$$\bar{J}^\nu = \bar{D}_\mu \bar{F}^{\mu\nu}. \tag{18}$$

Here, $\bar{D}_\mu$ is the covariant derivative with respect to the classical field $\bar{A}_\mu$, i.e., $\bar{D}_\mu = \nabla_\mu + ig[\bar{A}_\mu,]$, and $\bar{F}^{\mu\nu}$ is the classical field strength tensor $\bar{F}^{\mu\nu} = \partial^\mu\bar{A}^\nu - \partial^\nu\bar{A}^\mu + ig[\bar{A}^\mu, \bar{A}^\nu]$. Equation (18) does not fix the gauge completely, and there still remains a residual gauge freedom. In the following discussion, we fix the residual gauge freedom by $\bar{A}_\tau = 0$ (temporal gauge), which is convenient for the canonical quantization procedure we adopt in Sec. II E. As a boundary condition of Eq. (18), we require that $\bar{A}_\mu$ becomes a pure gauge $\bar{A}_\mu = \mathrm{const}$ at the asymptotic times $(t \to \pm\infty)$; i.e., we assume that there is no external source

$\bar{J}^\mu$ nor classical color electromagnetic field at the asymptotic times.

As we will see in Sec. II D, in order to ease some difficulties coming from the non-Abelian nature of QCD, we furthermore assume that the color direction of the classical source $\bar{J}^\mu$ and the classical field $\bar{A}_\mu$ is constant; i.e., it is independent of spacetime coordinates x and the spacetime vector index μ. For this case, there always exists a constant color vector n^a such that

$$\bar{A}_\mu(x) = \tilde{A}_\mu(x) \sum_{a=1}^{N_c^2-1} n^a t^a. \tag{19}$$

Here, $\tilde{A}_\mu$ is a scalar in the color space. The matrix t^a ($a = 1, \ldots, N_c^2 - 1$) is a generator of $SU(N_c)$, and n^a (normalized as $\sum_{a=1}^{N_c^2-1} n^a n^a = 1$) characterizes the color direction of the classical field $\bar{A}_\mu$. Under this assumption, the commutators of $\bar{A}_\mu$ exactly vanish as $[\bar{A}_\mu, \bar{A}_\nu] = 0$, and only the Abelian part of the classical field strength $\bar{F}_{\mu\nu}$ becomes nonvanishing as

$$\bar{F}_{\mu\nu} = (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) \sum_{a=1}^{N_c^2-1} n^a t^a \equiv \tilde{F}_{\mu\nu} \sum_{a=1}^{N_c^2-1} n^a t^a. \tag{20}$$

Thus, our assumption is essentially the same as the Abelian dominance assumption: $[\bar{A}_\mu, \bar{A}_\nu] \sim 0$ and $\bar{F}_{\mu\nu} \sim \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$.

Notice that we have made no restrictions on the spacetime x^μ dependence of $\tilde{A}_\mu$ as long as it satisfies the classical Yang-Mills equation (18).

C. Lagrangian

Let us consider the QCD Lagrangian with N_c colors and N_f flavors of quarks in the presence of the classical background field $\bar{A}_\mu$ described in Sec. II B. By separating the (total) gauge field A_μ into the classical field $\bar{A}_\mu$ and quantum fluctuations around it $\mathcal{A}_\mu$ as $A_\mu = \bar{A}_\mu + \mathcal{A}_\mu$, we obtain the QCD Lagrangian for the fluctuation in the τ-η coordinates as[1]

$$\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - gA\!\!\!/ - M]\psi - \frac{1}{2}\mathrm{tr}_c F_{\mu\nu}F^{\mu\nu} + 2\mathrm{tr}_c \bar{J}^\mu A_\mu - \frac{1}{\alpha}\mathrm{tr}_c(\bar{D}_\mu \mathcal{A}^\mu)^2 - 2i\mathrm{tr}_c(\bar{D}^\mu \bar{c})(D_\mu c). \tag{21}$$

Here, ψ is the fermion field, and c and $\bar{c}$ are the ghost and antighost fields to be quantized. $X\!\!\!/ \equiv \gamma^\mu X_\mu$ is the Feynman slash notation, and $\bar{\psi}$ is a shorthand for $\bar{\psi} \equiv \psi^\dagger \gamma^t$. tr_c is the trace operator in the color space. M represents fermion masses, which is given by an $N_f \times N_f$ diagonal matrix $M = \mathrm{diag}(m_1, m_2, \ldots, m_{N_f})$ in the flavor space. D_μ is the covariant derivative with respect to the total gauge field A_μ: $D_\mu = \nabla_\mu + ig[A_\mu,]$. The total field strength tensor $F_{\mu\nu}$ is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. The term $(1/\alpha)\mathrm{tr}_c(\bar{D}_\mu \mathcal{A}^\mu)^2$ is a covariant background gauge fixing term [17]. Hereafter, we shall take $\alpha = 1$ for simplicity. One can show that a choice of the gauge parameter α is irrelevant to the particle spectra [39].

We further expand the Lagrangian, Eq. (21), up to the quadratic order in the quantum fluctuations to obtain

$$\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - g\bar{A}\!\!\!/ - M]\psi - 2i\mathrm{tr}_c(\bar{D}_\mu \bar{c})(\bar{D}^\mu c) - \mathrm{tr}_c\left[\frac{1}{2}(\bar{D}_\mu \mathcal{A}_\nu - \bar{D}_\nu \mathcal{A}_\mu)^2 + (\bar{D}_\mu \mathcal{A}^\mu)^2 + 2ig\bar{F}_{\mu\nu}\mathcal{A}^\mu \mathcal{A}^\nu\right], \tag{22}$$

where constant and surface terms are omitted. Here, we treat the interactions with the classical field $\bar{A}_\mu$ nonperturbatively. This treatment is justified when the quantum fluctuations $\psi, \mathcal{A}_\mu, c$, and $\bar{c}$ are small enough compared to the strength of the classical field $\bar{A}_\mu$. The ignored terms $\mathcal{O}(g\bar{\psi}\mathcal{A}\!\!\!/\psi, g\mathcal{A}^3, g\bar{c}c\mathcal{A})$ are responsible for the screening of the classical field $\bar{A}_\mu$ by produced particles and elastic gg ↔ gg and inelastic g ↔ gg, g ↔ q$\bar{\text{q}}$, q ↔ qg scattering processes of produced particles. It is very interesting to see how the quark and/or gluon production is modified when these higher order quantum corrections are included; see Sec. IV for the discussion. We also note that the classical source $\bar{J}^\mu$ does not directly couple to the quantum fluctuations; it couples to them only indirectly through the classical field $\bar{A}_\mu$, which is generated by the classical Yang-Mills equation sourced by $\bar{J}_\mu$ [Eq. (18)]. In this sense, the particle production mechanism is not directly affected by the presence of the classical source $\bar{J}^\mu$.

D. Abelianization

It is difficult to handle the Lagrangian, Eq. (22), as it is because of its non-Abelian nature. Indeed, the equation of motion of the Lagrangian, Eq. (22), are complicated matrix equations in the color space. With the help of the Abelian dominance assumption for the classical field $\bar{A}_\mu$ made in Sec. II B, one can Abelianize, i.e., diagonalize the Lagrangian, Eq. (22), in the color space and obtain a set of Abelian equations of motion as below [20].

First, we diagonalize the classical field $\bar{A}_\mu = \tilde{A}_\mu \sum_{a=1}^{N_c^2-1} n^a t^a$ in the color space. Since $\sum_{a=1}^{N_c^2-1} n^a t^a$ is a constant Hermitian matrix in the color space, there always exists a *global* unitary transformation U that diagonalizes $n^a t^a$ as

$$\sum_{a=1}^{N_c^2-1} n^a t^a \to U^{-1}\left(\sum_{a=1}^{N_c^2-1} n^a t^a\right)U = \sum_{\alpha=1}^{N_c-1} w^\alpha H^\alpha, \tag{23}$$

[1]In general curved spacetime coordinates, there is an additional term coming from spin connections Γ_μ in the fermion covariant derivative, which is zero in the τ-η coordinates.

where w^α is constant normalized as $1 = \sum_\alpha |w^\alpha|^2$. H^α is a diagonal matrix that belongs to the Cartan subalgebra of $\mathrm{SU}(N_\mathrm{c})$ such that $[H^\alpha, H^\beta] = 0$ with a normalization $\mathrm{tr}_\mathrm{c}[H^\alpha H^\beta] = \delta^{\alpha\beta}/2$. In accordance with this transformation U, let us also redefine the quantum fluctuations $\psi, \mathcal{A}_\mu, c$, and $\bar{c}$ as

$$U^\dagger \psi \to \psi, \tag{24}$$

$$U^\dagger \mathcal{A}_\mu U \to \mathcal{A}_\mu, \tag{25}$$

$$U^\dagger \begin{pmatrix} c \\ \bar{c} \end{pmatrix} U \to \begin{pmatrix} c \\ \bar{c} \end{pmatrix}. \tag{26}$$

Second, we expand the color space by the Cartan-Weyl basis of $\mathrm{SU}(N_\mathrm{c})$: $\{H^\alpha, E^{\pm A}\}$ $(\alpha = 1, \ldots, N_\mathrm{c} - 1;\ A = 1, \ldots, N_\mathrm{c}(N_\mathrm{c} - 1)/2)$, where E^A is an off-diagonal matrix satisfying the following algebra:

$$E^{A\dagger} = E^{-A}, \tag{27}$$

$$\mathrm{tr}[E^A E^{B\dagger}] = \frac{\delta^{AB}}{2}, \tag{28}$$

$$[H^\alpha, E^{\pm A}] = \pm (v^\alpha)^A E^{\pm A}, \tag{29}$$

where $(v^\alpha)^A$ is the root vector of $\mathrm{SU}(N_\mathrm{c})$. By using this Cartan-Weyl basis, instead of the generator t^a, we expand the gluon field $\mathcal{A}_\mu$, and ghost and antighost fields c and $\bar{c}$ as (Cartan decomposition)

$$\mathcal{A}_\mu \equiv \sum_{\alpha=1}^{N_\mathrm{c}-1} \mathcal{W}_{\mu,\alpha} H^\alpha + \sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} [W_{\mu,A} E^{+A} + W^\dagger_{\mu,A} E^{-A}], \tag{30}$$

$$\begin{pmatrix} c \\ \bar{c} \end{pmatrix} \equiv \sum_{\alpha=1}^{N_\mathrm{c}-1} \begin{pmatrix} \mathcal{C}_\alpha \\ \bar{\mathcal{C}}_\alpha \end{pmatrix} H^\alpha + \sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} \left[\begin{pmatrix} C_A \\ \bar{C}_A \end{pmatrix} E^{+A} + \begin{pmatrix} C_A^\dagger \\ \bar{C}_A^\dagger \end{pmatrix} E^{-A} \right]. \tag{31}$$

After completing these two steps, one can rewrite the Lagrangian equation (22) in an Abelianized form as

$$\begin{aligned} \mathcal{L} = & \sum_{f=1}^{N_\mathrm{f}} \sum_{i=1}^{N_\mathrm{c}} \bar{\psi}_{i,f} [i\partial\!\!\!/ - q_i^{(\mathrm{q})} \tilde{A\!\!\!/} - m_f] \psi_{i,f} - \sum_{\alpha=1}^{N_\mathrm{c}-1} \frac{1}{4} |\nabla_\mu \mathcal{W}_{\nu,\alpha} - \nabla_\nu \mathcal{W}_{\mu,\alpha}|^2 - i \sum_{\alpha=1}^{N_\mathrm{c}-1} (\nabla_\mu \bar{\mathcal{C}}_\alpha)(\nabla^\mu \mathcal{C}_\alpha) \\ & - \sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} \left[\frac{1}{2} |(\nabla_\mu + i q_A^{(\mathrm{g})} \tilde{A}_\mu) W_{\nu,A} - (\nabla_\nu + i q_A^{(\mathrm{g})} \tilde{A}_\nu) W_{\mu,A}|^2 + |(\nabla_\mu + i q_A^{(\mathrm{g})} \tilde{A}_\mu) W_A^\mu|^2 + i q_A^{(\mathrm{g})} \tilde{F}_{\mu\nu} W_A^\mu W_A^{\nu\dagger} \right] \\ & - i \sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} [((\nabla_\mu + i q_A^{(\mathrm{gh})} \tilde{A}_\mu) \bar{C}_A)((\nabla^\mu + i q_A^{(\mathrm{gh})} \tilde{A}^\mu) C_A)^\dagger + ((\nabla_\mu + i q_A^{(\mathrm{gh})} \tilde{A}_\mu) \bar{C}_A)^\dagger ((\nabla^\mu + i q_A^{(\mathrm{gh})} \tilde{A}^\mu) C_A)]. \end{aligned} \tag{32}$$

Here, the color indices $i, j, \ldots$ and the flavor indices $f, f', \ldots$ for the quark field ψ are explicitly written. The gluon $\mathcal{W}_{\mu,\alpha}$, ghost $\mathcal{C}_\alpha$, and antighost $\bar{\mathcal{C}}_\alpha$ fields, which belong to the Cartan subalgebra of $\mathrm{SU}(N_\mathrm{c})$, do not couple to the classical field $\bar{A}_\mu$. Thus, no particle production occurs for these fluctuations, and hence we do not consider them hereafter. On the other hand, the quark $\psi_{i,f}$, gluon $W_{\mu,A}$, and ghost C_A and antighost $\bar{C}_A$ fields do couple to the classical field $\bar{A}_\mu$, whose effective color charges, $q_i^{(\mathrm{q})}$, $q_A^{(\mathrm{g})}$, and $q_A^{(\mathrm{gh})}$, respectively, are given by

$$q_i^{(\mathrm{q})} = g \sum_{\alpha=1}^{N_\mathrm{c}-1} w^\alpha (H^\alpha)_{ii}, \tag{33}$$

$$q_A^{(\mathrm{g})} = q_A^{(\mathrm{gh})} = g \sum_{\alpha=1}^{N_\mathrm{c}-1} w^\alpha (v^\alpha)^A. \tag{34}$$

The ghost charge is identical to the gluon charge $q_A^{(\mathrm{gh})} = q_A^{(\mathrm{g})}$ because both gluon $W_{\mu,A}$ and ghost C_A, $\bar{C}_A$ fields belong to the adjoint representation of $\mathrm{SU}(N_\mathrm{c})$. Although the effective color charges, $q_i^{(\mathrm{q})}$, $q_A^{(\mathrm{g})}$, and $q_A^{(\mathrm{gh})}$, depend on the color direction n^a and the gauge choice of the background field $\bar{A}_\mu$, the traces of the squared charges are independent of them,

$$\sum_{i=1}^{N_\mathrm{c}} |q_i^{(\mathrm{q})}|^2 = \frac{g^2}{2}, \tag{35}$$

$$\sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} |q_A^{(\mathrm{g})}|^2 = \sum_{A=1}^{\frac{N_\mathrm{c}(N_\mathrm{c}-1)}{2}} |q_A^{(\mathrm{gh})}|^2 = \frac{g^2 N_\mathrm{c}}{2}. \tag{36}$$

The trace of the squared charge in the adjoint representation is N_c times as large as that in the fundamental representation. These relations are generalizations of the SU(3) results [39,46–48].

One readily obtains Abelianized equations of motion from the Lagrangian, Eq. (32). They read

$$[i\tilde{\partial} - q_i^{(\mathrm{q})}\tilde{A} - m_f]\psi_{i,f} = 0, \tag{37}$$

$$[(\nabla_\rho + iq_A^{(\mathrm{g})}\tilde{A}_\rho)^2 g^{\mu\nu} + 2iq_A^{(\mathrm{g})}\tilde{F}^{\mu\nu}]W_{\nu,A} = 0, \tag{38}$$

$$(\nabla_\nu + iq_A^{(\mathrm{gh})}\tilde{A}_\nu)^2 \begin{pmatrix} C_A \\ \bar{C}_A \end{pmatrix} = 0. \tag{39}$$

E. Quantization and particle spectrum

Now, we canonically quantize the fluctuations, $\psi_{i,f}$, $W_{\mu,A}$, C_A, and $\bar{C}_A$, under the classical background field $\bar{A}_\mu$, and compute particle spectra produced from the classical field.

To be more concrete, we first define positive/negative mode functions at the asymptotic times ($t \to \pm\infty$) for the fluctuations. At the asymptotic times, as the classical field $\bar{A}_\mu$ becomes merely a pure gauge and no interaction occurs (see the assumptions made in Sec. II B), one can *uniquely*[2] define the positive/negative frequency mode functions at the corresponding asymptotic time by plane wave solutions. With this boundary condition at $t \to \pm\infty$, we solve the equations of motion, Eqs. (37)–(39), nonperturbatively with respect to the classical field, and hereby we obtain the positive/negative mode functions at the corresponding asymptotic time. By expanding the fluctuations with the mode functions and imposing canonical commutation relations, one obtains creation/annihilation operators for the positive/negative frequency modes at each asymptotic time ($t \to \pm\infty$). An important point here is that the mode functions do fully include multiple interactions with the classical field, and hence the positive (or negative) frequency mode at $t \to -\infty$ will evolve into a linear combination of the positive and negative frequency modes at $t \to \infty$. This linear relation is described by a Bogoliubov transformation, and we will see that the particle spectrum which will be observed at $t \to \infty$ evolved from a given initial state at $t \to -\infty$ is determined by the Bogoliubov coefficients. In the following, we shall assume that the initial state is given by a vacuum for simplicity, although one can equally formulate more generic initial states as well.

We remark that our formalism, which takes into account the interactions with the classical field $\bar{A}_\mu$ nonperturbatively by fully solving the equations of motion, does include perturbative contributions that can be computed by, for instance, the usual diagrammatic techniques of the S matrix [49]. For a specific type of electric fields, one can explicitly check this [30,50].

[2]One can quantize the fluctuations even if there are interactions in principle; however, the definition of positive/negative mode functions, i.e., the notion of particle, becomes ambiguous.

1. Quark

We canonically quantize the quark field $\psi_{i,f}$ at the asymptotic times ($t \to \pm\infty$) in order to compute the quark spectrum produced from the classical field.

To do this, we first expand the quark fields $\psi_{i,f}$ with the mode functions as

$$\psi_{i,f}(x) = \sum_s \int d\boldsymbol{p}_\perp^2 dp_\eta [{}_+\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}(x) a^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} + {}_-\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}(x) b^{(\mathrm{as})\dagger}_{i,f,-\boldsymbol{p}_\perp,-p_\eta,s}]. \tag{40}$$

Here, as = in, out specifies the asymptotic time $t \to \pm\infty$, respectively, at which we define a particle picture by employing the canonical quantization. The subscripts $\pm$ specify the positive and the negative frequency modes. The momentum labels $\boldsymbol{p}_\perp$ and p_η are the Fourier conjugate to the positions $\boldsymbol{x}_\perp$ and η, respectively; we label the longitudinal momentum by p_η, instead of the p_z conjugate to z, so as to treat the longitudinal expansion of the system manifestly with the η coordinate. The label $s = 1, 2$ is for the spin degree of freedom. We identify the mode functions ${}_\pm\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ (${}_\pm\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$) with plane wave solutions with positive/negative frequency at $t \to -\infty$ ($t \to \infty$),

$${}_\pm\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} \underset{t\to-\infty}{\longrightarrow} {}_\pm\psi^{(\mathrm{free})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}[\bar{A}_\mu(t \to -\infty)], \tag{41}$$

$${}_\pm\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} \underset{t\to\infty}{\longrightarrow} {}_\pm\psi^{(\mathrm{free})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}[\bar{A}_\mu(t \to \infty)], \tag{42}$$

where the plane wave solutions ${}_\pm\psi^{(\mathrm{free})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}[\breve{A}_\mu]$ satisfy the free field equation of motion under a pure gauge background $\breve{A}_\mu = \bar{A}_\mu(t \to \pm\infty)$. For details of the plane wave solutions ${}_\pm\psi^{(\mathrm{free})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$, see Appendix A 1 a. We also normalize the positive/negative frequency mode functions ${}_\pm\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ for each as = in, out as

$$\left({}_\pm\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} | {}_\pm\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_{\mathrm{F}} = \delta_{ss'}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta), \tag{43}$$

$$({}_\pm\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} | {}_\mp\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} = 0, \tag{44}$$

where the inner product for fermion fields $(\psi_1|\psi_2)_{\mathrm{F}}$ in the τ-η coordinates is given by

$$(\psi_1|\psi_2)_{\mathrm{F}} = \tau \int_{\tau=\mathrm{const}} d^2\boldsymbol{x}_\perp d\eta \bar{\psi}_1 \gamma^\tau \psi_2. \tag{45}$$

Next, we impose canonical commutation relations to complete the canonical quantization. Since we are working in the τ-η coordinates, we impose canonical commutation relations on an equal τ surface, instead of on an equal t surface as in the Cartesian coordinates,

$$\{\psi_{i,f}(\tau,\boldsymbol{x}_\perp,\eta), \pi_{i',f'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = i\delta_{ii'}\delta_{ff'}\delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)\frac{\delta(\eta-\eta')}{\tau}, \tag{46}$$

$$\{\pi_{i,f}(\tau,\boldsymbol{x}_\perp,\eta), \pi_{i',f'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = \{\psi_{i,f}(\tau,\boldsymbol{x}_\perp,\eta), \psi_{i',f'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = 0, \tag{47}$$

where the canonical conjugate $\pi_{i,f}$ to the quark field $\psi_{i,f}$ is given by $\pi_{i,f} = \delta\mathcal{L}/\delta(\partial_\tau\psi_{i,f}) = i\bar{\psi}_{i,f}\gamma^\tau$. The factor $1/\tau$ in Eq. (46) comes from the Jacobian $\sqrt{-g} = \tau$ of the τ-η coordinates. The canonical commutation relations, Eqs. (46) and (47), are equivalent to require that the operators $a^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$, $b^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ anticommute as

$$\left\{a^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}, a^{(\mathrm{as})\dagger}_{i',f',\boldsymbol{p}'_\perp,p'_\eta,s'}\right\} = \left\{b^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}, b^{(\mathrm{as})\dagger}_{i',f',\boldsymbol{p}'_\perp,p'_\eta,s'}\right\} = \delta_{ii'}\delta_{ff'}\delta_{ss'}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta), \tag{48}$$

$$(\text{otherwise}) = 0. \tag{49}$$

From these anticommutation relations, Eqs. (48) and (49), one can understand as usual that the operator $a^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ ($b^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$) acts as an annihilation operator of a quark (an antiquark) at the corresponding asymptotic time with the momentums $\boldsymbol{p}_\perp$, p_η, the spin s, the color charge $q^{(\mathrm{q})}_i$ ($-q^{(\mathrm{q})}_i$), and the flavor f.

As is stated in the beginning of this section, the creation/annihilation operators for different asymptotic times do not coincide with each other because of the interactions with the classical field. The linear relation is described by the following Bogoliubov transformation:

$$\begin{pmatrix} a^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} \\ b^{(\mathrm{out})\dagger}_{i,f,-\boldsymbol{p}_\perp,-p_\eta,s} \end{pmatrix} = \begin{pmatrix} ({}_+\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\psi_{i,f})_{\mathrm{F}} \\ ({}_-\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\psi_{i,f})_{\mathrm{F}} \end{pmatrix}$$
$$= \sum_{s'}\int d^2\boldsymbol{p}'_\perp dp'_\eta \begin{pmatrix} ({}_+\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|{}_+\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} & ({}_+\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|{}_-\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} \\ ({}_-\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|{}_+\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} & ({}_-\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|{}_-\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} \end{pmatrix} \begin{pmatrix} a^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'} \\ b^{(\mathrm{in})\dagger}_{i,f,-\boldsymbol{p}'_\perp,-p'_\eta,s'} \end{pmatrix}. \tag{50}$$

In order to obtain the (anti)quark spectrum at $t\to\infty$ produced from the background field $\bar{A}_\mu$, let us introduce a (anti)quark number density operator $n^{(\mathrm{q})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ ($n^{(\bar{\mathrm{q}})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$) by

$$n^{(\mathrm{q})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} \equiv a^{(\mathrm{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} a^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s},$$
$$n^{(\bar{\mathrm{q}})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} \equiv b^{(\mathrm{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} b^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}. \tag{51}$$

The quark and antiquark spectra are derived as an expectation value of the number density operators by a given initial state at $t\to-\infty$. Hereafter, let us assume that the initial state is given by a vacuum $|\mathrm{vac};\mathrm{in}\rangle$. By noting that the initial vacuum is a state that is annihilated by the annihilation operators at $t\to-\infty$ as $0 = a^{(\mathrm{in})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac};\mathrm{in}\rangle = b^{(\mathrm{in})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac};\mathrm{in}\rangle$ and by using the Bogoliubov transformation, Eq. (50), one immediately obtains

$$\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dp_\eta} \equiv \frac{\langle\mathrm{vac};\mathrm{in}|n^{(\mathrm{q})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac};\mathrm{in}\rangle}{\langle\mathrm{vac};\mathrm{in}|\mathrm{vac};\mathrm{in}\rangle} = \sum_{s'}\int d^2\boldsymbol{p}'_\perp dp'_\eta |({}_+\psi^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|{}_-\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}}|^2, \tag{52}$$

$$\frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dp_\eta} \equiv \frac{\langle\mathrm{vac};\mathrm{in}|n^{(\bar{\mathrm{q}})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac};\mathrm{in}\rangle}{\langle\mathrm{vac};\mathrm{in}|\mathrm{vac};\mathrm{in}\rangle} = \sum_{s'}\int d^2\boldsymbol{p}'_\perp dp'_\eta |({}_-\psi^{(\mathrm{out})}_{i,f,-\boldsymbol{p}_\perp,-p_\eta,s}|{}_+\psi^{(\mathrm{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}}|^2. \tag{53}$$

An important point of these formulas Eqs. (52) and (53) is that deriving the particle spectrum is thus reduced to finding out the mode functions ${}_\pm\psi^{(\mathrm{as})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ by solving the Dirac equation, Eq. (37), nonperturbatively with respect to the classical field.

As is expected from the Pauli principle, one can explicitly show that the phase space density does not exceed unity. Indeed, the anticommutation relation, Eq. (48), yields that the Bogoliubov coefficients are normalized as

$$\frac{1}{(2\pi)^3}\int d^2\boldsymbol{x}_\perp \int d\eta = \sum_{s'}\int d^2\boldsymbol{p}'_\perp dp'_\eta \left[\left|\left({}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}\middle|{}_{\pm}\psi^{(\text{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_\text{F}\right|^2 + \left|\left({}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}\middle|{}_{\mp}\psi^{(\text{in})}_{i,f,\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_\text{F}\right|^2\right], \tag{54}$$

where we have used $\delta^2(\boldsymbol{p}_\perp = \boldsymbol{0})\delta(p_\eta = 0) = 1/(2\pi)^3 \int d^2\boldsymbol{x}_\perp \int d\eta$. From Eq. (54), one immediately finds $(2\pi)^3 d^6 N^{(\text{q},\bar{\text{q}})}_{i,f,s}/d\boldsymbol{x}^2_\perp d\eta d\boldsymbol{p}^2_\perp dp_\eta \leq 1$.

So far, we have characterized the longitudinal momentum of produced quarks by the label p_η because it is a natural quantum number conjugate to the spacetime rapidity η and that manifestly respects the boost invariance of the system. Consequently, what we have obtained for the quark spectra in Eqs. (52) and (53) are the p_η spectrum. However, what we actually observe in experiments is not the p_η spectrum, but the p_z spectrum and/or the momentum rapidity $y_{\boldsymbol{p}}$ spectrum, where

$$y_{\boldsymbol{p}} \equiv \frac{1}{2}\ln\frac{\omega_{\boldsymbol{p}} + p_z}{\omega_{\boldsymbol{p}} - p_z} \tag{55}$$

with $\omega_{\boldsymbol{p}}$ being an on-shell energy $\omega_{\boldsymbol{p}} \equiv \sqrt{m^2 + \boldsymbol{p}^2_\perp + p^2_z}$. The p_z spectrum and/or the momentum rapidity $y_{\boldsymbol{p}}$ spectrum can be obtained from the p_η spectrum in the following way [28]: As in the p_η spectrum Eqs. (52) and (53), the p_z spectrum and/or the momentum rapidity $y_{\boldsymbol{p}}$ spectrum are obtained as an expectation value of the number operators, $n^{(\text{q})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ and $n^{(\bar{\text{q}})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$, which are labeled by p_z instead of p_η as

$$\frac{d^3N^{(\text{q})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dp_z} = \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\text{q})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dy_{\boldsymbol{p}}} \equiv \frac{\langle\text{vac; in}|n^{(\text{q})}_{i,f,\boldsymbol{p}_\perp,p_z,s}|\text{vac; in}\rangle}{\langle\text{vac; in}|\text{vac; in}\rangle}, \tag{56}$$

$$\frac{d^3N^{(\bar{\text{q}})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dp_z} = \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\bar{\text{q}})}_{i,f,s}}{d\boldsymbol{p}^2_\perp dy_{\boldsymbol{p}}} \equiv \frac{\langle\text{vac; in}|n^{(\bar{\text{q}})}_{i,f,\boldsymbol{p}_\perp,p_z,s}|\text{vac; in}\rangle}{\langle\text{vac; in}|\text{vac; in}\rangle}, \tag{57}$$

where the number operators are defined by the annihilation operators $a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$, $b^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ as

$$n^{(\text{q})}_{i,f,\boldsymbol{p}_\perp,p_z,s} \equiv a^{(\text{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_z,s}a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}, \quad n^{(\bar{\text{q}})}_{i,f,\boldsymbol{p}_\perp,p_z,s} \equiv b^{(\text{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_z,s}b^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}. \tag{58}$$

The annihilation operators $a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$, $b^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ are defined by expanding the fermion operator $\psi_{i,f}$ in terms of positive/negative frequency mode functions ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ in the Cartesian coordinates, which is labeled by p_z being the Fourier conjugate to z as

$$\psi_{i,f}(x) = \sum_s\int d\boldsymbol{p}^2_\perp dp_z\left[{}_{+}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}(x)a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s} + {}_{-}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}(x)b^{(\text{out})\dagger}_{i,f,-\boldsymbol{p}_\perp,-p_z,s}\right]. \tag{59}$$

Here, we adopt the same boundary condition as what we have required for ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$; i.e., we require ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ to coincide with the plane wave solutions at $t\to\infty$. As is shown in Appendix A 1 a, the mode functions in the Cartesian coordinates ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ and those in the τ-η coordinates ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$ are related with each other by an integral transformation described by[3]

$${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} = \int dp_z\frac{\text{e}^{\pm ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}}\,{}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}. \tag{60}$$

Using this integral transformation, Eq. (60), and comparing the expansion in the Cartesian coordinates, Eq. (59), with that in the τ-η coordinates, Eq. (40), one finds

$$a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s} = \int dp_\eta\frac{\text{e}^{ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}}a^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}, \tag{61}$$

$$b^{(\text{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_z,s} = \int dp_\eta\frac{\text{e}^{-ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}}b^{(\text{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}. \tag{62}$$

[3]Strictly speaking, what we show in Appendix A 1 a is that the plane wave solutions in the Cartesian coordinates ${}_{\pm}\psi^{(\text{free})}_{i,f,\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]$ and those in the τ-η coordinates ${}_{\pm}\psi^{(\text{free})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$ are related with each other by the integral transformation, Eq. (60). One can safely say that the same integral relation equally holds for the mode functions, ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ and ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$: Since the two sets of mode functions, ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s}$ and ${}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}$, obey the same differential equation $[i\gamma^m(\partial_m - iq^{(\text{q})}_i\tilde{A}_m) - m_f]{}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_z,s} = [i\gamma^\mu(\partial_\mu - iq^{(\text{q})}_i\tilde{A}_\mu) - m_f]{}_{\pm}\psi^{(\text{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s} = 0$ and that the linear relation between them is conserved in the time evolution, it is sufficient to show that the integral relation at the boundary $t,\tau\to\infty$, where both solutions become plane waves. Hence, the integral relation actually holds. The same argument can be applied for the integral transformation for gluons [Eq. (93)], which we will discuss in the next subsection.

Inserting these relations, Eqs. (61) and (62), back into Eqs. (56) and (57), one obtains

$$\begin{aligned}\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta dp'_\eta \frac{\mathrm{e}^{iy_{\boldsymbol{p}}(p_\eta-p'_\eta)}}{2\pi} \\ &\quad\times\frac{\langle\mathrm{vac;in}|a^{(\mathrm{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p'_\eta,s}a^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac;in}\rangle}{\langle\mathrm{vac;in}|\mathrm{vac;in}\rangle},\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta dp'_\eta \frac{\mathrm{e}^{iy_{\boldsymbol{p}}(p_\eta-p'_\eta)}}{2\pi} \\ &\quad\times\frac{\langle\mathrm{vac;in}|b^{(\mathrm{out})\dagger}_{i,f,\boldsymbol{p}_\perp,p'_\eta,s}b^{(\mathrm{out})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}|\mathrm{vac;in}\rangle}{\langle\mathrm{vac;in}|\mathrm{vac;in}\rangle}.\end{aligned} \tag{64}$$

When the system is perfectly boost invariant, the expectation values in Eqs. (63) and (64) for $p_\eta \neq p'_\eta$ vanish because p_η is a good quantum number and it never mixes with other values of p_η during the time evolution. In this case, one can further simplify Eqs. (63) and (64) as

$$\begin{aligned}\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{2\pi\delta(p_\eta=0)}\times\frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta \frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_\eta},\end{aligned} \tag{65}$$

$$\begin{aligned}\frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{2\pi\delta(p_\eta=0)}\times\frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta \frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d\boldsymbol{p}_\perp^2 dp_\eta},\end{aligned} \tag{66}$$

which are manifestly boost invariant in the sense that the $y_{\boldsymbol{p}}$ spectrum does not depend on the momentum rapidity $y_{\boldsymbol{p}}$. We note that we have derived the formulas, Eqs. (65) and (66), in a quantum field theoretical manner by following Ref. [28], but one can also obtain the same formulas within classical mechanics [32,34], though these two derivations agree with each other only if the system is perfectly boost invariant.

2. Gluon

Next, we turn to the canonical quantization of the gluon field $W_{\mu,A}$ and compute the gluon spectrum at $t\to\infty$. We do essentially the same procedure as what we have done in the quark case although there are slight differences due to the vector nature of gluons.

First, we expand the gluon field $W_{\mu,A}$ as

$$W_{\mu,A} = \sum_\sigma \int d\boldsymbol{p}_\perp^2 dp_\eta \Big[{}_+W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}c^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma} + {}_-W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}d^{(\mathrm{as})\dagger}_{A,-\boldsymbol{p}_\perp,-p_\eta,\sigma}\Big]. \tag{67}$$

$\sigma = 0, 1, 2, 3$ labels the polarization, and the other labels are the same as in the quark case. The mode functions ${}_\pm W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ are the solutions of the equations of motion, Eq. (38), with the plane wave boundary condition,

$${}_\pm W^{(\mathrm{in})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma} \xrightarrow[t\to-\infty]{} {}_\pm W^{(\mathrm{free})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}[\bar{A}_\mu(t\to-\infty)], \tag{68}$$

$${}_\pm W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma} \xrightarrow[t\to\infty]{} {}_\pm W^{(\mathrm{free})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}[\bar{A}_\mu(t\to\infty)], \tag{69}$$

where the plane wave solutions ${}_\pm W^{(\mathrm{free})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}[\breve{A}_\mu]$ satisfy the free field equation of motion under a pure gauge background field $\breve{A}_\mu = \bar{A}_\mu(t\to\pm\infty)$. For details of the plane wave solutions ${}_\pm W^{(\mathrm{free})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$, see Appendix A 2 a. The positive/negative frequency mode functions ${}_\pm W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ are normalized as

$$\begin{aligned}&-g^{\mu\nu}({}_\pm W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_\pm W^{(\mathrm{as})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} \\ &\quad= \pm\xi_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta),\end{aligned} \tag{70}$$

$$-g^{\mu\nu}({}_\pm W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_\mp W^{(\mathrm{as})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} = 0 \tag{71}$$

for each as = in, out. Here, the inner product for boson fields $(\phi_1|\phi_2)_{\mathrm{B}}$ in the τ-η coordinates is given by

$$(\phi_1|\phi_2)_{\mathrm{B}} = i\tau\int_{\tau=\mathrm{const}} d^2\boldsymbol{x}_\perp d\eta\,\phi_1^*\overleftrightarrow{\nabla}_\tau\phi_2, \tag{72}$$

where $\overleftrightarrow{\nabla}_\tau \equiv \overrightarrow{\nabla}_\tau - \overleftarrow{\nabla}_\tau$. The indefinite metric $\xi_{\sigma\sigma'}$ is introduced by

$$\xi_{\sigma\sigma'} \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{73}$$

which has symmetric off-diagonal elements $\xi_{03} = \xi_{30}$. Because of this property, the zeroth and the third polarization modes of gluons become unphysical and they do not appear in the physical spectrum as we will show later.

For later convenience, we decompose the positive/negative frequency mode functions ${}_{\pm}W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ by introducing a polarization vector $\varepsilon_{\mu,\sigma}$ and scalar amplitudes ${}_{\pm}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ as

$$ {}_{\pm}W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv \varepsilon_{\mu,\sigma}\,{}_{\pm}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}. \tag{74} $$

It is convenient to normalize the polarization vector as

$$ g^{\mu\nu}\varepsilon^*_{\mu,\sigma}\varepsilon_{\nu,\sigma'} = -\xi_{\sigma\sigma'}, \qquad \sum_{\sigma,\sigma'}\xi_{\sigma\sigma'}\varepsilon^*_{\mu,\sigma}\varepsilon_{\nu,\sigma'} = -g_{\mu\nu}, \tag{75} $$

and to require that the covariant derivatives vanish as[4]

$$ \nabla_\mu \varepsilon_{\nu,\sigma} = 0. \tag{77} $$

Then, the normalization conditions for ${}_{\pm}W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$, Eqs. (70) and (71), can be rewritten in terms of the scalar amplitudes as

$$ \sum_{\sigma'}\xi_{\sigma\sigma'}\left({}_{\pm}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}\middle|{}_{\pm}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} = \pm\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta), \tag{78} $$

$$ \sum_{\sigma'}\xi_{\sigma\sigma'}\left({}_{\pm}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}\middle|{}_{\mp}\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} = 0. \tag{79} $$

Next, we impose canonical commutation relations to complete the canonical quantization,

$$ [W_{\mu,A}(\tau,\boldsymbol{x}_\perp,\eta),\pi_{\nu,A'}(\tau,\boldsymbol{x}'_\perp,\eta')] = ig_{\mu\nu}\delta_{AA'}\delta^2(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)\frac{\delta(\eta-\eta')}{\tau}, \tag{80} $$

$$ [W_{\mu,A}(\tau,\boldsymbol{x}_\perp,\eta),W_{\nu,A'}(\tau,\boldsymbol{x}'_\perp,\eta')] = [\pi_{\mu,A}(\tau,\boldsymbol{x}_\perp,\eta),\pi_{\nu,A'}(\tau,\boldsymbol{x}'_\perp,\eta')] = 0, \tag{81} $$

where the canonical conjugate field $\pi_{\mu,A}$ to the gluon field $W_{\mu,A}$ is given by $\pi_{\mu,A} = \delta\mathcal{L}/\delta(\nabla_\tau W^\mu_A) = -\nabla_\tau W^\dagger_{\mu,A}$. The canonical commutation relations, Eqs. (80) and (81), are equivalent to requiring the operators $c^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}, d^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ to commute as

$$ \left[c^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}, c^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right] = \left[d^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}, d^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right] = \delta_{AA'}\xi_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta), \tag{82} $$

$$ (\text{otherwise}) = 0. \tag{83} $$

From these commutation relations, Eqs. (82) and (83), the operator $c^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ ($d^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$) can be understood as an annihilation operator of a gluon at the corresponding asymptotic time with the momentums $\boldsymbol{p}_\perp, p_\eta$, the polarization σ, and the color charge $q^{(\mathrm{g})}_A$ ($-q^{(\mathrm{g})}_A$).

As in the quark case, the creation/annihilation operators at different asymptotic times do not coincide with each other and the linear relation is described by a Bogoliubov transformation given by

$$ \begin{aligned} \begin{pmatrix} c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma} \\ d^{(\mathrm{out})\dagger}_{A,-\boldsymbol{p}_\perp,-p_\eta,\sigma} \end{pmatrix} &= \sum_{\sigma''}\xi_{\sigma\sigma''}(-g^{\mu\nu})\begin{pmatrix} ({}_{+}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|W_{\nu,A})_{\mathrm{B}} \\ -({}_{-}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|W_{\nu,A})_{\mathrm{B}} \end{pmatrix} \\ &= \sum_{\sigma'}\int d^2\boldsymbol{p}'_\perp dp'_\eta \Bigg\{\sum_{\sigma''}\xi_{\sigma\sigma''}(-g^{\mu\nu}) \\ &\quad\times \begin{pmatrix} ({}_{+}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{+}W^{(\mathrm{in})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} & ({}_{+}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{-}W^{(\mathrm{in})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} \\ -({}_{-}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{+}W^{(\mathrm{in})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} & -({}_{-}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{-}W^{(\mathrm{in})}_{\nu,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}} \end{pmatrix}\begin{pmatrix} c^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'} \\ d^{(\mathrm{in})\dagger}_{A,-\boldsymbol{p}'_\perp,-p'_\eta,\sigma'} \end{pmatrix}\Bigg\}. \end{aligned} \tag{84} $$

[4]One can always construct such a polarization vector by contracting the viervein matrix $e^m{}_\mu$ with a constant vector $\tilde{\varepsilon}_{m,\sigma}$ normalized as

$$ \eta^{mn}\tilde{\varepsilon}^*_{m,\sigma}\tilde{\varepsilon}_{n,\sigma'} = -\xi_{\sigma\sigma'}, \qquad \sum_{\sigma,\sigma'}\xi_{\sigma\sigma'}\tilde{\varepsilon}^*_{m,\sigma}\tilde{\varepsilon}_{n,\sigma'} = -\eta_{mn}. \tag{76} $$

In order to obtain the gluon spectrum at $t \to \infty$, let us introduce a gluon number density operator $n^{(\mathrm{g})}_{\pm A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ by

$$n^{(\mathrm{g})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv c^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma} c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}, \qquad n^{(\mathrm{g})}_{-A,\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv d^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma} d^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}. \tag{85}$$

As in the quark spectrum [Eqs. (52) and (53)], one can derive the gluon spectrum as an expectation value of the number density operators in the initial state as

$$\begin{aligned}\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_\eta} &\equiv \frac{\left\langle \mathrm{vac;in}|n^{(\mathrm{g})}_{\pm A,\boldsymbol{p}_\perp,p_\eta,\sigma}|\mathrm{vac;in}\right\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle} \\ &= \sum_{\sigma_1\sigma_2}\xi_{\sigma_1\sigma_2}\int d^2\boldsymbol{p}'_\perp dp'_\eta \Bigg\{\sum_{\sigma'_1\sigma'_2}\xi_{\sigma\sigma'_1}\xi_{\sigma\sigma'_2}(-g^{\mu_1\nu_1})(-g^{\mu_2\nu_2}) \\ &\quad\times \left({}_\pm W^{(\mathrm{out})}_{\mu_1,A,\boldsymbol{p}_\perp,p_\eta,\sigma'_1}\middle|{}_\mp W^{(\mathrm{in})}_{\nu_1,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma_1}\right)_{\mathrm{B}}\left({}_\mp W^{(\mathrm{in})}_{\nu_2,A,\boldsymbol{p}'_\perp,p'_\eta,\sigma_2}\middle|{}_\pm W^{(\mathrm{out})}_{\mu_2,A,\boldsymbol{p}_\perp,p_\eta,\sigma'_2}\right)_{\mathrm{B}}\Bigg\}.\end{aligned} \tag{86}$$

We note that only gluons from the quantum fluctuation are counted in Eq. (86) and there are no contributions from those from the classical background field. This treatment is justified only for the gluon spectrum at $t \to \infty$, where the classical background field is vanishing. If one is interested in the gluon spectrum at transient times $|t| < \infty$, where the classical background field is still present, then one has to count not only quantum gluons but also classical gluons in some way.

One can perform the polarization sum in this formula, Eq. (86), with the help of the decomposition, Eq. (74). Inserting the decomposition, Eq. (74), into Eq. (86), one obtains

$$\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_\eta} = \sum_{\sigma_1\sigma_2\sigma'_1\sigma'_2}\int d^2\boldsymbol{p}'_\perp dp'_\eta \xi_{\sigma_1\sigma_2}\xi_{\sigma\sigma'_1}\xi_{\sigma\sigma'_2}\xi_{\sigma_1\sigma'_1}\xi_{\sigma_2\sigma'_2}\left({}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma'_1}\middle|{}_\mp\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma_1}\right)_{\mathrm{B}}\left({}_\mp\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma_2}\middle|{}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma'_2}\right)_{\mathrm{B}}, \tag{87}$$

where use is made of the normalization condition for the polarization vector, Eq. (75). By noting that the indefinite metric $\xi_{\sigma\sigma'}$ has an off-diagonal structure as defined in Eq. (73), one finally finds

$$\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_\eta} = \begin{cases} \int d^2\boldsymbol{p}'_\perp dp'_\eta|({}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_\mp\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma})_{\mathrm{B}}|^2 & \text{for } \sigma = 1,2 \\ 0 & \text{for } \sigma = 0,3 \end{cases}. \tag{88}$$

It is now evident that gluons with the zeroth and the third polarizations vanish. This is consistent with our expectation that only two out of four polarization modes of gluons are physical. We stress that deriving the particle spectrum is thus reduced to finding out the mode functions ${}_\pm W^{(\mathrm{as})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$, or ${}_\pm\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$, by solving the equation of motion, Eq. (38), nonperturbatively with respect to the classical field.

Unlike the quark case, the phase space density can exceed unity because bosons are not subject to the Pauli principle. Indeed, one can show from the normalization condition for ${}_\pm\Phi^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ that the inner products between ${}_\pm\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ and ${}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ are normalized as

$$\frac{1}{(2\pi)^3}\int d^2\boldsymbol{x}_\perp\int d\eta = \int d^2\boldsymbol{p}'_\perp dp'_\eta\left[\left|\left({}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}\middle|{}_\pm\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma}\right)_{\mathrm{B}}\right|^2 - \left|\left({}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}\middle|{}_\mp\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma}\right)_{\mathrm{B}}\right|^2\right] \tag{89}$$

for the physical polarization modes $\sigma = 1, 2$. One finds that the inner products $|({}_\pm\Phi^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_\mp\Phi^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta,\sigma'})_{\mathrm{B}}|^2$ or the phase space density is not bounded because of the $-$ sign in Eq. (89). Notice that for the quark case, Eq. (54), we have a $+$ sign, which reflects the statistics of particles, i.e., $+$ for fermions and $-$ for bosons.

Finally, let us connect the p_η spectrum, Eq. (86), to the p_z spectrum and/or the momentum rapidity $y_{\boldsymbol{p}}$ spectrum as was done in the quark case. The p_z spectrum and/or the momentum rapidity $y_{\boldsymbol{p}}$ spectrum are obtained as an expectation value of the number operator $n^{(\mathrm{g})}_{\pm A,\boldsymbol{p}_\perp,p_z,\sigma}$, which are labeled by p_z instead of p_η, as

$$\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_z} = \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \equiv \frac{\langle \mathrm{vac;in}|n^{(\mathrm{g})}_{\pm A,\boldsymbol{p}_\perp,p_z,\sigma}|\mathrm{vac;in}\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle}, \quad (90)$$

where the number operator $n^{(\mathrm{g})}_{\pm A,\boldsymbol{p}_\perp,p_z,\sigma}$ is defined by the annihilation operators $c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}, d^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}$ as

$$\begin{aligned} n^{(\mathrm{g})}_{A,\boldsymbol{p}_\perp,p_z,\sigma} &\equiv c^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_z,\sigma} c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}, \\ n^{(\mathrm{g})}_{-A,\boldsymbol{p}_\perp,p_z,\sigma} &\equiv d^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_z,\sigma} d^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}. \end{aligned} \quad (91)$$

The annihilation operators $c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}, d^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma}$ are defined by expanding the gluon operator $W_{m,A} = e^{\mu}{}_{m} W_{\mu,A}$ in terms of positive/negative frequency mode functions ${}_{\pm}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}$ in the Cartesian coordinates, which are labeled by p_z being the Fourier conjugate to z as

$$\begin{aligned} W_{m,A}(x) = \sum_\sigma \int d\boldsymbol{p}_\perp^2 dp_z \Big[{}_{+}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}(x) c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma} \\ + {}_{-}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}(x) d^{(\mathrm{out})\dagger}_{A,-\boldsymbol{p}_\perp,-p_z,\sigma}\Big]. \end{aligned} \quad (92)$$

Here, we again require the plane wave boundary condition ${}_{\pm}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}$ at $t \to \infty$. As is shown in Appendix A 2 a, if properly normalized, the mode functions in the Cartesian coordinates ${}_{\pm}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}$ and those in the τ-η coordinates ${}_{\pm}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma}$ are related with each other by an integral transformation described by

$${}_{\pm}W^{(\mathrm{out})}_{\mu,A,\boldsymbol{p}_\perp,p_\eta,\sigma} = e^{m}{}_{\mu} \int dp_z \frac{\mathrm{e}^{\pm i p_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}} {}_{\pm}W^{(\mathrm{out})}_{m,A,\boldsymbol{p}_\perp,p_z,\sigma}. \quad (93)$$

Using this integral transformation, Eq. (93), and comparing the expansion in the Cartesian coordinates, Eq. (92), with that in the τ-η coordinates, Eq. (67), one finds

$$c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_z,\sigma} = \int dp_\eta \frac{\mathrm{e}^{i p_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}} c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}, \quad (94)$$

$$d^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_z,\sigma} = \int dp_\eta \frac{\mathrm{e}^{-i p_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}} d^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}. \quad (95)$$

Inserting these relations, Eqs. (94) and (95), back into Eq. (90), one obtains

$$\begin{aligned} \frac{d^3N^{(\mathrm{g})}_{A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{g})}_{A,\sigma}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta dp'_\eta \frac{\mathrm{e}^{iy_{\boldsymbol{p}}(p_\eta - p'_\eta)}}{2\pi} \\ &\quad \times \frac{\left\langle \mathrm{vac;in}|c^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p'_\eta,\sigma} c^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}|\mathrm{vac;in}\right\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle}, \end{aligned} \quad (96)$$

$$\begin{aligned} \frac{d^3N^{(\mathrm{g})}_{-A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{g})}_{-A,\sigma}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta dp'_\eta \frac{\mathrm{e}^{iy_{\boldsymbol{p}}(p_\eta - p'_\eta)}}{2\pi} \\ &\quad \times \frac{\left\langle \mathrm{vac;in}|d^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p'_\eta,\sigma} d^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}|\mathrm{vac;in}\right\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle}. \end{aligned} \quad (97)$$

When the system is perfectly boost invariant, the expectation values in Eqs. (96) and (97) for $p_\eta \neq p'_\eta$ vanish as in the quark case, and one finally obtains

$$\begin{aligned} \frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_z} &= \frac{1}{\omega_{\boldsymbol{p}}}\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dy_{\boldsymbol{p}}} \\ &= \frac{1}{2\pi\delta(p_\eta = 0)}\frac{1}{\omega_{\boldsymbol{p}}}\int dp_\eta \frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d\boldsymbol{p}_\perp^2 dp_\eta}. \end{aligned} \quad (98)$$

3. Ghost

Finally, we consider the canonical quantization of the ghost and antighost fields, C_A and $\bar{C}_A$, and show that ghosts are never produced from the classical field. We do essentially the same procedure as what we did in the previous quark and gluon cases.

We first expand the ghost and antighost fields, C_A and $\bar{C}_A$, as

$$\begin{aligned} \begin{pmatrix} C_A \\ \bar{C}_A \end{pmatrix} = \int d\boldsymbol{p}_\perp^2 dp_\eta \Bigg[{}_{+}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta} \begin{pmatrix} e^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta} \\ \bar{e}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta} \end{pmatrix} \\ + {}_{-}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta} \begin{pmatrix} f^{(\mathrm{as})\dagger}_{A,-\boldsymbol{p}_\perp,-p_\eta} \\ \bar{f}^{(\mathrm{as})\dagger}_{A,-\boldsymbol{p}_\perp,-p_\eta} \end{pmatrix} \Bigg], \end{aligned} \quad (99)$$

where the labels are the same as in the previous two cases. The mode functions ${}_{\pm}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$ are the solutions of the equations of motion, Eq. (39), with the plane wave boundary condition at $t \to \pm\infty$,

$${}_{\pm}\Theta^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta} \underset{t\to-\infty}{\longrightarrow} {}_{\pm}\Theta^{(\mathrm{free})}_{A,\boldsymbol{p}_\perp,p_\eta}[\bar{A}_\mu(t \to -\infty)], \quad (100)$$

$$ {}_{\pm}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta} \xrightarrow[t\to\infty]{} {}_{\pm}\Theta^{(\mathrm{free})}_{A,\boldsymbol{p}_\perp,p_\eta}[\bar{A}_\mu(t\to\infty)], \tag{101} $$

where the plane wave solutions ${}_{\pm}\Theta^{(\mathrm{free})}_{A,\boldsymbol{p}_\perp,p_\eta}[\breve{A}_\mu]$ satisfy the free field equation of motion under a pure gauge background field $\breve{A}_\mu = \bar{A}_\mu(t\to\pm\infty)$. For details of the plane wave solutions ${}_{\pm}\Theta^{(\mathrm{free})}_{A,\boldsymbol{p}_\perp,p_\eta}$, see Appendix A 3 a. The normalization conditions for the positive/negative frequency mode functions ${}_{\pm}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$ are

$$ \left({}_{\pm}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{\pm}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}} = \pm\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta), \tag{102} $$

$$ \left({}_{\pm}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{\mp}\Theta^{(\mathrm{as})}_{A,\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}} = 0. \tag{103} $$

Next, we canonically quantize the fluctuations by imposing canonical commutation relations,

$$ \{\overset{(-)}{C}_A(\tau,\boldsymbol{x}_\perp,\eta), \overset{(-)}{\pi}_{A'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = i\delta_{AA'}\delta^2(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)\frac{\delta(\eta-\eta')}{\tau}, \tag{104} $$

$$ \{\overset{(-)}{C}_A(\tau,\boldsymbol{x}_\perp,\eta), \overset{(-)}{C}_{A'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = \{\overset{(-)}{\pi}_A(\tau,\boldsymbol{x}_\perp,\eta), \overset{(-)}{\pi}_{A'}(\tau,\boldsymbol{x}'_\perp,\eta')\} = 0, \tag{105} $$

where the canonical conjugate fields to the ghost and antighost fields, C_A and $\bar{C}_A$, are given by $\pi_A = \delta\mathcal{L}/\delta(\partial_\tau C_A) = -i\partial_\tau\bar{C}^\dagger_A$ and $\bar{\pi}_A = \delta\mathcal{L}/\delta(\partial_\tau\bar{C}_A) = i\partial_\tau C^\dagger_A$, respectively. As a result of the canonical commutation relations, one finds the following anticommutation relations for the operators $e^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, \bar{e}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, f^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, \bar{f}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$ given by

$$ \left\{e^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, \bar{e}^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta}\right\} = \left\{f^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, \bar{f}^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta}\right\} = i\delta_{AA'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta), \tag{106} $$

$$ (\text{otherwise}) = 0. \tag{107} $$

Now, one can understand that the operators, $e^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, f^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$ ($\bar{e}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}, \bar{f}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$), act as annihilation operators of a ghost (an antighost) at the corresponding asymptotic time with the momentums $\boldsymbol{p}_\perp$, p_η and the color charges $q^{(\mathrm{gh})}_A, -q^{(\mathrm{gh})}_A$.

As is seen in the previous two cases, the creation/annihilation operators at different asymptotic times do not coincide with each other, and the linear relation is given by the following Bogoliubov transformation:

$$ \begin{pmatrix} \overset{(-)}{e}{}^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta} \\ \overset{(-)}{f}{}^{(\mathrm{out})\dagger}_{A,-\boldsymbol{p}_\perp,-p_\eta} \end{pmatrix} = \begin{pmatrix} ({}_{+}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|\overset{(-)}{C}_A)_{\mathrm{B}} \\ -({}_{-}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|\overset{(-)}{C}_A)_{\mathrm{B}} \end{pmatrix} $$
$$ = \int d^2\boldsymbol{p}'_\perp dp'_\eta \begin{pmatrix} \left({}_{+}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{+}\Theta^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta}\right)_{\mathrm{B}} & \left({}_{+}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{-}\Theta^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta}\right)_{\mathrm{B}} \\ -\left({}_{-}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{+}\Theta^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta}\right)_{\mathrm{B}} & -\left({}_{-}\Theta^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}|{}_{-}\Theta^{(\mathrm{in})}_{A,\boldsymbol{p}_\perp,p_\eta}\right)_{\mathrm{B}} \end{pmatrix} \begin{pmatrix} \overset{(-)}{e}{}^{(\mathrm{in})}_{A,\boldsymbol{p}'_\perp,p'_\eta} \\ \overset{(-)}{f}{}^{(\mathrm{in})\dagger}_{A,-\boldsymbol{p}'_\perp,-p'_\eta} \end{pmatrix}. \tag{108} $$

In order to obtain the ghost and antighost spectra at $t\to\infty$, let us introduce ghost and antighost number density operators $n^{(\mathrm{gh})}_{\pm A,\boldsymbol{p}_\perp,p_\eta}$ and $n^{(\bar{\mathrm{gh}})}_{\pm A,\boldsymbol{p}_\perp,p_\eta}$, respectively, by

$$ n^{(\mathrm{gh})}_{A,\boldsymbol{p}_\perp,p_\eta} \equiv e^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta}e^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}, \qquad n^{(\mathrm{gh})}_{-A,\boldsymbol{p}_\perp,p_\eta} \equiv f^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta}f^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta} \tag{109} $$

and

$$ n^{(\bar{\mathrm{gh}})}_{A,\boldsymbol{p}_\perp,p_\eta} \equiv \bar{e}^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta}\bar{e}^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}, \qquad n^{(\bar{\mathrm{gh}})}_{-A,\boldsymbol{p}_\perp,p_\eta} \equiv \bar{f}^{(\mathrm{out})\dagger}_{A,\boldsymbol{p}_\perp,p_\eta}\bar{f}^{(\mathrm{out})}_{A,\boldsymbol{p}_\perp,p_\eta}. \tag{110} $$

As in the previous two cases, the ghost and the antighost spectra can be derived as an expectation value of the number density operators. By using the Bogoliubov transformation, Eq. (108), and the fact that the commutation relation between $\overset{(-)}{e}{}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$ ($\overset{(-)}{f}{}^{(\mathrm{as})}_{A,\boldsymbol{p}_\perp,p_\eta}$) and $\overset{(-)}{e}{}^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta}$ ($\overset{(-)}{f}{}^{(\mathrm{as})\dagger}_{A',\boldsymbol{p}'_\perp,p'_\eta}$) vanishes because of the anticommutation relations, Eqs. (106) and (107), one finds

$$ \frac{d^3N^{(\mathrm{gh})}_{\pm A}}{d\boldsymbol{p}^2_\perp dp_\eta} = \frac{\langle \mathrm{vac;in}|n^{(\mathrm{gh})}_{\pm A,\boldsymbol{p}_\perp,p_\eta}|\mathrm{vac;in}\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle} = 0, \tag{111} $$

$$\frac{d^3N^{(\bar{\mathrm{gh}})}_{\pm A}}{d\boldsymbol{p}_\perp^2 dp_\eta} = \frac{\langle \mathrm{vac;in}| n^{(\bar{\mathrm{gh}})}_{\pm A,\boldsymbol{p}_\perp,p_\eta g} |\mathrm{vac;in}\rangle}{\langle \mathrm{vac;in}|\mathrm{vac;in}\rangle} = 0. \qquad (112)$$

That is, ghosts and antighosts are never produced from the classical field $\bar{A}_\mu$. This is a reasonable result because ghosts and antighosts are unphysical particles and they never appear in the physical spectrum. In general, the right-hand side (RHS) is always zero for any physical initial state $|\mathrm{phys;in}\rangle$ because any physical state does not contain ghosts or antighosts.

III. PARTICLE PRODUCTION FROM AN EXPANDING COLOR ELECTRIC FIELD

In Sec. II, we have shown, at the one-loop level quantum calculation and within the Abelian dominance assumption for the classical background field $\bar{A}_\mu$, that the particle spectra are obtained by solving the equations of motion of QCD nonperturbatively with respect to the classical field. In principle, the equations of motion are solvable; i.e., the particle spectra are computable for any $\bar{A}_\mu$ with arbitrary spacetime dependence, for instance, by using numerical methods. However, before going into more realistic calculations, where $\bar{A}_\mu$ has a complicated spacetime dependence, we consider a simple situation, where analytic solutions of the equations of motion are available. This enables us to get more insights on the particle production in QCD in an expanding system. In particular, we consider a spatially homogeneous and constant classical color electric background field with finite lifetime T in a boost-invariantly expanding geometry, i.e., $\boldsymbol{E} = \boldsymbol{e}_z E\theta(\tau)\theta(T-\tau), \boldsymbol{B} = \boldsymbol{0}$ given by a gauge potential

$$\tilde{A}_\tau, \tilde{A}_x, \tilde{A}_y = 0, \quad \tilde{A}_\eta = \begin{cases} E\tau^2/2 & (0<\tau<T) \\ ET^2/2 & (T<\tau) \end{cases}. \qquad (113)$$

As is explained in Appendix A, the analytical formula for the particle spectra become

$$\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}} = \frac{d^3N^{(\bar{\mathrm{q}})}_{i,f,s}}{d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}} = \frac{S_\perp}{(2\pi)^3}\int dp_\eta \left| A^{(\mathrm{q})}_{i,f,\boldsymbol{p}_\perp,p_\eta - q_i^{(\mathrm{q})}ET^2/2,s}(0) B^{(\mathrm{q})*}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}(T) - B^{(\mathrm{q})*}_{i,f,\boldsymbol{p}_\perp,p_\eta - q_i^{(\mathrm{q})}ET^2/2,s}(0) A^{(\mathrm{q})}_{i,f,\boldsymbol{p}_\perp,p_\eta,s}(T)\right|^2 \qquad (114)$$

for quarks and antiquarks,

$$\frac{d^3N^{(\mathrm{g})}_{\pm A,\sigma}}{d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}} = \frac{S_\perp}{(2\pi)^3}\int dp_\eta \left| A^{(\mathrm{g})}_{A,\boldsymbol{p}_\perp,p_\eta - q_A^{(\mathrm{g})}ET^2/2,\sigma}(0) B^{(\mathrm{g})*}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}(T) - B^{(\mathrm{g})*}_{A,\boldsymbol{p}_\perp,p_\eta - q_A^{(\mathrm{g})}ET^2/2,\sigma}(0) A^{(\mathrm{g})}_{A,\boldsymbol{p}_\perp,p_\eta,\sigma}(T)\right|^2 \qquad (115)$$

for physical gluons ($\sigma = 1, 2$). Here, we have used $\delta^2(\boldsymbol{p}_\perp = \boldsymbol{0}) = S_\perp/(2\pi)^2$ with $S_\perp$ being the transverse area. For the explicit expressions for the Bogoliubov coefficients $A^{(\mathrm{q})}, B^{(\mathrm{q})}$ and $A^{(\mathrm{g})}, B^{(\mathrm{g})}$, see Appendix A 1 c and Appendix A 2 c, respectively. Notice that unphysical gluons ($\sigma = 0, 3$) and ghosts are never produced as shown in Eq. (88), and in Eqs. (111) and (112), and hence we do not consider them hereafter. In the following, we numerically carry out the p_η integration and show the momentum-rapidity $y_{\boldsymbol{p}}$ spectra for quarks and gluons.

A. Features of particle production

In this subsection, we investigate specific features of quark and gluon production focusing on impacts of the longitudinal expansion and of finite lifetime effects. For this purpose, we treat the quark mass m_f, the coupling g, the field strength E, and the lifetime T as free parameters, and we compute the quark and gluon spectra without taking the summation of colors, i and A.

We stress that the particle spectra for a fixed color discussed here are very useful in understanding the specific features of the particle production. However, the spectra are apparently gauge-dependent, and hence one has to take the color summation in order to get physically meaningful results, which are discussed in Sec. III B.

1. Transverse distribution $d^3N/dy_{\boldsymbol{p}}d\boldsymbol{p}_\perp^2$

Figure 1 shows the transverse spectrum of quarks $d^3N^{(\mathrm{q})}_{i,f,s}/d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}$ (left) and of gluons $d^3N^{(\mathrm{g})}_{A,\sigma}/d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}$ (right). We observe that, for long lifetimes $\sqrt{|q_i^{(\mathrm{q})}E|}T$, $\sqrt{|q_A^{(\mathrm{g})}E|}T \gtrsim 1$, both spectra approach Gaussian distributions multiplied by a square of the lifetime T. This is consistent with what we naively expect from the Schwinger formula,

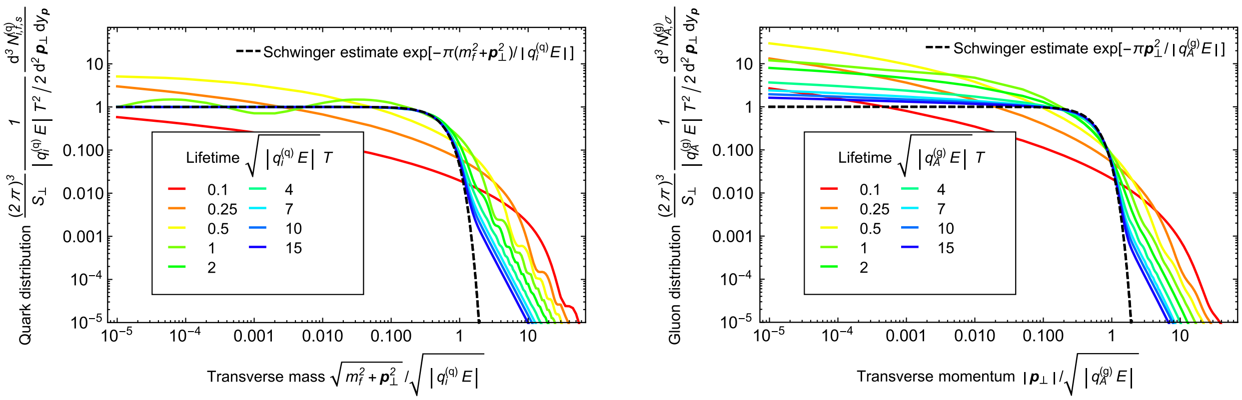

FIG. 1. Transverse distribution of quarks (left) and gluons (right) for various lifetimes T. The dashed lines are expectations from the Schwinger formula, Eqs. (118) and (119).

$$\left.\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d^2\boldsymbol{p}_\perp dp_z}\right|_{\text{Schwinger}} = \frac{V}{(2\pi)^3}\exp\left[-\pi\frac{m_{\mathrm{f}}^2+\boldsymbol{p}_\perp^2}{|q_i^{(\mathrm{q})}E|}\right], \quad (116)$$

$$\left.\frac{d^3N^{(\mathrm{g})}_{A,\sigma}}{d^2\boldsymbol{p}_\perp dp_z}\right|_{\text{Schwinger}} = \frac{V}{(2\pi)^3}\exp\left[-\pi\frac{\boldsymbol{p}_\perp^2}{|q_A^{(\mathrm{g})}E|}\right]. \quad (117)$$

Indeed, for large values of T, the produced particles are sufficiently accelerated by the electric field as $\omega_{\boldsymbol{p}} \sim |p_z| \sim |qA_z|$ ($q = q_i^{(\mathrm{q})}$ for quarks and $q = q_A^{(\mathrm{g})}$ for gluons), and $A_z \sim A_\eta/T = ET/2$ and $V \sim S_\perp T$ hold. Thus, we find

$$\left.\frac{d^3N^{(\mathrm{q})}_{i,f,s}}{d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}}\right|_{\text{Schwinger}} \sim \frac{S_\perp}{(2\pi)^3}\frac{|q_i^{(\mathrm{q})}E|T^2}{2}\exp\left[-\pi\frac{m_{\mathrm{f}}^2+\boldsymbol{p}_\perp^2}{|q_i^{(\mathrm{q})}E|}\right], \quad (118)$$

$$\left.\frac{d^3N^{(\mathrm{g})}_{A,\sigma}}{d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}}\right|_{\text{Schwinger}} \sim \frac{S_\perp}{(2\pi)^3}\frac{|q_A^{(\mathrm{g})}E|T^2}{2}\exp\left[-\pi\frac{\boldsymbol{p}_\perp^2}{|q_A^{(\mathrm{g})}E|}\right], \quad (119)$$

which are plotted in the dashed lines as the "Schwinger estimate" in Fig. 1. On the other hand, for short lifetimes $\sqrt{|q_i^{(\mathrm{q})}E|}T, \sqrt{|q_A^{(\mathrm{g})}E|}T \lesssim 1$, the spectra are harder compared to those for larger lifetimes and do not decay exponentially in $|\boldsymbol{p}_\perp|$ because the typical frequency $\omega \sim 1/T$ of the classical electric field is hard enough to excite hard particles. In other words, a naive application of the Schwinger formula is valid only for large values of the lifetime T, while one should take care of finite lifetime effects for small values of T.

In the low momentum region $|\boldsymbol{p}_\perp| \lesssim \sqrt{|q_i^{(\mathrm{q})}E|}$, $\sqrt{|q_A^{(\mathrm{g})}E|}$, gluons are more abundant than quarks. This is because the quark production is subject to the Pauli principle but the gluon production is not. The gluon spectrum shows a weak divergence for $|\boldsymbol{p}_\perp| \to 0$ but its inverse power is smaller than one, and it approaches zero with increasing the lifetime T.

2. Number density dN/dy

We numerically integrate the transverse distributions over $\boldsymbol{p}_\perp$ to compute the total number of produced particles per unit rapidity for quarks $dN^{(\mathrm{q})}_{i,f,s}/dy_{\boldsymbol{p}}$ and for gluons $dN^{(\mathrm{g})}_{A,\sigma}/dy_{\boldsymbol{p}}$.

The left panel of Fig. 2 shows the total number of massless quarks $dN^{(\mathrm{q})}_{i,f,s}/dy_{\boldsymbol{p}}$ and of gluons $dN^{(\mathrm{g})}_{A,\sigma}/dy_{\boldsymbol{p}}$. Here, we artificially set $|q| = |q_i^{(\mathrm{q})}| = |q_A^{(\mathrm{g})}|$ for the quark and gluon charges. For long lifetimes $\sqrt{|qE|}T \gtrsim 1$, one finds that the quark and gluon productions are consistent with the Schwinger formula: By integrating Eqs. (118) and (119) over $\boldsymbol{p}_\perp$, the Schwinger formula gives

$$\left.\frac{dN^{(\mathrm{q})}_{i,f,s}}{dy_{\boldsymbol{p}}}\right|_{\text{Schwinger}} \sim \frac{S_\perp}{(2\pi)^3}\frac{|q_i^{(\mathrm{q})}E|^2T^2}{2}\exp\left[-\pi\frac{m_{\mathrm{f}}^2}{|q_i^{(\mathrm{q})}E|}\right], \quad (120)$$

$$\left.\frac{dN^{(\mathrm{g})}_{\pm A,\sigma}}{dy_{\boldsymbol{p}}}\right|_{\text{Schwinger}} \sim \frac{S_\perp}{(2\pi)^3}\frac{|q_A^{(\mathrm{g})}E|^2T^2}{2}, \quad (121)$$

which are plotted in the dashed lines in the left panel of Fig 2. For short lifetimes $\sqrt{|qE|}T \lesssim 1$, one observes that the quark and gluon production are more abundant than the Schwinger estimates. This is because the typical frequency $\omega \sim 1/T$ of the electric field for such small values of T becomes so hard that a large number of hard particles are produced as was discussed in Fig. 1, for which the phase space is larger than those for soft particles expected from the Schwinger formula. For classical background fields with such hard frequencies,

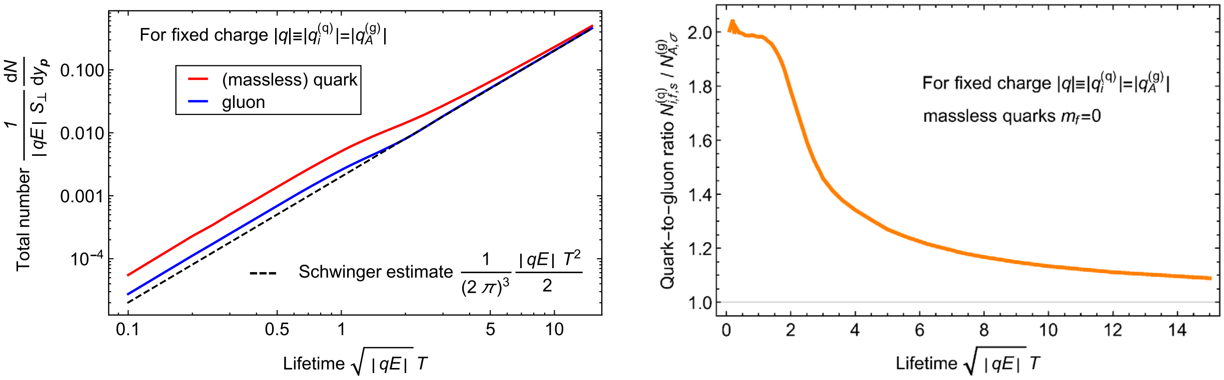

FIG. 2. Left: Total number of massless quarks (red line) and of gluons (blue line) per unit rapidity for $|q_i^{(\mathrm{q})}| = |q_A^{(\mathrm{g})}|$. The dashed line is an expectation from the Schwinger formula, Eqs. (120) and (121). Right: A ratio of the total number of produced massless quarks to that of gluons $N_{i,f,s}^{(\mathrm{q})}/N_{A,\sigma}^{(\mathrm{g})}$ for $|q_i^{(\mathrm{q})}| = |q_A^{(\mathrm{g})}|$.

perturbative particle production from a single classical background field gives a better description than Schwinger's nonperturbative particle production mechanism [30], and hence this enhancement is purely a perturbative effect.

For all values of T, we observe that the massless quark production is more abundant than the gluon production for $|q_i^{(\mathrm{q})}| = |q_A^{(\mathrm{g})}|$. This aspect is more clearly illustrated in the right panel of Fig. 2: For long lifetimes $\sqrt{|qE|}T \gtrsim 1$, the ratio of the produced quarks to that of gluons $N_{i,f,s}^{(\mathrm{q})}/N_{A,\sigma}^{(\mathrm{g})}$ approaches unity because both quarks and gluons are produced via Schwinger's nonperturbative particle production mechanism, in which the statistics of particles are irrelevant as is seen in Eqs. (120) and (121). For short lifetimes $\sqrt{|qE|}T \lesssim 1$, however, the ratio deviates from unity. This is because, for such small values of T, Schwinger's nonperturbative particle production mechanism is not efficient but perturbative particle production occurs, which depends on the statistics of particles in general. It is interesting to point out that the ratio is always larger than unity so that quarks are more abundantly produced than gluons. This is because quark spectrum is harder than gluon one for small values of T due to the statistics of particles as we saw in Fig. 1 and the phase space for produced quarks becomes larger than that of gluons.

In the $T \to 0$ limit, the ratio amounts to nearly two. In order to convince ourselves that this number "two" is correct and that this enhancement is indeed a perturbative phenomenon due to the finite lifetime effects, we consider a nonexpanding, spatially homogeneous but time-dependent electric field, $\boldsymbol{E} = E(t)\boldsymbol{e}_z$, as an example for a moment. In this case, one can analytically compute S-matrix elements, $\langle \mathrm{q}_{i,f,\boldsymbol{p}_\perp,p_z,s}\bar{\mathrm{q}}_{i',f',\boldsymbol{p}'_\perp,p'_z,s'};\mathrm{in}|S|\mathrm{vac};\mathrm{in}\rangle$ and $\langle \mathrm{g}_{A,\boldsymbol{p}_\perp,p_z,\sigma}\mathrm{g}_{A',\boldsymbol{p}'_\perp,p'_z,\sigma'};\mathrm{in}|S|\mathrm{vac};\mathrm{in}\rangle$, in the lowest order perturbation theory with respect to the classical background field $E(t)$. After some manipulations, one obtains

$$\begin{aligned}\frac{N_{i,f,s}^{(\mathrm{q})}}{V} &= \sum_{i',f',s'}\int d^3\boldsymbol{p}_\perp dp_z \int d^2\boldsymbol{p}'_\perp dp'_z \\ &\quad\times |\langle \mathrm{q}_{i,f,\boldsymbol{p}_\perp,p_z,s}\bar{\mathrm{q}}_{i',f',\boldsymbol{p}'_\perp,p'_z,s'};\mathrm{in}|S|\mathrm{vac};\mathrm{in}\rangle|^2 \\ &= \frac{1}{24\pi^2}\int_{2m_\mathrm{f}}^\infty d\omega\sqrt{1-\frac{4m_\mathrm{f}^2}{\omega^2}}\left(1+\frac{2m_\mathrm{f}^2}{\omega^2}\right)|q_i^{(\mathrm{q})}\tilde{E}(\omega)|^2 \\ &\xrightarrow[m_\mathrm{f}=0]{} \frac{1}{24\pi^2}\int_0^\infty d\omega |q_i^{(\mathrm{q})}\tilde{E}(\omega)|^2 \end{aligned} \tag{122}$$

and

$$\begin{aligned}\frac{N_{A,\sigma}^{(\mathrm{g})}}{V} &= \sum_{\pm A',\sigma'}\int d^3\boldsymbol{p}_\perp dp_z \int d^2\boldsymbol{p}'_\perp dp'_z \\ &\quad\times |\langle \mathrm{g}_{A,\boldsymbol{p}_\perp,p_z,\sigma}\mathrm{g}_{A',\boldsymbol{p}'_\perp,p'_z,\sigma'};\mathrm{in}|S|\mathrm{vac};\mathrm{in}\rangle|^2 \\ &= \frac{1}{48\pi^2}\int_0^\infty d\omega |q_A^{(\mathrm{g})}\tilde{E}(\omega)|^2, \end{aligned} \tag{123}$$

where $\tilde{E}(\omega)$ is the Fourier transformation of the electric field $\tilde{E}(\omega) \equiv \int dt E(t)\mathrm{e}^{i\omega t}$. Thus, $N_{i,f,s}^{(\mathrm{q})}/N_{A,\sigma}^{(\mathrm{g})} = 2|q_i^{(\mathrm{q})}|^2/|q_A^{(\mathrm{g})}|^2$ holds for massless quarks.

Figure 3 shows quark mass dependences of the quark production: The total quark number $dN_{i,f,s}^{(\mathrm{q})}/dy_{\boldsymbol{p}}$ (left) and the ratio of the total number of massive quarks to that of massless quarks (right) for several different values of the quark mass are plotted. One finds the following:

(i) For short lifetimes $\sqrt{|q_i^{(\mathrm{q})}E|}T \lesssim 1$, the total quark production number becomes independent of the quark mass and the ratio comes close to one because the typical energy scale of the classical electric field, which is characterized by its typical frequency $\omega \sim 1/T$, is much larger than the quark mass scale.

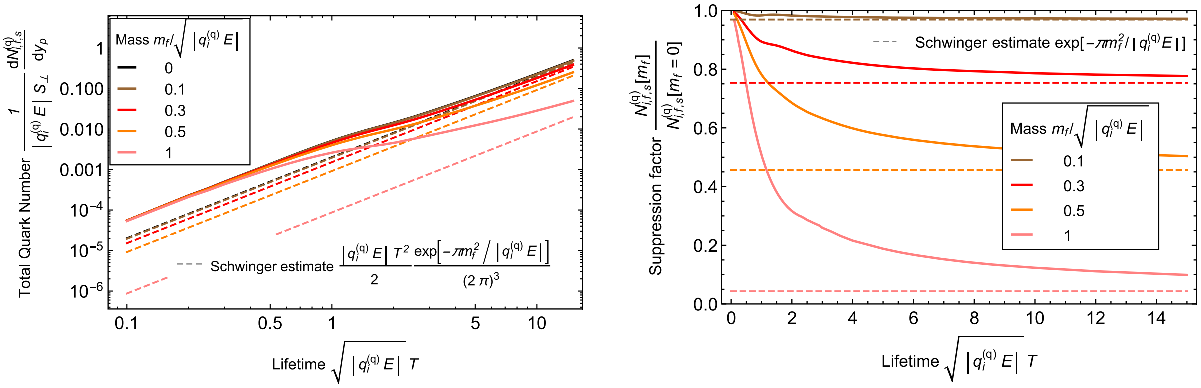

FIG. 3. Left: Total number of produced quarks per unit rapidity for various quark masses. The dashed lines are expectations from the Schwinger formula, Eq. (120). Right: The ratio of the total number of produced massive quarks to that of massless quarks $N^{(q)}_{i,f,s}[m_f]/N^{(q)}_{i,f,s}[m_f=0]$. The dashed lines are expectations from the Schwinger formula: $\exp[-\pi m_f^2/|q_i^{(q)}E|]$.

(ii) For long lifetimes $\sqrt{|q_i^{(q)}E|}T \gtrsim 1$, the total quark production number approaches the expectation of the Schwinger formula, Eq. (120), and the ratio starts to be suppressed exponentially with respect to the quark mass as $\exp[-\pi m_f^2/|q_i^{(q)}E|]$. Notice that the quark production is still always larger than Schwinger's value. This is because it needs a long lifetime T to justify Schwinger's nonperturbative particle production mechanism because of the finite lifetime effects.

(iii) The larger lifetime T is required for heavier quark production to converge to Schwinger's estimate, compared to that required for lighter quarks. One can understand this observation in terms of the Keldysh parameter $\gamma_{\rm Keldysh} = q_i^{(q)}ET/m_f$ [29,30], which is one of the dimensionless parameters characterizing the interplay between Schwinger's nonperturbative particle production ($\gamma_{\rm Keldysh} \gg 1$) and perturbative particle production ($\gamma_{\rm Keldysh} \ll 1$): The Keldysh parameter $\gamma_{\rm Keldysh}$ becomes smaller for larger values of m_f, and thus it requires larger lifetimes T to realize $\gamma_{\rm Keldysh} \gg 1$.

B. Phenomenology of particle production

For discussions on more phenomenological implications, let us consider particle production with physical parameter settings: $N_c = 3$, and $m_u, m_d = 0$ GeV, $m_s = 0.1$ GeV, and $m_c = 1.2$ GeV representing the mass of up, down, strange and charm quarks, respectively. We set $gE = 1\ {\rm GeV}^2$ as a typical value at RHIC energy scale. Under this setting, we consider the inclusive particle production by summing up the color degrees of freedom, i and A. Here, we assume for simplicity that the Abelianized classical electric field [see Eq. (23)] is always directing to the t^3 direction in the color space. The particle spectra depend on this color direction in general; however, one can numerically demonstrate that its dependence is rather small [48,51].

Before showing results, let us make some remarks on the validity of our results to the early stage dynamics of HIC:

(1) Our formalism assumes the Abelian dominance (see Sec. II B). This assumption is nontrivial because one can naively expect in HIC that the non-Abelian part for the classical field strength $g\bar{A}\bar{A} \sim Q_s^2/g$ is about the same order as the Abelian one, $\partial\bar{A} \sim Q_s^2/g$. Nevertheless, it is known that the full numerical simulation of the classical Yang-Mills evolution [Eq. (18)] [5] can be understood well within the Abelian dominance assumption; i.e., effects of the non-Abelian part are rather small [45]. Hence, it may be good to assume the Abelian dominance for the first approximation.

(2) Our formalism neglects higher order quantum effects beyond one-loop order. Hence, one cannot treat scatterings and screening effects of produced particles, which are essential for the thermalization of the system. Strictly speaking, this treatment works fine when the lifetime T is not so long, where the fluctuations are small enough compared to the strength of the classical field.

(3) The classical field configuration, Eq. (113), is very simple compared to the one in realistic situations:
   (a) The classical field is assumed to be constant in time for $\tau < T$ and suddenly switched off at $\tau = T$. A realistic classical field is also finite in time; however, it should smoothly decay in time and not experience such a sudden switching off.
   (b) We only consider a purely longitudinal electric field. In realistic situations, however, not only a longitudinal electric field but also a longitudinal magnetic field can exist.
   (c) The spatial homogeneity is assumed for the classical field. A realistic classical field, however, should have spatial structure with a typical length scale $\sim 1/Q_s$ due to CGC.

Thus, the quark and gluon spectrum presented below are just a first-order approximation. To get more reliable results for the phenomenology, one has to consider the above points for the field configuration. This is numerically possible within our formalism, although we leave it for a future study. Nevertheless, we stress that the simple field configuration, Eq. (113), does capture some essential features of the strong color electromagnetic field that exists just after a collision such as the boost invariance, the finite lifetime, and the existence of the longitudinal color electric field.

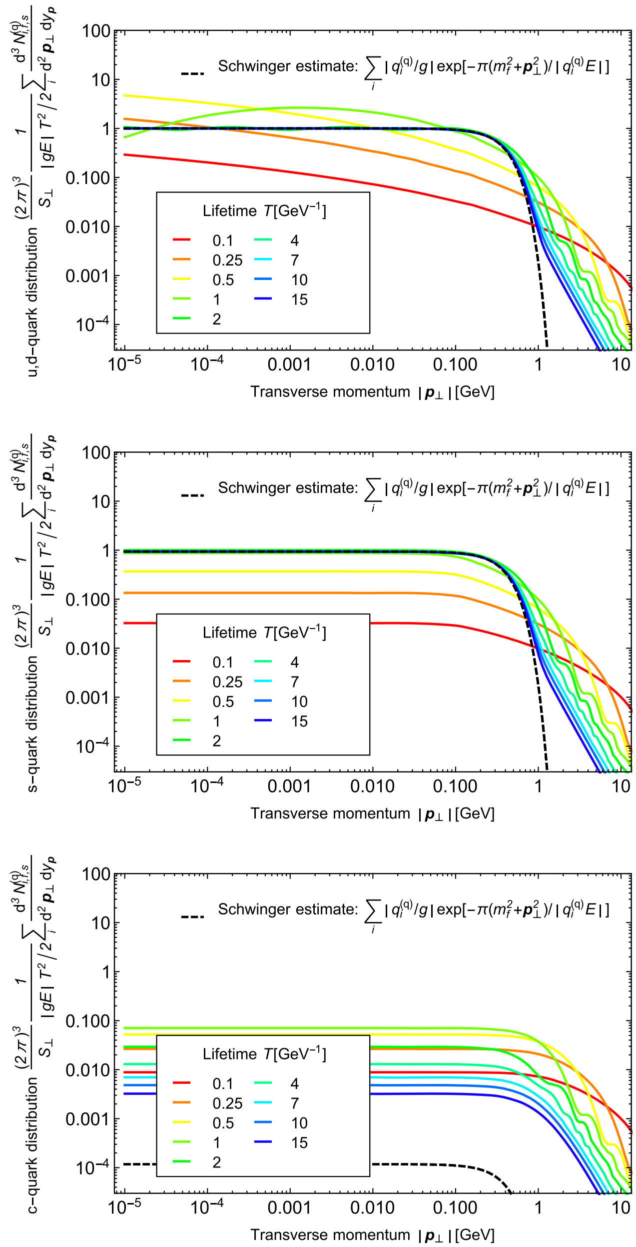

FIG. 4. Transverse distribution of quarks (top for up and down, middle for strange, and bottom for charm) for various lifetimes T. The dashed lines are expectations from the Schwinger formula, Eq. (118).

1. Transverse distribution $d^3N/dy_p dp_\perp^2$

Figures 4 and 5 show the transverse momentum spectrum of up and down (top), strange (middle), and charm quark (bottom) $\sum_i d^3N^{(\mathrm{q})}_{i,f,s}/d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}$, and that of gluons $\sum_A d^3N^{(\mathrm{g})}_{A,\sigma}/d^2\boldsymbol{p}_\perp dy_{\boldsymbol{p}}$, respectively. The dashed line in the figures represents the expectation from the Schwinger formula, Eqs. (118) and (119). We again recognize the interplay between Schwinger's nonperturbative particle production (long lifetimes $T \gtrsim 1\ \mathrm{GeV}^{-1}$) and perturbative particle production (short lifetimes $T \lesssim 1\ \mathrm{GeV}^{-1}$). This implies that finite lifetime effects are very relevant to the early stage dynamics of HIC, where the typical lifetime of the strong field is short as $T \sim 1/Q_\mathrm{s} \lesssim 1\ \mathrm{GeV}^{-1}$.

One also finds that the quark mass value largely affects the transverse spectrum:

(i) For small transverse momentum $|\boldsymbol{p}_\perp| \lesssim m_\mathrm{f}$, the spectra become constant in $|\boldsymbol{p}_\perp|$. This is because the $\boldsymbol{p}_\perp$ dependence of the particle production always appears in the combination of the transverse mass $\sqrt{m_\mathrm{f}^2 + \boldsymbol{p}_\perp^2}$ in homogeneous systems (see the explicit expressions of the mode functions given in Appendix A 1 c). Thus, one can neglect the $\boldsymbol{p}_\perp$ dependence and that the spectra are determined solely by the quark mass m_f for $|\boldsymbol{p}_\perp| \lesssim m_\mathrm{f}$.

(ii) For large transverse momentum $|\boldsymbol{p}_\perp| \gtrsim m_\mathrm{f}$, the spectra become independent of the quark mass m_f because now the transverse mass is determined by $|\boldsymbol{p}_\perp|$ only, and hence the m_f dependence can be neglected.

(iii) The larger lifetime T is required for the heavier (charm) quark production spectrum to converge to the Schwinger estimate, compared to that requiredfor lighter quarks (up, down, and strange quarks) as was discussed in Fig. 3 for the total quark number $dN^{(\mathrm{q})}_{i,f,s}/dy_{\boldsymbol{p}}$.

2. Number density dN/dy

Figure 6 shows the total number of quarks and antiquarks $\sum_{i,f,s} d(N^{(\mathrm{q})}_{i,f,s} + N^{(\bar{\mathrm{q}})}_{i,f,s})/dy_{\boldsymbol{p}}$, and that of gluons $\sum_{\pm A,\sigma} dN^{(\mathrm{g})}_{A,\sigma}/dy_{\boldsymbol{p}}$. Here, we consider the three flavor case ($N_\mathrm{f} = 3$); i.e., up, down, and strange quarks are considered. (The number-of-flavor N_f dependence of the particle production will be discussed in Fig. 7 below.) As in Fig. 2, we observe the following points:

(i) For long lifetimes $T \gtrsim 1\ \mathrm{GeV}^{-1}$, the total number of quarks and antiquarks, and that of gluons, approaches the Schwinger estimates, which are given by

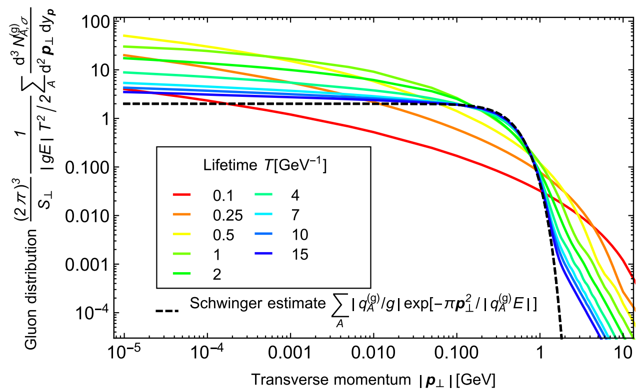

FIG. 5. Transverse distribution of gluons for various lifetimes T. The dashed line is an expectation from the Schwinger formula, Eq. (119).

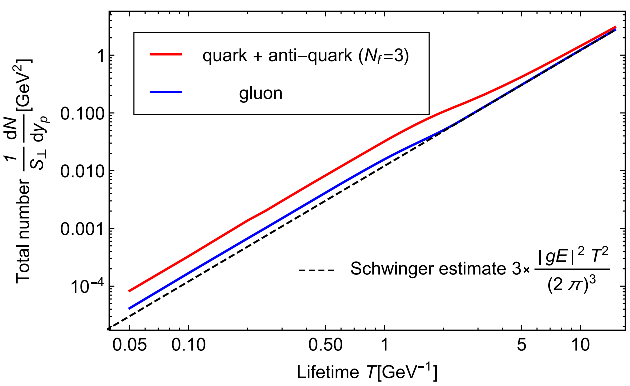

FIG. 6. Total number of quarks and antiquarks (red line) and gluons (blue line) per unit rapidity for $N_{\mathrm{f}} = 3$. The dashed line is an expectation from the Schwinger formula, Eqs. (124) and (125).

$$\sum_{i,f,s,\mathrm{q\bar{q}}} \frac{dN^{(\mathrm{q})}_{i,f,s}}{dy_{\boldsymbol{p}}}\bigg|_{\mathrm{Schwinger}} \sim N_s N_{\mathrm{q\bar{q}}} \times \sum_{i,f} \frac{S_\perp}{(2\pi)^3} \frac{|q^{(\mathrm{q})}_i E|^2 T^2}{2} \times \exp\left[-\pi \frac{m_{\mathrm{f}}^2}{|q^{(\mathrm{q})}_i E|}\right] \sim N_s N_{\mathrm{q\bar{q}}} N_{\mathrm{lq}} \times \frac{S_\perp}{(2\pi)^3} \frac{|gE|^2 T^2}{4}, \tag{124}$$

$$\sum_{\pm A,\sigma} \frac{dN^{(\mathrm{g})}_{\pm A,\sigma}}{dy_{\boldsymbol{p}}}\bigg|_{\mathrm{Schwinger}} \sim 2 N_c N_\sigma \times \frac{S_\perp}{(2\pi)^3} \frac{|gE|^2 T^2}{4}, \tag{125}$$

where $N_s = 2$, $N_{\mathrm{q\bar{q}}} = 2$, $N_\sigma = 2$ count the number of the spin, the quark and antiquark degeneracy, and the physical polarization of gluons. N_{lq} represents the number of "light" quarks satisfying $m_{\mathrm{f}}^2 \ll |gE|$.

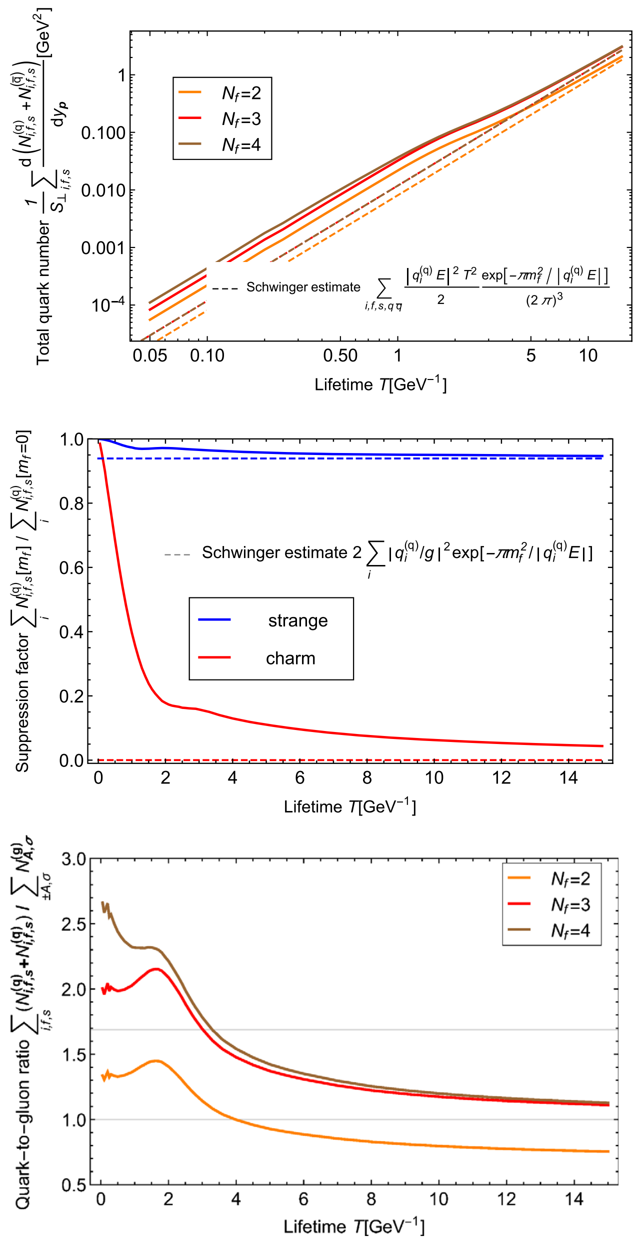

FIG. 7. The number-of-flavor $N_{\mathrm{f}} = 2$ (u,d), 3 (u,d,s), 4 (u,d,s,c) dependences: Top: The total quark and antiquark number. The dashed lines are expectations from the Schwinger formula, Eq. (124). Middle: A ratio of the total number of massive charm and strange quarks to that of massless up and down quarks, $\sum_i N^{(\mathrm{q})}_{i,f,s}[m_{\mathrm{f}} \neq 0] / \sum_i N^{(\mathrm{q})}_{i,f,s}[m_{\mathrm{f}} = 0]$. The dashed lines are expectations from the Schwinger formula, Eq. (126). Bottom: The ratio of the total number of quarks and antiquarks to that of gluons $\sum_{i,f,s}(N^{(\mathrm{q})}_{i,f,s} + N^{(\mathrm{\bar{q}})}_{i,f,s}) / \sum_{\pm A,\sigma} N^{(\mathrm{g})}_{A,\sigma}$. The upper horizontal line indicates the thermal ratio 27/16.

We have used the color charge formulas, Eqs. (35) and (36). As the strange quark mass is much smaller than the strength of the electric field, $m_{\mathrm{s}}^2 \ll |gE|$, we regard the strange quark as a "light" quark and set $N_{\mathrm{lq}} = 3$ (see the middle panel of Fig. 7 for justification of this consideration). Then, the Schwinger estimates for quark and antiquark production and for gluon production accidentally coincide with each other for $N_c = 3$ because the prefactors in both cases give the same value, $N_s N_{q\bar{q}} N_{lq} = 2N_c N_\sigma = 12$.

(ii) For short lifetimes $T \lesssim 1\ \mathrm{GeV}^{-1}$, quarks and antiquarks are more abundantly produced than gluons because of the finite lifetime effects. For details of the ratio of the total number of quarks and antiquarks to that of gluons (see the bottom panel of Fig. 7), which we will discuss later.

(iii) The particle production is very *fast*, which is consistent with an earlier work on the quark production [35]. For the typical value of the transverse area in HIC $S_\perp \sim \pi(7\ \mathrm{fm})^2$, about 1000 particles per unit rapidity (650 quarks and antiquarks plus 350 gluons) are produced at about $T \sim 0.5\ \mathrm{fm}/c$.

Figure 7 shows dependence of particle production on the number of flavors, $N_f = 2$ (u,d), 3 (u,d,s), 4 (u,d,s,c): The total quark and antiquark number, $\sum_{i,f,s} d(N^{(q)}_{i,f,s} + N^{(\bar{q})}_{i,f,s})/dy_{\boldsymbol{p}}$, for several different values of N_f is plotted in the top panel. There is a significant change from $N_f = 2$ to $N_f = 3$ for all values of T. This means that the inclusion of strangeness degree of freedom is inevitable in understanding the early stage dynamics of HIC quantitatively, whereas the change of the quark multiplicity from $N_f = 3$ to $N_f = 4$, i.e., by inclusion of charm quarks, is negligible (noticeable) for long (short) lifetimes T.

In order to see more clearly how this difference appears, we plot a ratio of the total number of massive charm and strange quarks to that of massless up and down quarks, $\sum_i N^{(q)}_{i,f,s}[m_f \neq 0]/\sum_i N^{(q)}_{i,f,s}[m_f = 0]$, in the middle panel. From this panel, one can understand that the strange quark production is comparable to the production of up and down quarks (the ratio is almost unity) for all values of T because the strange quark mass is sufficiently "light" compared to the strength of the electric field $m_s^2 \ll |gE|$. On the other hand, one finds for the charm quark production that it is comparable to the production of massless quarks for smaller values of $T \lesssim 1\ \mathrm{GeV}^{-1}$ because of the perturbative enhancement of the particle production discussed in Fig. 2, while it is negligible for larger values of $T \gtrsim 1\ \mathrm{GeV}^{-1}$ because Schwinger's nonperturbative particle production is strongly suppressed by the mass effect,

$$\left.\frac{\sum_i N^{(q)}_{i,f,s}[m_f \neq 0]}{\sum_i N^{(q)}_{i,f,s}[m_f = 0]}\right|_{\text{Schwinger}} \sim 2\sum_i \left|\frac{q_i^{(q)}E}{gE}\right|^2 \exp\left[-\pi\frac{m_f^2}{|q_i^{(q)}E|}\right]. \tag{126}$$

We note that the enhancement of the charm quark production from a pulsed electric field in a nonexpanding system was previously discussed in Refs. [30,31].

Finally, in the bottom panel of Fig. 7, we plot a ratio of the total number of quarks and antiquarks to that of gluons, $\sum_{i,f,s}(N^{(q)}_{i,f,s} + N^{(\bar{q})}_{i,f,s})/\sum_{\pm A,\sigma} N^{(g)}_{A,\sigma}$, for several different values of N_f. As was discussed in the top and middle panels of Fig. 7, there is a significant change from $N_f = 2$ to $N_f = 3$ for all values of T, while the change from $N_f = 3$ to $N_f = 4$ is negligible (noticeable) for long (short) lifetimes T depending on the strange and charm quark masses. In the short lifetime limit $T \to 0$, the ratio approaches $2N_f/N_c$. This is because the quark masses become irrelevant to the quark production for such small values of T and that the (lowest order) perturbative particle production, which becomes efficient for small values of lifetimes T, says that twice as many quarks than gluons will be produced as was discussed in Fig. 3. On the other hand, in the long lifetime limit $T \to \infty$, the ratio approaches the value N_{lq}/N_c because particle production is dominated by Schwinger's nonperturbative mechanism whose contribution is estimated by Eqs. (124) and (125). One of the important points here is that the ratio is always larger than unity for realistic values, $N_f \geq 3$, $N_{lq} = 3$, $N_c = 3$; i.e., quarks and antiquarks in total are more abundant than gluons. This means that not only gluons but also quarks are important in understanding the early stage dynamics of HIC. We note that this result is based on our one-loop order treatment as was remarked in the beginning of this section. Thus, the above result does not take into account effects of scatterings such as $g \to q\bar{q}$, which is important in the chemical equilibration of the system and could substantially change the ratio to $\sum_{i,f,s}(N^{(q)}_{i,f,s} + N^{(\bar{q})}_{i,f,s})/\sum_{\pm A,\sigma} N^{(g)}_{A,\sigma}|_{\text{chem eq}} \sim 27/16$.

In the figure, we see bumps at around $T \sim 0.5\ \mathrm{fm}/c$ while we did not find such bumps in the right panel of Fig. 2. These bumps appear because gluons typically have a larger effective charge than quarks have. Indeed, one can estimate the typical magnitude of the effective charge of a quark $\langle|q^{(q)}|\rangle$ and a gluon $\langle|q^{(g)}|\rangle$ from Eqs. (35) and (36) as

$$\langle|q^{(q)}|\rangle \equiv \sqrt{\frac{1}{N_c}\sum_{i=1}^{N_c}|q_i^{(q)}|^2} = g \times \frac{1}{\sqrt{2N_c}}, \tag{127}$$

$$\langle|q^{(g)}|\rangle \equiv \sqrt{\frac{1}{N_c(N_c-1)/2}\sum_{i=1}^{N_c(N_c-1)/2}|q_A^{(g)}|^2} = g \times \frac{1}{\sqrt{N_c - 1}} > \langle|q^{(q)}|\rangle. \tag{128}$$

As was discussed in Sec. III A, quark and gluon production numbers approach the expectation of the Schwinger formula quicker for larger values of $|q|T$. Thus, we can say that the gluon production number approaches the expectation of the Schwinger formula quicker than the quark production number does because $\langle |q^{(\mathrm{g})}| \rangle > \langle |q^{(\mathrm{q})}| \rangle$. In other words, the interplay between the perturbative and nonperturbative particle production mechanism for quarks and gluons does not occur at the same time and the time needed for gluon's interplay is smaller than that for quark's one. (This is not the case in Fig. 2, where $|q^{(\mathrm{q})}| = |q^{(\mathrm{g})}|$ is assumed.) By noting that Schwinger's nonperturbative particle production mechanism produces smaller numbers of particles compared to that the perturbative particle production mechanism does, we understand that the production number of gluons is relatively smaller compared to that of quarks at transient times $T \sim 0.5$ fm$/c$, where the perturbative (nonperturbative) particle production mechanism dominates for quarks (gluons). This is the reason why the bumps appear.

IV. SUMMARY AND OUTLOOK

We have extensively studied the quark and gluon production from an expanding classical color electric field, motivated by the early stage dynamics of HIC. First, we have formulated the particle production from classical color electromagnetic fields in an expanding system in the one-loop level quantum calculation within the Abelian dominance assumption for the classical fields.

Then, we compute the quark and gluon spectra within this formalism for the simplest case of the classical background field in an expanding geometry. That is, the classical field is assumed to be purely electric and boost-invariantly expanding, homogeneous, and constant within finite duration (lifetime) T, $\boldsymbol{E} = \boldsymbol{e}_z E\theta(\tau)\theta(T-\tau)$. In this setup, analytical solutions for the equations of motion of QCD are available; this enables us to develop a clear understanding of the particle production from the classical fields in an expanding system in QCD.

In this way, we have explicitly demonstrated for the first time in an expanding system that there is a significant interplay between Schwinger's nonperturbative particle production (long lifetimes T) and perturbative particle production (short lifetimes T), which results in that the transverse momentum $\boldsymbol{p}_\perp$ spectrum becomes harder (softer) for smaller (larger) values of T and in an enhancement of the particle production for small values of T, compared to the estimate of the Schwinger formula.

In addition to this, we have studied the difference in the production of quarks and of gluons. We have found that quarks are more abundantly produced than gluons, and that the difference of the statistics results in the increase of soft gluons and in an efficiency of the perturbative enhancement $N^{(\mathrm{q})}_{i,f,s}/N^{(\mathrm{g})}_{A,\sigma} \sim 2$ for small values of T.

The quark mass dependence of the quark production is also studied by examining the ratio $N^{(\mathrm{q})}_{i,f,s}[m_\mathrm{f} \neq 0]/N^{(\mathrm{q})}_{i,f,s}[m_\mathrm{f} = 0]$: We found that it varies from one for short lifetimes T to the expected value of the Schwinger formula $\exp[-\pi m_\mathrm{f}^2/|q_i^{(\mathrm{q})}E|]$ for long lifetimes T, and that it needs longer lifetime T for heavier quark production to be described by the Schwinger formula.

As implications to the heavy ion phenomenology, we have argued that (i) the naive use of the Schwinger formula may be inappropriate in describing the early stage dynamics of HIC because of the finite lifetime effects; (ii) very *fast* particle production occurs, which results in about 1000 particles per unit rapidity (650 quarks and antiquarks plus 350 gluons) produced at about $T \sim 0.5$ fm$/c$ at the RHIC energy scale; (iii) since quark production is more abundant than gluon production from the classical electric fields in the early times, it is very important to study the dynamics, not of the pure gluonic system, but of the *quark*-gluon system in understanding the early stage dynamics of HIC; and (iv) the strange quark production is comparable to the light (up and down) quarks for any value of the lifetime T, while the charm quark production rate heavily depends on the lifetime T, which is noticeable (negligible) for small (large) values of T.

There are many possible future directions of this work. The first direction is to improve our formalism to include the higher order quantum corrections. These terms are responsible for scatterings and screening effects, which are essential in understanding thermalization, i.e., isotropization, hydrodynamization, and chemical equilibration of the system. Besides, it is discussed vigorously that momentum exchanges due to the scatterings induce spectral cascades (for a recent review covering this topic, see [40]), which result in some interesting behaviors such as a formation of gluonic Bose-Einstein condensates [41–44]. We note for completeness that recently the screening effects from quark currents in a nonexpanding system were discussed in Refs. [48,52].

Another direction is to improve configurations of the classical field: In realistic situations, the classical field has a finite extent in the transverse direction and has random fluctuations with a typical transverse correlation length $\sim Q_\mathrm{s}^{-1}$. Time dependence of the classical field should also affect the particle spectra. Besides, it is known that the classical color field has magnetic components in addition to electric components in the longitudinal direction [5]. The existence of longitudinal magnetic fields may enhance the particle production rate [36,37,48,53,54]. In addition to this, such field configuration is known to invoke the Nielsen-Olesen type instability [55–57], although its typical time scale is rather slow. It is interesting to study the particle production under the presence of such instabilities; for nonexpanding, static ($T \to \infty$) color electromagnetic fields, it was discussed that the instability may enhance the gluon production [58].

The last direction that we would like to mention is about the quark dynamics. As was discussed so far, quarks are abundantly produced at very early times, and hence they may have important information about and/or an important role in the early stage dynamics of HIC. Since quarks have an U(1) electromagnetic charge, which does not suffer from the strong interactions, one can investigate the quark dynamics by using U(1) electromagnetic probes such as photons [59] and dileptons [60,61]. Another interesting topic involving the quark dynamics is an existence of strong U(1) electromagnetic fields just after a collision of nuclei [62,63]. Although such strong U(1) electromagnetic fields die away immediately after a collision within less than 1 fm/c, they could significantly influence the quark dynamics because the strong U(1) electromagnetic fields are as strong as the pion mass scale and the quark production is fast enough. Thus, one can expect some experimental traces of them, for instance, in U(1) charge dependences in observables. In particular, a U(1) charge dependent directed flow $v_1^\pm$ in asymmetric heavy ion collisions [64,65] has recently been measured by the STAR Collaboration [66]. This should provide important insights into the quark production, i.e., the early stage dynamics of HIC, although a theoretical understanding of this observable is still lacking. Another interesting physics that involves the strong U(1) electromagnetic fields is the chiral magnetic effect [67], whose real time dynamics from the microscopic point of view is still incomplete (although there are some earlier works on this topic [68,69]) and hence is worthwhile to be investigated further by extending our work.

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APPENDIX: ANALYTIC SOLUTIONS OF THE ABELIANIZED EQUATION OF MOTION

In this appendix, we analytically obtain mode functions for the equations of motion, Eqs. (37)–(39), under (a) a pure gauge background field (i.e., $\boldsymbol{E}=\boldsymbol{0}$, $\boldsymbol{B}=\boldsymbol{0}$),

$$\tilde{A}_\mu = \text{const}, \tag{A1}$$

(b) a spatially homogeneous and constant color electric background field (i.e., $\boldsymbol{E}=\mathbf{e}_z E$, $\boldsymbol{B}=\boldsymbol{0}$),

$$\tilde{A}_\tau, \tilde{A}_x, \tilde{A}_y = 0, \qquad \tilde{A}_\eta = \tau^2 E/2, \tag{A2}$$

and (c) a spatially homogeneous and constant color electric background field for a finite duration T [i.e., $\boldsymbol{E}=\mathbf{e}_z E\theta(\tau-\tau_0)\theta(\tau_0+T-\tau)$, $\boldsymbol{B}=\boldsymbol{0}$],

$$\tilde{A}_\tau, \tilde{A}_x, \tilde{A}_y = 0,$$
$$\tilde{A}_\eta = \begin{cases} \tau_0^2 E/2 & (\tau<\tau_0) \\ \tau^2 E/2 & (\tau_0<\tau<\tau_0+T) \\ (\tau_0+T)^2 E/2 & (\tau_0+T<\tau) \end{cases}. \tag{A3}$$

1. Quark

We consider the equation of motion for the quark field ψ under the Abelianized background gauge field in the τ-η coordinates [see Eq. (37)],

$$[i\partial\!\!\!/ - q\tilde{A}\!\!\!/ - m]\psi(x) = 0. \tag{A4}$$

Here, we have suppressed the indices for color i and flavor f for simplicity.

To avoid complexities coming from the spinor structure of Eq. (A4), we consider a solution of the form [36,37]

$$\psi \equiv [i\partial\!\!\!/ - q\tilde{A}\!\!\!/ + m]\phi. \tag{A5}$$

One can readily find a differential equation for ϕ as

$$0 = \left[(\partial_\mu + iq\tilde{A}_\mu)^2 + \frac{\partial_\tau + iq\tilde{A}_\tau}{\tau} + \frac{iq}{2}\gamma^\mu\gamma^\nu\tilde{F}_{\mu\nu} + m^2\right]\phi. \tag{A6}$$

Since we are interested in the situations where a color electric field pointing to the z direction exists at most in this appendix, one can simplify Eq. (A6) as

$$0 = \left[(\partial_\mu + iq\tilde{A}_\mu)^2 + \frac{\partial_\tau + iq\tilde{A}_\tau}{\tau} + iqE\gamma^t\gamma^z + m^2\right]\phi. \tag{A7}$$

Next, we expand ϕ in terms of eigenvectors[5] of $\gamma^t\gamma^z$ as

$$\phi \equiv \sum_{s=1}^{2}\phi_s\Gamma_s, \tag{A8}$$

[5]Strictly speaking, the matrix $\gamma^t\gamma^z$ has four eigenvectors Γ_s ($s=1, 2, 3, 4$) in total; $\Gamma_{1,2}$ with eigenvalues $\lambda_{1,2}=1$ and $\Gamma_{3,4}$ with eigenvalues $\lambda_{3,4}=-1$. Solutions $\psi_{3,4}$ for the original equation, Eq. (A4), constructed from $\phi_3\Gamma_3$ and $\phi_4\Gamma_4$ are linearly dependent on solutions $\psi_{1,2}$ constructed from $\phi_1\Gamma_1$ and $\phi_2\Gamma_2$ [36]. Thus, it is sufficient to consider $s=1, 2$ only in order to obtain all the independent solutions of the differential equation, Eq. (A4), for ψ.

where ϕ_s are scalar functions and the eigenvectors Γ_s ($s = 1, 2$) satisfy

$$\gamma^t\gamma^z\Gamma_s = \lambda_s\Gamma_s, \qquad \Gamma_s^\dagger\Gamma_{s'} = \delta_{ss'} \tag{A9}$$

with the eigenvalues λ_s given by $\lambda_1 = \lambda_2 = 1$. Physically, we have defined the spin of quarks by the direction of the background field $\tilde{F}_{\mu\nu}$ because $\gamma^t\gamma^z$ is proportional to the tz component of the background field as $\tilde{F}_{tz} = E\gamma^t\gamma^z$. Now, a differential equation for ϕ_s reads

$$0 = \left[(\partial_\mu + iq\tilde{A}_\mu)^2 + \frac{\partial_\tau + iq\tilde{A}_\tau}{\tau} + iqE + m^2\right]\phi_s, \tag{A10}$$

which are free from the cumbersome spinor structure in the original equation, Eq. (A4), for ψ.

a. Under a pure gauge background field (plane wave solutions)

Let us construct all the mode functions for the equation of motion, Eq. (A4), under a pure gauge background field $\tilde{A}_\mu$ given by Eq. (A1), which we write $\psi^{(\mathrm{free})}$. We first consider solving the differential equation for $\phi_s^{(\mathrm{free})}$ [Eq. (A10)]. For this, we make an ansatz of the form

$$\begin{aligned}\phi_s^{(\mathrm{free})}(x) &\equiv \int d^2\boldsymbol{p}_\perp dp_\eta \phi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}(x) \\ &\equiv \int d^2\boldsymbol{p}_\perp dp_\eta \Omega(x)\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau) \\ &\quad\times \frac{\mathrm{e}^{-\eta/2}}{\sqrt{m^2+\boldsymbol{p}_\perp^2}}\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}.\end{aligned} \tag{A11}$$

Here, the momentum labels $\boldsymbol{p}_\perp$, p_η are introduced, and Ω is a Wilson-line gauge factor denoted by

$$\Omega(x) \equiv \exp\left[-iq\int^x dx^\mu\tilde{A}_\mu\right]. \tag{A12}$$

The factor $\mathrm{e}^{-\eta/2}/\sqrt{m^2+\boldsymbol{p}_\perp^2}$ is inserted so as to properly normalize $\psi^{(\mathrm{free})}$ in the τ-η coordinates as we shall see. Now, one can readily find that $\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ satisfies the Bessel differential equation,

$$\begin{aligned}0 = \Big[\tau^2\partial_\tau^2 + \tau\partial_\tau + \Big\{\Big(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau\Big)^2 \\ - (ip_\eta+1/2)^2\Big\}\Big]\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}.\end{aligned} \tag{A13}$$

Since the differential equation, Eq. (A13), is a second order differential equation, there are two independent solutions, which we write ${}_k\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ ($k = 1, 2$). It is convenient for our purpose to choose

$$\begin{pmatrix} {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} \equiv \frac{\sqrt{\pi}}{2}(m^2+\boldsymbol{p}_\perp^2)^{1/4}\mathrm{e}^{\pi p_\eta/2}\mathrm{e}^{-i\pi/4} \times \begin{pmatrix} H^{(2)}_{ip_\eta+1/2}(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau) \\ H^{(1)}_{-ip_\eta-1/2}(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau)\end{pmatrix}, \tag{A14}$$

where $H^{(n)}_\nu(z)$ ($n = 1, 2$) are the Hankel function of the nth kind, and we have normalized the solutions ${}_k\chi_{\boldsymbol{p}_\perp,p_\eta,s}$ by

$$|{}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2 + |{}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2 = 1/\tau. \tag{A15}$$

It is also useful to point out that the solutions ${}_k\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ satisfy the following simultaneous differential equation:

$$\begin{aligned}&\frac{i}{\sqrt{m^2+\boldsymbol{p}_\perp^2}}\left[\partial_\tau + \frac{ip_\eta+1/2}{\tau}\right]\begin{pmatrix} {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} \\ &= \begin{pmatrix} {}_2\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,s} \\ -{}_1\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}.\end{aligned} \tag{A16}$$

We are ready to construct all the mode functions ${}_\pm\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$. Using the definition of $\phi^{(\mathrm{free})}$ [Eq. (A5)], one has

$$\begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]\end{pmatrix} \equiv [i\partial\!\!\!/ - q\tilde{A}\!\!\!/ + m]\begin{pmatrix} {}_1\phi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\phi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}\Gamma_s. \tag{A17}$$

Here, we have changed the left subscript $k = 1, 2$ into $\pm$ for a notational simplicity because ${}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$ (${}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$) corresponds to the positive (negative) frequency mode function in the τ-η coordinates as we will explain soon. With the help of Eqs. (A9) and (A16), one finds that Eq. (A17) can be more explicitly written as

$$\begin{aligned}\begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]\end{pmatrix} = \Omega\Bigg[&\begin{pmatrix} {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}V_{s,1} \\ &+ \begin{pmatrix} {}_2\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,s} \\ -{}_1\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}V_{s,2}\Bigg]\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}.\end{aligned} \tag{A18}$$

Here, we have introduced four-spinors, $V_{s,1}$ and $V_{s,2}$, by

$$V_{s,1} \equiv \mathrm{e}^{\eta/2}\frac{-\boldsymbol{p}_\perp\cdot\boldsymbol{\gamma}_\perp + m}{\sqrt{m^2+\boldsymbol{p}_\perp^2}}\Gamma_s, \quad V_{s,2} \equiv \mathrm{e}^{-\eta/2}\gamma^t\Gamma_s, \tag{A19}$$

which are normalized as

$$\bar{V}_{s,i}\gamma^\tau V_{s,j} = \delta_{ij}. \quad \text{(A20)}$$

From the normalization conditions for $_k\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ [Eq. (A15)] and $V_{s,i}$ [Eq. (A20)], it is evident that the mode functions $_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$ [Eq. (A17)] satisfy the correct normalization condition for spinor fields in the τ-η coordinates [see also Eqs. (43) and (44) in the main text] as

$$\begin{aligned}&({}_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]|{}_\pm\psi^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}[\tilde{A}_\mu])_{\text{F}} = \delta_{ss'}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta),\\ &({}_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]|{}_\mp\psi^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}[\tilde{A}_\mu])_{\text{F}} = 0, \end{aligned} \quad \text{(A21)}$$

where the fermion inner product $(\psi_1|\psi_2)_{\text{F}}$ is the same as is defined in Eq. (45).

Our mode functions $_+\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$ ($_-\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$) defined in Eq. (A17) can actually be understood as the positive (negative) frequency mode function in the τ-η coordinates because it can be written as a superposition of the positive (negative) frequency mode function in the Cartesian coordinates [28,70]. To see this, we use the integral representations for the Hankel functions $H^{(n)}_\nu(z)$ $(n = 1, 2)$,

$$\begin{aligned}H^{(1)}_\nu(z) &= \frac{\mathrm{e}^{-i\nu\pi/2}}{i\pi}\int_{-\infty}^{\infty} dt \mathrm{e}^{iz\cosh t - \nu t},\\ H^{(2)}_\nu(z) &= -\frac{\mathrm{e}^{i\nu\pi/2}}{i\pi}\int_{-\infty}^{\infty} dt \mathrm{e}^{-iz\cosh t - \nu t},\end{aligned} \quad \text{(A22)}$$

to obtain

$$_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu] = \int dp_z \frac{\mathrm{e}^{\pm ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}}\,{}_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]. \quad \text{(A23)}$$

Here, $\omega_{\boldsymbol{p}}$ is on-shell energy $\omega_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}_\perp^2 + p_z^2}$ and $y_{\boldsymbol{p}}$ is the momentum rapidity as was introduced in Eq. (55). $_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]$ are the positive/negative frequency mode functions in the Cartesian coordinates, which satisfy the free field equation of motion in the Cartesian coordinates, $0 = [i\gamma^m(\partial_m + iq\tilde{A}_m) - m]\,{}_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]$, and that are labeled by p_z being the Fourier conjugate to z: They are given by

$$\begin{pmatrix} _+\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m] \\ _-\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]\end{pmatrix} = \Omega\left[\begin{pmatrix} _1\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s} \\ _2\chi_{\boldsymbol{p}_\perp,p_z,s}\end{pmatrix} v_{s,1} + \begin{pmatrix} _2\chi^{(\text{free})*}_{\boldsymbol{p}_\perp,p_z,s} \\ -_1\chi^{*}_{\boldsymbol{p}_\perp,p_z,s}\end{pmatrix} v_{s,2}\right]\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_z z}}{(2\pi)^{3/2}}, \quad \text{(A24)}$$

where we have defined the spin label s by the direction of the background electric field $\tilde{F}_{tz} = E\gamma^t\gamma^z$ as in $_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$ by expanding the spinor space in terms of the eigenvectors of the background field Γ_s. Ω is the same as the Wilson-line gauge factor introduced in Eq. (A12), which does not depend on a choice of coordinates

$$\Omega(x) = \exp\left[-iq\int^x dx^\mu \tilde{A}_\mu\right] = \exp\left[-iq\int^x d\xi^m \tilde{A}_m\right], \quad \text{(A25)}$$

where ξ^m represent the Cartesian coordinates $\xi^m = (t, x, y, z)$ with m running through t, x, y, z as in the main text. The functions $_k\chi_{\boldsymbol{p}_\perp,p_z,s}$ $(k = 1, 2)$ can be explicitly written as

$$\begin{aligned}_1\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s} &\equiv \frac{i}{\sqrt{2}}\sqrt{1 + \frac{p_z}{\omega_{\boldsymbol{p}}}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t},\\ _2\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s} &\equiv \frac{-i}{\sqrt{2}}\sqrt{1 - \frac{p_z}{\omega_{\boldsymbol{p}}}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t},\end{aligned} \quad \text{(A26)}$$

and the four-spinors $v_{s,1}$ and $v_{s,2}$ are given by

$$\begin{aligned}v_{s,1} &\equiv \frac{-\boldsymbol{p}_\perp\cdot\boldsymbol{\gamma}_\perp + m}{\sqrt{m^2 + \boldsymbol{p}_\perp^2}}\Gamma_s = \mathrm{e}^{-\eta/2}V_{s,1},\\ v_{s,2} &\equiv \gamma^t\Gamma_s = \mathrm{e}^{\eta/2}V_{s,2}.\end{aligned} \quad \text{(A27)}$$

$_\pm\psi^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]$ are properly normalized in the Cartesian coordinates as

$$\begin{aligned}\int_{t=\text{const}} d^2\boldsymbol{x}_\perp dz\,{}_\pm\bar{\psi}^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]\gamma^t\,{}_\pm\psi^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_z,s'}[\tilde{A}_m]\\ = \delta_{ss'}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_z - p'_z),\end{aligned} \quad \text{(A28)}$$

$$\int_{t=\text{const}} d^2\boldsymbol{x}_\perp dz\,{}_\pm\bar{\psi}^{(\text{free})}_{\boldsymbol{p}_\perp,p_z,s}[\tilde{A}_m]\gamma^t\,{}_\mp\psi^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_z,s'}[\tilde{A}_m] = 0. \quad \text{(A29)}$$

b. Under a spatially homogeneous and constant color electric background field

We consider a spatially homogeneous and constant color electric background field given by Eq. (A2), and analytically obtain all the mode functions for the equation of motion, Eq. (A4), which we write $\psi^{(\text{const})}$. Let us begin with the differential equation, Eq. (A10), for $\phi^{(\text{const})}_s$. To solve this equation, we make an ansatz of the form

$$\phi_s^{(\mathrm{const})}(x) \equiv \int d^2\boldsymbol{p}_\perp dp_\eta \phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}(x)$$
$$\equiv \int d^2\boldsymbol{p}_\perp dp_\eta \chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau)\frac{\mathrm{e}^{-\eta/2}}{\sqrt{m^2+\boldsymbol{p}_\perp^2}}\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}. \quad \text{(A30)}$$

As in the pure gauge case (Appendix A 1 a), the momentum labels $\boldsymbol{p}_\perp, p_\eta$ and the normalization factor $\mathrm{e}^{-\eta/2}/\sqrt{m^2+\boldsymbol{p}_\perp^2}$ are introduced. We note that even if there are pure gauge potentials in addition to the electric field, the following computations do not change by simply adding the Wilson-line gauge factor Ω [Eq. (A25)] into the above ansatz [Eq. (A30)]. Now, the differential equation for $\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ reads

$$0 = \left[\partial_\tau^2 + \frac{\partial_\tau}{\tau} + \left(\frac{p_\eta - i/2 + qE\tau^2/2}{\tau}\right)^2 + iqE + m^2 + \boldsymbol{p}_\perp^2\right]\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}. \quad \text{(A31)}$$

Two independent solutions of Eq. (A31), which we write ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ ($k=1,2$), can be written in terms of the Tricomi confluent hypergeometric function $U(a;b;z)$. Here, we consider the following particular solutions:

$${}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \equiv \frac{1}{\sqrt{\tau}}\exp\left[-\pi\frac{m^2+\boldsymbol{p}_\perp^2}{4|qE|} - i\frac{qE\tau^2}{4}\right]\left(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau\right)^{-ip_\eta}U\left(i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE};\frac{1}{2}-ip_\eta;i\frac{qE\tau^2}{2}\right), \quad \text{(A32)}$$

$${}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \equiv \frac{-i}{2}\frac{1}{\sqrt{\tau}}\exp\left[-\pi\frac{m^2+\boldsymbol{p}_\perp^2}{4|qE|} + i\frac{qE\tau^2}{4}\right]\left(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau\right)^{1+ip_\eta}U\left(1-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE};\frac{3}{2}+ip_\eta;-i\frac{qE\tau^2}{2}\right), \quad \text{(A33)}$$

which are normalized as

$$|{}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2 + |{}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2 = 1/\tau. \quad \text{(A34)}$$

It is useful to note that these particular solutions ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ satisfy a simultaneous differential equation, which is similar to that for the pure gauge case, Eq. (A16), as

$$\frac{i}{\sqrt{m^2+\boldsymbol{p}_\perp^2}}\left[\partial_\tau + \frac{ip_\eta + iq\tilde{A}_\eta + 1/2}{\tau}\right]\begin{pmatrix} {}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} = \begin{pmatrix} {}_2\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,s} \\ -{}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}. \quad \text{(A35)}$$

Thanks to this property, one finds that the mode functions ${}_k\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ constructed from ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ have the same spinor structure as what we have for the plane wave solutions ${}_\pm\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ [Eq. (A18)] as we will see soon.

Now, one can readily construct the mode functions ${}_k\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ ($k=1$, 2) for the equation of motion, Eq. (A4), under a spatially homogeneous and constant color electric background field $\tilde{A}_\mu$ [Eq. (A2)] as[6]

$$\begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} \equiv [i\partial\!\!\!/ - q\tilde{A}\!\!\!/ + m]\begin{pmatrix} {}_1\phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}\Gamma_s, \quad \text{(A36)}$$

where we have used the definition of $\phi^{(\mathrm{const})}$ [Eq. (A5)]. With the help of Eqs. (A9) and (A35), one finds that Eq. (A36) can be more explicitly written as

$$\begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} = \left[\begin{pmatrix} {}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}V_{s,1} + \begin{pmatrix} {}_2\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,s} \\ -{}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix}V_{s,2}\right]\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}. \quad \text{(A37)}$$

From the normalization conditions for ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ [Eq. (A34)] and $V_{s,i}$ [Eq. (A20)], one can confirm that the mode functions ${}_\pm\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ [Eq. (A37)] are correctly normalized as

$$({}_\pm\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_\pm\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} = \delta_{ss'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta),$$
$$({}_\pm\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_\mp\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'})_{\mathrm{F}} = 0. \quad \text{(A38)}$$

c. Under a spatially homogeneous and constant color electric background field with lifetime *T*

We consider a spatially homogeneous and constant color electric background field with lifetime T [Eq. (A3)] and find out all the mode functions $\psi^{(\mathrm{finite})}$ for the equation of motion, Eq. (A4). The problem is equivalent to solving the

[6] Unlike the plane wave solutions ${}_\pm\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ [Eq. (A17)] studied in Appendix A 1 a, we have not renamed the left subscript $k=1$, 2 of ${}_k\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ into $\pm$ because one cannot identify the positive and the negative frequency mode functions in principle when there are interactions, which mix up the positive and the negative frequency mode functions.

equation of motion, Eq. (A4), under a pure gauge background field for $0 < \tau < \tau_0$ and $\tau_0 + T < \tau$, and under a spatially homogeneous and constant color electric background field for $\tau_0 < \tau < \tau_0 + T$. All the mode functions for respective regions are already derived in Appendix A 1 a and Appendix A 1 b, respectively. Thus, all we have to do is to connect these solutions smoothly at the boundary $\tau = \tau_0$ and $\tau = \tau_0 + T$. Namely, we require

$$\psi^{(\mathrm{finite})}|_{\tau=\tau_0-0,\tau_0+T-0} = \psi^{(\mathrm{finite})}|_{\tau=\tau_0+0,\tau_0+T+0}. \tag{A39}$$

In making this connection, it is useful to use a linear relation between ${}_\pm\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}$ and $k\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}$ $(k = 1, 2)$ at fixed time $\tau = \tau_1$ described by

$$\begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)] \end{pmatrix} = \sum_{s'} \int d^2\boldsymbol{p}'_\perp dp'_\eta \begin{pmatrix} ({}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}|{}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)])_{\mathrm{F}} & ({}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}|{}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)])_{\mathrm{F}} \\ ({}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s}|{}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)])_{\mathrm{F}} & ({}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}|{}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu(\tau_1)])_{\mathrm{F}} \end{pmatrix} \begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}'_\perp,p'_\eta,s'} \end{pmatrix}$$
$$\equiv U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1) \begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s} \end{pmatrix}. \tag{A40}$$

The matrix elements are given by

$$\begin{aligned} A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1) &\equiv (U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1))_{11} = \left[\left(U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)\right)_{22}\right]^* \\ &= \tau_1\left[{}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s}(\tau_1)\,{}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1) + {}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s}(\tau_1)\,{}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)\right], \end{aligned} \tag{A41}$$

$$\begin{aligned} B^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1) &\equiv \left(U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)\right)_{21} = \left[-(U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1))_{12}\right]^* \\ &= \tau_1\left[{}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s}(\tau_1)\,{}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1) - {}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,s}(\tau_1)\,{}_1\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)\right], \end{aligned} \tag{A42}$$

and are normalized as

$$1 = |A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2 + |B^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}|^2, \tag{A43}$$

which means that the transformation $U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)$ is unitary: $1 = U^{(\mathrm{q})\dagger}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_1)$. Although the mode functions diverge at $\tau \to 0$ because of the coordinate singularity at $\tau = 0$ of the τ-η coordinates, one can safely take the limit $\tau_1 \to 0$ of the transformation $U^{(\mathrm{q})}$, i.e., the matrix elements $A^{(\mathrm{q})}, B^{(\mathrm{q})}$. By using

$$H^{(1)}_\nu(z) \underset{|z|\to 0}{\longrightarrow} \left(\frac{z}{2}\right)^{-\nu}\left(\frac{\Gamma(\nu)}{i\pi} + \mathcal{O}(|z|)\right) + \left(\frac{z}{2}\right)^{\nu}\left(\frac{1 + i\cot(\nu\pi)}{\Gamma(1+\nu)} + \mathcal{O}(|z|)\right), \tag{A44}$$

$$H^{(2)}_\nu(z) \underset{|z|\to 0}{\longrightarrow} \left(\frac{z}{2}\right)^{-\nu}\left(-\frac{\Gamma(\nu)}{i\pi} + \mathcal{O}(|z|)\right) + \left(\frac{z}{2}\right)^{\nu}\left(\frac{1 - i\cot(\nu\pi)}{\Gamma(1+\nu)} + \mathcal{O}(|z|)\right), \tag{A45}$$

$$U(a;b;z) \underset{|z|\to 0}{\longrightarrow} z^{1-b}\left(\frac{\Gamma(-1+b)}{\Gamma(a)} + \mathcal{O}(|z|)\right) + \left(\frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \mathcal{O}(|z|)\right), \tag{A46}$$

one obtains

$$A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau) \underset{\tau\to 0}{\longrightarrow} \frac{\sqrt{\pi}}{2}\frac{\mathrm{e}^{-\frac{i\pi}{4}(1-\frac{qE}{|qE|})}\mathrm{e}^{-\frac{\pi p_\eta}{2}(1+\frac{qE}{|qE|})}\mathrm{e}^{-\pi\frac{m^2+\boldsymbol{p}_\perp^2}{4|qE|}}\left(\frac{|qE|}{m^2+\boldsymbol{p}_\perp^2}\right)^{-ip_\eta-1/2}}{\cosh(\pi p_\eta)\Gamma\left(1-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE}\right)} \times\left[1+\mathrm{e}^{-\frac{i\pi}{4}(\frac{qE}{|qE|}-2)}\mathrm{e}^{\frac{\pi p_\eta}{2}(\frac{qE}{|qE|}+2)}\left(\frac{2|qE|}{m^2+\boldsymbol{p}_\perp^2}\right)^{ip_\eta+1/2}\frac{\Gamma(1-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE})}{\Gamma\left(\frac{1}{2}-ip_\eta-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE}\right)}\right], \tag{A47}$$

$$B^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau) \underset{\tau\to 0}{\longrightarrow} \frac{\sqrt{\pi}}{2}\frac{\mathrm{e}^{\frac{i\pi}{4}(1+\frac{qE}{|qE|})}\mathrm{e}^{\frac{\pi p_\eta}{2}(1-\frac{qE}{|qE|})}\mathrm{e}^{-\pi\frac{m^2+\boldsymbol{p}_\perp^2}{4|qE|}}\left(\frac{|qE|}{m^2+\boldsymbol{p}_\perp^2}\right)^{-ip_\eta-1/2}}{\cosh(\pi p_\eta)\Gamma\left(1-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE}\right)} \times\left[1+\mathrm{e}^{-\frac{i\pi}{4}(\frac{qE}{|qE|}+2)}\mathrm{e}^{\frac{\pi p_\eta}{2}(\frac{qE}{|qE|}-2)}\left(\frac{2|qE|}{m^2+\boldsymbol{p}_\perp^2}\right)^{ip_\eta+1/2}\frac{\Gamma\left(1-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE}\right)}{\Gamma\left(\frac{1}{2}-ip_\eta-i\frac{m^2+\boldsymbol{p}_\perp^2}{2qE}\right)}\right]. \tag{A48}$$

We consider two kinds of boundary conditions for the mode functions: We define mode functions ${}_\pm\psi^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta,s}$ (${}_\pm\psi^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}$) by a boundary condition at $\tau<\tau_0$ ($\tau>\tau_0+T$) to coincide with the plane wave solutions ${}_\pm\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]$. Using the linear relation, Eq. (A40), one can easily construct such mode functions, ${}_\pm\psi^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta,s}$ and ${}_\pm\psi^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}$, as

$$\begin{pmatrix} {}_+\psi^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_-\psi^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} = \begin{cases} \begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]\end{pmatrix} & \text{for } 0<\tau<\tau_0 \\ U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0)\begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2,s} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2,s}\end{pmatrix} & \text{for } \tau_0<\tau<\tau_0+T \\ U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0)U^{(\mathrm{q})\dagger}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,s}(\tau_0+T) & \\ \quad\times\begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,s}[\tilde{A}_\mu]\end{pmatrix} & \text{for } \tau_0+T<\tau \end{cases}, \tag{A49}$$

$$\begin{pmatrix} {}_+\psi^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s} \\ {}_-\psi^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}\end{pmatrix} = \begin{cases} U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)U^{(\mathrm{q})\dagger}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}(\tau_0) & \\ \quad\times\begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}[\tilde{A}_\mu]\end{pmatrix} & \text{for } 0<\tau<\tau_0 \\ U^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)\begin{pmatrix} {}_1\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE(\tau_0+T)^2/2,s} \\ {}_2\psi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE(\tau_0+T)^2/2,s}\end{pmatrix} & \text{for } \tau_0<\tau<\tau_0+T \\ \begin{pmatrix} {}_+\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu] \\ {}_-\psi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,s}[\tilde{A}_\mu]\end{pmatrix} & \text{for } \tau_0+T<\tau \end{cases}. \tag{A50}$$

These two sets of mode functions are not independent but related with each other by a Bogoliubov transformation discussed in the main text [see Eq. (50)]. Now, one can analytically compute the Bogoliubov coefficients as

$$\begin{aligned}\left({}_+\psi^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_+\psi^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_{\mathrm{F}} &= \left[\left({}_-\psi^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_-\psi^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_{\mathrm{F}}\right]^*\\ &= \delta_{ss'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2))\\ &\quad\times\left[A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}(\tau_0)A^{(\mathrm{q})*}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)+B^{(\mathrm{q})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}(\tau_0)B^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)\right], \end{aligned}\tag{A51}$$

$$\begin{aligned}\left({}_-\psi^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_+\psi^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_{\mathrm{F}} &= \left[-\left({}_+\psi^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta,s}|{}_-\psi^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta,s'}\right)_{\mathrm{F}}\right]^*\\ &= \delta_{ss'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2))\\ &\quad\times\left[A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}(\tau_0)B^{(\mathrm{q})*}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)-B^{(\mathrm{q})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,s}(\tau_0)A^{(\mathrm{q})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau_0+T)\right]. \end{aligned}\tag{A52}$$

2. Gluon

We consider the Abelianized equation of motion for the gluon field W_μ in the τ-η coordinates [see Eq. (38)],

$$[(\nabla_\rho+iq\tilde{A}_\rho)^2g^{\mu\nu}+2iq\tilde{F}^{\mu\nu}]W_\nu=0. \tag{A53}$$

Here, we omit the color indices A for simplicity.

In solving Eq. (A53), we first expand the gluon field W_μ in terms of a polarization vector in the τ-η coordinates $\varepsilon_{\mu,\sigma}$ ($\sigma=0$, 1, 2, 3) and a scalar amplitude ϕ_σ as

$$W_\mu\equiv\sum_{\sigma=1}^4\varepsilon_{\mu,\sigma}\phi_\sigma. \tag{A54}$$

By noting that the choice of the polarization vector is arbitrary in principle, we assume here that the polarization vectors $\varepsilon_{\mu,\sigma}$ are constructed from a constant vector $\tilde{\varepsilon}_{m,\sigma}$ by contracting the viervein matrix $e^\mu{}_m$ as

$$\varepsilon_{\mu,\sigma}=e^m{}_\mu\tilde{\varepsilon}_{m,\sigma}. \tag{A55}$$

Under this assumption, the covariant derivative of the polarization vector $\varepsilon_{\nu,\sigma}$ vanishes as $\nabla_\mu\varepsilon_{\nu,\sigma}=0$.

In this appendix, we only consider the cases where a constant color electric field pointing to the z direction is present at most. For such cases, it is convenient to choose $\varepsilon_{\mu,\sigma}$ ($\tilde{\varepsilon}_{m,\sigma}$) to be an eigenvector of the background field strength tensor $\tilde{F}_{\mu\nu}$ ($\tilde{F}_{mn}$) as

$$\tilde{F}^\nu_\mu\varepsilon_{\nu,\sigma}=\Lambda_\sigma\varepsilon_{\mu,\sigma},\Leftrightarrow\tilde{F}^n_m\tilde{\varepsilon}_{n,\sigma}=\Lambda_\sigma\tilde{\varepsilon}_{m,\sigma}, \tag{A56}$$

where four eigenvalues Λ_σ are given by

$$\Lambda_0=-E,\qquad\Lambda_1=\Lambda_2=0,\qquad\Lambda_3=E. \tag{A57}$$

In other words, we have defined the polarization of gluons by the direction of the background field. We also normalize the polarization vector $\varepsilon_{\mu,\sigma}$ ($\tilde{\varepsilon}_{\mu,\sigma}$) as

$$g^{\mu\nu}\varepsilon^*_{\mu,\sigma}\varepsilon_{\nu,\sigma'}=-\xi_{\sigma\sigma'},\qquad\sum_{\sigma,\sigma'}\xi_{\sigma\sigma'}\varepsilon^*_{\mu,\sigma}\varepsilon_{\nu,\sigma'}=-g_{\mu\nu}, \tag{A58}$$

and

$$\eta^{mn}\tilde{\varepsilon}^*_{m,\sigma}\tilde{\varepsilon}_{n,\sigma'}=-\xi_{\sigma\sigma'},\quad\sum_{\sigma,\sigma'}\xi_{\sigma\sigma'}\tilde{\varepsilon}^*_{m,\sigma}\tilde{\varepsilon}_{n,\sigma'}=-\eta_{mn}, \tag{A59}$$

where $\xi_{\sigma\sigma'}$ is the indefinite metric introduced in Eq. (73).

Now, one obtains a differential equation for ϕ_σ as

$$0=\left[(\partial_\mu+iq\tilde{A}_\mu)^2+\frac{\partial_\tau+iq\tilde{A}_\tau}{\tau}+2iq\Lambda_\sigma\right]\phi_\sigma. \tag{A60}$$

a. Under a pure gauge background field (plane wave solutions)

In order to construct all the mode functions for the equation of motion, Eq. (A53), under a pure gauge background field $\tilde{A}_\mu$ [Eq. (A1)], which we write $W^{(\text{free})}_\mu$, we first consider solving the differential equation for $\phi^{(\text{free})}_\sigma$ [Eq. (A60)]. For this, we make an ansatz of the form

$$\phi^{(\text{free})}_\sigma(x)\equiv\int d^2\boldsymbol{p}_\perp dp_\eta\phi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(x), \tag{A61}$$

$$\phi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\equiv\Omega(x)\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau)\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}. \tag{A62}$$

Here, the momentum labels $\boldsymbol{p}_\perp$, p_η are introduced. Ω is the Wilson-line gauge factor, which is the same as what we have defined in Eq. (A12). One readily finds that $\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$ satisfies the Bessel differential equation,

$$0=\left[\tau^2\partial_\tau^2+\tau\partial_\tau+\left\{\left(\sqrt{m^2+\boldsymbol{p}_\perp^2}\tau\right)^2-(ip_\eta)^2\right\}\right]\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}. \tag{A63}$$

Since Eq. (A63) is a second order differential equation, there are two independent solutions, which we write ${}_k\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$ ($k=1,2$). In this appendix, we consider

$$ {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv \frac{\sqrt{\pi}}{2i} \mathrm{e}^{\pi p_\eta/2} H^{(2)}_{ip_\eta}(|\boldsymbol{p}_\perp|\tau), \quad {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv ({}_1\chi_{\boldsymbol{p}_\perp,p_\eta,\sigma})^*, \tag{A64} $$

where we have normalized the solutions ${}_k\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$ by

$$ \frac{1}{\tau} = \sum_{\sigma'} \xi_{\sigma\sigma'} \left[i \left({}_1\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \overleftrightarrow{\partial}_\tau {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'} \right) \right] = -\sum_{\sigma'} \xi_{\sigma\sigma'} \left[i \left({}_2\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \overleftrightarrow{\partial}_\tau {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'} \right) \right], \tag{A65} $$

$$ 0 = \sum_{\sigma'} \xi_{\sigma\sigma'} \left[i \left({}_1\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \overleftrightarrow{\partial}_\tau {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'} \right) \right] = \sum_{\sigma'} \xi_{\sigma\sigma'} \left[i \left({}_2\chi^{(\mathrm{free})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \overleftrightarrow{\partial}_\tau {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'} \right) \right]. \tag{A66} $$

Now, we are ready to construct all the mode functions ${}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$. By using the definition of $\phi^{(\mathrm{free})}_\sigma$ [Eq. (A54)], one can construct the ${}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$ as

$$ \begin{pmatrix} {}_+W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \\ {}_-W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \end{pmatrix} \equiv \varepsilon_{\mu,\sigma} \begin{pmatrix} {}_1\phi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_2\phi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \end{pmatrix} = \varepsilon_{\mu,\sigma}\Omega \begin{pmatrix} {}_1\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_2\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \end{pmatrix} \frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}. \tag{A67} $$

Here, we have changed the left subscript $k=1,2$ into $\pm$ for a notational simplicity because ${}_+W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$ (${}_-W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$) corresponds to the positive (negative) frequency mode function in the τ-η coordinates as we will explain soon. With the help of the normalization conditions, Eq. (A58) for $\varepsilon_{\mu,\sigma}$ and Eqs. (A65) and (A66) for ${}_k\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$, one can easily check that, in the temporal gauge $\tilde{A}_\tau=0$, the mode functions satisfy the correct normalization condition for vector fields in the τ-η coordinates [see also Eqs. (70) and (71) in the main text],

$$ -g^{\mu\nu}\left({}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho] | {}_\pm W^{(\mathrm{free})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}[\tilde{A}_\rho] \right)_{\mathrm{B}} = \pm\xi_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_\eta-p'_\eta), \tag{A68} $$

$$ -g^{\mu\nu}\left({}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho] | {}_\mp W^{(\mathrm{free})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}[\tilde{A}_\rho] \right)_{\mathrm{B}} = 0. \tag{A69} $$

The mode function ${}_+W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$ (${}_-W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$) can be written as a superposition of the positive (negative) frequency mode function in the Cartesian coordinates, and hence one can understand that ${}_+W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$ (${}_-W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$) can be understood as the positive (negative) frequency mode function in the τ-η coordinates. In order to see this, we again use the integral representations for the Hankel functions $H^{(n)}_\nu(z)$ ($n=1,2$) [Eq. (A22)] to find

$$ {}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] = e^m{}_\mu \int dp_z \frac{\mathrm{e}^{\pm ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}} {}_\pm W^{(\mathrm{free})}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_n]. \tag{A70} $$

Here, ${}_+W^{(\mathrm{free})}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_n]$ (${}_-W^{(\mathrm{free})}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_n]$) is the positive (negative) frequency mode function in the Cartesian coordinates satisfying the free field equation of motion in the Cartesian coordinates as $0=(\partial_l+iq\tilde{A}_l)^2{}_\pm W^{(\mathrm{free})}_{m,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_n]$ labeled by p_z conjugate to z,

$$ {}_\pm W^{(\mathrm{free})}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_n] = \tilde{\varepsilon}_{m,\sigma}\Omega \frac{\mathrm{e}^{\mp i\omega_{\boldsymbol{p}}t}}{\sqrt{2\omega_{\boldsymbol{p}}}} \frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_z z}}{(2\pi)^{3/2}}. \tag{A71} $$

These mode functions are properly normalized in the Cartesian coordinates. In the temporal gauge $\tilde{A}_t=0$, it reads

$$ -\eta^{mn}\int_{t=\mathrm{const}} d^2\boldsymbol{x}_\perp dz {}_\pm W^{(\mathrm{free})*}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_l] \overleftrightarrow{\partial}_t {}_\pm W^{(\mathrm{free})}_{n,\boldsymbol{p}'_\perp,p'_z,\sigma'}[\tilde{A}_l] = \pm\xi_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p_z-p'_z), \tag{A72} $$

$$ -\eta^{mn}\int_{t=\mathrm{const}} d^2\boldsymbol{x}_\perp dz {}_\pm W^{(\mathrm{free})*}_{m,\boldsymbol{p}_\perp,p_z,\sigma}[\tilde{A}_l] \overleftrightarrow{\partial}_t \times {}_\mp W^{(\mathrm{free})}_{n,\boldsymbol{p}'_\perp,p'_z,\sigma'}[\tilde{A}_l] = 0. \tag{A73} $$

b. Under a spatially homogeneous and constant color electric background field

We consider a spatially homogeneous and constant color electric background field [Eq. (A2)] and construct all the mode functions for the equation of motion, Eq. (A53), which we write $W^{(\mathrm{const})}_\mu$. First, we solve Eq. (A60) for $\phi^{(\mathrm{const})}_\sigma$ by making an ansatz,

$$ \phi^{(\mathrm{const})}_\sigma(x) \equiv \int d^2\boldsymbol{p}_\perp dp_\eta \phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(x), \tag{A74} $$

$$\phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(x) \equiv \chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,s}(\tau)\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}}. \tag{A75}$$

As in the pure gauge case (Appendix A 2 a), the momentum labels $\boldsymbol{p}_\perp$, p_η are introduced. We note that even if there are pure gauge potentials in addition to the electric field, the following computations do not change by simply adding the Wilson-line gauge factor Ω [Eq. (A25)] into the above ansatz [Eq. (A75)]. Now, the differential equation for $\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$ becomes

$$0 = \left[\partial_\tau^2 + \frac{\partial_\tau}{\tau} + \left(\frac{p_\eta + qE\tau^2/2}{\tau}\right)^2 + \boldsymbol{p}_\perp^2 + 2iq\Lambda_\sigma\right]\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}, \tag{A76}$$

where the eigenvalues Λ_σ are given by Eq. (A57). Two independent solutions of Eq. (A76), which we write ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$ ($k = 1, 2$), can be written in terms of the Tricomi confluent hypergeometric function $U(a;b;z)$. Here, we consider the following particular solutions:

$$\begin{aligned}{}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv{}& \frac{1}{\sqrt{2}}\exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|} + p_\eta + i\frac{q\Lambda_\sigma}{qE}\right) - i\frac{|qE|\tau^2}{4}\right] \\ &\times\left(\frac{|qE|\tau^2}{2}\right)^{ip_\eta/2} U\left(\frac{1}{2} + i\frac{\boldsymbol{p}_\perp^2}{2|qE|} + ip_\eta\right. \\ &\left.- \frac{q\Lambda_\sigma}{qE}; 1 + ip_\eta; i\frac{|qE|\tau^2}{2}\right),\end{aligned} \tag{A77}$$

$${}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \equiv \sum_{\sigma'}\xi_{\sigma\sigma'}\,{}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}, \tag{A78}$$

which are normalized as

$$\begin{aligned}\frac{1}{\tau} &= \sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\left({}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\overleftrightarrow{\partial}_\tau\,{}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right)\right] \\ &= -\sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\left({}_2\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\overleftrightarrow{\partial}_\tau\,{}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right)\right],\end{aligned} \tag{A79}$$

$$\begin{aligned}0 &= \sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\left({}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\overleftrightarrow{\partial}_\tau\,{}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right)\right] \\ &= \sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\left({}_2\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\overleftrightarrow{\partial}_\tau\,{}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right)\right].\end{aligned} \tag{A80}$$

Now, one readily obtains the mode functions ${}_kW^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ ($k = 1, 2$) as

$$\begin{aligned}\begin{pmatrix}{}_1W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \\ {}_2W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]\end{pmatrix} &\equiv \varepsilon_{\mu,\sigma}\begin{pmatrix}{}_1\phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_2\phi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\end{pmatrix} \\ &= \varepsilon_{\mu,\sigma}\begin{pmatrix}{}_1\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_2\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\end{pmatrix}\frac{\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\mathrm{e}^{ip_\eta\eta}}{(2\pi)^{3/2}},\end{aligned} \tag{A81}$$

where the definition of $\phi^{(\mathrm{const})}$ [Eq. (A54)] is used. With the normalization conditions, Eq. (A58) for $\varepsilon_{\mu,\sigma}$ and Eqs. (A79) and (A80) for ${}_k\chi^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}$, one finds that the mode functions ${}_kW^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ are correctly normalized as

$$\begin{aligned}&-g^{\mu\nu}\left({}_1W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_1W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} \\ &\quad= +g^{\mu\nu}\left({}_2W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_2W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} \\ &\quad= \xi_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta),\end{aligned} \tag{A82}$$

$$\begin{aligned}&-g^{\mu\nu}\left({}_1W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_2W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} \\ &\quad= -g^{\mu\nu}\left({}_2W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}|{}_1W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} = 0.\end{aligned} \tag{A83}$$

c. Under a spatially homogeneous and constant color electric background field with lifetime T

We consider a constant color electric background field with lifetime T [Eq. (A3)] and find out all the mode functions $W^{(\mathrm{finite})}_\mu$ for the equation of motion, Eq. (A53). The problem is equivalent to solving the equation of motion, Eq. (A53), under a pure gauge background field for $0 < \tau < \tau_0$ and $\tau_0 + T < \tau$, and under a spatially homogeneous and constant color electric background field for $\tau_0 < \tau < \tau_0 + T$. All the mode functions for respective regions are already derived in Appendix A 2 a and Appendix A 2 b, respectively. Thus, all we have to do is to connect these solutions smoothly at the boundaries $\tau = \tau_0$ and $\tau = \tau_0 + T$. Namely, we require

$$W^{(\mathrm{finite})}_\mu\Big|_{\tau=\tau_0-0,\tau_0+T-0} = W^{(\mathrm{finite})}_\mu\Big|_{\tau=\tau_0+0,\tau_0+T+0}, \tag{A84}$$

$$\nabla_\tau W^{(\mathrm{finite})}_\mu\Big|_{\tau=\tau_0-0,\tau_0+T-0} = \nabla_\tau W^{(\mathrm{finite})}_\mu\Big|_{\tau=\tau_0+0,\tau_0+T+0}. \tag{A85}$$

In making this connection, it is useful to use a linear relation between ${}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ and ${}_kW^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ ($k = 1, 2$) at fixed time $\tau = \tau_1$ described by

$$\begin{pmatrix} {}_{+}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)] \\ {}_{-}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)] \end{pmatrix}$$

$$= -g^{\nu\lambda}\sum_{\sigma'\sigma''}\xi_{\sigma'\sigma''}\int d^2\boldsymbol{p}'_\perp dp'_\eta$$

$$\times \begin{pmatrix} \left({}_1W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma''}|{}_+W^{(\mathrm{free})}_{\lambda,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)]\right)_{\mathrm{B}} & -\left({}_2W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma''}|{}_+W^{(\mathrm{free})}_{\lambda,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)]\right)_{\mathrm{B}} \\ \left({}_1W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma''}|{}_-W^{(\mathrm{free})}_{\lambda,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)]\right)_{\mathrm{B}} & -\left({}_2W^{(\mathrm{const})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma''}|{}_-W^{(\mathrm{free})}_{\lambda,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\rho(\tau_1)]\right)_{\mathrm{B}} \end{pmatrix} \begin{pmatrix} {}_1W^{(\mathrm{const})}_{\mu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'} \\ {}_2W^{(\mathrm{const})}_{\mu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'} \end{pmatrix}$$

$$\equiv U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1)\begin{pmatrix} {}_1W^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,\sigma} \\ {}_2W^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,\sigma} \end{pmatrix}. \tag{A86}$$

The matrix elements are given by

$$A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1) \equiv (U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1))_{11} = \sum_{\sigma'}\xi_{\sigma\sigma'}\left[\left(U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}(\tau_1)\right)_{22}\right]^* = \sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\tau_1\left({}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,\sigma'}\overleftrightarrow{\partial}_{\tau 1}\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\right)\Big|_{\tau=\tau_1}\right], \tag{A87}$$

$$B^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1) \equiv \left(U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1)\right)_{21} = \sum_{\sigma'}\xi_{\sigma\sigma'}\left[(U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}(\tau_1))_{12}\right]^* = \sum_{\sigma'}\xi_{\sigma\sigma'}\left[i\tau_1\left({}_1\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_1^2/2,\sigma'}\overleftrightarrow{\partial}_{\tau 2}\chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\right)\Big|_{\tau=\tau_1}\right] \tag{A88}$$

and are normalized as

$$1 = \sum_{\sigma'}\xi_{\sigma\sigma'}\left[A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\left[A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right]^* - B^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}\left[B^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma'}\right]^*\right] \tag{A89}$$

so that $\det U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_1) = 1$ holds. Although the mode functions diverge at $\tau \to 0$ because of the coordinate singularity at $\tau = 0$ of the τ-η coordinates, one can safely take the limit $\tau_1 \to 0$ of the transformation $U^{(\mathrm{g})}$, i.e., the coefficients $A^{(\mathrm{g})}, B^{(\mathrm{g})}$. By using the asymptotic formulas for the special functions Eqs. (A44)–(A46), one finds

$$A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau) \underset{\tau\to 0}{\longrightarrow} \sum_{\sigma'}\xi_{\sigma\sigma'}\left\{-\sqrt{\frac{\pi}{2}}\frac{\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{-ip_\eta/2}\exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|}-i\frac{q\Lambda_{\sigma'}}{qE}\right)\right]}{\sinh(\pi p_\eta)\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-\frac{q\Lambda_{\sigma'}}{qE}\right)}\mathrm{e}^{-\pi p_\eta}\right.$$
$$\left.\times\left[1-\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{ip_\eta}\mathrm{e}^{3\pi p_\eta/2}\frac{\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-\frac{q\Lambda_{\sigma'}}{qE}\right)}{\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-ip_\eta-\frac{q\Lambda_{\sigma'}}{qE}\right)}\right]\right\}, \tag{A90}$$

$$B^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau) \underset{\tau\to 0}{\longrightarrow} \sum_{\sigma'}\xi_{\sigma\sigma'}\left\{-\sqrt{\frac{\pi}{2}}\frac{\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{-ip_\eta/2}\exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|}-i\frac{q\Lambda_{\sigma'}}{qE}\right)\right]}{\sinh(\pi p_\eta)\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-\frac{q\Lambda_{\sigma'}}{qE}\right)}\right.$$
$$\left.\times\left[1-\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{ip_\eta}\mathrm{e}^{-\pi p_\eta/2}\frac{\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-\frac{q\Lambda_{\sigma'}}{qE}\right)}{\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-ip_\eta-\frac{q\Lambda_{\sigma'}}{qE}\right)}\right]\right\}. \tag{A91}$$

We consider two kinds of boundary conditions for the mode functions: We define mode functions ${}_\pm W^{(\mathrm{finite;in})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ (${}_\pm W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$) by a boundary condition at $\tau < \tau_0$ ($\tau > \tau_0 + T$) to coincide with the plane wave solutions ${}_\pm W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu]$. With the linear relation, Eq. (A86), one can construct such mode functions ${}_\pm W^{(\mathrm{finite;in})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ and ${}_\pm W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}$ as

$$\begin{pmatrix} {}_{+}W^{(\mathrm{finite;in})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_{-}W^{(\mathrm{finite;in})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma} \end{pmatrix} = \begin{cases} \begin{pmatrix} {}_{+}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \\ {}_{-}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \end{pmatrix} & \text{for } 0 < \tau < \tau_0 \\ U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0) \begin{pmatrix} {}_{1}W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2,\sigma} \\ {}_{2}W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2,\sigma} \end{pmatrix} & \text{for } \tau_0 < \tau < \tau_0+T \\ U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0) U^{(\mathrm{g})-1}_{\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,\sigma}(\tau_0+T) & \\ \quad \times \begin{pmatrix} {}_{+}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,\sigma}[\tilde{A}_\nu] \\ {}_{-}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE\tau_0^2/2+qE(\tau_0+T)^2/2,\sigma}[\tilde{A}_\nu] \end{pmatrix} & \text{for } \tau_0+T < \tau \end{cases}, \tag{A92}$$

$$\begin{pmatrix} {}_{+}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma} \\ {}_{-}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma} \end{pmatrix} = \begin{cases} U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0+T) U^{(\mathrm{g})-1}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma}(\tau_0) & \\ \quad \times \begin{pmatrix} {}_{+}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma}[\tilde{A}_\nu] \\ {}_{-}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma}[\tilde{A}_\nu] \end{pmatrix} & \text{for } 0 < \tau < \tau_0 \\ U^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0+T) \begin{pmatrix} {}_{1}W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE(\tau_0+T)^2/2,\sigma} \\ {}_{2}W^{(\mathrm{const})}_{\mu,\boldsymbol{p}_\perp,p_\eta-qE(\tau_0+T)^2/2,\sigma} \end{pmatrix} & \text{for } \tau_0 < \tau < \tau_0+T \\ \begin{pmatrix} {}_{+}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \\ {}_{-}W^{(\mathrm{free})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma}[\tilde{A}_\nu] \end{pmatrix} & \text{for } \tau_0+T < \tau \end{cases}. \tag{A93}$$

These two sets of mode functions are not independent but related with each other by a Bogoliubov transformation discussed in the main text [see Eq. (84)]. Now, one can analytically compute the Bogoliubov coefficients as

$$\begin{aligned} &\sum_{\sigma''}\xi_{\sigma\sigma''}(-g^{\mu\nu})\left({}_{+}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{+}W^{(\mathrm{finite;in})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} \\ &= \sum_{\sigma'''}\xi_{\sigma\sigma'''}\left[-\sum_{\sigma''}\xi^{-1}_{\sigma'''\sigma''}(-g^{\mu\nu})\left({}_{-}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{-}W^{(\mathrm{finite;in})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}}\right]^* \\ &= \delta_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2)) \\ &\quad\times\sum_{\sigma''}\xi_{\sigma\sigma''}\left[A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma}(\tau_0)A^{(\mathrm{g})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma''}(\tau_0+T)-B^{(\mathrm{g})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma''}(\tau_0)B^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0+T)\right], \end{aligned} \tag{A94}$$

$$\begin{aligned} &-\sum_{\sigma''}\xi_{\sigma\sigma''}(-g^{\mu\nu})\left({}_{-}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{+}W^{(\mathrm{finite;in})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}} \\ &= \sum_{\sigma'''}\xi_{\sigma\sigma'''}\left[\sum_{\sigma''}\xi_{\sigma'''\sigma''}(-g^{\mu\nu})\left({}_{+}W^{(\mathrm{finite;out})}_{\mu,\boldsymbol{p}_\perp,p_\eta,\sigma''}|{}_{-}W^{(\mathrm{finite;in})}_{\nu,\boldsymbol{p}'_\perp,p'_\eta,\sigma'}\right)_{\mathrm{B}}\right]^* \\ &= \delta_{\sigma\sigma'}\delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2)) \\ &\quad\times\sum_{\sigma''}\xi_{\sigma\sigma''}\left[-A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma}(\tau_0)B^{(\mathrm{g})*}_{\boldsymbol{p}_\perp,p_\eta,\sigma''}(\tau_0+T)+B^{(\mathrm{g})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2,\sigma''}(\tau_0)A^{(\mathrm{g})}_{\boldsymbol{p}_\perp,p_\eta,\sigma}(\tau_0+T)\right]. \end{aligned} \tag{A95}$$

3. Ghost

We consider the Abelianized equation of motion for ghost and antighost fields, C and $\bar{C}$, in the τ-η coordinates [see Eq. (39)]. It is sufficient for this purpose to consider a differential equation of the type

$$0 = (\nabla_\mu + iq\tilde{A}_\mu)^2\Theta = \left[(\partial_\mu + iq\tilde{A}_\mu)^2 + \frac{\partial_\tau + iq\tilde{A}_\tau}{\tau}\right]\Theta. \tag{A96}$$

This equation is exactly the same as Eq. (A60) for the gluon fields ϕ_σ for $\Lambda_\sigma = 0$, and so one can solve Eq. (A96) in the same way as we did in Appendix A 2 a. Therefore, we just write down the results without repeating the derivation and/or discussions in the following.

a. Under a pure gauge background field (plane wave solutions)

Under a pure gauge background field $\tilde{A}_\mu$ given by Eq. (A1), the positive and the negative frequency mode functions ${}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu]$ are given by

$$\begin{aligned} {}_+\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] &= \Omega_1 \chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}\frac{e^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}e^{ip_\eta\eta}}{(2\pi)^{3/2}}, \\ {}_-\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] &= \Omega_2 \chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}\frac{e^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}e^{ip_\eta\eta}}{(2\pi)^{3/2}}, \end{aligned} \tag{A97}$$

where

$$\begin{aligned} {}_1\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta} &= \frac{\sqrt{\pi}}{2i}e^{\pi p_\eta/2}H^{(2)}_{ip_\eta}(|\boldsymbol{p}_\perp|\tau), \\ {}_2\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta} &= [-\chi^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}]^*. \end{aligned} \tag{A98}$$

The mode functions satisfy the correct normalization conditions for scalar fields in the τ-η coordinates [see also Eqs. (102) and (103) in the main text]. For temporal gauge $\tilde{A}_\tau = 0$, it reads

$$({}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu]|{}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_\eta}[\tilde{A}_\mu])_{\text{B}} = \pm\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta), \tag{A99}$$

$$({}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu]|{}_\mp\Theta^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_\eta}[\tilde{A}_\mu])_{\text{B}} = 0. \tag{A100}$$

To see the mode functions ${}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu]$ defined in Eq. (A97) are actually the positive/negative frequency mode functions in the τ-η coordinates, we again use the integral representation for the Hankel functions $H^{(n)}_\nu(z)$ ($n = 1, 2$) [Eq. (A22)] to get the same integral relation as that for quarks Eq. (A23) and for gluons Eq. (A70) as

$${}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] = \int dp_z \frac{e^{\pm ip_\eta y_{\boldsymbol{p}}}}{\sqrt{2\pi\omega_{\boldsymbol{p}}}}{}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_z}[\tilde{A}_m]. \tag{A101}$$

All notations are the same as in the previous two cases except ${}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_z}[\tilde{A}_m]$ being the positive/negative frequency mode functions in the Cartesian coordinates,

$${}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}_\perp,p_z}[\tilde{A}_m] = \frac{e^{\mp i\omega_{\boldsymbol{p}}t}}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{e^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}e^{ip_z z}}{(2\pi)^{3/2}}, \tag{A102}$$

which are properly normalized in the Cartesian coordinates with the temporal gauge condition $\tilde{A}_t = 0$ as

$$\int_{t=\text{const}} d^2\boldsymbol{x}_\perp dz\, {}_\pm\Theta^{(\text{free})*}_{\boldsymbol{p}_\perp,p_z}[\tilde{A}_m]\overleftrightarrow{\partial}_t\, {}_\pm\Theta^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_z}[\tilde{A}_m] = \pm\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_z - p'_z), \tag{A103}$$

$$\int_{t=\text{const}} d^2\boldsymbol{x}_\perp dz\, {}_\pm\Theta^{(\text{free})*}_{\boldsymbol{p}_\perp,p_z}[\tilde{A}_m]\overleftrightarrow{\partial}_t\, {}_\mp\Theta^{(\text{free})}_{\boldsymbol{p}'_\perp,p'_z}[\tilde{A}_m] = 0. \tag{A104}$$

b. Under a spatially homogeneous and constant color electric background field

Under a spatially homogeneous and constant color electric field [Eq. (A2)], the positive/negative frequency mode functions ${}_\pm\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}$ are given by

$$\begin{aligned} {}_1\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta} &= {}_1\chi^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}\frac{e^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}e^{ip_\eta\eta}}{(2\pi)^{3/2}}, \\ {}_2\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta} &= {}_2\chi^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}\frac{e^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}e^{ip_\eta\eta}}{(2\pi)^{3/2}}, \end{aligned} \tag{A105}$$

where

$$\begin{aligned} {}_1\chi^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta} &= \frac{1}{\sqrt{2}}\exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|}+p_\eta\right) - i\frac{|qE|\tau^2}{4}\right]\left(\frac{|qE|\tau^2}{2}\right)^{ip_\eta/2} \\ &\quad\times U\left(\frac{1}{2}+i\frac{\boldsymbol{p}_\perp^2}{2|qE|}+ip_\eta; 1+ip_\eta; i\frac{|qE|\tau^2}{2}\right), \end{aligned} \tag{A106}$$

$${}_2\chi^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta} = [-\chi^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}]^*. \tag{A107}$$

The mode functions are correctly normalized as

$$\delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp)\delta(p_\eta - p'_\eta) = \left({}_1\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}|{}_1\Theta^{(\text{const})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\text{B}} = -\left({}_2\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}|{}_2\Theta^{(\text{const})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\text{B}}, \tag{A108}$$

$$0 = \left({}_1\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}|{}_2\Theta^{(\text{const})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\text{B}} = \left({}_2\Theta^{(\text{const})}_{\boldsymbol{p}_\perp,p_\eta}|{}_1\Theta^{(\text{const})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\text{B}}. \tag{A109}$$

c. Under a spatially homogeneous and constant color electric background field with lifetime T

For a spatially homogeneous and constant color electric background field with lifetime T [Eq. (A3)], the mode functions ${}_{\pm}\Theta^{(\mathrm{finite;as})}_{\boldsymbol{p}_\perp,p_\eta}$ (as = in, out) are given by

$$\begin{pmatrix} {}_{+}\Theta^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta} \\ {}_{-}\Theta^{(\mathrm{finite;in})}_{\boldsymbol{p}_\perp,p_\eta} \end{pmatrix} = \begin{cases} \begin{pmatrix} {}_{+}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] \\ {}_{-}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] \end{pmatrix} & \text{for } 0<\tau<\tau_0 \\ U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0) \begin{pmatrix} {}_{1}\Theta^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_0^2/2} \\ {}_{2}\Theta^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_0^2/2} \end{pmatrix} & \text{for } \tau_0<\tau<\tau_0+T \\ U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0) U^{(\mathrm{gh})-1}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_0^2/2 + qE(\tau_0+T)^2/2}(\tau_0+T) & \\ \quad\times \begin{pmatrix} {}_{+}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_0^2/2 + qE(\tau_0+T)^2/2}[\tilde{A}_\mu] \\ {}_{-}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_0^2/2 + qE(\tau_0+T)^2/2}[\tilde{A}_\mu] \end{pmatrix} & \text{for } \tau_0+T<\tau \end{cases}, \tag{A110}$$

$$\begin{pmatrix} {}_{+}\Theta^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta} \\ {}_{-}\Theta^{(\mathrm{finite;out})}_{\boldsymbol{p}_\perp,p_\eta} \end{pmatrix} = \begin{cases} U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T) U^{(\mathrm{gh})-1}_{\boldsymbol{p}_\perp,p_\eta + qE\tau_0^2/2 - qE(\tau_0+T)^2/2}(\tau_0) & \\ \quad\times \begin{pmatrix} {}_{+}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta + qE\tau_0^2/2 - qE(\tau_0+T)^2/2}[\tilde{A}_\mu] \\ {}_{-}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta + qE\tau_0^2/2 - qE(\tau_0+T)^2/2}[\tilde{A}_\mu] \end{pmatrix} & \text{for } 0<\tau<\tau_0 \\ U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T) \begin{pmatrix} {}_{1}\Theta^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta - qE(\tau_0+T)^2/2} \\ {}_{2}\Theta^{(\mathrm{const})}_{\boldsymbol{p}_\perp,p_\eta - qE(\tau_0+T)^2/2} \end{pmatrix} & \text{for } \tau_0<\tau<\tau_0+T \\ \begin{pmatrix} {}_{+}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] \\ {}_{-}\Theta^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta}[\tilde{A}_\mu] \end{pmatrix} & \text{for } \tau_0+T<\tau \end{cases}. \tag{A111}$$

Here, the matrix $U^{(\mathrm{gh})}$ is given by

$$U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta} = \begin{pmatrix} A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta} & B^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta} \\ B^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta} & A^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta} \end{pmatrix}, \tag{A112}$$

where the matrix elements $A^{(\mathrm{gh})}, B^{(\mathrm{gh})}$ are

$$A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_1) = i\tau_1 \left({}_{1}\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_1^2/2} \overleftrightarrow{\partial}_{\tau 1} \chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta} \right)\Big|_{\tau=\tau_1}, \qquad B^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_1) = i\tau_1 \left({}_{1}\chi^{(\mathrm{const})*}_{\boldsymbol{p}_\perp,p_\eta - qE\tau_1^2/2} \overleftrightarrow{\partial}_{\tau 2} \chi^{(\mathrm{free})}_{\boldsymbol{p}_\perp,p_\eta} \right)\Big|_{\tau=\tau_1}. \tag{A113}$$

The normalization condition for $A^{(\mathrm{gh})}, B^{(\mathrm{gh})}$ is

$$1 = |A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}|^2 + |B^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}|^2 \tag{A114}$$

so that $\det U^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta} = 1$ holds. In the limit of $\tau \to 0$, $A^{(\mathrm{gh})}, B^{(\mathrm{gh})}$ behaves as

$$A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau) \underset{\tau\to 0}{\longrightarrow} -\sqrt{\frac{\pi}{2}} \frac{\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{-ip_\eta/2} \exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|}\right)\right]}{\sinh(\pi p_\eta)\Gamma\left(\frac{1}{2} - i\frac{\boldsymbol{p}_\perp^2}{2|qE|}\right)} \mathrm{e}^{-\pi p_\eta} \left[1 - \left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{ip_\eta} \mathrm{e}^{3\pi p_\eta/2} \frac{\Gamma\left(\frac{1}{2} - i\frac{\boldsymbol{p}_\perp^2}{2|qE|}\right)}{\Gamma\left(\frac{1}{2} - i\frac{\boldsymbol{p}_\perp^2}{2|qE|} - ip_\eta\right)}\right], \tag{A115}$$

$$B^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau) \underset{\tau\to 0}{\longrightarrow} -\sqrt{\frac{\pi}{2}}\frac{\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{-ip_\eta/2}\exp\left[-\frac{\pi}{2}\left(\frac{\boldsymbol{p}_\perp^2}{2|qE|}\right)\right]}{\sinh(\pi p_\eta)\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}\right)}\left[1-\left(\frac{2|qE|}{\boldsymbol{p}_\perp^2}\right)^{ip_\eta}\mathrm{e}^{-\pi p_\eta/2}\frac{\Gamma(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|})}{\Gamma\left(\frac{1}{2}-i\frac{\boldsymbol{p}_\perp^2}{2|qE|}-ip_\eta-\frac{q\Lambda_{\sigma'}}{qE}\right)}\right]. \tag{A116}$$

The Bogoliubov coefficients between the two sets of mode functions [see Eq. (108)] are given by

$$\begin{aligned}\left({}_+\Theta^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta}|{}_+\Theta^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}} &= \left[-\left({}_-\Theta^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta}|{}_-\Theta^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}}\right]^* \\ &= \delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2)) \\ &\quad\times\left[A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2}(\tau_0)A^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T)-B^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2}(\tau_0)B^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T)\right],\end{aligned} \tag{A117}$$

$$\begin{aligned}-\left({}_-\Theta^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta}|{}_+\Theta^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}} &= \left[\left({}_+\Theta^{(\text{finite;out})}_{\boldsymbol{p}_\perp,p_\eta}|{}_-\Theta^{(\text{finite;in})}_{\boldsymbol{p}'_\perp,p'_\eta}\right)_{\mathrm{B}}\right]^* \\ &= \delta^2(\boldsymbol{p}_\perp-\boldsymbol{p}'_\perp)\delta(p'_\eta-(p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2)) \\ &\quad\times\left[-A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2}(\tau_0)B^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T)+B^{(\mathrm{gh})*}_{\boldsymbol{p}_\perp,p_\eta+qE\tau_0^2/2-qE(\tau_0+T)^2/2}(\tau_0)A^{(\mathrm{gh})}_{\boldsymbol{p}_\perp,p_\eta}(\tau_0+T)\right].\end{aligned} \tag{A118}$$

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