

Reexamining the photon polarization in $B \rightarrow K\pi\pi\gamma$

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We reexamine, update and extend a suggestion we made fifteen years ago for measuring the photon polarization in $b \rightarrow s\gamma$ by observing in $B \rightarrow K\pi\pi\gamma$ an asymmetry of the photon with respect to the $K\pi\pi$ plane. Asymmetries are calculated for different charged final states due to intermediate $K_1(1400)$ and $K_1(1270)$ resonant states. Three distinct interference mechanisms are identified contributing to asymmetries at different levels for these two kaon resonances. For $K_1(1400)$ decays including a final state π^0 an asymmetry around +30% is calculated, dominated by interference of two intermediate $K^*\pi$ states, while an asymmetry around +10% in decays including final $\pi^+\pi^-$ is dominated by interference of S and D wave $K^*\pi$ amplitudes. In decays via $K_1(1270)$ to final states including a π^0 a negative asymmetry is favored up to -10% if one assumes S wave dominance in decays to $K^*\pi$ and $K\rho$, while in decays involving $\pi^+\pi^-$ the asymmetry can vary anywhere in the range -13% to $+24\%$ depending on unknown phases. For more precise asymmetry predictions in the latter decays we propose studying phases in $K_1 \rightarrow K^*\pi, K\rho$ by performing dedicated amplitude analyses of $B \rightarrow J/\psi(\psi')K\pi\pi$. In order to increase statistics in studies of $B \rightarrow K\pi\pi\gamma$ we suggest using isospin symmetry to combine in the same analysis samples of charged and neutral B decays.

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I. INTRODUCTION

Flavor-changing radiative B meson decays provide important tests for the standard model. A crucial feature, which has not yet been tested experimentally in these processes, is the dominantly left-handed polarization of the photon in $b \rightarrow s\gamma$. In several extensions of the standard model the photon in $b \rightarrow s\gamma$ acquires a sizable right-handed component due to chirality flip along a heavy fermion line in the electroweak loop process [1]. A very early test for probing the dominantly left-handed photon polarization through time-dependent CP asymmetries, induced by interference of a large left-handed b amplitude and a small left-handed \bar{b} amplitude, was suggested in Ref. [2] and pursued experimentally by the BABAR [3] and Belle [4] Collaborations. Several years later a second test, reminiscent of a method measuring the tau neutrino helicity in $\tau \rightarrow a_1\nu_\tau, a_1 \rightarrow \rho\pi$ [5,6], was proposed based on measuring final particle momenta in $B^{0,+} \rightarrow K\pi\pi\gamma$ [7,8]. The photon polarization, a parity-odd quantity, was shown to be related to an asymmetry between the number of photons emitted in the two sides of the plane defined by $K\pi\pi$ in their center-of-mass frame. Since this asymmetry is odd also under time-reversal, a potentially large asymmetry requires that the decay amplitude acquires a nontrivial sizable phase due to final state interactions. Such a large calculable phase was shown to be produced in $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ and $B^0 \rightarrow K^+\pi^-\pi^0\gamma$ by two interfering amplitudes involving K^{*+} and K^{*0} intermediate resonances [7,8].

A calculation of the decay $B \rightarrow K_1(1400)\gamma \rightarrow K\pi\pi^0\gamma$, through interfering amplitudes for intermediate $K^{*0}\pi$ and $K^{*+}\pi$ states, was shown to lead to a sizable integrated asymmetry around 34% [7,8]. The feasibility of observing such a large asymmetry in future experiments has been discussed in this work, assuming a branching ratio $\mathcal{B}(B \rightarrow K_1(1400)\gamma) = 0.7 \times 10^{-5}$ as estimated in some models [9]. The process $B \rightarrow K_1(1270)\gamma$, observed a few years later with a considerably larger branching ratio [see Eqs. (3) and (6) below], was studied subsequently [10] under model-dependent assumptions about the strong decay $K_1(1270) \rightarrow K\pi\pi$, thereby introducing a considerable uncertainty in the polarization analysis [11]. Quite recently the same authors proposed an alternative approach for obtaining this hadronic information by studying the process $B \rightarrow J/\psi K_1 \rightarrow J/\psi K\pi\pi$ in parallel with $B \rightarrow K_1\gamma \rightarrow K\pi\pi\gamma$ [12]. A photon polarization analysis combining contributions from several kaon resonances with $J^P = 1^+, 1^-, 2^+$ has been outlined in Ref. [8], but would have to be treated further by experimental methods due to its complexity.

The purpose of this paper is to reexamine the situation in $B \rightarrow K_1(1270)\gamma \rightarrow K\pi\pi\gamma$ while drawing a comparison with $B \rightarrow K_1(1400)\gamma \rightarrow K\pi\pi\gamma$ which we studied only partially in Refs. [7,8]. In contrast to Ref. [10] which applied a quark pair creation model for describing the strong decay $K_1(1270) \rightarrow K\pi\pi$, our approach will be purely phenomenological using as much information as

possible from experiments. We will discuss a few sources for the photon up-down asymmetry with respect to the K_1 decay plane, that are related to different types of interference occurring in K_1 decays.

In Sec. II we summarize the current relevant experimental data, including branching ratios and certain final state interaction phases for K_1 decays to $K^*\pi$ and ρK leading to $K\pi\pi$ final states. A detailed derivation of relations between covariant and partial wave amplitudes describing the latter processes is presented in Sec. III in order to resolve a discrepancy between relations used in Refs. [7,8] and Refs. [10,11]. General expressions for decay amplitudes of $K_1 \rightarrow K\pi\pi$ are obtained in Sec. IV, distinguishing between hadronic final states involving $\pi^+\pi^-$ and $\pi^\pm\pi^0$. The photon up-down asymmetry in $B \rightarrow K\pi\pi\gamma$ with respect to the $K\pi\pi$ plane is calculated in Sec. V for these final states, separately for intermediate $K_1(1400)$ and $K_1(1270)$ resonant states. We discuss the role of three potential sources for an asymmetry. Section VI uses approximate isospin symmetry in radiative B decays to suggest combining charged and neutral $B \rightarrow K\pi\pi\gamma$ decays in order to increase statistics in studies of the photon polarization. Finally we conclude in Sec. VII.

II. EXPERIMENTAL SITUATION

A. $B \rightarrow K\pi\pi\gamma$

Following the suggestions made in Refs. [7,8] for measuring the photon polarization in $B \rightarrow K\pi\pi\gamma$ several experiments reported measuring these processes. Inclusive branching ratios were measured in four charged modes, $B^+ \rightarrow K^+\pi^-\pi^+\gamma$, $B^0 \rightarrow K^0\pi^+\pi^-\gamma$, $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ and $B^0 \rightarrow K^+\pi^-\pi^0\gamma$, for an hadronic invariant mass $m(K\pi\pi)$ in a range between 1 GeV/ c^2 and 1.8 or 2 GeV/ c^2 . Both the Belle [13,14] and BABAR [15] collaborations have observed the first two charged and neutral B decay modes involving a pair of charged pions resulting in the following averaged branching ratios [16]:

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow K^+\pi^-\pi^+\gamma) &= (2.76 \pm 0.22) \times 10^{-5}, \\ \mathcal{B}(B^0 \rightarrow K^0\pi^+\pi^-\gamma) &= (1.95 \pm 0.22) \times 10^{-5}. \end{aligned} \quad (1)$$

BABAR has also measured branching ratios for decay modes involving a neutral pion [15]:

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow K^0\pi^+\pi^0\gamma) &= (4.6 \pm 0.5) \times 10^{-5}, \\ \mathcal{B}(B^0 \rightarrow K^+\pi^-\pi^0\gamma) &= (4.1 \pm 0.4) \times 10^{-5}. \end{aligned} \quad (2)$$

Exclusive radiative B^+ decays involving the charged kaon resonance $K_1^+(1270)$ decaying to $K^+\pi^-\pi^+$ have been reported by Belle [14],

$$\mathcal{B}(B^+ \rightarrow K_1^+(1270)\gamma) = (4.3 \pm 1.3) \times 10^{-5}. \quad (3)$$

Radiative B decays to $K_2^*(1430)$, first reported by the CLEO collaboration [17],

$$\mathcal{B}(B \rightarrow K_2^*(1430)\gamma) = (1.7 \pm 0.6) \times 10^{-5}, \quad (4)$$

were observed subsequently by BABAR at a similar rate [18],

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow K_2^{*+}(1430)\gamma) &= (1.4 \pm 0.4) \times 10^{-5}, \\ \mathcal{B}(B^0 \rightarrow K_2^{*0}(1430)\gamma) &= (1.24 \pm 0.24) \times 10^{-5}. \end{aligned} \quad (5)$$

We note that so far none of the $K\pi\pi\gamma$ modes observed by Belle included a π^0 in the final state, in contrast to several of the above measurements by BABAR. Belle also obtained upper bounds at 90% confidence level for decays involving $K_1(1400)$ to final states including $\pi^+\pi^-$, using only about 18% of their final data set [14],

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow K_1^+(1400)\gamma) &< 1.5 \times 10^{-5}, \\ \mathcal{B}(B^0 \rightarrow K_1^0(1400)\gamma) &< 1.2 \times 10^{-5}. \end{aligned} \quad (6)$$

These upper bounds are a factor of two larger than the branching ratio assumed in Ref. [7].

A first attempt for measuring the photon polarization in $B \rightarrow K\pi\pi\gamma$ was made by the LHCb collaboration [19,20]. Nearly 14,000 signal events were reconstructed in the all charged mode $B^+ \rightarrow K^+\pi^-\pi^+\gamma$. The formalism developed in Refs. [7,8], extended to include interference of a few kaon resonances, was applied to decay distributions for four $K\pi\pi$ mass intervals in the overall range 1.1–1.9 GeV/ c^2 . The final result, a nonzero up-down asymmetry at 5.2σ , was insufficient for providing a significantly quantitative measurement of the photon polarization.

B. $K_1 \rightarrow K\pi\pi$

An analysis of the photon polarization in $B \rightarrow K\pi\pi\gamma$ via intermediate $K_1(1400)$ and $K_1(1270)$ resonances requires knowledge of branching ratios for these kaon resonances decaying into $K^*\pi$ and ρK states, and of magnitudes and relative phases between corresponding partial wave decay amplitudes. The situation in decays of $K_1(1400)$ is described in Table I. This information is based solely on a thirty-six-year-old experiment [21] performing a partial wave analysis for $J^P = 1^+ K\pi\pi$ states produced by K^-p diffractive scattering with couplings to $K^*\pi$ and ρK in both

TABLE I. Branching fractions and particle momenta for the main decay modes of $K_1(1400)$ [16].

Mode	\mathcal{B}	Γ_D/Γ_S	δ_{DS} [21]	$ \vec{p} $ (MeV)
$K^*\pi$	$(94 \pm 6)\%$	0.04 ± 0.01	$\sim 260^\circ$	401
ρK	$(3 \pm 3)\%$	291

TABLE II. Branching fractions and particle momenta for the main decay modes of $K_1(1270)$ [16,22].

Mode	\mathcal{B} [16]	Γ_D/Γ_S [16]	δ_{DS}	$ \vec{p} $ (MeV)	\mathcal{B} Fit 1 [22]	\mathcal{B} Fit 2 [22]	Average
ρK	$(42 \pm 6)\%$	46	$(57.3 \pm 3.5)\%$	$(58.4 \pm 4.3)\%$	57.9%
$K^*\pi$	$(16 \pm 5)\%$	1.0 ± 0.7	...	302	$(26.0 \pm 2.1)\%$	$(17.1 \pm 2.3)\%$	21.6%

S and D waves. In addition to measuring the ratio of S and D wave $K_1(1400)$ branching ratios into $K^*\pi$, some tantalizing information, $\delta_{DS}^{(K^*\pi)} \sim 260^\circ$, $\alpha_S \sim 40^\circ$, has been obtained for two relevant phases, between $K^*\pi S$ and D partial wave amplitudes and between S wave amplitudes for $K^*\pi$ and ρK , respectively.

The situation in decays of $K_1(1270)$ is displayed in Table II. The left-hand side is based on the same K^-p scattering experiment [21], while the right-hand side quotes results obtained much more recently by the Belle Collaboration through an amplitude analysis determining the resonant structure of the $K^+\pi^-\pi^+$ final state in $B^+ \rightarrow J/\psi K^+\pi^-\pi^+$ [22]. The difference between the $K_1(1270)$ decay branching ratios obtained in these two different methods seems to be associated with a third decay channel of $K_1(1270)$ involving $K_0^*(1430)\pi$, for which a sizable branching ratio of $(28 \pm 4)\%$ was claimed in [21] in contrast to a negligible branching ratio around two percent reported in [22]. A rather crude measurement exists for the ratio of S and D wave branching ratios into $K^*\pi$ [21]. However no direct information exists on two relevant phases, between $K^*\pi$ partial wave amplitudes and between S wave amplitudes for $K^*\pi$ and ρK . A relative phase around $\phi(\rho K) - \phi(K^*\pi) \sim -40^\circ$ has been measured between total ρK and $K^*\pi$ decay amplitudes [22]. Assuming that these two amplitudes are dominated by an S wave, this would imply $\alpha_S \sim -40^\circ$.

III. COVARIANT AND PARTIAL WAVE $K_1 \rightarrow K^*\pi, \rho K$ AMPLITUDES

The amplitude for an axial-vector meson decaying to a vector meson and a pseudoscalar meson has two equivalent descriptions, in terms of two covariant amplitudes and in terms of S and D partial wave amplitudes. The polarization analysis for $B \rightarrow K\pi\pi\gamma$ is based on covariant amplitudes [7,8] while data are given in terms of partial wave amplitudes. In this section we will prove relations between these two descriptions which will be used in our forthcoming analysis. While these relations were given briefly in Refs. [7,8], different relations have been used by the authors of [10,11] quoting Ref. [23] with no detail. Here we wish to settle this discrepancy by proving these relations in some detail.

Consider, for instance $K_1 \rightarrow K^*\pi$. The covariant amplitude for $K_1^+(p, \epsilon) \rightarrow K^{*0}(p', \epsilon')\pi^+(p_\pi)$, involving particles with four-momenta p, p', p_π and polarization vectors ϵ, ϵ' , is given by:

$$\mathcal{M}^1 = A_{K^*\pi}(\epsilon \cdot \epsilon'^*) + B_{K^*\pi}(\epsilon \cdot p_\pi)(\epsilon'^* \cdot p_\pi). \quad (7)$$

In the K_1 rest frame ($\vec{p} = 0$) we define z as the direction of the K^* momentum, while the pion moves in the direction $-z$. The three possible initial spin-one K_1 states involving spin projection $\lambda = +1, 0, -1$ along z are denoted $|1, \lambda\rangle$. The three polarization vectors ϵ and ϵ' for these three states $\lambda = \pm 1, 0$ are

$$\lambda = 1: \epsilon = \epsilon' = \left(0, -\sqrt{\frac{1}{2}}(\vec{e}_1 + i\vec{e}_2)\right), \quad (8)$$

$$\lambda = -1: \epsilon = \epsilon' = \left(0, \sqrt{\frac{1}{2}}(\vec{e}_1 - i\vec{e}_2)\right), \quad (9)$$

$$\lambda = 0: \epsilon = (0, \vec{e}_3),$$

$$\epsilon' = (|\vec{p}_\pi|/m_{K^*}, (E_{K^*}/m_{K^*})\vec{e}_3). \quad (10)$$

For $\lambda = 1$ this is the form of ϵ in the K_1 rest frame. The same form in this frame, identical to its form in the K^* rest frame, applies to ϵ' because a Lorentz transformation along z does not change the x, y components, mixing only the t, z components. For $\lambda = 0$ ϵ' is obtained from $\epsilon'(K^*) = (0, \vec{e}_3)$ in the rest frame of K^* by a Lorentz transformation to the rest frame of K_1 using $\gamma = E_{K^*}/m_{K^*}$, $\gamma\beta = -|\vec{p}_\pi|/m_{K^*}$,

$$\epsilon'_0(K_1) = \gamma[\epsilon'_0(K^*) - \beta\epsilon'_3(K^*)] = |\vec{p}_\pi|/m_{K^*},$$

$$\epsilon'_3(K_1) = \gamma[\epsilon'_3(K^*) - \beta\epsilon'_0(K^*)] = E_{K^*}/m_{K^*}. \quad (11)$$

We note that the transversity condition $\epsilon' \cdot p' = 0$ is satisfied for all polarization states λ of the K^* meson. In particular, for $\lambda = 0$ we have, using $p'(K^*) = (E_{K^*}, |\vec{p}_\pi|\vec{e}_3)$,

$$p'(K^*) \cdot \epsilon'(K^*) = E_{K^*}|\vec{p}_\pi|/m_{K^*} - |\vec{p}_\pi|E_{K^*}/m_{K^*} = 0. \quad (12)$$

The covariant decay amplitude (7) can now be calculated for these three polarization states:

$$\lambda = \pm 1: (\epsilon \cdot \epsilon'^*) = -\frac{1}{2}(\vec{e}_1 \pm i\vec{e}_2)(\vec{e}_1 \mp i\vec{e}_2) = -1;$$

$$\epsilon \cdot p_\pi = 0 \Rightarrow \mathcal{M}_{\lambda=\pm 1}^1 = -A_{K^*\pi}. \quad (13)$$

$$\begin{aligned}
\lambda = 0: \epsilon \cdot \epsilon^{J*} &= -\frac{E_{K^*}}{m_{K^*}}; \quad \epsilon \cdot p_\pi = |\vec{p}_\pi|; \\
\epsilon^{J*} \cdot p_\pi &= \frac{E_\pi |\vec{p}_\pi|}{m_{K^*}} + \frac{E_{K^*} |\vec{p}_\pi|}{m_{K^*}} = \frac{m_{K_1} |\vec{p}_\pi|}{m_{K^*}} \\
\Rightarrow \mathcal{M}_{\lambda=0}^1 &= -A_{K^*\pi} \frac{E_{K^*}}{m_{K^*}} + B_{K^*\pi} \frac{m_{K_1} |\vec{p}_\pi|^2}{m_{K^*}}. \quad (14)
\end{aligned}$$

Let us now write decay amplitudes for the three polarization states $|1, \lambda\rangle$ in terms of amplitudes for S and D waves, $L = 0, 2$, noting that the angular momentum states carry $L_z = 0$ ($m = 0$) for K^* and π moving in $\pm z$ directions. Using SU(2) Clebsch-Gordan coefficients ($10; 1\lambda|1\lambda$) and absorbing a factor $1/\sqrt{5}$ in the definition of the D-wave amplitude, we have:

$$\begin{aligned}
\mathcal{M}_{\lambda=\pm 1}^1 &= (00; 1 \pm 1|1 \pm 1) C_S^{(K^*\pi)} \\
&\quad + (20; 1 \pm 1|1 \pm 1) \sqrt{5} C_D^{(K^*\pi)} \\
&= C_S^{(K^*\pi)} + \frac{1}{\sqrt{2}} C_D^{(K^*\pi)}, \quad (15)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{\lambda=0}^1 &= (00; 10|10) C_S^{(K^*\pi)} + (20; 10|10) \sqrt{5} C_D^{(K^*\pi)} \\
&= C_S^{(K^*\pi)} - \sqrt{2} C_D^{(K^*\pi)}. \quad (16)
\end{aligned}$$

Squaring magnitudes of these amplitudes and averaging over the three polarizations states of the K_1 meson, one obtains

$$\frac{1}{3} \sum_{\lambda=0, \pm 1} |\mathcal{M}_\lambda^1|^2 = |C_S^{(K^*\pi)}|^2 + |C_D^{(K^*\pi)}|^2, \quad (17)$$

implying a decay rate

$$\Gamma(K_1 \rightarrow K^*\pi) = \frac{1}{8\pi m_{K_1}^2} (|C_S^{(K^*\pi)}|^2 + |C_D^{(K^*\pi)}|^2) |\vec{p}_\pi|. \quad (18)$$

Comparing Eqs. (13) and (14) with (15) and (16) one obtains

$$-A_{K^*\pi} = C_S^{(K^*\pi)} + \frac{1}{\sqrt{2}} C_D^{(K^*\pi)}, \quad (19)$$

$$\begin{aligned}
-B_{K^*\pi} \frac{m_{K_1} |\vec{p}_\pi|^2}{m_{K^*}} &= C_S^{(K^*\pi)} \left(\frac{E_{K^*}}{m_{K^*}} - 1 \right) \\
&\quad + \frac{1}{\sqrt{2}} C_D^{(K^*\pi)} \left(\frac{E_{K^*}}{m_{K^*}} + 2 \right), \quad (20)
\end{aligned}$$

or

$$\begin{aligned}
-B_{K^*\pi} |\vec{p}_\pi|^2 &= C_S^{(K^*\pi)} \left(\frac{E_{K^*} - m_{K^*}}{m_{K_1}} \right) \\
&\quad + \frac{1}{\sqrt{2}} C_D^{(K^*\pi)} \left(\frac{E_{K^*} + 2m_{K^*}}{m_{K_1}} \right). \quad (21)
\end{aligned}$$

These relations agree with those applied in Refs. [7,8] using a different convention for partial wave amplitudes. [The amplitudes $C_{S,D}^{(K^*\pi)}$ are related to $c_{S,D}$ occurring in Eq. (20) of [8] by $C_S^{(K^*\pi)} = -c_S$ and $C_D^{(K^*\pi)} = -\sqrt{2} \frac{m_{K_1} |\vec{p}_\pi|^2}{2m_{K^*} + E_{K^*}} c_D$.]

While the expression for $A_{K^*\pi}$ agrees with the one quoted by the authors of Ref. [10], these authors used a different relation for $B_{K^*\pi}$. Their Eq. (27) reads in our notation [24],

$$\begin{aligned}
-B_{K^*\pi} |\vec{p}_\pi|^2 &= \frac{E_{K^*}}{m_{K^*}} \left[C_S^{(K^*\pi)} \left(\frac{m_{K^*} - E_{K^*}}{m_{K_1}} \right) \right. \\
&\quad \left. + \frac{1}{\sqrt{2}} C_D^{(K^*\pi)} \left(\frac{m_{K^*} + 2E_{K^*}}{m_{K_1}} \right) \right]. \quad (22)
\end{aligned}$$

We find that this relation is in disagreement with (21), and is therefore incorrect.

Relations similar to (19) and (21) apply to $K_1 \rightarrow K\rho$:

$$-A_{K\rho} = C_S^{(K\rho)} + \frac{1}{\sqrt{2}} C_D^{(K\rho)}, \quad (23)$$

$$\begin{aligned}
-B_{K\rho} |\vec{p}_K|^2 &= C_S^{(K\rho)} \left(\frac{E_\rho - m_\rho}{m_{K_1}} \right) + \frac{1}{\sqrt{2}} C_D^{(K\rho)} \left(\frac{E_\rho + 2m_\rho}{m_{K_1}} \right). \quad (24)
\end{aligned}$$

IV. DECAY AMPLITUDES FOR $K_1 \rightarrow K\pi\pi$

The two pairs of processes in Eqs. (1) and (2) obtain contributions from $K_1(1270)$ and $K_1(1400)$ resonances. The decays of these resonances to different $K\pi\pi$ charged modes may be divided into two distinct pairs distinguished by their intermediate resonant decay channels [8]. The first pair involves two single decay channels into $K^*\pi$ and $K\rho$,

$$K_1^+ \rightarrow \left\{ \begin{array}{c} K^{*0}\pi^+ \\ K^+\rho^0 \end{array} \right\} \rightarrow K^+\pi^-\pi^+, \quad (25)$$

$$K_1^0 \rightarrow \left\{ \begin{array}{c} K^{*+}\pi^- \\ K^0\rho^0 \end{array} \right\} \rightarrow K^0\pi^+\pi^-, \quad (26)$$

while the second pair obtains contributions from two interfering $K^*\pi$ decay channels in addition to a single $K\rho$ channel,

$$K_1^+ \rightarrow \left\{ \begin{array}{c} K^{*+}\pi^0 \\ K^{*0}\pi^+ \\ K^0\rho^+ \end{array} \right\} \rightarrow K^0\pi^+\pi^0, \quad (27)$$

$$K_1^0 \rightarrow \left\{ \begin{array}{c} K^{*+}\pi^- \\ K^{*0}\pi^0 \\ K^+\rho^- \end{array} \right\} \rightarrow K^+\pi^-\pi^0. \quad (28)$$

The decays of K_1^+ and K_1^0 within each pair are related to each other by isospin reflection $u \leftrightarrow d$, ($\bar{d} \leftrightarrow -\bar{u}$), implying equal momenta-dependent amplitudes in the isospin symmetry limit:

$$A(K_1^+ \rightarrow K^+\pi^-\pi^+) = A(K_1^0 \rightarrow K^0\pi^+\pi^-), \quad (29)$$

$$A(K_1^+ \rightarrow K^0\pi^+\pi^0) = A(K_1^0 \rightarrow K^+\pi^-\pi^0). \quad (30)$$

These two amplitudes, for final states characterized by two charged pions in one case and by a pair of charged and neutral pions in the other, will be studied separately. Only the second pair of amplitudes has been analyzed for $K_1(1400)$ in Refs. [7,8].

A. Decays involving two charged pions

$$K_1^+ \rightarrow K^+\pi^-\pi^+, K_1^0 \rightarrow K^0\pi^+\pi^-$$

The $K^{*0}\pi^+$ contribution to the decay amplitude for $K_1^+(p, \epsilon) \rightarrow K^+(p_3)\pi^+(p_1)\pi^-(p_2)$ is obtained by convoluting the amplitude (7) with the amplitude for $K^{*0} \rightarrow K^+\pi^-$,

$$A(K^{*0} \rightarrow K^+\pi^-) = g_{K^*K\pi}\epsilon' \cdot (p_2 - p_3), \quad (31)$$

including a Breit-Wigner propagator for the K^* ,

$$\begin{aligned} & \left[g^{\mu\nu} - \frac{(p_2 + p_3)^\mu (p_2 + p_3)^\nu}{m_{K^*}^2} \right] \mathcal{B}_{23}^{(K^*)}, \\ & \mathcal{B}_{23}^{(K^*)} \equiv (s_{23} - m_{K^*}^2 - im_{K^*}\Gamma_{K^*})^{-1}, \\ & s_{23} \equiv (p_2 + p_3)^2. \end{aligned} \quad (32)$$

A similar contribution due to $K_1^+ \rightarrow K^+\rho^0, \rho^0 \rightarrow \pi^+\pi^-$ involves invariant amplitudes $A_{\rho K}, B_{\rho K}$ describing $K_1 \rightarrow K\rho$, the strong coupling $g_{\rho\pi\pi}$ and a Breit-Wigner propagator for the ρ , $\mathcal{B}_{12}^{(\rho)} \equiv (s_{12} - m_\rho^2 - im_\rho\Gamma_\rho)^{-1}$. Specifically, we define the amplitude

$$A(\rho^0 \rightarrow \pi^+\pi^-) = g_{\rho\pi\pi}\epsilon' \cdot (p_1 - p_2). \quad (33)$$

Adding these two contributions and neglecting a non-resonant term (which is justified in $K_1(1400)$ more than in $K_1(1270)$ —see Tables I and II and the last sentence in a paragraph below Table VII), the total covariant amplitude for $K_1^+(p, \epsilon) \rightarrow K^+(p_3)\pi^+(p_1)\pi^-(p_2)$ is given by

$$\mathcal{M} = C_1(\epsilon \cdot p_1) - C_2(\epsilon \cdot p_2), \quad (34)$$

where

TABLE III. Up-down photon asymmetry $\tilde{\mathcal{A}}$ in $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ from intermediate $K_1(1400)$. The asymmetry $\tilde{\mathcal{A}}_a$ neglects a contribution of a ρK amplitude as described in the text. For the total asymmetry we use $\alpha_S = 40^\circ$, a value favored by the analysis of [21].

$\delta_{DS}^{(K^*\pi)}$ (degrees)	0	45	90	135	180	225	270	315
$\tilde{\mathcal{A}}_a$	0.30	0.21	0.14	0.14	0.19	0.28	0.34	0.35
$\tilde{\mathcal{A}}_{\text{total}}$	0.30	0.21	0.15	0.14	0.20	0.29	0.35	0.36

$$C_i = C_i^{(K^*\pi)} + C_i^{(K\rho)} (i = 1, 2), \quad (35)$$

$$\begin{aligned} C_1^{(K^*\pi)} &= g_{K^*K\pi}\mathcal{B}_{23}^{(K^*)} \left\{ A_{K^*\pi} + B_{K^*\pi}p_1 \cdot (p_2 - p_3) \right. \\ &\quad \left. - \frac{m_K^2 - m_\pi^2}{m_{K^*}^2} [A_{K^*\pi} - B_{K^*\pi}((p \cdot p_1) - m_\pi^2)] \right\}, \\ C_1^{(K\rho)} &= g_{\rho\pi\pi}\mathcal{B}_{12}^{(\rho)} [A_{K\rho} - B_{K\rho}(p \cdot p_1 - p \cdot p_2)], \\ C_2^{(K^*\pi)} &= -g_{K^*K\pi}\mathcal{B}_{23}^{(K^*)}(2A_{K^*\pi}), \\ C_2^{(K\rho)} &= -g_{\rho\pi\pi}\mathcal{B}_{12}^{(\rho)} [A_{K\rho} + B_{K\rho}(p \cdot p_1 - p \cdot p_2)]. \end{aligned} \quad (36)$$

The four scalar products of two momenta in (36), $p_1 \cdot p_2$, $p_1 \cdot p_3$, $p \cdot p_1$ and $p \cdot p_2$, may all be written in terms of s_{13} and s_{23} . That is, C_i are functions of these two variables and the decay amplitude has the explicitly covariant form

$$\mathcal{M} = C_1(s_{13}, s_{23})(\epsilon \cdot p_1) - C_2(s_{13}, s_{23})(\epsilon \cdot p_2). \quad (37)$$

B. Decays involving a neutral pion

$$K_1^+ \rightarrow K^0\pi^+\pi^0, K_1^0 \rightarrow K^+\pi^-\pi^0$$

In these decays the amplitude has the same structure as (34),

$$\mathcal{M}' = C'_1(s_{13}, s_{23})(\epsilon \cdot p_1) - C'_2(s_{13}, s_{23})(\epsilon \cdot p_2), \quad (38)$$

with two contributions to $C'_{1,2}$ from $K^{*0}\pi$ and $K^{*+}\pi$ and one contribution from $K\rho^\pm$. The overall contribution from $K^{*}\pi$ is antisymmetric under the exchange of the two pion momenta and, using isospin, is expressed in terms of the same quantities $C_i^{(K^*\pi)}$ given in (36):

$$C'_i^{(K^*\pi)} = \sqrt{\frac{1}{2}} [C_i^{(K^*\pi)} - C_i^{(K^*\pi)}(p_1 \leftrightarrow p_2)]. \quad (39)$$

The single contribution from $K\rho$ is

$$C'_i^{(K\rho)} = \sqrt{2}C_i^{(K\rho)}. \quad (40)$$

C. Experimental information on ratios of amplitude

In the next section studying the photon polarization in $B \rightarrow K_1 \gamma$, $K_1 \rightarrow K \pi \pi$, which depends on interference of amplitudes, we will need ratios of certain quantities which we calculate now.

The strong couplings $g_{K^* K \pi}$ and $g_{\rho \pi \pi}$ occurring in (36) (for which we used a slightly different convention in [8]) are obtained from the K^* and ρ widths. Using

$$\begin{aligned} \Gamma(K^{*0} \rightarrow K^+ \pi^-) &= \frac{2}{3} \Gamma_{K^*} \mathcal{B}(K^* \rightarrow K \pi) \\ &= \frac{1}{6\pi m_{K^*}^2} |g_{K^* K \pi}|^2 |\vec{p}_\pi|^3, \end{aligned} \quad (41)$$

$$\Gamma(\rho^0 \rightarrow \pi^+ \pi^-) = \Gamma_\rho \mathcal{B}(\rho \rightarrow \pi \pi) = \frac{1}{6\pi m_\rho^2} |g_{\rho \pi \pi}|^2 |\vec{p}_\pi|^3, \quad (42)$$

where $\Gamma_{K^*} = 51$ MeV, $\Gamma_\rho = 150$ MeV, $|\vec{p}_\pi|_{K^* \rightarrow K \pi} = 289$ MeV, $|\vec{p}_\pi|_{\rho \rightarrow \pi \pi} = 364$ MeV [16], we calculate

$$\frac{|g_{\rho \pi \pi}|}{|g_{K^* K \pi}|} = 1.29. \quad (43)$$

This compares well with an SU(3) prediction

$$\frac{g_{\rho \pi \pi}}{g_{K^* K \pi}} = -\sqrt{2}. \quad (44)$$

The quantities $A_{K^* \pi}$, $B_{K^* \pi}$, $A_{K \rho}$, $B_{K \rho}$ in Eqs. (36) may be obtained from S and D wave amplitudes measured in $K_1^+ \rightarrow K^{*0} \pi^+$ and $K_1^+ \rightarrow K^+ \rho^0$ decays, denoted $C_{S,D}^{(K^* \pi)}$ and $C_{S,D}^{(K \rho)}$, using Eqs. (19) (21) for the first process and (23) (24) for the second. Branching ratios for $K_1 \rightarrow K^* \pi$ and $K_1 \rightarrow K \rho$ summed over all charged modes and corresponding ratios of decay rates for S and D waves were given in Tables I and II for $K_1(1400)$ and $K_1(1270)$, respectively. We will denote by $\delta_{DS}^{(K^* \pi)}$ and $\delta_{DS}^{(K \rho)}$ relative phases between S and D wave amplitudes in $K_1^+ \rightarrow K^{*0} \pi^+$ and $K_1^+ \rightarrow K^+ \rho^0$, respectively, and by κ_S and α_S the magnitude and phase of the ratio of S wave amplitudes for these decays,

$$\begin{aligned} \delta_{DS}^{(K^* \pi)} &\equiv \arg(C_D^{(K^* \pi)} / C_S^{(K^* \pi)}), \\ \delta_{DS}^{(K \rho)} &\equiv \arg(C_D^{(K \rho)} / C_S^{(K \rho)}), \\ \kappa_S e^{i\alpha_S} &\equiv C_S^{(K \rho)} / C_S^{(K^* \pi)}. \end{aligned} \quad (45)$$

Ratios of amplitude will now be calculated separately for $K_1(1400)$ and $K_1(1270)$ applying Eqs. (19) (21) to $K_1^+ \rightarrow K^{*0} \pi^+$ and (23) (24) to $K_1^+ \rightarrow K^+ \rho^0$. Meson masses will be taken from [16].

(i) $K_1(1400)$

Using $|C_D^{(K^* \pi)}|^2 / |C_S^{(K^* \pi)}|^2 = 0.04 \pm 0.01$ and since the branching ratio $K_1 \rightarrow \rho K$ is very small, we calculate:

$$\frac{B_{K^* \pi}}{A_{K^* \pi}} = \frac{0.38 + 1.73 e^{i\delta_{DS}^{(K^* \pi)}}}{1 + 0.14 e^{i\delta_{DS}^{(K^* \pi)}}}, \quad (46)$$

$$\frac{B_{K \rho}}{A_{K \rho}} = \frac{0.45 + C_D^{(K \rho)} / C_S^{(K \rho)}}{1 + 0.05 C_D^{(K \rho)} / C_S^{(K \rho)}} \sim 0.45, \quad (47)$$

The ratio $|C_S^{(K \rho)}| / |C_S^{(K^* \pi)}|$ may be obtained from

$$\frac{\mathcal{B}(K_1 \rightarrow [K \rho]_S)}{\mathcal{B}(K_1 \rightarrow [K^* \pi]_S)} = \frac{2 |C_S^{(K \rho)}|^2 |\vec{p}_K|}{|C_S^{(K^* \pi)}|^2 |\vec{p}_\pi|}, \quad (48)$$

implying together with (43) and assuming a central value $\mathcal{B}(K_1 \rightarrow K \rho) = 3\%$,

$$\kappa_S \equiv \frac{g_{\rho \pi \pi}}{g_{K^* K \pi}} \frac{C_S^{(K \rho)}}{C_S^{(K^* \pi)}} = 0.19 e^{i\alpha_S}. \quad (49)$$

[The relative phase α_S between the $[K^* \pi]_S$ and $[\rho K]_S$ amplitudes was quoted as $20^\circ < \alpha_S < 60^\circ$ in [7,8], following the ACCMOR paper [21].] The factor of 2 on the right-hand side of (48) is due to the specific choice of the modes $K_1^+ \rightarrow K^{*0} \pi^+$ and $K_1^+ \rightarrow \rho^0 K^+$ used to define the couplings $C_S^{(K^* \pi)}$ and $C_S^{(K \rho)}$, while the branching ratios on the left-hand side are for final states summed over all charges.

(ii) $K_1(1270)$

Taking the central value in $|C_D^{(K^* \pi)}|^2 / |C_S^{(K^* \pi)}|^2 = 1.0 \pm 0.7$ and assuming that S wave dominates $K_1 \rightarrow \rho K$ because of an extremely small available phase space, we find:

$$\frac{B_{K^* \pi}}{A_{K^* \pi}} = \frac{0.43 + 16.6 e^{i\delta_{DS}^{(K^* \pi)}}}{1 + 0.71 e^{i\delta_{DS}^{(K^* \pi)}}}, \quad (50)$$

$$\frac{B_{K \rho}}{A_{K \rho}} = 0.51. \quad (51)$$

Using for branching ratios of $K_1 \rightarrow K^* \pi$ and $K_1 \rightarrow K \rho$ averages of the two Belle fits in Table II, we calculate

$$\kappa_S \equiv \frac{g_{\rho \pi \pi}}{g_{K^* K \pi}} \frac{C_S^{(K \rho)}}{C_S^{(K^* \pi)}} = 5.42 e^{i\alpha_S}. \quad (52)$$

A relative phase $\phi(\rho K) - \phi(K^* \pi) \sim -40^\circ$ between total amplitudes has been measured by the Belle

collaboration [22]. However, translating this into a constraint on α_S requires information about the partial wave amplitudes which is not available.

V. PHOTON POLARIZATION AND ASYMMETRY IN $B \rightarrow K\pi\pi\gamma$

We have shown that the decay amplitude for $K_1(\vec{p}=0, \vec{\epsilon}) \rightarrow \pi(p_1)\pi(p_2)K(p_3)$ in the K_1 rest frame has the general structure

$$\mathcal{M}_{\text{rest}} = C_1(s_{13}, s_{23})\vec{p}_1 \cdot \vec{\epsilon} - C_2(s_{13}, s_{23})\vec{p}_2 \cdot \vec{\epsilon} = \vec{J} \cdot \vec{\epsilon}, \quad (53)$$

where

$$\vec{J} \equiv C_1(s_{13}, s_{23})\vec{p}_1 - C_2(s_{13}, s_{23})\vec{p}_2. \quad (54)$$

Considering now $B \rightarrow K_1\gamma$ followed by $K_1 \rightarrow K\pi\pi$ we wish to study the angular distribution of the photon with respect to the K_1 decay plane as function of the photon polarization. For completeness we will derive this relation, although some parts of the derivation can be found in Refs. [7,8]. One reason for presenting this complete analysis is correcting a sign error in defining a specific direction in this previous work.

Working in the rest frame of K_1 , we take the photon momentum \vec{p}_γ along the $-z$ direction, and the B meson momentum along the $+z$ direction. There are two amplitudes for $\bar{B} \rightarrow K_1\gamma$ decays, corresponding to left- and right-handed photons

$$\mathcal{M}_L \equiv \mathcal{A}(\bar{B} \rightarrow K_1\gamma_L), \quad \mathcal{M}_R \equiv \mathcal{A}(\bar{B} \rightarrow K_1\gamma_R). \quad (55)$$

Defining the photon polarization parameter,

$$\lambda_\gamma \equiv \frac{|\mathcal{M}_R|^2 - |\mathcal{M}_L|^2}{|\mathcal{M}_R|^2 + |\mathcal{M}_L|^2}, \quad (56)$$

we would like to determine λ_γ through the angular distribution of the decay products of the K_1 meson.

The amplitude for $\bar{B} \rightarrow K\pi\pi\gamma_{L,R}$ is proportional to the decay amplitude $K_1(p, \epsilon) \rightarrow K\pi\pi$ with $\epsilon = \epsilon_{L,R}$ corresponding to the transverse polarization states $|\lambda = \mp 1\rangle$ of the K_1 meson in its rest frame [see Eqs. (8) (9)],

$$\begin{aligned} \mathcal{A}(\bar{B} \rightarrow K\pi\pi\gamma_{L,R}) &= \mathcal{M}_{L,R}(\vec{\epsilon}_{\mp 1} \cdot \vec{J}) \\ &= \pm \frac{1}{\sqrt{2}} \mathcal{M}_{L,R}(J_x \mp iJ_y). \end{aligned} \quad (57)$$

Squaring the amplitude,

$$\begin{aligned} |\mathcal{A}(\bar{B} \rightarrow K\pi\pi\gamma_{L,R})|^2 &= \frac{1}{2} |\mathcal{M}_{L,R}|^2 (J_x \mp iJ_y)(J_x \pm iJ_y) \\ &= \frac{1}{2} |\mathcal{M}_{L,R}|^2 \{ |J_x|^2 \\ &\quad + |J_y|^2 \mp 2\text{Im}(J_x J_y^*) \}, \end{aligned} \quad (58)$$

and summing over the two photon polarization states, one obtains

$$\begin{aligned} \sum_{\lambda=L,R} |\mathcal{A}(\bar{B} \rightarrow K\pi\pi\gamma_\lambda)|^2 \\ = \frac{1}{2} \{ |\mathcal{M}_L|^2 + |\mathcal{M}_R|^2 \} [(|J_x|^2 + |J_y|^2) + 2\lambda_\gamma \text{Im}(J_x J_y^*)]. \end{aligned} \quad (59)$$

We denote by $\hat{n} = (\vec{p}_1 \times \vec{p}_2)/|\vec{p}_1 \times \vec{p}_2|$ the normal to the K_1 decay plane defined by the two pions momenta. The orientation of the $K\pi\pi$ plane with respect to the $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ axes is determined by three Euler-like angles (θ, ϕ, ψ) . The polar angles (θ, ψ) define the orientation of \hat{n} with respect to \hat{e}_z such that $\cos\theta = \hat{e}_z \cdot \hat{n}$, and the third angle ϕ parameterizes rotations of the $K\pi\pi$ plane around \hat{n} . The intersection of the $K\pi\pi$ plane with the (\hat{e}_x, \hat{e}_y) plane is the nodal line, and its angle with respect to \hat{e}_x is ψ . We denote unit vectors in the $K\pi\pi$ plane by (\hat{e}_1, \hat{e}_2) such that $\hat{e}_3 \equiv \hat{n}$, and define ϕ as the angle between the nodal line and \hat{e}_1 .

The vector \vec{J} lies in the (\hat{e}_1, \hat{e}_2) plane. Its components in the $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ coordinates can be expressed in terms of the angles introduced above:

$$\begin{aligned} J_x &= (J_1 \cos\phi + J_2 \sin\phi) \cos\psi \\ &\quad - (-J_1 \sin\phi + J_2 \cos\phi) \sin\psi \cos\theta, \\ J_y &= (J_1 \cos\phi + J_2 \sin\phi) \sin\psi \\ &\quad + (-J_1 \sin\phi + J_2 \cos\phi) \cos\psi \cos\theta, \\ J_z &= (-J_1 \sin\phi + J_2 \cos\phi) \sin\theta. \end{aligned} \quad (60)$$

These equations are obtained by noting that the projections of \vec{J} in the $K\pi\pi$ plane, J_\parallel along the nodal line and J_\perp perpendicular to it, are

$$\begin{aligned} J_\parallel &= J_1 \cos\phi + J_2 \sin\phi, \\ J_\perp &= -J_1 \sin\phi + J_2 \cos\phi. \end{aligned} \quad (61)$$

The components along the (\hat{e}_x, \hat{e}_y) directions are

$$J_x = J_\parallel \cos\psi - J_\perp \sin\psi \cos\theta, \quad (62)$$

$$J_y = J_\parallel \sin\psi + J_\perp \cos\psi \cos\theta. \quad (63)$$

Substituting (61) in these relations leads to (60).

Using

$$|J_x|^2 + |J_y|^2 = |\vec{J}|^2 - |J_z|^2, \quad (64)$$

and averaging over ϕ implies for the first term in (59),

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} d\phi (|\vec{J}_x|^2 + |J_y|^2) &= |\vec{J}|^2 \left(1 - \frac{1}{2} \sin^2 \theta\right) \\ &= \frac{1}{2} |\vec{J}|^2 (1 + \cos^2 \theta). \end{aligned} \quad (65)$$

The second term multiplying λ_γ is

$$\begin{aligned} \text{Im}(J_x J_y^*) &= \text{Im}(J_{\parallel} J_{\perp}^*) \cos^2 \psi \cos \theta - \text{Im}(J_{\parallel}^* J_{\perp}) \sin^2 \psi \cos \theta \\ &= 2\text{Im}(J_{\parallel} J_{\perp}^*) \cos \theta = 2\text{Im}(J_1 J_2^*) \cos \theta \\ &= \text{Im}[\hat{n} \cdot (\vec{J} \times \vec{J}^*)] \cos \theta. \end{aligned} \quad (66)$$

Thus, after averaging over rotations in the $K\pi\pi$ decay plane (angle ϕ) and around the \hat{e}_z axis (angle ψ), the decay distribution in the angle θ is given by

$$\begin{aligned} \frac{d\Gamma}{ds_{13} ds_{23} d\cos\theta} &= C(s_{13}, s_{23}) \{ |\vec{J}|^2 (1 + \cos^2 \theta) \\ &\quad + \lambda_\gamma 2\text{Im}[\hat{n} \cdot (\vec{J} \times \vec{J}^*)] \cos \theta \}. \end{aligned} \quad (67)$$

The second term in this decay distribution is sensitive to the photon polarization parameter λ_γ . Its contribution can be isolated by forming an up-down asymmetry with respect to the angle θ . At each point (s_{13}, s_{23}) in the Dalitz plot one may define an up-down asymmetry with respect to the \hat{e}_z axis

$$\begin{aligned} \mathcal{A}(s_{13}, s_{23}) &\equiv \frac{1}{d\Gamma/(ds_{13} ds_{23})} \left(\int_0^{\pi/2} d\cos\theta \frac{d\Gamma}{ds_{13} ds_{23} d\cos\theta} \right. \\ &\quad \left. - \int_{\pi/2}^{\pi} d\cos\theta \frac{d\Gamma}{ds_{13} ds_{23} d\cos\theta} \right). \end{aligned} \quad (68)$$

We have seen in (39) that for $K\pi\pi$ final states including a π^0 the overall contribution from the two $K^*\pi$ intermediate states to $C_{1,2}$, which enters the definition of \vec{J} in (54), is antisymmetric under an exchange of the two pion momenta. Consequently the interference of the two $K^*\pi$ contributions, which for an intermediate $K_1(1400)$ is a dominant source for a photon up-down asymmetry in $B \rightarrow K\pi\pi^0\gamma$ (see next subsection), is antisymmetric under $s_{13} \leftrightarrow s_{23}$, and thus vanishes when being integrated over the entire Dalitz plot. For this reason one redefines a slightly modified integrated up-down asymmetry by multiplying the numerator with $\text{sgn}(s_{13} - s_{23})$ which is also antisymmetric in (s_{13}, s_{23}) ,

$$\begin{aligned} \tilde{A} &\equiv \frac{1}{\frac{8}{3} \langle |\vec{J}|^2 \rangle} 2\lambda_\gamma \langle \text{sgn}(s_{13} - s_{23}) \text{Im}(\hat{n} \cdot (\vec{J} \times \vec{J}^*)) \rangle \\ &= \frac{3}{4} \frac{\langle \text{sgn}(s_{13} - s_{23}) \text{Im}(\hat{n} \cdot (\vec{J} \times \vec{J}^*)) \rangle}{\langle |\vec{J}|^2 \rangle} \lambda_\gamma. \end{aligned} \quad (69)$$

The angular brackets denote integration over the Dalitz plot, $\langle \cdots \rangle = \int \cdots ds_{13} ds_{23}$. This asymmetry may be formulated also as an up-down asymmetry with respect to an angle $\tilde{\theta}$ defined by $\cos \tilde{\theta} \equiv \text{sgn}(s_{13} - s_{23}) \cos \theta$, where $\tilde{\theta}$ is the angle between \hat{e}_z and the normal to the plane determined by $\vec{p}_{\text{fast}} \times \vec{p}_{\text{slow}}$ [25].

A. Three mechanisms for a photon asymmetry

Given the expressions of $C_i(s_{13}, s_{23})$ occurring in amplitudes for $K_1 \rightarrow K\pi\pi$ decays and the experimental information about these amplitudes as described in Sec. IV, we are now ready to calculate the photon up-down asymmetry with respect to the $K\pi\pi$ decay plane.

As mentioned in the introduction, a nonzero up-down asymmetry which is odd under time-reversal requires two interfering amplitudes with a nonzero relative phase due to final state interactions. We identify three types of interference which involve such potential phases:

- (i) (a) Interference of amplitudes for two $K^*\pi$ intermediate states. Such interference, involving $K^{*0}\pi^+, K^{*+}\pi^0$ and $K^{*0}\pi^0, K^{*+}\pi^-$ in $K_1^+ \rightarrow K^0\pi^+\pi^0$ and $K_1^0 \rightarrow K^+\pi^-\pi^0$ respectively, occurs only in decays involving a final neutral pion. The amplitude for these $K_1^{+,0} \rightarrow K\pi\pi^0$ decays is given in Sec. IV B. The relevant strong phase originates in an overlap of two isospin-related Breit-Wigner K^{*0} and K^{*+} resonance bands in the Dalitz plot. The contribution of this interference to the asymmetry includes also interference of S and D wave $K^*\pi$ amplitudes which depends on $\delta_{DS}^{(K^*\pi)}$ and vanishes for $\delta_{DS}^{(K^*\pi)} = 0$. We denote an asymmetry from interference of this kind by \tilde{A}_a .
- (ii) (b) Interference between $K^*\pi$ and $K\rho$ amplitudes. Such interference occurs in all $K_1 \rightarrow K\pi\pi$ decays including both $K_1^+ \rightarrow K^+\pi^+\pi^-, K_1^0 \rightarrow K^0\pi^+\pi^-$ and $K_1^+ \rightarrow K^0\pi^+\pi^0, K_1^0 \rightarrow K^+\pi^-\pi^0$. This contribution to an asymmetry is affected by an overlap in the Dalitz plot of the K^* and ρ bands and depends on the two relative phases $\delta_{DS}^{(K^*\pi)}$ and α_S . We denote this contribution to an asymmetry by \tilde{A}_b .
- (iii) (c) Interference of S and D wave amplitudes in $K_1 \rightarrow K^*\pi$. This kind of interference occurs in all four $K_1 \rightarrow K\pi\pi$ charged modes. Because of an assumed negligible D wave amplitude in $K_1 \rightarrow K\rho$ due to very limited available phase space (in particular in $K_1(1270) \rightarrow K\rho$), we neglect a similar interference in these decays. The interference

between S and D wave $K^*\pi$ amplitudes does not depend on overlapping bands in the Dalitz plot and on α_S . The resulting asymmetry depends on α_S (through the asymmetry denominator) and on $\delta_{DS}^{(K^*\pi)}$ and vanishes for $\delta_{DS}^{(K^*\pi)} = 0$. This contribution to an asymmetry will be denoted \tilde{A}_c .

Results will now be presented for up-down photon asymmetries with respect to the $K\pi\pi$ decay plane, which we calculate separately for decays involving $K_1(1400)$ and $K_1(1270)$ resonant states. In addition to total asymmetries we will present asymmetries due to interference of type (a) in decays involving a final neutral pion, and due to interference of types (b) and (c) for decays involving a $\pi^+\pi^-$ pair. We point out that the total asymmetry in the latter decays is the sum $\tilde{A}_{\text{total}} = \tilde{A}_b + \tilde{A}_c$, in which \tilde{A}_b and \tilde{A}_c depend on both $\delta_{DS}^{(K^*\pi)}$ and α_S .

B. Photon asymmetry due to $B \rightarrow K_1(1400)\gamma$

1. $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ and $B^0 \rightarrow K^+\pi^-\pi^0\gamma$

Table III shows total asymmetries and asymmetries of type (a) calculated for a large range of phases $\delta_{DS}^{(K^*\pi)}$, assuming for the total asymmetry a value $\alpha_S = 40^\circ$ favored by [21]. We note that in decays involving a final state π^0 the total asymmetry is completely dominated by interference of type (a) of two amplitudes for two $K^*\pi$ intermediate states and is therefore practically independent on α_S . This follows from the dominance of the $K^*\pi$ mode and the negligible $K_1(1400)$ decay branching ratio into $K\rho$. The asymmetry $\tilde{A}_{\text{total}} = 0.30$ at $\delta_{DS}^{(K^*\pi)} = 0$ is purely due to an overlap of two equal strength (by isospin) Breit-Wigner K^{*0} and K^{*+} bands in the Dalitz plot. Using a value of $\delta_{DS}^{(K^*\pi)}$ around 260° , as indicated by the partial wave analysis performed in Ref. [21], one expects a slightly larger asymmetry of 34% [7,8].

2. $B^+ \rightarrow K^+\pi^+\pi^-\gamma$ and $B^0 \rightarrow K^0\pi^+\pi^-\gamma$

Table IV presents asymmetries of types (b) and (c) and total asymmetries for the same range of values of $\delta_{DS}^{(K^*\pi)}$ as in Table III, assuming $\alpha_S = 40^\circ$ as mentioned above and $|C_D^{(K^*\pi)}|/|C_S^{(K^*\pi)}| = 0.2$ (See Table I.) The total asymmetry is seen to be dominated by terms of type (c) due to interference of S and D wave amplitudes in

$K_1(1400) \rightarrow K^*\pi$, while terms of type (b) originating in interference between $K^*\pi$ and $K\rho$ amplitudes are negligible. This can be traced back to the very small $K\rho$ branching ratio of K_1 decay which is completely dominated by $K^*\pi$. (See Table I.) While for arbitrary $\delta_{DS}^{(K^*\pi)}$ the total asymmetry may be positive or negative, it is predicted to be about +10% for $\delta_{DS}^{(K^*\pi)} \sim 260^\circ$ which is favored by the analysis in Ref. [21].

C. Photon asymmetry due to $B \rightarrow K_1(1270)\gamma$

1. $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ and $B^0 \rightarrow K^+\pi^-\pi^0\gamma$

Table V shows total asymmetries and asymmetries of type (a) due to interference of amplitudes for two $K^*\pi$ intermediate states for $B^+ \rightarrow K^0\pi^+\pi^0\gamma$ decays via $K_1(1270)$ as functions of $\delta_{DS}^{(K^*\pi)}$. The total asymmetry is predicted to lie in a narrow range between -7% and -10% , considerably smaller than the corresponding asymmetry via $K_1(1400)$ given in Table III. While the latter was shown to be positive the former is negative. Unlike the situation we encountered with $K_1(1400)$, the total asymmetry via $K_1(1270)$ is not dominated by interference of type (a). This can be traced back to the small branching ratio of $K_1(1270)$ decay into $K^*\pi$ relative to its considerably larger decay rate into ρK .

2. $B^+ \rightarrow K^+\pi^+\pi^-\gamma$ and $B^0 \rightarrow K^0\pi^+\pi^-\gamma$

Table VI shows photon asymmetries calculated for $B^+ \rightarrow K^+\pi^+\pi^-\gamma$ from intermediate $K_1(1270)$ assuming $|C_D^{(K^*\pi)}|/|C_S^{(K^*\pi)}| = 1$. In the absence of experimental information on $\delta_{DS}^{(K^*\pi)}$ and α_S we varied these two phases over their entire range of $0^\circ - 360^\circ$ searching for an overall range of \tilde{A}_{total} . The asymmetries presented in the table correspond to four cases: The largest positive and negative total asymmetries, $+24\%$ and -13% , a vanishing total asymmetry (obtained also for other values of the two phases) and a fourth case involving arbitrarily chosen two phases of 30° each. In Table VII we present asymmetries assuming $|C_D^{(K^*\pi)}|/|C_S^{(K^*\pi)}| = 0.2$, calculated for the same four pairs of phases ($\delta_{DS}^{(K^*\pi)}, \alpha_S$) as in Table VI. We also found two extreme values of the total asymmetry, $+15\%$ and -13% , obtained for phases $(270^\circ, 135^\circ)$ and $(90^\circ, 315^\circ)$, respectively, and asymmetries $\leq 1\%$ obtained

TABLE IV. Up-down photon asymmetry \tilde{A} in $B^+ \rightarrow K^+\pi^+\pi^-\gamma$ from intermediate $K_1(1400)$. Asymmetries \tilde{A}_b and \tilde{A}_c are defined in the text. The asymmetries are calculated for $\alpha_S = 40^\circ$, a value favored by the analysis in [21].

$\delta_{DS}^{(K^*\pi)}$ (degrees)	0	45	90	135	180	225	270	315
\tilde{A}_b	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.00
\tilde{A}_c	0	-0.07	-0.10	-0.07	0.0	0.07	0.10	0.07
\tilde{A}_{total}	0	-0.07	-0.10	-0.07	0.01	0.08	0.11	0.07

TABLE V. Up-down photon asymmetry in $B^+ \rightarrow K^0 \pi^+ \pi^0 \gamma$ from intermediate $K_1(1270)$. The asymmetry \tilde{A}_a neglects a contribution of a ρK amplitude as described in the text. For the total asymmetry we assume $\alpha_S = -40^\circ$, an approximate value obtained from Ref. [22] by assuming S wave dominance of $K_1(1270)$ decays to $K^* \pi$ and $K \rho$.

$\delta_{DS}^{(K^* \pi)}$ (degrees)	0	45	90	135	180	225	270	315
\tilde{A}_a	0.02	-0.00	-0.04	-0.10	-0.05	0.08	0.06	0.04
\tilde{A}_{total}	-0.09	-0.10	-0.10	-0.10	-0.07	-0.07	-0.08	-0.09

TABLE VI. Up-down photon asymmetry in $B^+ \rightarrow K^+ \pi^+ \pi^- \gamma$ from intermediate $K_1(1270)$ assuming $|C_D^{(K^* \pi)}|/|C_S^{(K^* \pi)}| = 1$. The asymmetries \tilde{A}_b and \tilde{A}_c are defined in the text. A full range of values for \tilde{A}_{total} is calculated for selected values of $\delta_{DS}^{(K^* \pi)}$ and α_S .

$(\delta_{DS}^{(K^* \pi)}, \alpha_S)$ (degrees)	(90, 0)	(270, 270)	(225, 135)	(30, 30)
\tilde{A}_b	-0.05	-0.08	+0.12	-0.05
\tilde{A}_c	-0.08	+0.08	+0.12	-0.02
\tilde{A}_{total}	-0.13	+0.00	+0.24	-0.07

TABLE VII. Up-down photon asymmetry in $B^+ \rightarrow K^+ \pi^+ \pi^- \gamma$ from intermediate $K_1(1270)$ assuming $|C_D^{(K^* \pi)}|/|C_S^{(K^* \pi)}| = 0.2$. The asymmetries \tilde{A}_b and \tilde{A}_c are defined in the text. Asymmetries are calculated for the same four pairs $(\delta_{DS}^{(K^* \pi)}, \alpha_S)$ as in Table VI.

$(\delta_{DS}^{(K^* \pi)}, \alpha_S)$ (degrees)	(90, 0)	(270, 270)	(225, 135)	(30, 30)
\tilde{A}_b	-0.05	-0.08	+0.09	-0.04
\tilde{A}_c	-0.05	+0.05	+0.05	-0.02
\tilde{A}_{total}	-0.11	-0.03	+0.14	-0.06

for other values including a continuum range ($90^\circ - 135^\circ, 90^\circ - 135^\circ$). We checked that a nonresonant K_1 decay amplitude proportional to $\epsilon \cdot (p_1 + p_2)$, in phase with the $K^* \pi$ amplitude and with equal rate [22], has a relatively small effect on the total asymmetry.

We conclude that without further phase information about α_S and $\delta_{DS}^{(K^* \pi)}$ the total asymmetry can have any value ranging from -13% to $+24\%$. Typical contributions of asymmetries of types (b) and (c) have comparable magnitudes which may enhance or cancel each other in the total asymmetry. Comparing the entries in Tables VI and VII shows that for certain phases the value of $|C_D^{(K^* \pi)}|/|C_S^{(K^* \pi)}|$ may have a significant effect on the photon asymmetry.

VI. ISOSPIN SYMMETRY IN $B \rightarrow K \pi \pi \gamma$

We have pointed out that the two pairs of strong decay amplitudes of $K_1^+ \rightarrow K \pi \pi$ and $K_1^0 \rightarrow K \pi \pi$ in Eqs. (29) and

(30), for final states related by isospin reflection $u \leftrightarrow d$, are equal in the isospin symmetry limit. Does a similar relation hold approximately for corresponding weak decay amplitudes $B^+ \rightarrow K \pi \pi \gamma$ and $B^0 \rightarrow K \pi \pi \gamma$?

Isospin breaking in radiative B meson decays has been studied in the literature and found to be small. For two recent brief reviews including theoretical and experimental references see Refs. [26,27]. Isospin asymmetry at a level of 5%, consistent with zero at 2σ , was measured by the Belle and BABAR collaborations for $B \rightarrow K^* \gamma$ [16,28,29],

$$A_I(B \rightarrow K^* \gamma) \equiv \frac{\Gamma(B^0 \rightarrow K^{*0} \gamma) - \Gamma(B^+ \rightarrow K^{*+} \gamma)}{\Gamma(B^0 \rightarrow K^{*0} \gamma) + \Gamma(B^+ \rightarrow K^{*+} \gamma)} = 0.052 \pm 0.026. \quad (70)$$

Isospin breaking in inclusive radiative decays $B \rightarrow X_s \gamma$ is expected to be further suppressed and has been measured at this level by BABAR [30],

$$A_I(B \rightarrow X_s \gamma) \equiv \frac{\Gamma(B^0 \rightarrow X_s^0 \gamma) - \Gamma(B^+ \rightarrow X_s^+ \gamma)}{\Gamma(B^0 \rightarrow X_s^0 \gamma) + \Gamma(B^+ \rightarrow X_s^+ \gamma)} = -0.006 \pm 0.058 \pm 0.009 \pm 0.024. \quad (71)$$

Thus one may assume that the following two approximate isospin equalities hold at a few percent level also for radiative decays to the K_1 resonances:

$$\begin{aligned} A(B^+ \rightarrow K_1^+ \gamma \rightarrow K^+ \pi^+ \pi^- \gamma) &\approx A(B^0 \rightarrow K_1^0 \gamma \rightarrow K^0 \pi^- \pi^+ \gamma), \\ A(B^+ \rightarrow K_1^+ \gamma \rightarrow K^0 \pi^+ \pi^0 \gamma) &\approx A(B^0 \rightarrow K_1^0 \gamma \rightarrow K^+ \pi^- \pi^0 \gamma). \end{aligned} \quad (72)$$

At this level of approximation one may therefore study the photon polarization by combining data for charged and neutral B decays. This should double the statistics. One must pay some attention to the definition of an up-down asymmetry for these two pairs of processes by considering the isospin reflection, $u \leftrightarrow d$, which relates the final kaon and two pions in B^+ decays to corresponding final mesons in B^0 decays.

VII. CONCLUSION

In this paper we reexamined, updated and extended a suggestion made fifteen years ago to measure the photon polarization in $b \rightarrow s \gamma$ by observing in $B \rightarrow K \pi \pi \gamma$ an asymmetry of the photon with respect to the $K \pi \pi$ plane. Asymmetries were calculated for different charged final states due to intermediate $K_1(1400)$ and $K_1(1270)$ resonant states. Three interference mechanisms were identified playing different roles in decays involving these two kaon resonances.

- (i) The situation is quite simple in decays via $K_1(1400)$, for which an upper bound $\mathcal{B}(B \rightarrow K_1(1400) \gamma) < (1.2 - 1.5) \times 10^{-5}$ has been measured using less

than 20% of the Belle total data sample involving $\pi^+\pi^-$. As $K_1(1400)$ is dominated by $K^*\pi$ decays, the total symmetry in decays involving a final state π^0 is large and positive favoring values around 30% from an overlap of two Breit-Wigner K^* bands of equal strength. The asymmetry in decays involving a final state $\pi^+\pi^-$ pair, dominated by interference of S and D wave amplitudes in $K_1(1400) \rightarrow K^*\pi$, is considerably smaller favoring a value around 10%. As these asymmetries show some dependence on the phase $\delta_{DS}^{(K^*\pi)}$ between S and D wave amplitudes in $K_1(1400) \rightarrow K^*\pi$ which has only been measured in [21], an independent measurement of this phase in dedicated amplitude analyses of $B \rightarrow J/\psi(\psi')K\pi\pi$ decays would be useful.

- (ii) The situation is considerably more involved in decays via $K_1(1270)$, for which a branching ratio $\mathcal{B}(B \rightarrow K_1(1270)\gamma) \sim 4 \times 10^{-5}$ has been measured. There are two reasons for this situation. First, the $K_1(1270)$ decays more frequently to $K\rho$ than to $K^*\pi$, for which the branching ratio is only around 20%. Consequently, the total asymmetry in decays involving a final state π^0 is not dominated by interference of two intermediate $K^*\pi$ states. A second reason for being unable to predict an asymmetry in decays involving an intermediate $K_1(1270)$ is lack of information about final state interaction phases in its decays to $K^*\pi$ and $K\rho$. Assuming S wave dominance of $K_1(1270)$ decays to $K^*\pi$ and $K\rho$ an analysis in [22] implies a value $\alpha_S \sim -40^\circ$ for the relative phase between these two amplitudes. Using this value of α_S the asymmetry is predicted to be negative and at most -10% for a final state involving a π^0 .

The situation in decays via $K_1(1270)$ involving a final state $\pi^+\pi^-$ pair is more uncertain because there

is no information about the two relevant phases, α_S and $\delta_{DS}^{(K^*\pi)}$, and there exists only a crude measurement of $|C_D^{(K^*\pi)}|/|C_S^{(K^*\pi)}|$. Varying these phases over their entire range of 0° – 360° we calculated total asymmetries between -13% and $+24\%$, depending to some extent on the ratio of D and S amplitudes. The asymmetry obtains comparable contributions from interference of $K^*\pi$ and $K\rho$ amplitudes and interference of S and D wave $K^*\pi$ amplitudes, which may act constructively or destructively with respect to one another. Major progress in predicting these asymmetries would be achieved by measuring the phases α_S and $\delta_{DS}^{(K^*\pi)}$ and improving the current measurement of the D to S ratio in $K_1(1270) \rightarrow K^*\pi$. This could be achieved in dedicated amplitude analyses of $B \rightarrow J/\psi(\psi')K\pi\pi$ decays to be performed in the future by the Belle II Collaboration at SuperKEKB [31,32].

Finally, in order to increase statistics in studies of the photon polarization, we suggest using approximate isospin symmetry (72) for combining in the same analysis $B \rightarrow K\pi\pi\gamma$ decays for charged and neutral B mesons. So far the Belle collaboration used less than 20% of their total data sample to obtain the branching ratio (3) for $B^+ \rightarrow K_1^+(1270)\gamma$ and the separate upper bounds (6) on $B^+ \rightarrow K_1^+(1400)\gamma$ and $B^0 \rightarrow K_1^0(1400)\gamma$ for final states involving $\pi^+\pi^-$ [14]. We urge the Belle collaboration to combine B^+ and B^0 decays when analyzing their full data sample for these decays, and to study also final states including a π^0 in combined B^+ and B^0 decay samples.

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