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# Quantum backreaction on classical dynamics

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Motivated by various systems in which quantum effects occur in classical backgrounds, we consider the dynamics of a classical particle as described by a coherent state that is coupled to a quantum bath via biquadratic interactions. We evaluate the resulting quantum dissipation of the motion of the classical particle. We also find classical initial conditions for the bath that effectively lead to the same dissipation as that due to quantum effects, possibly providing a way to approximately account for quantum backreaction within a classical analysis.

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## I. INTRODUCTION

Several systems of interest involve the coupling of classical backgrounds to quantum fields. The dynamics of the classical system radiates quantum excitations and thus dissipates. We are interested in evaluating the backreaction of the quantum excitations on the classical dynamics.

This study is particularly relevant to gravitational systems where we do not yet have a full quantum theory and in which context this problem has already received some attention [1–5]. For example, in inflationary cosmology, classical dynamics of the inflaton field excites quantum fields that then become observable cosmological density perturbations. The inflaton field denoted $\Phi(t)$ is assumed to be homogeneous and initially displaced from its minimum. As the field rolls towards its minimum, it can excite a second field, $\phi$, that is coupled to it. Generally symmetries under $\Phi \to -\Phi$ and $\phi \to -\phi$ are assumed so that the lowest order coupling term is $\lambda\Phi^2\phi^2$. The classical evolution of $\phi$ will be governed by

$$\Box\phi + m^2\phi + 2\lambda\Phi^2\phi = 0 \tag{1}$$

and the initial condition $\phi = 0$, $\dot{\phi} = 0$, gives $\phi = 0$ for all times. In quantum theory, however, if $\phi$ is assumed in its ground state initially, it gets excited by the dynamics of the $\Phi$ field. Then the quantum evolution of $\phi$ is nontrivial and it backreacts on the dynamics of $\Phi$ and dissipates its motion. We are interested in evaluating this quantum dissipation. We are also interested in finding a set of classical initial conditions different from $\phi = 0 = \dot{\phi}$ for which the classical dissipation closely agrees with the quantum result.

These questions are of interest beyond inflationary cosmology. Gravitational collapse leads to Hawking radiation that is purely quantum and this will cause the collapsing body to evaporate. The collapsing body is a large object that is most conveniently treated classically, as is its gravitational field. But the radiation is quantum. Can the backreaction on the collapse be estimated on the basis of a classical calculation?

There are nongravitational settings where similar questions arise. For example, what is the backreaction of Schwinger pair production on the electric field? A full treatment of this problem in $1+1$ dimensions for the special case of massless fermions leads to an interesting $t^{-1/2}$ decay of the electric field and an effective electrical conductivity of the vacuum [6] but the case of massive fermions is still open. Another setting where classical and quantum descriptions confront each other is when discussing the production of topological solitons in particle collisions [7,8]. Solitons are solutions of the classical field theory equations and this is the most convenient framework to discuss them. In studying the creation of solitons by scattering particles, if the initial condition involves a large number of particles, they too can be described by classical equations. Thus one may be inclined to think that classical evolution is sufficient to study the creation of solitons in (many) particle collisions. However this is not true in general because, depending on the initial conditions, the classical evolution may be restricted to an embedded subspace of the model [9,10], just as $\phi = 0$ is the dynamical subspace in the example of Eq. (1). Solitons, by their topological nature, involve a very large part of the dynamical space of field configurations and, in certain situations, quantum effects could be crucial for the dynamics to explore the full space of fields necessary to create solitons.

A concrete example helps to explain this issue better. Consider light-on-light collisions. These involve the collisions of a large number of photons and a classical description via Maxwell's equations should suffice. However, then the collision is trivial since Maxwell's equations are linear. In quantum theory, photon collisions will sometimes produce charged particle-antiparticle pairs (e.g. $W^{\pm}$, electrons, and other standard model particles). These will create a plasma that will backreact on the dynamics of the light-on-light collisions. Only the quantum dynamics will explore the full standard model and possibly produce electroweak strings [11] or sphalerons [12] that are solutions of the classical electroweak equations.

The problem outlined above is very difficult to address in field theory and we will only solve a simpler quantum mechanical problem. We first expand the fields in modes. For example for a scalar field,

$$\phi(t,\mathbf{x}) = \sum_{\mathbf{k}} c_{\mathbf{k}}(t) f_{\mathbf{k}}(\mathbf{x}) \tag{2}$$

where $f_{\mathbf{k}}(\mathbf{x})$ are a set of orthonormal mode functions, $c_{\mathbf{k}}(t)$ are mode coefficients, and the sum is an integral if the modes form a continuum. Then, as is standard in quantum field theory (for example see [13]), the free field part of the theory is equivalent to an infinite set of simple harmonic oscillators (SHOs) given by the variables $c_{\mathbf{k}}(t)$ and these can be quantized. The interaction terms in the field theory lead to couplings between the modes and are equivalent to couplings between the SHOs. An interaction term of the type $\lambda\Phi^2\phi^2$, as discussed above, will be equivalent to coupling four SHOs, two corresponding to mode coefficients of $\Phi$ and two to those of $\phi$. In general the couplings will be of the form $C_{\mathbf{k}_1}C_{\mathbf{k}_2}c_{\mathbf{k}_3}c_{\mathbf{k}_4}$ with $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0$, where $C_{\mathbf{k}}$ denotes a mode coefficient of $\Phi$. The biquadratic terms, $C^2_{\mathbf{K}}c^2_{\mathbf{k}}$, are the only ones that are symmetric under $C_{\mathbf{k}_1} \to -C_{\mathbf{k}_1}$ and also, separately, $c_{\mathbf{k}_3} \to -c_{\mathbf{k}_3}$ and hence are the only ones that will survive if we evaluate the expectation value of the coupling term. This suggests that the biquadratic couplings may dominate and our simplification in what follows will be to only consider this coupling. However, this simplification should be examined further because there are many more terms that are not biquadratic and fluctuations, not just the expectation value, may be important. (Systems with bilinear couplings, $C_{\mathbf{K}}c_{\mathbf{k}}$, can be diagonalized and have been analyzed in early work [14,15].) Since $C_{\mathbf{K}}$ represents a classical degree of freedom, we take it to be in a coherent state initially in our quantum analysis, while $c_{\mathbf{k}}$'s are quantum variables that are taken to be in their ground state initially.

To summarize this discussion, we consider a heavy SHO coupled to bath of light SHOs via biquadratic couplings. A solution of the classical equations is that the heavy SHO oscillates and the light SHOs remain at rest. This picture changes in the quantum analysis in which the heavy SHO is initially described by a coherent state and the light SHOs are in their ground state. Oscillations of the heavy SHO excite the light SHOs and there are two forms of backreaction on the heavy SHO. First the heavy SHO motion gets damped. Second, the state of the heavy SHO is no longer a coherent state and the heavy SHO state changes towards becoming less classical, more quantum. In the present paper we focus on the backreaction that causes dissipation. The backreaction that takes the heavy SHO out of its coherent state is interesting but not directly relevant to the dynamical question and we postpone it for the time being.

We start out by describing the quantum mechanical model in Sec. II. Section III contains our classical analysis which we perform with action-angle variables, first studying the dynamics for a single light SHO, followed by a calculation of the classical dissipation for a bath of SHOs. The bath is essential to obtain dissipation because otherwise there is energy exchange between the heavy and light SHOs but no dissipation. In Sec. IV we analyze the quantum model, first for a single light SHO, then for a bath of light SHOs, and we then evaluate the quantum dissipation. Our final result for the quantum vs classical backreaction is discussed in Sec. V and the reader who is not interested in the details of the calculations can directly go to Sec. V. We conclude in Sec. VI. The Appendix contains a discussion of quantization of the SHO using action-angle variables.

## II. MODEL

The heavy SHO position and momentum variables are $(X,P)$; the light SHO variables are $(x_i,p_i)$ for $i=1,\ldots,N$. Traditionally, we would write the Hamiltonian

$$H = \frac{P^2}{2M} + \frac{1}{2}M\Omega^2X^2 + \sum_{i=1}^{N}\left(\frac{p_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2x_i^2\right) + \frac{1}{2N}X^2\sum_{i=1}^{N}\frac{\epsilon_i}{l_i^4}x_i^2 \tag{3}$$

where $l_i$ is a length scale and $\epsilon_i$ has dimensions of energy. Rescaling

$$(M\Omega)^{1/2}X \to X, \qquad (m_i\omega_i)^{1/2}x_i \to x_i, \tag{4}$$

$$P \to (M\Omega)^{1/2}P, \qquad p_i \to (m_i\omega_i)^{1/2}p_i \tag{5}$$

and assuming a universal coupling, i.e. $\epsilon_i/l_i^4$ are independent of $i$, and dividing throughout by a factor of $\Omega$, we get the Hamiltonian in the form

$$H = \frac{P^2}{2} + \frac{X^2}{2} + \sum_{i=1}^{N}\omega_i\left(\frac{p_i^2}{2} + \frac{x_i^2}{2}\right) + \frac{\epsilon}{2N}X^2\sum_{i=1}^{N}x_i^2. \tag{6}$$

Note that we do *not* use the Einstein summation convention.

## III. CLASSICAL ANALYSIS

### A. Single light SHO

A neat method to do the classical calculation is to perform a canonical transformation so that the phase of the SHO is the coordinate variable and the amplitude is related to the momentum variable,

$$q \to \sqrt{\frac{2I}{m\omega}}\sin\theta, \qquad p \to \sqrt{2Im\omega}\cos\theta. \tag{7}$$

The new Hamiltonian is

$$H_{\rm new} = I_1 + \omega I_2 + 2\epsilon I_1 I_2 \sin^2\theta_1 \sin^2\theta_2 \tag{8}$$

where $(\theta_1, I_1)$ are variables for the heavy SHO and $(\theta_2, I_2)$ are for the light SHO. The equations of motion are

$$\begin{aligned}\dot\theta_1 &= 1 + 2\epsilon I_2 \sin^2\theta_1 \sin^2\theta_2\\ \dot I_1 &= -2\epsilon I_1 I_2 \sin(2\theta_1)\sin^2\theta_2\\ \dot\theta_2 &= \omega + 2\epsilon I_1 \sin^2\theta_1\sin^2\theta_2\\ \dot I_2 &= -2\epsilon I_1 I_2 \sin^2\theta_1 \sin(2\theta_2).\end{aligned} \tag{9}$$

The unperturbed solution (with $\epsilon \to 0$) is

$$\begin{aligned}\theta_1 &= t + \phi_1 \qquad I_1 = K_1\\ \theta_2 &= \omega t + \phi_2 \qquad I_2 = K_2\end{aligned} \tag{10}$$

where $\phi_1$, $\phi_2$, $K_1$ and $K_2$ are constants.

To first order in $\epsilon$,

$$\theta_1 = t + \phi_1 + 2\epsilon K_2 \int_0^t dt' \sin^2(t'+\phi_1)\sin^2(\omega t' + \phi_2) \tag{11}$$

$$I_1 = K_1 - 2\epsilon K_1 K_2 \int_0^t dt' \sin(2(t'+\phi_1))\sin^2(\omega t'+\phi_2) \tag{12}$$

$$\theta_2 = \omega t + \phi_2 + 2\epsilon K_1 \int_0^t dt' \sin^2(t'+\phi_1)\sin^2(\omega t'+\phi_2) \tag{13}$$

$$I_2 = K_2 - 2\epsilon K_1 K_2 \int_0^t dt' \sin^2(t'+\phi_1)\sin(2(\omega t'+\phi_2)). \tag{14}$$

To connect with the usual position of the heavy SHO we use

$$\begin{aligned}X &= \sqrt{2I_1}\sin\theta_1\\ &= \sqrt{2K_1}\left[1 - 2\epsilon K_2 \int_0^t dt' \sin(2(t'+\phi_1))\sin^2(\omega t'+\phi_2)\right]^{1/2}\\ &\quad\times \sin\left[t + \phi_1 + 2\epsilon K_2 \int_0^t dt' \sin^2(t'+\phi_1)\sin^2(\omega t'+\phi_2)\right].\end{aligned} \tag{15}$$

In terms of the oscillation amplitudes, $X_0$ and $A$, we take $K_1 = X_0^2/2$, $K_2 = A^2/2$. If the initial condition is that the heavy SHO is displaced but at rest, we take $\phi_1 = \pi/2$; for the phase of the light SHO we write $\phi_2 = \phi$. Then,

$$\begin{aligned}X &= X_0\left[1 + \epsilon A^2 \int_0^t dt' \sin(2t')\sin^2(\omega t'+\phi)\right]^{1/2}\\ &\quad\times\cos\left[t + \epsilon A^2\int_0^t dt' \cos^2(t')\sin^2(\omega t'+\phi)\right].\end{aligned} \tag{16}$$

These integrals can be done in closed form but the expressions are not illuminating.

The modified frequency of oscillation can be found by identifying the linearly growing phase of the cosine in Eq. (16) and is obtained by using

$$\int_0^t dt' \cos^2(t')\sin^2(\omega t'+\phi) = \frac{t}{4} + \text{oscillating terms}. \tag{17}$$

This gives the oscillation frequency to first order in $\epsilon$,

$$\Omega = 1 + \frac{\epsilon}{4}A^2. \tag{18}$$

In Sec. V we will find $A$ for which this modified frequency agrees with the modified frequency in the quantum analysis.

### B. Classical dissipation for bath of light SHOs

To obtain dissipation we have to work out $\dot I_i$ to second order in $\epsilon$. In the equation,

$$\dot I_1 = -2\epsilon I_1 I_2 \sin(2\theta_1)\sin^2\theta_2 \tag{19}$$

we insert the first order expressions in Eqs. (11)–(14). It is convenient to define

$$\begin{aligned}J &\equiv -\frac{t}{4} + \int_0^t dt' \sin^2(t'+\phi_1)\sin^2(\omega t'+\phi_2)\\ &= -\frac{[\sin(2\alpha) - \sin(2\phi_1)]}{8} - \frac{[\sin(2\beta)-\sin(2\phi_2)]}{8\omega}\\ &\quad + \frac{[\sin(2(\alpha+\beta)) - \sin(2\phi_+)]}{16(1+\omega)}\\ &\quad + \frac{[\sin(2(\alpha-\beta)) - \sin(2\phi_-)]}{16(1-\omega)}\end{aligned} \tag{20}$$

where $\alpha = t + \phi_1$, $\beta = \omega t + \phi_2$, and $\phi_\pm = \phi_1 \pm \phi_2$.

Then

$$\frac{\partial J}{\partial \phi_1} = \int_0^t dt' \sin(2(t'+\phi_1))\sin^2(\omega t'+\phi_2) \tag{21}$$

$$\frac{\partial J}{\partial \phi_2} = \int_0^t dt' \sin^2(t'+\phi_1)\sin(2(\omega t'+\phi_2)) \tag{22}$$

and

$$\dot{I}_1 = -2\epsilon K_1 K_2 \left[1 - 2\epsilon\left(K_1 \frac{\partial J}{\partial \phi_2} + K_2 \frac{\partial J}{\partial \phi_1}\right)\right] \times [\sin(2\alpha') + 4\epsilon K_2 J \cos(2\alpha')] \times [\sin^2(\beta') + 2\epsilon K_1 J \sin(2\beta')] \quad (23)$$

where $\alpha' = (1 + \epsilon K_2/2)t + \phi_1$, $\beta' = (\omega + \epsilon K_1/2)t + \phi_2$. We have discarded terms of higher order than $\epsilon^2$ except to show linear order corrections to the oscillation frequencies even if these corrections lead to higher order corrections in $\dot{I}_1$.

We ignore the order $\epsilon$ terms since they are oscillating and do not lead to dissipation. With some algebra

$$\dot{I}_1 \to \epsilon^2 4 K_1 K_2 \left[\frac{1}{2}\left(K_1 \frac{\partial J}{\partial \phi_2} + K_2 \frac{\partial J}{\partial \phi_1}\right) \times \left\{\sin(2\alpha) - \frac{1}{2}\sin(2\alpha_+) - \frac{1}{2}\sin(2\alpha_-)\right\} + J\left\{\frac{K_+}{2}\cos(2\alpha_+) - \frac{K_-}{2}\cos(2\alpha_-) - K_2\cos(2\alpha)\right\}\right] \quad (24)$$

where $K_\pm = K_1 \pm K_2$ and $\alpha_\pm = \alpha \pm \beta = (1 \pm \omega)t + \phi_\pm$ with $\phi_\pm = \phi_1 \pm \phi_2$.

We want to find the dissipation when the classical SHO is coupled to a bath of independent, incoherent, light SHOs. Let us assume that the bath of light SHOs has a spectral distribution of frequencies given by a function $n(\omega)$. In other words, the number of light SHOs with frequencies between $\omega$ and $\omega + d\omega$ is $n(\omega)d\omega$. Therefore we will calculate

$$\dot{E}_{1,\mathrm{classical}} \equiv \langle \dot{I} \rangle = \int_0^\infty d\omega n(\omega) \dot{I}. \quad (25)$$

Further, we are only interested in the dissipatory terms, not in the oscillatory terms. We will also assume $n(0) = 0$. Then the terms that dominate have $(1 - \omega)$ in the denominator and we can effectively replace

$$J \to \frac{\sin(2((1-\omega)t + \phi_-)) - \sin(2\phi_-)}{16(1-\omega)} = -\frac{\sin^2((1-\omega)t)\sin(2\phi_-)}{8(1-\omega)} + \frac{\sin(2(1-\omega)t)\cos(2\phi_-)}{16(1-\omega)} \to \frac{\sin(2(1-\omega)t)\cos(2\phi_-)}{16(1-\omega)} \quad (26)$$

since, in the last step, the first term tends to zero as $1 - \omega \to 0$, while the second term goes to a finite value. Similarly

$$\frac{\partial J}{\partial \phi_1} \to -\frac{\sin(2(1-\omega)t)\sin(2\phi_-)}{8(1-\omega)} = -2J\tan(2\phi_-) \quad (27)$$

$$\frac{\partial J}{\partial \phi_2} \to +\frac{\sin(2(1-\omega)t)\sin(2\phi_-)}{8(1-\omega)} = +2J\tan(2\phi_-). \quad (28)$$

Recognizing that the integration over $\omega$ in Eq. (25) will be dominated by $\omega \approx 1$ and that the oscillating terms do not contribute to the dissipation, we obtain

$$\dot{I}_1 \to \frac{\epsilon^2}{8} K_1 K_2 K_- \frac{\sin(2(1-\omega)t)}{(1-\omega)} \quad (29)$$

where we have replaced $J$ using Eq. (26). Next we use

$$\int_0^\infty dx \frac{\sin(x - x_0)}{x - x_0} \approx \int_{-\infty}^\infty dx \frac{\sin(x - x_0)}{x - x_0} = \pi \quad (30)$$

for $x_0 \gg 1$, and get

$$\dot{E}_{1,\mathrm{classical}} \approx -\epsilon^2 \frac{\pi}{8} K_1 K_2 K_- n(1) \quad (31)$$

for $t \gg 1$. In terms of the initial amplitudes of the SHOs, we take $K_1 = X_0^2/2$, $K_2 = A^2/2$, to get

$$\dot{E}_{1,\mathrm{classical}} \approx -\frac{\pi}{64}\epsilon^2 n(1) X_0^4 A^2 \left(1 - \frac{A^2}{X_0^2}\right) \quad (32)$$

where $A$ is the amplitude of the bath of SHOs at the resonant frequency $\omega = 1$. A surprising feature of this result is that the phases of the SHOs have dropped out.

## IV. QUANTUM ANALYSIS

The action-angle variables $(\theta, I)$ used in the classical analysis were more convenient as they enabled a direct calculation of the change in the energy of the heavy SHO due to backreaction. Quantization in these variables is described in the Appendix and is subtle because of operator ordering issues. Also, since the perturbation term involves the SHO positions, action-angle variables do not lead to any obvious simplifications in the quantum analysis and we work with the conventional $(x, p)$ coordinates.

We write the wave function in SHO Fock basis states

$$\psi(t, X, x) = \sum_{n,m=0}^\infty c_{nm}(t) f_n(t) |n\rangle_X |m\rangle_x \quad (33)$$

where

$$f_n(t) = e^{-it/2} e^{-|z|^2/2} \frac{z^n}{\sqrt{n!}} = e^{-it/2} e^{-|z_0|^2/2} \frac{z_0^n e^{-\mathrm{int}}}{\sqrt{n!}}. \quad (34)$$

In the second equality, we have used the coherent state solution $z = z_0 e^{-it}$.

The initial state is taken to be a direct product of a coherent state for $X$ and ground state for $x$, i.e.,

$$c_{nm}(0) = \delta_{m0}. \tag{35}$$

For convenience, we shall also use the notation

$$b_{nm}(t) = c_{nm}(t) f_n(t). \tag{36}$$

In terms of creation and annihilation operators

$$A = \frac{1}{\sqrt{2}}(X + iP), \qquad A^\dagger = \frac{1}{\sqrt{2}}(X - iP), \tag{37}$$

$$a = \frac{1}{\sqrt{2}}(x + ip), \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \tag{38}$$

we have

$$H = \left(A^\dagger A + \frac{1}{2}\right) + \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\epsilon}{2}\left(\frac{A^\dagger + A}{\sqrt{2}}\right)^2 \left(\frac{a^\dagger + a}{\sqrt{2}}\right)^2. \tag{39}$$

Then the Schrödinger equation gives

$$i\partial_t b_{nm} = \left[\left(n + \frac{1}{2}\right) + \omega\left(m + \frac{1}{2}\right)\right] b_{nm} + \frac{\epsilon}{8}\sum_{l,k=0}^{\infty} \langle n|(A^\dagger + A)^2|l\rangle\langle m|(a^\dagger + a)^2|k\rangle b_{lk}. \tag{40}$$

Now we use

$$\langle n|(A^\dagger + A)^2|l\rangle = \sqrt{n(n-1)}\delta_{n,l+2} + (2n+1)\delta_{n,l} + \sqrt{(n+2)(n+1)}\delta_{n,l-2} \tag{41}$$

$$\langle m|(a^\dagger + a)^2|k\rangle = \sqrt{m(m-1)}\delta_{m,k+2} + (2m+1)\delta_{m,k} + \sqrt{(m+2)(m+1)}\delta_{m,k-2} \tag{42}$$

to get

$$\begin{aligned} i\partial_t b_{nm} = {} & \left[\left(n + \frac{1}{2}\right) + \omega\left(m + \frac{1}{2}\right) + \frac{\epsilon}{8}(2n+1)(2m+1)\right] b_{nm} \\ & + \frac{\epsilon}{8}\Big[\sqrt{n(n-1)}\Big\{\sqrt{m(m-1)}b_{n-2,m-2} + (2m+1)b_{n-2,m} + \sqrt{(m+2)(m+1)}b_{n-2,m+2}\Big\} \\ & + (2n+1)\Big\{\sqrt{m(m-1)}b_{n,m-2} + \sqrt{(m+2)(m+1)}b_{n,m+2}\Big\} \\ & + \sqrt{(n+2)(n+1)}\Big\{\sqrt{m(m-1)}b_{n+2,m-2} + (2m+1)b_{n+2,m} + \sqrt{(m+2)(m+1)}b_{n+2,m+2}\Big\}\Big]. \end{aligned} \tag{43}$$

Note that this equation for $b_{nm}$ also has a term proportional to $b_{nm}$ on the right-hand side. This term is responsible for changing the frequency of oscillations and is better brought over to the left-hand side, leading to

$$\begin{aligned} \partial_t(e^{i\tilde{E}_{nm}t} b_{nm}) = {} & -i\frac{\epsilon}{8}e^{i\tilde{E}_{nm}t}\Big[\sqrt{n(n-1)}\Big\{\sqrt{m(m-1)}b_{n-2,m-2} + (2m+1)b_{n-2,m} + \sqrt{(m+2)(m+1)}b_{n-2,m+2}\Big\} \\ & + (2n+1)\Big\{\sqrt{m(m-1)}b_{n,m-2} + \sqrt{(m+2)(m+1)}b_{n,m+2}\Big\} \\ & + \sqrt{(n+2)(n+1)}\Big\{\sqrt{m(m-1)}b_{n+2,m-2} + (2m+1)b_{n+2,m} + \sqrt{(m+2)(m+1)}b_{n+2,m+2}\Big\}\Big] \end{aligned} \tag{44}$$

where

$$\tilde{E}_{nm} \equiv \left(n + \frac{1}{2}\right) + \omega\left(m + \frac{1}{2}\right) + \frac{\epsilon}{8}(2n+1)(2m+1). \tag{45}$$

Equation (44) is our master equation for $b_{nm}(t)$ that we will solve perturbatively.

### A. Perturbative treatment of single light SHO case

To first order in $\epsilon$, we can replace $b_{lk}$ on the right-hand side of Eq. (44) by its unperturbed value

$$b_{nm} = f_n(t)e^{-i\omega t/2}\delta_{m0} + \mathcal{O}(\epsilon) \tag{46}$$

to get

$$\partial_t(e^{i\tilde{E}_{nm}t}b_{nm}) = -i\frac{\epsilon}{8}e^{i(\tilde{E}_{nm}-\omega/2)t}\left[\left(z^2+(2n+1)+\frac{n(n-1)}{z^2}\right)\sqrt{2}\delta_{m,2} + \left(z^2+\frac{n(n-1)}{z^2}\right)\delta_{m,0}\right]f_n. \tag{47}$$

Therefore only $b_{n0}$ and $b_{n2}$ are nontrivial. For $b_{n0}$ we get

$$b_{n0}(t) = e^{-i\omega t/2}\left[e^{-i\epsilon(2n+1)t/8} - i\frac{\epsilon}{8}\left\{z_0^2 e^{-it}+\frac{n(n-1)}{z_0^2}e^{+it}\right\}\sin(t)\right]f_n(t). \tag{48}$$

Note that a perturbation expansion in powers of $\epsilon$ would mean that we series expand the $\exp(-i\epsilon(2n+1)t/8)$ term. However, then there is a term that is linear in $t$ and the expansion is valid only for very short times, in fact in an $n$ dependent way. The way we have done the calculation here separates out changes in the frequency of oscillation and then the result is valid for all times, as we have also seen in the classical case. Also, we will see that although the correction in Eq. (48) has a term that goes like $\epsilon n(n-1)/z_0^2$, this contribution is of the same order (and cancels) the term that goes like $z_0^2$.

Another peculiarity is that the correction term to $b_{n0}$ does not vanish when $z_0 = 0$ if $n = 2$. This suggests that even if the heavy SHO coherent state is not oscillating, it will excite the second SHO. This can be seen directly from Eq. (44) in which the term $(2m+1)b_{n-2,m}$ is nonzero for $n = 2$, $m = 0$ even if $z_0 = 0$ because $f_{n-2} = 1$ for $n = 2$ and $z_0 = 0$. Excitations of the light SHO in the background of a static coherent state are to be expected since the chosen initial state is an eigenstate only of the unperturbed Hamiltonian, not of the full Hamiltonian.

The solution for $b_{n2}$ is

$$b_{n2}(t) = -i\frac{\epsilon}{4\sqrt{2}}e^{-i3\omega t/2} \times\left[e^{-it}z_0^2\frac{\sin((\omega-1)t)}{\omega-1}+(2n+1)\frac{\sin(\omega t)}{\omega} + e^{+it}\frac{n(n-1)}{z_0^2}\frac{\sin((\omega+1)t)}{\omega+1}\right]f_n(t). \tag{49}$$

### B. Expectation values

#### 1. Energy of heavy SHO

The Hamiltonian of the heavy SHO is

$$H_1 = A^\dagger A + \frac{1}{2}. \tag{50}$$

We will calculate the time derivative of $\langle H_1\rangle$,

$$\frac{d}{dt}\langle H_1\rangle = i\langle[H,H_1]\rangle. \tag{51}$$

Now

$$[H,H_1] = \frac{\epsilon}{2}x^2[X^2,A^\dagger A] = \frac{\epsilon}{2}x^2(A^2-(A^\dagger)^2). \tag{52}$$

We use

$$\langle n|A^2-(A^\dagger)^2|l\rangle = \sqrt{(n+1)(n+2)}\delta_{n+2,l} - \sqrt{n(n-1)}\delta_{n-2,l} \tag{53}$$

$$\langle 0|x^2|0\rangle = \frac{1}{2}, \qquad \langle 0|x^2|2\rangle = \frac{1}{\sqrt{2}} = \langle 2|x^2|0\rangle. \tag{54}$$

Therefore

$$\frac{d}{dt}\langle H_1\rangle = -\frac{\epsilon}{2}\sum_n\sqrt{(n+1)(n+2)} \times \mathrm{Im}[b^*_{n,0}b_{n+2,0}+\sqrt{2}(b^*_{n,0}b_{n+2,2}+b^*_{n,2}b_{n+2,0})]. \tag{55}$$

We need the coefficients $b_{n,m}$ only to first order in $\epsilon$ to get the time derivative of $\langle H_1\rangle$ to second order in $\epsilon$.

We insert $b_{n,0}$ and $b_{n,2}$ from Eqs. (48) and (49) to obtain

$$\sum_n\sqrt{(n+1)(n+2)}\,\mathrm{Im}(b^*_{n0}b_{n+2,0}) = -z_0^2\sin((2+\epsilon/2)t) - \frac{\epsilon}{8}(2z_0^2+1)\sin(2t) \tag{56}$$

where we have used

$$\sum_n|f_n|^2 = 1, \qquad \sum_n n|f_n|^2 = z_0^2, \qquad \sum_n n(n-1)|f_n|^2 = z_0^4, \tag{57}$$

that can be derived from the identity,

$$\left(x\frac{d}{dx}\right)^k e^x = \sum_{n=0}^{\infty} n^k\frac{x^n}{n!}. \tag{58}$$

Next we calculate the middle term on the right-hand side of Eq. (55)

$$\sqrt{2}\sum_n\sqrt{(n+1)(n+2)}\,\mathrm{Im}(b^*_{n,0}b_{n+2,2}) = -\frac{\epsilon}{8}\left[\frac{\{(4z_0^2+1)(z_0^2+2)\omega^2-2(2z_0^2+1)\omega-(2z_0^2+5)z_0^2\}}{\omega(\omega^2-1)} \times\sin(2(\omega+1)t) + \frac{(2z_0^2+5)z_0^2}{\omega}\sin(2t)+\frac{z_0^4}{\omega-1}\sin(4t)\right] \tag{59}$$

and the final term of Eq. (55) is

$$\sqrt{2}\sum_n \sqrt{(n+1)(n+2)}\,\mathrm{Im}(b^*_{n,2}b_{n+2,0}) = \frac{\epsilon}{8}z_0^2\left[\frac{\{(4z_0^2+1)\omega^2-(2z_0^2+1)\}}{\omega(\omega^2-1)}\sin(2(\omega-1)t) - \frac{(2z_0^2+1)}{\omega}\sin(2t) - \frac{z_0^2}{\omega+1}\sin(4t)\right]. \tag{60}$$

Therefore

$$\frac{d}{dt}\langle H_1\rangle = \frac{\epsilon z_0^2}{2}\sin((2+\epsilon/2)t) + \frac{\epsilon^2}{16}\left[\left(2z_0^2+1+\frac{2z_0^2}{\omega}(2z_0^2+3)\right)\sin(2t) + \frac{2\omega z_0^4}{\omega^2-1}\sin(4t) + \frac{P_1(z_0,\omega)}{\omega(\omega^2-1)}\sin(2(\omega+1)t) - \frac{z_0^2P_2(z_0,\omega)}{\omega(\omega^2-1)}\sin(2(\omega-1)t)\right] \tag{61}$$

where

$$P_1(z_0,\omega) = (4z_0^2+1)(z_0^2+2)\omega^2 - 2(2z_0^2+1)\omega - (2z_0^2+5)z_0^2 \tag{62}$$

$$P_2(z_0,\omega) = (4z_0^2+1)\omega^2 - (2z_0^2+1). \tag{63}$$

At this level there is no dissipation since energy is simply exchanged back and forth between the two SHOs. To obtain dissipation we introduce a bath of incoherent, light SHOs.

### C. Bath of light SHOs

As in the classical case [see Eq. (25)], we now integrate over a spectrum of incoherent, light SHOs with spectral function $n(\omega)$. The rate of energy loss of the heavy SHO will be

$$\dot{E}_{1,\mathrm{quantum}} \equiv \frac{d}{dt}\int_0^\infty d\omega n(\omega)\langle H_1\rangle = \text{oscillatory terms} - \frac{\epsilon^2}{16}\int_0^\infty d\omega n(\omega)\frac{z_0^2P_2(z_0,\omega)}{\omega(\omega^2-1)}\sin(2(\omega-1)t). \tag{64}$$

We will ignore the nondissipative oscillating terms. Since $\omega\in[0,\infty)$, the terms that are not oscillating are the ones that are inversely proportional to $1-\omega$ and whose oscillation frequency is also $1-\omega$. This means that we only need to keep the last term in Eq. (64). We assume that the integral in the last term is dominated by the region $\omega\approx 1$ and take $t\gg 1$ to get

$$\dot{E}_{1,\mathrm{quantum}} \approx -\frac{\epsilon^2}{8}n(1)z_0^4\int_0^\infty d\omega\frac{\sin(2(\omega-1)t)}{2(\omega-1)} \approx -\frac{\pi}{16}\epsilon^2 n(1)z_0^4 = -\frac{\pi}{64}\epsilon^2 n(1)X_0^4 \tag{65}$$

where we have used the relation $z_0 = X_0/\sqrt{2}$.

## V. COMPARISON OF CLASSICAL AND QUANTUM SYSTEMS

Comparison of the quantum result in Eq. (65) with the classical result in Eq. (32) gives

$$\dot{E}_{1,\mathrm{classical}} = \dot{E}_{1,\mathrm{quantum}}A^2\left(1-\frac{A^2}{X_0^2}\right) = \dot{E}_{1,\mathrm{quantum}}\frac{E_2}{(\omega/2)}\left(1-\frac{E_2}{E_1}\right) \tag{66}$$

where $E_1$ is the energy of the heavy SHO and $E_2$ is the energy of the light SHO in the bath that is at the resonant frequency $\omega=\Omega$. (By rescalings in Sec. II we had set $\Omega=1$.) Next, to determine suitable values of $A^2$, equivalently $E_2$, we consider the dynamics of the heavy SHO.

The expectation value of the position of the heavy SHO is given by

$$\langle X\rangle = \frac{1}{\sqrt{2}}\sum_{n,m=0}^{\infty}(zc_{n+1,m}c^*_{n,m} + z^*c^*_{n+1,m}c_{n,m})|f_n|^2 \tag{67}$$

where we used $\sqrt{n}f_n = zf_{n-1}$. This expression will be evaluated to first order in $\epsilon$ in which case only $c_{n0}$ (not $c_{n2}$) is relevant. From Eq. (48) we write

$$c_{n0} = e^{-i\omega t/2}\left[e^{-i\epsilon(2n+1)t/8} - i\frac{\epsilon}{8}\left\{z_0^2e^{-it}+\frac{n(n-1)}{z_0^2}e^{+it}\right\}\sin(t)\right]. \tag{68}$$

We use Eq. (57) to do the sum over $n$ in Eq. (67) and find

$$z\sum_{n=0}^{\infty}c_{n+1,0}c^*_{n,0}|f_n|^2 = z_0e^{-i(1+\epsilon/4)t} - i\frac{\epsilon}{4}z_0\sin(t). \tag{69}$$

Then, to leading order in $\epsilon$

$$\langle X\rangle = X_0\cos\left[\left(1+\frac{\epsilon}{4}\right)t\right] \tag{70}$$

which comes from the first term in the square brackets in Eq. (68). The remaining terms all cancel.

Comparing Eq. (70) to (18) we see that the classical and quantum results for the oscillation frequency agree to $\mathcal{O}(\epsilon)$ if we take $A = 1$. This is a natural value because then the classical energy ($E_2 = 1/2$) is precisely the energy of the ground state for the light SHO at the resonant frequency $\omega = 1$. Now, with $A = 1$, Eq. (66) gives

$$\dot{E}_{1,\text{classical}} = \dot{E}_{1,\text{quantum}}\left(1 - \frac{\Omega/2}{E_1}\right) \tag{71}$$

where we have reinserted $\Omega$, the frequency of the heavy SHO. The dissipation rates are identical for coherent states with large occupation number (given by $\mathcal{N}_1 = E_1/\Omega$) up to $\mathcal{O}(\epsilon^2)$.

## VI. CONCLUSIONS

Our final results in Eqs. (70) and (71) are quite remarkable. They show that the classical and quantum oscillation frequencies and dissipation rates both agree provided the classical analysis is done with the light SHOs in a classical analog of the quantum ground state and if the coherent state has large occupation number. This suggests that quantum vacuum dissipation may be studied classically by giving each of the bath SHOs their ground state energy.

Another surprising conclusion that we mentioned in the Introduction is that backreaction on the classical SHO will make it more quantum. The reason is that the initial coherent state is the most classical state, defined by its minimum uncertainty $\Delta X \Delta P = \hbar/2$, and backreaction can only increase the uncertainty and make the state more quantum. This is opposite of the usual role of interactions that cause quantum states to decohere and become more classical. In a similar way, the initial state is taken to be a product state but it evolves into a mixed state and the SHOs becomes more entangled with time.

Our calculations are valid only in leading (second) order in perturbation theory. We plan to study the system at higher order in perturbation theory and at strong coupling in the future, where the classical and quantum analyses may deviate from each other. We also plan to study the rate at which the coherent state "incoheres" due to backreaction, and the rate at which the heavy and light degrees of freedom get entangled.

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## APPENDIX: QUANTIZATION OF SHO IN ACTION-ANGLE VARIABLES

Consider a quantum SHO

$$H = \frac{p^2}{2} + \frac{x^2}{2} = a^\dagger a + \frac{1}{2} \tag{A1}$$

where

$$a = \frac{x + ip}{\sqrt{2}}, \qquad a^\dagger = \frac{x - ip}{\sqrt{2}} \tag{A2}$$

and

$$[a, a^\dagger] = 1 \tag{A3}$$

follows from $[x, p] = i$.

Now consider the transformation

$$a = e^{-i\theta}\sqrt{I}, \qquad a^\dagger = \sqrt{I}e^{+i\theta} \tag{A4}$$

where we assume that $\sqrt{I}$ is a Hermitian operator and will shortly discuss the meaning of this operator. Then

$$H = I + \frac{1}{2} \tag{A5}$$

and also Eq. (A3) leads to

$$[\theta, I] = i, \tag{A6}$$

which has the representation

$$I = -i\frac{\partial}{\partial\theta}. \tag{A7}$$

Therefore the normalized eigenstates with energy $n + 1/2$ are

$$\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}} \tag{A8}$$

with $n = 0, 1, 2, \ldots$ because the wave functions are periodic under $\theta \to \theta + 2\pi$. Eigenstates with negative integer values of $n$ are not allowed in the physical spectrum because of the assumed Hermiticity of $\sqrt{I}$ and the definition of $\sqrt{I}$ below.

To interpret $\sqrt{I}$ we define

$$\sqrt{I}e^{in\theta} = \sqrt{n}e^{in\theta} \tag{A9}$$

and work in the basis $\{e^{in\theta}\}$, assuming that $\sqrt{I}$ acts linearly. For example, if

$$\psi(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta} \tag{A10}$$

where $c_n$ are expansion coefficients, then

$$\sqrt{I}\psi(\theta) = \sum_{n=0}^{\infty} c_n \sqrt{n} e^{in\theta}. \tag{A11}$$

To recover the usual SHO wave functions in position space, we need to find the eigenstates of the position operator, $\hat{x}$, in the $e^{in\theta}$ basis. That is, we need to solve

$$\hat{x} \sum_{n=0}^{\infty} c_n \psi_n(\theta) = x \sum_{n=0}^{\infty} c_n \psi_n(\theta). \tag{A12}$$

Note that the coefficients $c_n$ will depend on the c-number position $x$. In Dirac notation, $c_n(x) = \langle x|n\rangle_\theta$, and these are the wave functions in position space. Using

$$\hat{x} = \frac{a + a^\dagger}{\sqrt{2}} = \frac{1}{\sqrt{2}}[e^{-i\theta}\sqrt{I} + \sqrt{I}e^{+i\theta}] \tag{A13}$$

leads to the recursion relation

$$\sqrt{n+1}c_{n+1} - x\sqrt{2}c_n + \sqrt{n}c_{n-1} = 0. \tag{A14}$$

The recursion relation for Hermite polynomials,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \tag{A15}$$

can be used to check that

$$c_n(x) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} H_n(x) e^{-x^2/2} \tag{A16}$$

satisfies Eq. (A14). These $c_n$'s are the usual normalized wave functions of the excited states of the SHO in the $x$-representation.


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