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Dynamical analysis of an integrable cubic Galileon cosmological model

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Recently a cubic Galileon cosmological model was derived by the assumption that the field equations are invariant under the action of point transformations. The cubic Galileon model admits a second conservation law which means that the field equations form an integrable system. The analysis of the critical points for this integrable model is the main subject of this work. To perform the analysis, we work on dimensionless variables that are different from those of the Hubble normalization. New critical points are derived, while the gravitational effects which follow from the cubic term are studied.

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I. INTRODUCTION

A theory which has drawn the attention of the scientific society in the last few years is Galileon gravity [1,2]. It belongs to the modified theories of gravity in which a noncanonical scalar field is introduced and the field equations are invariant under the Galilean transformation. The action integral of Galileon gravity belongs to the Horndeski theories [3], which means that the gravitational theory is of second order [4]. The vast applications of study for Galileons in the literature cover all the areas of gravitation physics, including neutron stars, black holes, and acceleration of the Universe (for instance, see [4–21] and references therein).

In this work we are interested in the cosmological scenario and specifically in the so-called Galileon cosmology [22–24]. In cosmology, the Galileon field has been applied in order to explain various phases of the evolution of the Universe [25–29]. Specifically, the new terms in the gravitational action integral can force the dynamics in such a way that the model fits the observations. The mechanics can also explain the inflation era [30–36] as the late-time acceleration of the Universe [37–42]. Last but not least, the growth index of matter perturbations have been constrained in [43].

As we mentioned in the previous paragraph, Galileon gravity belongs to the Horndeski theories, and specifically, there is an infinite number of different models which can be constructed from a general Lagrangian. A simple model is the cubic Galileon model [11–17] where the action integral is that of a canonical scalar field plus a new term which has a cubic derivative on the Galileon field. The theory can be seen as a first extension of the scalar-field cosmology. Due to this cubic term, the nonlinearity and the complexity of the field equations is increased dramatically. Recently in [44], a cubic Galileon model was derived which admits an additional conservation law and where the field equations formed an integrable dynamical system.

Integrability is an important issue in all areas of physics and mathematical sciences. The reason for this is that while a dynamical system can be studied numerically, it is unknown if an actual solution which describes the "orbits" exists. The integrable cubic Galileon model admits special solutions which describe an ideal gas universe, that is, power-law scale factors. While this is similar to the solution of the canonical scalar field, we found that the power of the power-law solution is not strongly constrained by the Galileon field; this is because of the cubic term. On the other hand, a special property of that model is that when the potential in the action integral dominates, then the cubic term disappears, which mean that the theory approach is that of a canonical scalar field. However, as we shall see from our analysis, the existence of the conservation law provides new dynamics which have not been investigated previously.

The aim of this work is to study the evolution of the integrable cubic Galileon cosmological model. For that, we perform an analysis of the critical points. In particular, in Sec. II we briefly discuss the cubic Galileon cosmology and review the integrable case that was derived before in [44]. Section III includes the main material of our analysis, where we rewrite the field equations in dimensionless variables. We define variables different from that of the H-normalization, where we find

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that the dynamical system is not bounded. Because of the latter property, the critical points at the finite and the infinite regions are studied. At the finite region we find various critical points which can describe the expansion history of our Universe as the matter-dominated era. Appendixes A, B, C, D, E and F include important mathematical material which justify our analysis. One important property of the integrable model that we study is that there is a limit in which the terms in the field equations (which follow from the cubic term of the Galileon Lagrangian) vanish, and the model is then reduced to that of a canonical scalar-field cosmological model. Hence, in order to study the effects of the cubic term in Sec. IV, we perform an asymptotic expansion of the solution when the cubic term dominates the Universe. In Sec. V we extend our analysis to the case where an extra matter term is included in the gravitational action integral. Finally, we discuss our results and draw our conclusions in Sec. VI.

II. CUBIC GALILEON COSMOLOGY

The cubic Galileon model is defined by the following action integral:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) - \frac{1}{2}g(\phi)\partial^\mu\phi\partial_\mu\phi\Box\phi\right) \tag{1}$$

which has various cosmological and gravitational applications.

In the cosmological scenario of a homogeneous and isotropic universe with zero spatial curvature, the line element of the spacetime is that of the Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{2}$$

where $a(t)$ is the scale factor of the universe.

Indeed, for this line element, variation with respect to the metric tensor in (1) provides the gravitational field equations

$$3H^2 = \frac{\dot{\phi}^2}{2}(1 - 6g(\phi)H\dot{\phi} + g'(\phi)\dot{\phi}^2) + V(\phi) \tag{3}$$

and

$$2\dot{H} + \dot{\phi}^2(1 + g'(\phi)\dot{\phi}^2 - 3g(\phi)H\dot{\phi} + g(\phi)\ddot{\phi}) = 0, \tag{4}$$

while variation with respect to the field ϕ provides the (modified) "Klein-Gordon" equation

$$\ddot{\phi}(2\dot{\phi}^2 g'(\phi) - 6Hg(\phi)\dot{\phi} + 1) + \dot{\phi}^2\left(\frac{1}{2}\dot{\phi}^2 g''(\phi) - 3g(\phi)\dot{H} - 9H^2 g(\phi)\right) + 3H\dot{\phi} + V'(\phi) = 0, \tag{5}$$

which describes the evolution of the field, and $H = \frac{\dot{a}}{a}$. Recall that we have assumed that the Galileon field inherits the symmetries of the spacetime; that is, if K^μ is an isometry of (2), i.e. $[K, g_{\mu\nu}] = 0$, then ϕ inherits the symmetries of the spacetime if and only if $[K, \phi] = 0$. Therefore, ϕ is only a function of the "t" parameter, i.e. $\phi(x^\mu) = \phi(t)$.

An equivalent way to write the field equations is by defining fluid components such as energy density and pressure which correspond to the Galileon field. Indeed, if we consider the energy density

$$\rho_G = \frac{\dot{\phi}^2}{2}(1 - 6g(\phi)H\dot{\phi} + g'(\phi)\dot{\phi}^2) + V(\phi) \tag{6}$$

and the pressure term

$$p_G = \frac{\dot{\phi}^2}{2}(1 + 2g\ddot{\phi} + g_{,\phi}\dot{\phi}^2) - V(\phi), \tag{7}$$

the field equations take the form $G^\mu{}_\nu = T_\nu^{(G)\mu}$, where $T_{\mu\nu}^{(G)}$ is the energy-momentum tensor corresponding to the Galileon field and

$$T_{\mu\nu}^{(G)} = \rho_G u_\mu u_\nu + p_G(g_{\mu\nu} + u_\mu u_\nu) \tag{8}$$

in which $u^\mu = \delta_t^\mu$ is the normalized comoving observer ($u^\mu u_\mu = -1$). Equation (5) is now equivalent to the Bianchi identity $T^{(G)\mu\nu}{}_{;\nu} = 0$, that is,

$$\dot{\rho}_{\rm DE} + 3H(\rho_{\rm DE} + p_{\rm DE}) = 0. \tag{9}$$

Last but not least, the dark-energy equation-of-state parameter is defined as follows:

$$w_{\rm DE} \equiv \frac{p_{\rm DE}}{\rho_{\rm DE}} = \frac{1}{3H^2}\left[\frac{\dot{\phi}^2}{2}(1 + 2g\ddot{\phi} + g_{,\phi}\dot{\phi}^2) - V(\phi)\right]. \tag{10}$$

It can be shown that with a proper election of $g(\phi)$, the equation-of-state parameter $w_{\rm DE}$ can realize the quintessence scenario (the phantom one) and cross the phantom divide during the evolution, which is one of the advantages of Galileon cosmology. In general, the specific functions of $g(\phi)$ and $V(\phi)$ are unknown, and for different functions, there will be a different evolution.

A. Extra conservation law

Recently, in [44] two unknown functions were specified by the requirement that the gravitational field equations form an integrable dynamical system. Specifically, the following functions were found:

$$V(\phi) = V_0 e^{-\lambda\phi} \quad \text{and} \quad g(\phi) = g_0 e^{\lambda\phi}. \tag{11}$$

There exists a symmetry vector which provides, with the use of Noether's second theorem, the following conservation law for the field equations:

$$I_1 = -\left(2a^2\dot{a} - \frac{2}{\lambda}a^3\dot{\phi}\right) + g_0 \mathrm{e}^{\lambda\phi} a^3\dot{\phi}^3 - \frac{6}{\lambda} g_0 a^2 e^{\lambda\phi}\dot{a}\dot{\phi}^2. \tag{12}$$

Because of the nonlinearity of the field equations, the general solution cannot be written in a closed form. However, from the symmetry, vector-invariant curves have been defined, and by using the zeroth-order invariants, some power-law (singular) solutions have been derived. In particular, the following solutions were obtained:

$$a_1(t) = a_0 t^p, \qquad \phi(t) = \frac{2}{\lambda}\ln(\phi_0 t),$$
$$g_0 = \frac{\lambda(2-\lambda^2 p)}{4(3p-1)\phi_0^2}, \qquad V_0 = \phi_0^2\left(\frac{2}{\lambda^2} + p(3p-2)\right) \tag{13}$$

and

$$a_{2,3}(t) = a_0 t^{\frac{1}{3}}, \qquad \phi_{2,3}(t) = \pm\frac{\sqrt{6}}{3}\ln(\phi_0 t),$$
$$V_0 = 0, \qquad \lambda_{2,3} = \pm\sqrt{6}. \tag{14}$$

These solutions are special solutions since they exist for specific initial conditions. In order to study the general evolution of the system, we perform an analysis in the phase space.

A phase-space analysis for this cosmological model has been performed previously in [16]; however, the integrable case with $g(\phi)$ and $V(\phi)$ given by the expressions (11) was excluded from [16]. Moreover, there is a special observation in the integrable case in the sense that $V(\phi)g(\phi) = \text{const}$. The latter means that when $V(\phi)$ dominates, $g(\phi)$ becomes very small and the cubic Galileon model reduces to that of a canonical scalar field which can also mimic the cosmological constant when $V(\phi) \gg \dot{\phi}^2$.

In the following we write the field equations in new dimensionless variables, and we perform our analysis.

III. EVOLUTION OF THE DYNAMICAL SYSTEM

From Eq. (3) one immediately sees that the Hubble function $H(t)$ can cross the value $H(t) = 0$, from negative to positive values, or vice versa. This means that the standard H-normalization is not useful, and new variables have to be defined. We introduce the new variables

$$x = \frac{\dot{\phi}}{\sqrt{6(H^2+1)}}, \quad y = \frac{V_0 e^{-\lambda\phi}}{3(H^2+1)}, \quad z = \frac{H}{\sqrt{H^2+1}}, \tag{15}$$

and the parameter $\alpha = g_0 V_0$, so we obtain the three-dimensional dynamical system

$$x' = \frac{f_1(\mathbf{x})}{f_4(\mathbf{x})}, \qquad y' = \frac{f_2(\mathbf{x})}{f_4(\mathbf{x})}, \qquad z' = \frac{f_3(\mathbf{x})}{f_4(\mathbf{x})} \tag{16}$$

where $\mathbf{x} = (x, y, z)$ and functions $f_1 - f_4$ are defined by the following expressions:

$$\begin{aligned} f_1(x,y,z) &= y^2(3x^3z + 3xz(-y+z^2-2) + \sqrt{6}\lambda y) \\ &\quad + 6\alpha^2x^5(2\lambda^2x^2z - \sqrt{6}\lambda x - 2\sqrt{6}\lambda xz^2 + 6z^3) \\ &\quad + \alpha x^2y(18\lambda x^3z - \sqrt{6}x^2(2\lambda^2+12z^2+3) \\ &\quad - 6\lambda xz(y-2z^2) + 3\sqrt{6}(2yz^2+y-2z^4+z^2)), \end{aligned} \tag{17a}$$

$$\begin{aligned} f_2(x,y,z) &= 2y(12\alpha^2\lambda^2x^6z - 6\sqrt{6}\alpha^2\lambda x^5(2z^2+1) \\ &\quad + 18\alpha x^4z(\lambda y + 2\alpha z^2) - 4\sqrt{6}\alpha x^3y(\lambda^2+3z^2) \\ &\quad + 3x^2yz(y - 2\alpha\lambda(y - 2(z^2+1))) \\ &\quad + \sqrt{6}xy(6\alpha z^2(y-z^2) - \lambda y) + 3y^2z(z^2-y)), \end{aligned} \tag{17b}$$

$$\begin{aligned} f_3(x,y,z) &= 3(z^2-1)(y^2(x^2 + 2\alpha x(\sqrt{6}z - \lambda x) + z^2) \\ &\quad + 4\alpha^2x^4(\lambda^2x^2 - \sqrt{6}\lambda xz + 3z^2) \\ &\quad + 2\alpha xy(3\lambda x^3 - 2\sqrt{6}x^2z + 2\lambda xz^2 \\ &\quad - \sqrt{6}z^3) - y^3), \end{aligned} \tag{17c}$$

$$f_4(x,y,z) = 2(2\alpha x(3\alpha x^3 + 2\lambda xy - \sqrt{6}yz) + y^2), \tag{17d}$$

and a prime denotes the new derivative $\frac{df}{d\tau} \equiv f' \equiv \frac{\dot{f}}{\sqrt{H^2+1}}$. Interestingly, the system (16) admits the first integral

$$2\alpha x^3(\lambda x - \sqrt{6}z) + y(x^2 - z^2) + y^2 = 0 \tag{18}$$

which is Friedmann's first equation, and it constrains the evolution of the solution. Thus, the dynamics are restricted to a surface given by (18). For a fixed value of y, the first and last equations in (16) are invariant under the discrete symmetry $(x, z, \tau) \to -(x, z, \tau)$. Thus, the fixed points related by this discrete symmetry have the opposite dynamical behavior. By definition, $y \geq 0$.

Let us compare with the variables introduced in [16] defined by $x_1 = \frac{\dot{\phi}}{\sqrt{6}H}$, $y_1 = \frac{\sqrt{V(\phi)}}{\sqrt{3}H}$, $z_1 = g(\phi)H\dot{\phi}$.

Since here we have chosen $V(\phi)$ and $g(\phi)$ such that $g(\phi)V(\phi) = \alpha$, we have the relations $x_1 = \frac{x}{z}$, $y_1 = \frac{\sqrt{y}}{z}$, $z_1 = \frac{\sqrt{\frac{2}{3}}\alpha x z}{y}$, and the extra relation $\alpha x_1 - \sqrt{\frac{3}{2}} y_1^2 z_1 = 0$. This implies that the fixed points $A^{\pm}$, $B^{\pm}$, C, and D, investigated in detail in [16], do not exist in our scenario since the values of their coordinates (x_1, y_1, z_1) do not satisfy the extra relation above.

Furthermore, some cosmological parameters with great physical significance are the effective equation-of-state parameter $w_{\mathrm{tot}} \equiv \frac{p_{\mathrm{tot}}}{\rho_{\mathrm{tot}}} = w_{\mathrm{DE}}$ (because we set $\rho_m = 0$) and the "deceleration parameter"

$$q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} + \frac{3}{2} w_{\mathrm{tot}}. \tag{19}$$

Some conditions for the cosmological viability of the most general scalar-tensor theories have to be satisfied by extended Galileon dark energy models; in other words, the model must be free of ghosts and Laplacian instabilities [45–47]. In the special case of the action (1) (in units where $\kappa \equiv 8\pi G = 1$), we require, for the avoidance of Laplacian instabilities associated with the scalar-field propagation speed, that [46]

$$c_S^2 \equiv \frac{6w_1 H - 3w_1^2 - 6\dot{w}_1}{4w_2 + 9w_1^2} \geq 0; \tag{20}$$

$$w_1 \equiv g\dot{\phi}^3 + 2H, \quad w_2 \equiv 3\dot{\phi}^2 \left[\frac{1}{2} + g_{,\phi}\dot{\phi}^2 - 6Hg\dot{\phi}\right] - 9H^2. \tag{21}$$

Meanwhile, for the absence of ghosts, it is required that

$$Q_S \equiv \frac{(4w_2 + 9w_1^2)}{3w_1^2} > 0. \tag{22}$$

Finally, we have from Eqs. (10), (20), and (22) that the phantom phase can be free of instabilities and thus cosmologically viable, as it was already shown for Galileon cosmology [46].

A. Analysis at the finite region

The fixed points or fixed lines at the finite region of the system (16) and a summary of their stability conditions are presented in Table I. In Table II we display several cosmological parameters for the fixed points at the finite region of the system (16). The discussion about the physical interpretation of these points and the points at infinity is left for Sec. III D.

We proceed with the determination of critical points at infinity.

B. Analysis at "infinity"

Because the phase space of the system is unbounded, we introduce the Poincarè compactification and a new time derivative f',

$$X = \frac{x}{\sqrt{1 + x^2 + y^2}}, \qquad Y = \frac{y}{\sqrt{1 + x^2 + y^2}},$$
$$f' \to (1 - X^2 - Y^2) f'. \tag{23}$$

The dynamics on the "cylinder at infinity" can be obtained by setting $X = \cos\theta$, $Y = \sin\theta$; the dynamics in the coordinates (θ, z) is governed by the equations

$$\theta' = h_1(\theta, z) \coloneqq \lambda^2 z \sin(\theta) \cos^3(\theta), \tag{24a}$$

$$z' = h_2(\theta, z) \coloneqq \lambda^2 (z^2 - 1) \cos^2(\theta). \tag{24b}$$

We linearize around a given fixed point on the cylinder at infinity by introducing $X = \cos\theta - \varepsilon_1$, $Y = \sin\theta - \varepsilon_2$, with $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$. Notice that $1 - X^2 - Y^2 \simeq 2\varepsilon_1 \cos\theta + 2\varepsilon_2 \sin\theta$, so to examine the stability of the fixed points at the cylinder, and from the interior of it, we have to estimate how $r = \varepsilon_1 \cos\theta + \varepsilon_2 \sin\theta$ evolves, not just the stability in the plane (θ, z). For $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$, we obtain the expansion rate

$$r' = r[\lambda^2 z \cos^2(\theta)(\cos(2\theta) - 3)]. \tag{25}$$

The stability condition of a fixed point along r is then $r'/r < 0$. The full stability of the above fixed points is summarized in Table III.

C. Numerical analysis

Let us complete our analysis by performing some numerical simulations. Specifically, we choose the constants of the model to satisfy the conditions

$$g_0 = -\frac{\lambda(\lambda^2 p - 2)}{4(3p - 1)\phi_0^2}, \qquad V_0 = \phi_0^2 \left(\frac{2}{\lambda^2} + p(3p - 2)\right);$$

hence, $\alpha = -\frac{\lambda(\frac{2}{\lambda^2} + p(3p-2))(\lambda^2 p - 2)}{4(3p-1)}$.

Moreover, we impose the condition $p > \frac{1}{3}$, which guarantees the stability of the perturbation of the scaling solution [44]. Because we have chosen $\alpha \geq 0$, this leads to the "allowed" region on the parameter space defined by

TABLE I. Summary of the stability conditions of the fixed points at the finite region of the system (16), where $\mathcal{P}(x) = 6(1+\alpha\lambda)x^3 + 3\sqrt{6}(2\alpha+\lambda)x^2 + 2(3+\lambda^2)x + \sqrt{6}\lambda$. We use the acronyms "A.S." for asymptotically stable and "A.U." for asymptotically unstable.

Label: Coordinates (x, y, z)	Existence	Eigenvalues	Stability
P_1: $(0,0,0)$	Always	Undetermined	Numerical analysis
P_2: $(-\frac{\sqrt{6}}{\lambda}, 0, -1)$	Always	$0, -3, -6$	Stable for $\lambda < -\sqrt{6}$ or $0 < \lambda < \sqrt{6}$, saddle for $-\sqrt{6} < \lambda < 0$ or $\lambda > \sqrt{6}$. (see Appendix C)
P_3: $(\frac{\sqrt{6}}{\lambda}, 0, 1)$	Always	$0, 3, 6$	Unstable for $\lambda < -\sqrt{6}$ or $0 < \lambda < \sqrt{6}$, saddle for $-\sqrt{6} < \lambda < 0$ or $\lambda > \sqrt{6}$. (see Appendix C)
P_4: $(0,1,1)$	$\lambda = 0$	$-3, -3, 0$	A.S. (see Appendix D)
P_5: $(0,1,-1)$	$\lambda = 0$	$3, 3, 0$	A.U. (see Appendix D)
$P_6(x_c)$: $\left(x_c, \sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1}, -1\right)$	Where $x_c \neq 0$ is a real root of $\mathcal{P}(x)$ such that $\sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1} \geq 0$	Numerical analysis	Numerical analysis
$P_7(x_c)$: $\left(-x_c, \sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1}, 1\right)$	Where $x_c \neq 0$ is a real root of $\mathcal{P}(x)$ such that $\sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1} \geq 0$	Numerical analysis	Numerical analysis
P_8: $(0,0,-1)$	Always	$-3, -3, -(6\alpha\lambda - \frac{3}{2})$	Sink for $\alpha\lambda > \frac{1}{4}$ Saddle otherwise
P_9: $(0,0,1)$	Always	$3, 3, (6\alpha\lambda - \frac{3}{2})$	Source for $\alpha\lambda > \frac{1}{4}$ Saddle otherwise
P_{10}: $(-\frac{\sqrt{\frac{3}{2}}}{\lambda}, 0, -1)$	$\alpha = 0,\ \lambda \neq 0$	$(-\frac{9}{2\lambda^2}, \frac{3}{4}(2 - \frac{9}{\lambda^2}), -\frac{9}{2\lambda^2} - 3)$	Sink for $0 < \lambda^2 < \frac{3}{2}$ Saddle otherwise
P_{11}: $(\frac{\sqrt{\frac{3}{2}}}{\lambda}, 0, 1)$	$\alpha = 0,\ \lambda \neq 0$	$(\frac{9}{2\lambda^2}, -\frac{3}{4}(2 - \frac{9}{\lambda^2}), \frac{9}{2\lambda^2} + 3)$	Source for $0 < \lambda^2 < \frac{3}{2}$ Saddle otherwise
P_{12}: $(-\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{3}{2\lambda^2}, -1)$	$\lambda^2 + 3\alpha\lambda - 3 = 0$	$0, 3, -\frac{3}{2}$	Saddle
P_{13}: $(\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{3}{2\lambda^2}, 1)$	$\lambda^2 + 3\alpha\lambda - 3 = 0$	$0, 3, -\frac{3}{2}$	Saddle
$P_{14}(z_c)$: $(0, z_c^2, z_c)$	$\lambda = 0$	$(0, -3z_c, -3z_c)$	A.S. for $0 < z_c \leq 1$ A.U. for $-1 \leq z_c < 0$ (see Appendix E).
$P_{15}(z_c)$: $(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$	$\lambda = 0, \beta = \frac{1}{\sqrt{2}}(\sqrt{3}\alpha - \sqrt{3\alpha^2 - 2})$, $\alpha > \sqrt{\frac{2}{3}}$	$(0, -3z_c, -3z_c)$	A.S. for $0 < z_c \leq 1$ A.U. for $-1 \leq z_c < 0$ (see Appendix F).
$P_{16}(z_c)$: $(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$	$\lambda = 0, \beta = \frac{1}{\sqrt{2}}(\sqrt{3}\alpha + \sqrt{3\alpha^2 - 2})$, $\alpha > \sqrt{\frac{2}{3}}$	$(0, -3z_c, -3z_c)$	A.S. for $0 < z_c \leq 1$ A.U. for $-1 \leq z_c < 0$ (see Appendix F).

(i) $\lambda < -\sqrt{6},\ p \geq \frac{1}{3}\sqrt{\frac{\lambda^2 - 6}{\lambda^2}} + \frac{1}{3}$, or

(ii) $\lambda = -\sqrt{6},\ p > \frac{1}{3}$, or

(iii) $-\sqrt{6} < \lambda < 0,\ p \geq \frac{2}{\lambda^2}$, or

(iv) $0 < \lambda < \sqrt{6},\ \frac{1}{3} < p \leq \frac{2}{\lambda^2}$, or

(v) $\lambda > \sqrt{6},\ \frac{1}{3} < p \leq \frac{1}{3}\sqrt{\frac{\lambda^2 - 6}{\lambda^2}} + \frac{1}{3}$,

as displayed in Fig. 1.

In Fig. 2, we present a Poincarè projection of the system (16) on the invariant set $z = -1$. The green dots correspond to the points $P_6(x_c)$ (that we solved numerically). In the special case $\lambda^2 = 6$, $p = \frac{2}{3}$, we have $1 + \alpha\lambda = 0$; thus, the polynomial $\mathcal{P}(x)$ is quadratic, and there are only two roots of $\mathcal{P}(x) = 0$. The blue contour is defined by $f_4(x, y, -1) = 0$. As shown in the figures, this line is singular, and it attracts some orbits. The brown solid line corresponds to the intersection of the invariant surface $2\alpha x^3(\lambda x - \sqrt{6}z) + y(x^2 - z^2) + y^2 = 0$ and the invariant set $z = -1$. In the top figures, P_8 attracts some orbits, but others are attracted by one of the green points associated

TABLE II. Cosmological parameters for the fixed points at the finite region of the system (16).

Label	$(c_s^2, Q_S, \Omega_{\mathrm{DE}}, \omega_{\mathrm{DE}}, q)$	Physical interpretation
P_1	$(-\frac{1}{3}, 3, 0, \frac{(4a_1\lambda^2+b_1(9-6\lambda^2))^2\tau}{81b_1^2(3b_1-2a_1)\lambda} + \mathcal{O}(\tau^{-1}),$ $\frac{(4a_1\lambda^2+b_1(9-6\lambda^2))^2\tau}{54b_1^2(3b_1-2a_1)\lambda} + \frac{1}{2} + \mathcal{O}(\tau^{-1}))$ $a_1 = -\frac{2\lambda(\lambda-2\alpha(\lambda^2-3))\pm\sqrt{6}\sqrt{\lambda(\alpha(6\alpha\lambda-2\lambda^2+3)+\lambda)}}{6\lambda(\alpha(2\lambda^2-3)-\lambda)},$ $b_1 = \frac{2}{9}$, or $a_1 = \frac{1}{3}, b_1 = \frac{4}{9}$, or $b_1 = \frac{4}{3}a_1, a_1(a_1\lambda(\alpha(2\lambda^2-15)-\lambda)+1) = 0$	$a(t) \approx (b_2 - b_1 t)^{3b_1/2}(a_0 + \mathcal{O}(t^{-1})))$. The Galileon mimics radiation for $b_1 = \frac{2}{9}$. The Galileon mimics matter for $b_1 = \frac{4}{9}$. Power-law solution for $b_1 = \frac{4}{3}a_1 \neq 0$ The Galileon mimics dust for $4a_1\lambda^2 + b_1(9-6\lambda^2) = 0$. $c_s^2 < 0$
$P_{2,3}$	$(-\frac{1}{3}, 3, \frac{6}{\lambda^2}, 0, \frac{1}{2})$	The Galileon mimics dust. $c_s^2 < 0$
$P_{4,5}$	$(-\frac{1}{9}, -9, 1, -1, -1)$	de Sitter solution $c_s^2 < 0, Q_S < 0$
$P_{6,7}(x_c)$	See Sec. III D.	Accelerated solution for $x_c < 0, \lambda < -\frac{3x_c^2+2}{\sqrt{6}x_c}$, or $x_c > 0, \lambda > -\frac{3x_c^2+2}{\sqrt{6}x_c}$
$P_{8,9}$	$(-\frac{4}{9}, -9, 0, 0, \frac{1}{2})$	The Galileon mimics dust. $c_s^2 < 0, Q_S < 0$
$P_{10,11}$	$(\frac{2}{9} - \frac{19}{18(\lambda^2-2)}, \frac{18}{\lambda^2} - 9, \frac{3}{2\lambda^2}, 0, \frac{1}{4}(\frac{9}{\lambda^2} + 2))$	The Galileon mimics dust. $c_s^2 \geq 0, Q_S > 0$ for $0 < \lambda^2 < 2$
$P_{12,13}$	$(\frac{2(\lambda-1)(\lambda+1)(\lambda^2-6)}{11\lambda^4-18\lambda^2-36}, -\frac{3(11\lambda^4-18\lambda^2-36)}{(\lambda^2-6)^2},$ $1, -\frac{3}{2\lambda^2}, \frac{1}{2})$	The Galileon mimics dust. $c_s^2 \geq 0, Q_S > 0$ for $1 \leq \lambda^2 < \frac{3}{11}(3 + \sqrt{53}) \approx 2.80367$
$P_{14}(z_c)$	$(-\frac{1}{9}, -9, 1, -1, -1)$	de Sitter solution $\dot{\phi} \approx 0, a(t) \approx a_0 e^{\frac{tz_c}{\sqrt{1-z_c{}^2}}}$, $z_c = \pm\sqrt{\frac{V_0}{3+V_0}}$ $c_s^2 < 0, Q_S < 0$
$P_{15,16}(z_c)$	See Sec. III D.	de Sitter solution $\Delta\phi \approx \frac{\sqrt{6}\beta z_c t}{\sqrt{1-z_c{}^2}}, a(t) \approx a_0 e^{\frac{tz_c}{\sqrt{1-z_c{}^2}}}$, $z_c = \pm\frac{\sqrt{V_0}}{\sqrt{3\sqrt{6}\alpha\beta+V_0}}$

with $P_6(x_c)$. Note that P_8 is not the attractor of the whole phase space since $\alpha\lambda < \frac{1}{4}$. In the bottom figures P_8 is the attractor, not just in this invariant set but in the whole phase space since $\alpha\lambda > \frac{1}{4}$.

D. Discussion

In this section we discuss the stability conditions, cosmological properties, and physical meaning of the (lines of) fixed points in both finite and infinite regions.

(1) P_1: $(0, 0, 0)$ always exists. To analyze the stability we resort to numerical examination.

TABLE III. Summary of the stability conditions of the fixed points at infinity of the system (16).

Label: Coordinates (θ, z)	Coordinates (X, Y, z)	r'/r	(λ_1, λ_2)	Stability
Q_1: $(\frac{\pi}{2}, z_c)$	$(0, 1, z_c)$	0	(0,0)	Nonhyperbolic
Q_2: $(0, -1)$	$(1, 0, -1)$	$2\lambda^2$	$(-2\lambda^2, -\lambda^2)$	Saddle
Q_3: $(\pi, -1)$	$(-1, 0, -1)$	$2\lambda^2$	$(-2\lambda^2, -\lambda^2)$	Saddle
Q_4: $(0, 1)$	(1,0,1)	$-2\lambda^2$	$(2\lambda^2, \lambda^2)$	Saddle
Q_5: $(\pi, 1)$	$(-1, 0, 1)$	$-2\lambda^2$	$(2\lambda^2, \lambda^2)$	Saddle

In Appendix B we prove, using normal form calculations, that the fixed point P_1 corresponds to the cosmological solution (B15). Moreover, in order to improve the range and accuracy, we calculate the diagonal first-order Padé approximants

$$[1/1]_{\dot{\phi}}(t), \qquad [1/1]_H(t), \qquad [1/1]_\phi(t),$$

around $t = \infty$. This yields the following approximate expressions:

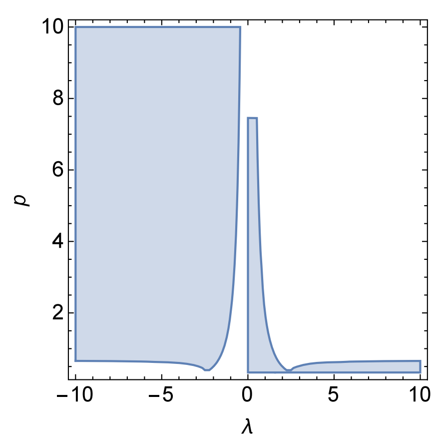

FIG. 1. "Allowed" region of the parameter space.

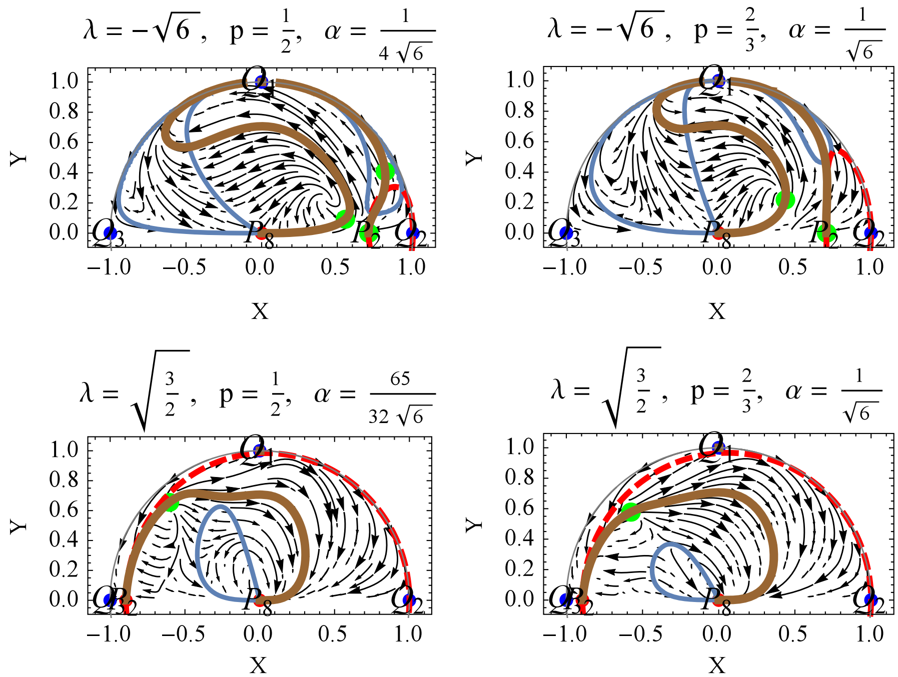

FIG. 2. Poincarè projection of the system (16) on the invariant set $z=-1$. The green dot corresponds to the points $P_6(x_c)$. In the special case $\lambda^2=6$, $p=\frac{2}{3}$, we have $1+\alpha\lambda=0$; thus, the polynomial $\mathcal{P}(x)$ is quadratic, and there are only two roots of $\mathcal{P}(x)=0$. The blue contour is defined by $f_4(x,y,-1)=0$. As shown in the figure, this line is singular and attracts some orbits. The brown solid line corresponds to the intersection of the invariant surface $2\alpha x^3(\lambda x-\sqrt{6}z)+y(x^2-z^2)+y^2=0$ and the invariant set $z=-1$. In the top figures, P_8 attracts some orbits, but others are attracted by one of the green points associated with $P_6(x_c)$. Note that P_8 is not the attractor of the whole phase space since $\alpha\lambda<\frac{1}{4}$. In the bottom figures P_8 is the attractor, not just in this invariant set but in the whole phase space since $\alpha\lambda>\frac{1}{4}$. The red thick dashed line denotes the local center manifold of P_2.

$$\dot{\phi}(t)\approx\frac{1}{4}\lambda(2a_1-3b_1)\times\left(\frac{8a_1-12b_1}{-2a_1t+2a_2+3b_1t-3b_2}+\frac{3b_1\ln t}{t^2}\right), \tag{26a}$$

$$H(t)\approx\frac{3}{8}b_1^2\left(-\frac{3\ln t}{t^2}-\frac{4}{b_2-b_1t}\right), \tag{26b}$$

$$\phi(t)\approx\frac{\ln\left(\frac{4V_0}{4\alpha\lambda^3(2a_1-3b_1)^2+2\lambda^2(2a_1-3b_1)^2+3b_1(9b_1-4)}\right)}{\lambda}+\frac{2\ln t}{\lambda}, \tag{26c}$$

while the scale factor is calculated to be

$$a(t)\approx a_0e^{\frac{9b_1^2}{8t}}\left(\frac{1}{t}\right)^{-\frac{9b_1^2}{8t}}(b_2-b_1t)^{3b_1/2}=(b_2-b_1t)^{3b_1/2}(a_0+\mathcal{O}(t^{-1})). \tag{27}$$

Due to the new conservation law (12), the allowed values of the constants a_1, a_2, b_1, b_2 are

(a)

$$a_1=-\frac{2\lambda(\lambda-2\alpha(\lambda^2-3))\pm\sqrt{6}\sqrt{\lambda(\alpha(6\alpha\lambda-2\lambda^2+3)+\lambda)}}{6\lambda(\alpha(2\lambda^2-3)-\lambda)},\qquad b_1=\frac{2}{9},$$
$$a_1(6-162b_2)+36a_2-1=0,\qquad 4a_1-\frac{2}{3}+\frac{I_1}{a_0{}^3}=0,\quad \text{or}$$

(b)

$$a_1=\frac{1}{3},\qquad b_1=\frac{4}{9},\qquad 2\alpha\lambda^3-15\alpha\lambda-\lambda^2=0,\qquad \frac{I_1}{a_0{}^3}+4a_2-3b_2=0,\quad \text{or}$$

(c)

$$a_1(a_1\lambda(\alpha(2\lambda^2-15)-\lambda)+1)=0,$$
$$b_1=\frac{4}{3}a_1, \qquad b_2=\frac{4}{3}a_2, \qquad I_1=0.$$

It is interesting to note that the power-law solution (13) satisfies the condition

$$(x(\tau),y(\tau),z(\tau)) = \left(\frac{\sqrt{\frac{2}{3}}}{\lambda\sqrt{p^2+t^2}}, \frac{V_0}{3\phi_0^2(p^2+t^2)}, \frac{p}{\sqrt{p^2+t^2}}\right). \tag{28}$$

Additionally, after the substitution of the functional forms of $(x(\tau),y(\tau),z(\tau))$ in (28), and the substitution of $g_0=-\frac{\lambda(\lambda^2p-2)}{4(3p-1)\phi_0^2}$, $V_0=\phi_0^2(\frac{2}{\lambda^2}+p(3p-2))$, it follows that the restriction (28) is satisfied for all the values of t.

There exists a relation between τ and t given by

$$\tau=\sqrt{p^2+t^2}-p\ln\left(\frac{p(\sqrt{p^2+t^2}+p)}{t}\right), \tag{29}$$

such that $t\to\infty$ implies $\tau\to\infty$. Thus, as $\tau\to\infty$, this power-law solution approaches P_1 as $t\to\infty$. For large τ we can invert this to get

$$t=\frac{1}{4}(\sqrt{8p^2+4(p\ln(p)+\tau)^2}+2p\ln(p)+2\tau) = \tau+p\ln(p)+\frac{p^2}{2\tau}-\frac{p^3\ln(p)}{2\tau^2}+\mathcal{O}(\tau^{-3}). \tag{30}$$

Thus, we can take, as an approximation for large τ,

$$x=\frac{\sqrt{\frac{2}{3}}}{\lambda\tau}-\frac{\sqrt{\frac{2}{3}}p\ln(p)}{\lambda\tau^2}+\mathcal{O}(\tau^{-3}),$$
$$y=\frac{V_0}{3\phi_0^2\tau^2}+\mathcal{O}(\tau^{-3}),$$
$$z=\frac{p}{\tau}-\frac{p^2\ln(p)}{\tau^2}+\mathcal{O}(\tau^{-3}).$$

Furthermore,

$$\lambda\phi=(2\ln(\phi_0)+2\ln\tau)+\frac{2p\ln(p)}{\tau}-\frac{p^2(\ln^2(p)-1)}{\tau^2}+\mathcal{O}(\tau^{-3}),$$
$$\dot{\phi}=\frac{2}{\lambda\tau}-\frac{2(p\ln(p))}{\lambda\tau^2}+\mathcal{O}(\tau^{-3}).$$

These features are represented in Fig. 3. There, the Poincarè projection of the system (16) is shown on the invariant surface (18) in the variables $(X,Y,z)=\left(\frac{x}{\sqrt{1+x^2+y^2}},\frac{y}{\sqrt{1+x^2+y^2}},z\right)$. The red continuous line corresponds to the exact solution (28). The origin (represented by a blue dot) attracts this line. The vector field (16) is projected onto the surface.

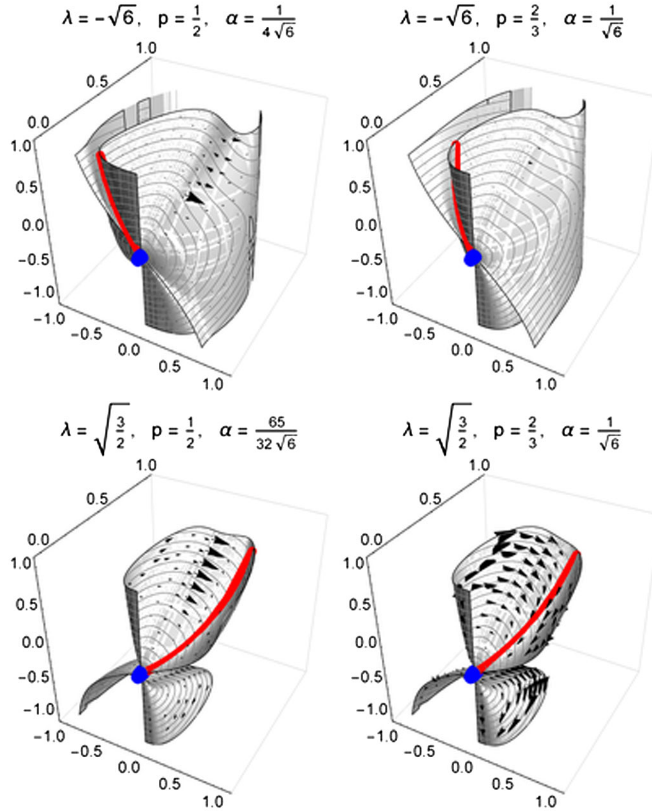

FIG. 3. Poincarè projection of the system (16) on the invariant surface (18) in the coordinates $(X,Y,z)=\left(\frac{x}{\sqrt{1+x^2+y^2}},\frac{y}{\sqrt{1+x^2+y^2}},z\right)$. The red continuous line corresponds to the exact solution (28). The origin (represented by a blue dot) attracts this line. The vector field (16) is projected onto the surface.

The values of $(c_s^2,Q_S,\Omega_{\rm DE},\omega_{\rm DE},q)$ for P_1 are $(-\frac{1}{3}$, 3, 0, $\frac{(4a_1\lambda^2+b_1(9-6\lambda^2))^2\tau}{81b_1^2(3b_1-2a_1)\lambda}+\mathcal{O}(\tau^{-1})$, $\frac{(4a_1\lambda^2+b_1(9-6\lambda^2))^2\tau}{54b_1^2(3b_1-2a_1)\lambda}+\frac{1}{2}+\mathcal{O}(\tau^{-1}))$. The scale factor satisfies $a(t)\approx(b_2-b_1t)^{3b_1/2}(a_0+\mathcal{O}(t^{-1}))$. The Galileon mimics radiation for $b_1=\frac{2}{9}$ and matter for $b_1=\frac{4}{9}$. It is a power-law solution for $b_1=\frac{4}{3}a_1\neq0$. Finally, the Galileon mimics dust for $4a_1\lambda^2+b_1(9-6\lambda^2)=0$. Furthermore, $c_s^2<0$. This point has not been obtained previously in [16] or [17] since in these works the authors used H-normalization, which obviously fails when $H=0$.

(2) P_2: $(-\frac{\sqrt{6}}{\lambda},0,-1)$ always exists. The eigenvalues are $0,-3,-6$, so the points are nonhyperbolic. Using the center manifold theory, we have proven that P_2 is stable for $\lambda<-\sqrt{6}$ or $0<\lambda<\sqrt{6}$, and saddle for $-\sqrt{6}<\lambda<0$ or $\lambda>\sqrt{6}$ (see Appendix C). This point represents kinetic-dominated solutions with

$H \to -\infty$. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_2 are $(-\frac{1}{3}, 3, \frac{6}{\lambda^2}, 0, \frac{1}{2})$. The Galileon mimics dust. Furthermore, $c_s^2 < 0$. So, the Laplacian instabilities associated with the scalar-field propagation speed cannot be avoided for this solution [46].

(3) P_3: $(\frac{\sqrt{6}}{\lambda}, 0, 1)$ always exists. The eigenvalues are 0,3,6, so the points are nonhyperbolic. This has the opposite dynamical behavior of P_2. Using the center manifold theory (in a similar way as in Appendix C, we can prove that P_3 is unstable for $\lambda < -\sqrt{6}$ or $0 < \lambda < \sqrt{6}$, and saddle for $-\sqrt{6} < \lambda < 0$ or $\lambda > \sqrt{6}$). This point represents kinetic-dominated solutions with $H \to +\infty$. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_3 are $(-\frac{1}{3}, 3, \frac{6}{\lambda^2}, 0, \frac{1}{2})$. The Galileon mimics dust. Furthermore, $c_s^2 < 0$, so the Laplacian instabilities associated with the scalar-field propagation speed cannot be avoided for this solution [46].

(4) The fixed point P_4: $(0, 1, 1)$ exists if $\lambda = 0$. As before, it corresponds to the special case $V = V_0$, and the coupling function becomes constant too, $g = g_0$; it is a de Sitter solution, but now $H \to +\infty$. The eigenvalues are $-3, -3$, 0, so it is nonhyperbolic. Using the center manifold theory we find that it is asymptotically stable (for details see Appendix D). Furthermore, perturbations from the equilibrium grow or decay algebraically in time, not exponentially as in the usual linear stability analysis. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_4 are $(-\frac{1}{9}, -9, 1, -1, -1)$. As can be seen, $c_s^2 < 0$, $Q_S < 0$. Thus, this solution suffers from Laplacian instabilities and the presence of ghosts.

(5) The fixed point P_5: $(0, 1, -1)$ exists if $\lambda = 0$. The eigenvalues are 3,3,0, so it is nonhyperbolic. To analyze their stability we resort to numerical examination or use the center manifold theory. This fixed point corresponds to a de Sitter solution driven by a cosmological constant, since $V = V_0$, and the coupling function becomes constant too, $g = g_0$ (although $H \to -\infty$, so it is not cosmological viable). This point has the opposite dynamical behavior of P_4; thus, it is asymptotically unstable. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_5 are $(-\frac{1}{9}, -9, 1, -1, -1)$. The associated cosmological solution corresponds to a de Sitter solution. However, $c_s^2 < 0$, $Q_S < 0$. Thus, this solution suffers from Laplacian instabilities and the presence of ghosts.

(6) For each choice $\epsilon = \pm 1$, there are 1, 2, or 3 isolated fixed points of the form $P_{6,7}(x_c)$: $(\epsilon x_c, \sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1}, -\epsilon)$, where x_c are the nonzero real roots of the polynomial $\mathcal{P}(x) = 6(1+\alpha\lambda)x^3 + 3\sqrt{6}(2\alpha+\lambda)x^2 + 2(3+\lambda^2)x + \sqrt{6}\lambda$, satisfying $\sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1} \geq 0$. To analyze their stability we resort to numerical examination.

The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for $P_{6,7}(x_c)$ are $c_s^{2*} = -P_1(x_c)/(Q_1(x_c)Q_2(x_c))$, $P_1(x_c) = 144\alpha^4 x_c^{10} + 24\alpha^3 x_c^7 y_c(\lambda x_c + 2\sqrt{6}) + 6\alpha^2 x_c^4 y_c^2(2(\lambda^2-7)x_c^2 + 6\sqrt{6}\lambda x_c + 18y_c + 11) + 2\alpha x_c y_c^3(3\lambda x_c^3 + \lambda x_c(21y_c - 8) + 12\sqrt{6}x_c^2 + \sqrt{6}(3y_c - 4)) + y_c^4(-3x_c^2 + 3y_c - 4)$, $Q_1(x_c) = 3(6\alpha^2 x_c^4 + 2\alpha x_c y_c(2\lambda x_c + \sqrt{6}) + y_c^2)$, and $Q_2(x_c) = 24\alpha^2 x_c^6 - 4\alpha x_c^3 y_c(\sqrt{6} - 4\lambda x_c) + (4x_c^2 - 3)y_c^2$, where $y_c = \sqrt{\frac{2}{3}\lambda x_c + x_c^2 + 1} \geq 0$. $Q_S^* = \frac{9(12(6\alpha^2+4\alpha\lambda+1)x_c^6 + 4\sqrt{6}(\alpha(4\lambda^2-3)+2\lambda)x_c^5 + (8\lambda(3\alpha+\lambda)+15)x_c^4 + 2\sqrt{6}(\lambda-6\alpha)x_c^3 - 6(\lambda^2+1)x_c^2 - 6\sqrt{6}\lambda x_c - 9)}{(-6\sqrt{6}\alpha x_c^3 + \sqrt{6}\lambda x_c + 3x_c^2 + 3)^2}$, $\Omega_{\rm DE}^* = \frac{x_c(x_c(6(\alpha\lambda+1)x_c^2 + 3\sqrt{6}(2\alpha+\lambda)x_c + 2\lambda^2 + 9) + 2\sqrt{6}\lambda) + 3}{\sqrt{6}\lambda x_c + 3x_c^2 + 3}$, $\omega_{\rm DE}^* = -\frac{1}{3}x_c(3x_c + \sqrt{6}\lambda) - 1$, and $q^* = \frac{1}{2}(-x_c(3x_c + \sqrt{6}\lambda) - 2)$.

The fixed points $P_{6,7}(x_c)$ represent accelerated solutions for $x_c < 0$, $\lambda < -\frac{3x_c^2+2}{\sqrt{6}x_c}$ or $x_c > 0$, $\lambda > -\frac{3x_c^2+2}{\sqrt{6}x_c}$. Since the above analytical expressions for c_s^2 and Q_S are quite complicated, we have resorted to numerical investigation. To represent the regions of physical interest, we proceed in the following way. Recall that the polynomial $\mathcal{P}(x) = 6(1+\alpha\lambda)x^3 + 3\sqrt{6}(2\alpha+\lambda)x^2 + 2(3+\lambda^2)x + \sqrt{6}\lambda$ has a discrete number of roots x_c (1, 2, or 3 depending on the parameters α and λ). Since this polynomial is linear in α, we have, for each value of x_c ($x_c \neq 0, x_c \neq -\frac{\sqrt{6}}{\lambda}$), the relation $\alpha \coloneqq \alpha(x_c, \lambda) = -\frac{6(x_c^3+x_c) + \sqrt{6}\lambda(3x_c^2+1) + 2\lambda^2 x_c}{6x_c^2(\lambda x_c + \sqrt{6})}$, $\alpha > 0$. So, we can represent the regions of physical interest on the parameter space (x_c, λ), rather than in the plane (α, λ) (to avoid solving a generically third-order polynomial in x_c using Cardano's formulas, with the subsequent numerical error issue). The above procedure leads to the conditions $c_s^2 \geq 0$, $Q_S > 0$ and $c_s^2 < 0$, $Q_S \leq 0$, for $P_{6,7}(x_c)$ as displayed in Fig. 4. In order to cover all the possible values of the parameters, we have made the representation in the (compact) variables $(\frac{x_c}{\sqrt{1+x_c^2+\lambda^2}}, \frac{\lambda}{\sqrt{1+x_c^2+\lambda^2}})$.

(7) The fixed point P_8: $(0, 0, -1)$ always exists. The eigenvalues are $-3, -3, -(6\alpha\lambda - \frac{3}{2})$. Thus, it is a sink for $\alpha\lambda > \frac{1}{4}$ or a saddle otherwise. This solution is dominated by the Hubble scalar with $H \to -\infty$. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_8 are $(-\frac{4}{9}, -9, 0, 0, \frac{1}{2})$. The Galileon mimics dust. However, the fluid satisfies the conditions $c_s^2 < 0$, $Q_S < 0$. Thus, this solution suffers from Laplacian instabilities and the presence of ghosts.

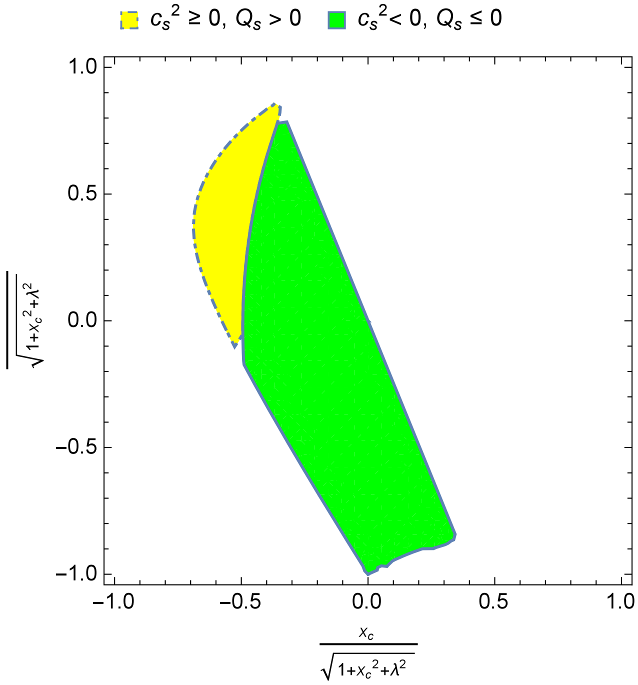

FIG. 4. Parameter space that leads to the conditions $c_s^2 \geq 0$, $Q_S > 0$ and $c_s^2 < 0$, $Q_S \leq 0$, for $P_{6,7}(x_c)$.

(8) The fixed point P_9: $(0,0,1)$ always exists. The eigenvalues are $3, 3, (6\alpha\lambda - \frac{3}{2})$. Thus, it is a source for $\alpha\lambda > \frac{1}{4}$ or a saddle otherwise. This solution is dominated by the Hubble scalar with $H \to +\infty$. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_9 are $(-\frac{4}{9}, -9, 0, 0, \frac{1}{2})$. The Galileon mimics dust. However, the fluid satisfies the conditions $c_s^2 < 0$, $Q_S < 0$. Thus, this solution suffers from Laplacian instabilities and the presence of ghosts.

(9) The fixed point P_{10}: $(-\frac{\sqrt{\frac{3}{2}}}{\lambda}, 0, -1)$ exists for $\alpha = 0$, $\lambda \neq 0$. The eigenvalues are $(-\frac{9}{2\lambda^2}, \frac{3}{4}(2 - \frac{9}{\lambda^2}), -\frac{9}{2\lambda^2} - 3)$. Thus, it is a sink for $0 < \lambda^2 < \frac{3}{2}$ and a saddle otherwise. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_{10} are $(\frac{2}{9} - \frac{19}{18(\lambda^2-2)}, \frac{18}{\lambda^2} - 9, \frac{3}{2\lambda^2}, 0, \frac{1}{4}(\frac{9}{\lambda^2} + 2))$. The Galileon mimics dust. Furthermore, $c_s^2 \geq 0$, $Q_S > 0$ for $0 < \lambda^2 < 2$. In this region of the parameter space, the cosmological solution is free of Laplacian instabilities and ghosts.

(10) The fixed point P_{11}: $(\frac{\sqrt{\frac{3}{2}}}{\lambda}, 0, 1)$ exists for $\alpha = 0$, $\lambda \neq 0$. The eigenvalues are $(\frac{9}{2\lambda^2}, -\frac{3}{4}(2 - \frac{9}{\lambda^2}), \frac{9}{2\lambda^2} + 3)$. It is a source for $0 < \lambda^2 < \frac{3}{2}$ or a saddle otherwise. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_{10} are $(\frac{2}{9} - \frac{19}{18(\lambda^2-2)}, \frac{18}{\lambda^2} - 9, \frac{3}{2\lambda^2}, 0, \frac{1}{4}(\frac{9}{\lambda^2} + 2))$. The Galileon mimics dust. Furthermore, $c_s^2 \geq 0$, $Q_S > 0$ for $0 < \lambda^2 < 2$. In this region of the parameter space, the cosmological solution is free of Laplacian instabilities and ghosts.

(11) The fixed point P_{12}: $(-\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{3}{2\lambda^2}, -1)$ exists for $\lambda^2 + 3\alpha\lambda - 3 = 0$. The eigenvalues are $0, 3, -\frac{3}{2}$. It is a saddle. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_{12} are $(\frac{2(\lambda-1)(\lambda+1)(\lambda^2-6)}{11\lambda^4-18\lambda^2-36}, -\frac{3(11\lambda^4-18\lambda^2-36)}{(\lambda^2-6)^2}, 1, -\frac{3}{2\lambda^2}, \frac{1}{2})$. The Galileon mimics dust. Furthermore, $c_s^2 \geq 0$, $Q_S > 0$ for $1 \leq \lambda^2 < \frac{3}{11}(3 + \sqrt{53}) \approx 2.80367$. In this region of the parameter space, the cosmological solution is free of Laplacian instabilities and ghosts.

(12) The fixed point P_{13}: $(\frac{\sqrt{\frac{3}{2}}}{\lambda}, \frac{3}{2\lambda^2}, 1)$ exists for $\lambda^2 + 3\alpha\lambda - 3 = 0$. The eigenvalues are $0, 3, -\frac{3}{2}$. It is a saddle. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for P_{12} are $(\frac{2(\lambda-1)(\lambda+1)(\lambda^2-6)}{11\lambda^4-18\lambda^2-36}, -\frac{3(11\lambda^4-18\lambda^2-36)}{(\lambda^2-6)^2}, 1, -\frac{3}{2\lambda^2}, \frac{1}{2})$. The Galileon mimics dust. Furthermore, $c_s^2 \geq 0$, $Q_S > 0$ for $1 \leq \lambda^2 < \frac{3}{11}(3 + \sqrt{53}) \approx 2.80367$. In this region of the parameter space, the cosmological solution is free of Laplacian instabilities and ghosts.

(13) The line of fixed points $P_{14}(z_c)$: $(0, z_c^2, z_c)$ exists for $\lambda = 0$. The eigenvalues are $(0, -3z_c, -3z_c)$. Thus, it is nonhyperbolic. Using the center manifold theory we find that it is asymptotically stable for $0 < z_c \leq 1$ and asymptotically unstable for $-1 \leq z_c < 0$ (for details see Appendix E). The associated cosmological solution corresponds to a de Sitter solution that satisfies ϕ, H, which are approximately constant, such that

$$\dot{\phi} \approx 0, \qquad H(t) \approx \frac{z_c}{\sqrt{1 - z_c{}^2}},$$

$$a(t) \approx a_0 e^{\frac{t z_c}{\sqrt{1-z_c{}^2}}}, \qquad z_c = \pm\sqrt{\frac{V_0}{3 + V_0}},$$

where the potential is constant $V(\phi) = V_0$ since $\lambda = 0$. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for $P_{14}(z_c)$ are $(-\frac{1}{9}, -9, 1, -1, -1)$. However, $c_s^2 < 0$, $Q_S < 0$. Thus, this solution suffers from Laplacian instabilities and the presence of ghosts.

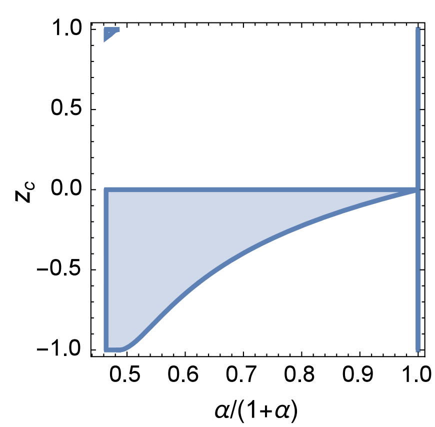

FIG. 5. Parameter space where $P_{15}(z_c)$ is free from Laplacian instabilities and ghosts.

(14) The line of fixed points $P_{15}(z_c)$: $(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$ exists for $\lambda = 0$, $\beta = \frac{1}{\sqrt{2}}(\sqrt{3}\alpha - \sqrt{3\alpha^2 - 2})$. The eigenvalues are $(0, -3z_c, -3z_c)$. Thus, it is nonhyperbolic. The stability has to be examined numerically or using center manifold calculations. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for $P_{15}(z_c)$ are $c_s^2 = \frac{(\alpha(19\sqrt{9\alpha^2-6}-3\alpha(51-8\alpha(3\alpha(2\alpha(\sqrt{9\alpha^2-6}-3\alpha)+7)-5\sqrt{9\alpha^2-6})))+6)z_c}{(\alpha(\sqrt{9\alpha^2-6}-3\alpha)+2)(3(24(\alpha(\sqrt{9\alpha^2-6}-3\alpha)+1)\alpha^2+7)z_c-8(\sqrt{9\alpha^2-6}-3\alpha)\sqrt{1-{z_c}^2})}$, $Q_S = \frac{8(\sqrt{9\alpha^2-6}-3\alpha)\sqrt{1-{z_c}^2}+3(24\alpha^2(3\alpha^2-\sqrt{9\alpha^2-6}\alpha-1)-7)z_c}{(2\alpha(\sqrt{9\alpha^2-6}-3\alpha)+1)^2 z_c}$, $\Omega_{\rm DE} = 1$, $\omega_{\rm DE} = \frac{(\sqrt{9\alpha^2-6}-3\alpha)(6\alpha z_c-\sqrt{1-{z_c}^2})}{6z_c}$, and $q = -1$. This solution is free of Laplacian instabilities and ghosts in the region displayed in Fig. 5 [we used the compact variable $\alpha/(1+\alpha)$ to cover all real values of α].

(15) The line of fixed points $P_{16}(z_c)$: $(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$ exists for $\lambda = 0$, $\beta = \frac{1}{\sqrt{2}}(\sqrt{3}\alpha + \sqrt{3\alpha^2 - 2})$. The eigenvalues are $(0, -3z_c, -3z_c)$. Thus, it is nonhyperbolic. The stability has to be examined numerically or using center manifold calculations. The values of $(c_s^2, Q_S, \Omega_{\rm DE}, \omega_{\rm DE}, q)$ for $P_{16}(z_c)$ are $c_s^2 = \frac{(6-\alpha(3\alpha(8\alpha(3\alpha(2\alpha(\sqrt{9\alpha^2-6}+3\alpha)-7)-5\sqrt{9\alpha^2-6})+51)+19\sqrt{9\alpha^2-6}))z_c}{(\alpha(\sqrt{9\alpha^2-6}+3\alpha)-2)(3(24\alpha^2(\alpha(\sqrt{9\alpha^2-6}+3\alpha)-1)-7)z_c-8(\sqrt{9\alpha^2-6}+3\alpha)\sqrt{1-{z_c}^2})}$, $Q_S = \frac{3(24\alpha^2(\alpha(\sqrt{9\alpha^2-6}+3\alpha)-1)-7)z_c-8(\sqrt{9\alpha^2-6}+3\alpha)\sqrt{1-{z_c}^2}}{(1-2\alpha(\sqrt{9\alpha^2-6}+3\alpha))^2 z_c}$, $\Omega_{\rm DE} = 1$, $\omega_{\rm DE} = \frac{(\sqrt{9\alpha^2-6}+3\alpha)(\sqrt{1-{z_c}^2}-6\alpha z_c)}{6z_c}$, and $q = -1$. The line of fixed points $P_{16}(z_c)$ always satisfies $c_s^2 Q_S < 0$. Thus, these solutions suffers either from Laplacian instabilities or from the presence of ghosts.

For the lines of fixed points $P_{15}(z_c)$ and $P_{16}(z_c)$, the cosmological solutions satisfy

$$\dot{\phi}(t) \approx \frac{\sqrt{6}\beta z_c}{\sqrt{1-{z_c}^2}}, \qquad H(t) \approx \frac{z_c}{\sqrt{1-{z_c}^2}},$$
$$\phi(t) - \phi_0 \approx \frac{\sqrt{6}\beta z_c t}{\sqrt{1-{z_c}^2}}, \qquad a(t) \approx a_0 e^{\frac{t z_c}{\sqrt{1-{z_c}^2}}},$$
$$z_c = \pm\frac{\sqrt{V_0}}{\sqrt{3\sqrt{6}\alpha\beta + V_0}},$$

where $\beta = \frac{\sqrt{3}\alpha \mp \sqrt{3\alpha^2-2}}{\sqrt{2}}$, respectively, and the potential is constant, $V(\phi) = V_0$, since $\lambda = 0$.

Finally, the fixed points or fixed lines at infinity are as follows:

(i) $(\frac{\pi}{2}, z_c)$ with eigenvalues (0,0). It is nonhyperbolic and, according to the analysis in Appendix A, it is generically a saddle.

(ii) $(0, -1)$ with eigenvalues $(-2\lambda^2, -\lambda^2)$. This point attracts nearby orbits lying on the cylinder for all $\lambda \neq 0$. However, if we take into account the stability along r, the point is generically a saddle.

(iii) $(\pi, -1)$ with eigenvalues $(-2\lambda^2, -\lambda^2)$. This point attracts nearby orbits lying on the cylinder for all $\lambda \neq 0$. However, if we take into account the stability along r, the point is generically a saddle.

(iv) (0,1) with eigenvalues $(2\lambda^2, \lambda^2)$. This point repels nearby orbits lying on the cylinder for all $\lambda \neq 0$. However, if we take into account the stability along r, the point is generically a saddle.

(v) $(\pi, 1)$ with eigenvalues $(2\lambda^2, \lambda^2)$. This point repels nearby orbits lying on the cylinder for all $\lambda \neq 0$. However, if we take into account the stability along r, the point is generically a saddle.

IV. ASYMPTOTIC EXPANSIONS IN THE REGIME WHERE THE CUBIC DERIVATIVE TERM DOMINATES

An important question about the present model is, in what phase of the cosmological evolution are the extra cubic interaction terms expected to play an important role? Another interesting question is, how do the cosmological predictions of our model differ from those of standard FRW cosmology driven by a scalar field with exponential potential? To answer these questions we proceed as follows.

To show in what phase of cosmological history the cubic derivatives are expected to play a role, let us consider $g_0 \gg 1$, and take the limit g_0 to infinity in the equations for $\dot{H}$, $\ddot{\phi}$. We obtain the approximations

$$\ddot{\phi} + \frac{1}{2}\lambda\dot{\phi}^2 = 0, \qquad \dot{H} - \lambda H\dot{\phi} + 3H^2 + \frac{1}{6}\lambda^2\dot{\phi}^2 = 0 \tag{31}$$

which admit the solution

$$H_s(t) = \frac{6H_0 - \lambda\dot{\phi}_0}{3((t-t_0)(6H_0 - \lambda\dot{\phi}_0) + 2)} + \frac{\lambda\dot{\phi}_0}{3\lambda\dot{\phi}_0(t-t_0) + 6},$$
$$\phi_s(t) = \phi_0 + \frac{2}{\lambda}\ln\left(1 + \frac{\lambda\dot{\phi}_0}{2}(t-t_0)\right), \tag{32}$$

where we have chosen the initial conditions $H(t_0) = H_0$, $\phi(t_0) = \phi_0$, $\dot{\phi}(t_0) = \dot{\phi}_0$. Furthermore, we have

$$a_s(t) = a_0\left(1 + \frac{1}{2}\lambda\dot{\phi}_0(t-t_0)\right)^{\frac{1}{3}} \times \left(1 + \frac{1}{2}(t-t_0)(6H_0 - \lambda\dot{\phi}_0)\right)^{\frac{1}{3}}. \tag{33}$$

Taking initial conditions such that $\dot{\phi}_0 = 0$ or $6H_0 - \lambda\dot{\phi}_0 = 0$, we obtain $a(t) \propto t^{\frac{1}{3}}$; that is, it corresponds to a radiation-dominated universe. On the other hand, if we choose initial conditions such that $3H_0 - \lambda\dot{\phi}_0 = 0$, we obtain $a(t) \propto t^{\frac{2}{3}}$; that is, it corresponds to a universe dominated by dark matter. For other values of the initial conditions, we can model the transition from a radiation-dominated universe to a matter-dominated one. However, from the leading terms in the Friedmann equations, as $g_0 \to \infty$, we obtain

$$-\frac{1}{2}a_0^3\dot{\phi}_0^3 e^{\lambda\phi_0}(\lambda\dot{\phi}_0 - 6H_0) = 0. \tag{34}$$

Since we are looking for universes with $a_0 > 0$, it immediately follows that Galileon modifications are particularly relevant for the radiation-dominated (early-time) universe as shown in [16]. On the other hand, for $g_0 \to 0$, the model is well suited for describing the late-time universe, and we recover the standard quintessence results found elsewhere, e.g., in the seminal work [48]. Now let us use the above solution as a seed solution to construct an asymptotic expansion for large g_0 when the potential is turned on.

We define

$$\begin{aligned} H &= H_s + \varepsilon h + \mathcal{O}(\varepsilon^2) \\ \phi &= \phi_s + \varepsilon\Phi + \mathcal{O}(\varepsilon^2), \\ \varepsilon &= g_0^{-1} \end{aligned} \tag{35}$$

and assume $\dot{\phi}_0 \neq 0$ (without loss of generality, we can set $\phi_0 = 0$). Then we have at zeroth-order the equation

$$\frac{4\dot{\phi}_0^3(\lambda\dot{\phi}_0 - 6H_0)}{(\lambda\dot{\phi}_0(t-t_0)+2)(-6H_0(t-t_0)+\lambda\dot{\phi}_0(t-t_0)-2)} = 0. \tag{36}$$

Since we have assumed $\dot{\phi}_0 \neq 0$, the solution is $H_0 = \frac{\lambda}{6}\dot{\phi}_0$. This implies that $a(t) \propto t^{\frac{2}{3}}$. That is, it corresponds to a universe dominated by a dust fluid or dark matter. Substituting back the value for H_0 in the Klein-Gordon, Raychaudhuri, and Friedmann equations, respectively, we have the following first-order equations:

$$\begin{aligned} &\dot{\phi}_0^2(-\lambda^2 + 3\lambda\dot{\phi}_0((t-t_0)\Phi''(t)(\lambda\dot{\phi}_0(t-t_0)+4) \\ &\quad + 2\Phi'(t)(\lambda\dot{\phi}_0(t-t_0)+2)) \\ &\quad + 12\Phi''(t) + 6) - 12V_0 = 0, \end{aligned} \tag{37a}$$

$$\begin{aligned} &9h'(t)(\lambda\dot{\phi}_0(t-t_0)+2)^2 \\ &\quad + \lambda(-\lambda^2 + 3\lambda\dot{\phi}_0\Phi'(t)(\lambda\dot{\phi}_0(t-t_0)+2)+6) = 0, \end{aligned} \tag{37b}$$

$$\begin{aligned} &2(\dot{\phi}_0^2(-3\dot{\phi}_0(\lambda\dot{\phi}_0(t-t_0)+2)(\lambda\Phi'(t)-6h(t)) + \lambda^2 - 6) \\ &\quad - 12V_0) = 0. \end{aligned} \tag{37c}$$

The solution of (37) that satisfies $\Phi(t_0) = \Phi_0$, $h(t_0) = h_0$ is given by

$$\begin{aligned} \Phi = \Phi_0 &+ \frac{4(2V_0 - 3h_0\dot{\phi}_0^3)}{\lambda^2\dot{\phi}_0^4(\frac{1}{2}\lambda\dot{\phi}_0(t-t_0)+1)} - \frac{4(2V_0 - 3h_0\dot{\phi}_0^3)}{\lambda^2\dot{\phi}_0^4} \\ &+ \frac{((\lambda^2-6)\dot{\phi}_0^2 + 12V_0)\ln(\frac{1}{2}\lambda\dot{\phi}_0(t-t_0)+1)}{3\lambda^2\dot{\phi}_0^4}, \end{aligned} \tag{38a}$$

$$h = \frac{3h_0\dot{\phi}_0^3 - 2V_0}{3\dot{\phi}_0^3(\frac{1}{2}\lambda\dot{\phi}_0(t-t_0)+1)^2} + \frac{2V_0}{3\dot{\phi}_0^3(\frac{1}{2}\lambda\dot{\phi}_0(t-t_0)+1)}, \tag{38b}$$

where we observe that $\lim_{t\to\infty}\Phi(t) = \infty$ unless $\frac{V_0}{\dot{\phi}_0^2} = \frac{6-\lambda^2}{12}$, and $\lim_{t\to\infty}h(t) = 0$. Furthermore, the relative errors are defined to be

$$\begin{aligned} E(H_s) &= \frac{|H - H_s|}{H}, \qquad E(\phi_s) = \frac{|\phi - \phi_s|}{\phi}, \\ E(\dot{H}_s) &= \frac{|\dot{H} - \dot{H}_s|}{\dot{H}}, \qquad E(\dot{\phi}_s) = \frac{|\dot{\phi} - \dot{\phi}_s|}{\dot{\phi}}, \end{aligned} \tag{39}$$

and satisfy the conditions

$$\begin{aligned} \lim_{t\to\infty} E(\Phi_s(t)) &= \lim_{t\to\infty} E(\dot{\Phi}_s(t)) \\ &= \frac{\epsilon((\lambda^2-6)\dot{\phi}_0^2 + 12V_0)}{6\lambda\dot{\phi}_0^4}, \\ \lim_{t\to\infty} E(H_s(t)) &= \lim_{t\to\infty} E(H_s(t)) = \frac{2V_0\epsilon}{\lambda\dot{\phi}_0^4}. \end{aligned} \tag{40a}$$

Thus, the relative errors can be small enough for $V_0 \ll \dot{\phi}_0^4$, $\lambda = \pm\sqrt{6}$ or $V_0 \ll \dot{\phi}_0^4$, $\dot{\phi}_0^2 \gg 1$ (the last condition is fulfilled for all finite values of V_0 and large values of $\dot{\phi}_0$). We see that for $V_0 \gg \dot{\phi}_0^4$ and finite $\dot{\phi}_0$, the relative errors are large, and the approximation fails. In the former case the potential makes the ideal gas solution unstable, as expected for a universe dominated by a dust fluid or dark matter. This is an important result since the cubic term in the Galileon field provides a matter epoch.

V. GALILEON MODEL WITH MATTER

Until now we have considered the case with vacuum (i.e. $\rho_m = 0$), and we have found, at some fixed points, $c_s^2 < 0$, and $Q_S < 0$ also. On the other hand, for the avoidance of Laplacian instabilities, we require $c_s^2 \geq 0$, and for the absence of ghosts, it is required that $Q_S > 0$. In this section we study if matter stabilizes the Galileon field, in the sense that it helps to restore the above conditions. To begin with, the definition of c_s^2 changes to $c_S^2 \equiv \frac{6w_1 H - 3w_1^2 - 6\dot{w}_1 - 6\rho_m}{4w_2 + 9w_1^2}$, when $\rho_m > 0$ and $w_{\rm tot} \neq w_{\rm DE}$.

By using the variables

$$x = \frac{\dot{\phi}}{\sqrt{6(H^2+1)}}, \quad y = \frac{V_0 e^{-\lambda\phi}}{3(H^2+1)}, \quad z = \frac{H}{\sqrt{H^2+1}}, \tag{41}$$

and the parameter $\alpha = g_0 V_0$, we obtain a dynamical system that is the same as (16):

$$x' = \frac{f_1(\mathbf{x})}{f_4(\mathbf{x})}, \quad y' = \frac{f_2(\mathbf{x})}{f_4(\mathbf{x})}, \quad z' = \frac{f_3(\mathbf{x})}{f_4(\mathbf{x})}, \tag{42}$$

where $\mathbf{x} = (x, y, z)$ and the functions $f_1 - f_4$ are defined by (17). We have an auxiliary evolution equation for

$$\tilde{\Omega}_m = \frac{\rho_m}{3(H^2+1)}, \qquad \rho_m = \rho_{m,0} a^{-3}. \tag{43}$$

The system can be written in the compact form

$$x' = \frac{2qz^2(6\alpha x^3 z + \sqrt{6}y) - 6\sqrt{6}\alpha\lambda x^4 + 12\alpha x^3 z^3 - 3\sqrt{6}x^2 y + \sqrt{6}y(3y - z^2)}{12\alpha x^2}, \tag{44a}$$

$$y' = y(2(q+1)z^3 - \sqrt{6}\lambda x), \tag{44b}$$

$$z' = (q+1)z^2(z^2 - 1), \tag{44c}$$

$$\tilde{\Omega}_m' = \tilde{\Omega}_m z(2(q+1)z^2 - 3) \tag{44d}$$

where q, the deceleration parameter, can be expressed in terms of the phase-space variables (x, y, z). Observe that the evolution equation for matter decouples from the other evolution equations. The subtle difference, including dust matter, is that the restriction (18) now becomes

$$y^2 + 2x^3\alpha(-\sqrt{6}z + x\lambda) + y(x^2 - z^2 + \tilde{\Omega}_m) = 0. \tag{45}$$

Since $y \geq 0$, $\tilde{\Omega}_m \geq 0$, the motion of the particle in the phase space is not just restricted to the given surface $y^2 + 2x^3\alpha(-\sqrt{6}z + x\lambda) + y(x^2 - z^2) = 0$, as is the case in (16) and (17). Instead, the phase space is now the set

$$\{(x,y,z) : y^2 + 2x^3\alpha(-\sqrt{6}z + x\lambda) + y(x^2 - z^2) \leq 0, y \geq 0\}. \tag{46}$$

The fixed points for the vacuum case are recovered, that is, with $\tilde{\Omega}_m = 0$; but some existence conditions change, which is the main difference between the two cases. The existence conditions for P_{10}, P_{11} change to $\alpha/\lambda \geq 0$; the existence conditions of P_{12}, P_{13} are changed by $\lambda^2 + 3\alpha\lambda - 3 \geq 0$. The stability conditions change accordingly. The eigenvalues and stability conditions for P_{10}, P_{11} remain unaffected when matter is included. The eigenvalues for $P_{12,13}$ change to $\pm(-3, \frac{3}{4} - \frac{3\sqrt{\lambda(6\alpha(\alpha\lambda-4)-7\lambda)+24}}{4\sqrt{6\alpha^2+1}\lambda}, \frac{3}{4} + \frac{3\sqrt{\lambda(6\alpha(\alpha\lambda-4)-7\lambda)+24}}{4\sqrt{6\alpha^2+1}\lambda})$, but the point will still be a saddle. The results shown in Table II remain unaltered. We conclude, then, that the matter background has no imprint in the Galileon field concerning the avoidance of Laplacian instabilities and the absence of ghosts. The conditions $c_s^2 \geq 0$, $Q_S > 0$ are not restored for the points that violate them.

VI. CONCLUSIONS

In this paper we have studied cubic Galileon cosmology with an extra conservation law from the dynamical systems perspective. The novelty of this model is that it was derived from the application of Noether's theorem in the gravitational Lagrangian. First, we have noticed that the fixed points $A^{\pm}$, $B^{\pm}$, C, and D investigated in detail in [16] do not exist in our scenario. So, our analysis has its own right, and it is complementary to all the analyses done before. This new scenario admits power-law solutions. We have found a new asymptotic solution given by the fixed point solution P_1, which satisfies

$$H \approx \frac{3}{8} b_1^2 \left(-\frac{3 \ln t}{t^2} - \frac{4}{b_2 - b_1 t} \right),$$

$$\phi \approx \frac{\ln\left(\frac{4V_0}{4\alpha\lambda^3(2a_1 - 3b_1)^2 + 2\lambda^2(2a_1 - 3b_1)^2 + 3b_1(9b_1 - 4)}\right)}{\lambda} + \frac{2 \ln t}{\lambda},$$

while the scale factor is calculated to be $a(t) \approx a_0 e^{\frac{9b_1^2}{8t}} t^{\frac{9b_1^2}{8t}} (b_2 - b_1 t)^{3b_1/2} = (b_2 - b_1 t)^{3b_1/2}(a_0 + \mathcal{O}(t^{-1}))$.

This point has not been obtained previously in [16] or [17] since in these works the authors used H-normalization, which obviously fails when $H = 0$. This solution attracts the exact power-law solution previously described in [44] for the proper choice of free parameters. Moreover, we have obtained several solutions [P_1-P_5, P_8, P_9, $P_{14}(x_c)$, $P_{16}(z_c)$], which violate the conditions $c_s^2 \geq 0$, $Q_S > 0$ (one or both) suffering from Laplacian instabilities and from the presence of ghosts, making them physically less interesting. Excluding the de Sitter solutions, all of them mimic dust solutions. However, we found other solutions that are free of Laplacian instabilities and ghosts. For instance,

(i) The solutions $P_{6,7}(x_c)$ can satisfy the conditions $c_s^2 \geq 0$, $Q_S > 0$, denoted by the yellow region with a dot-dashed boundary in Fig. 4. However, they can violate these conditions; for example, it is possible to have $c_s^2 < 0$, $Q_S \leq 0$ as represented by the green region surrounded by a solid line in Fig. 4.

(ii) The fine-tuned solutions P_{10} and P_{11} that exist in the unmodified (quintessence) scenario can be a sink (respectively, source for $0 \leq \lambda^2 \leq \frac{3}{2}$), and they satisfy $c_s^2 \geq 0$, $Q_S > 0$ for $0 < \lambda^2 < 2$.

(iii) The solutions P_{12} and P_{13} that exist for $\lambda^2 + 3\alpha\lambda - 3 = 0$ mimic dust, and they are saddles. We have proved that $c_s^2 \geq 0$, $Q_S > 0$ for $1 \leq \lambda^2 < \frac{3}{11}(3 + \sqrt{53}) \approx 2.80367$. In this region of the parameter space, the cosmological solution is free of Laplacian instabilities and ghosts.

(iv) The line of fixed points $P_{15}(z_c)$: $(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$ exists for $\lambda = 0$, $\beta = \frac{1}{\sqrt{2}}(\sqrt{3}\alpha - \sqrt{3\alpha^2 - 2})$, and the cosmological solution is given by

$$\dot{\phi}(t) \approx \frac{\sqrt{6}\beta z_c}{\sqrt{1 - {z_c}^2}}, \qquad H(t) \approx \frac{z_c}{\sqrt{1 - {z_c}^2}},$$

$$\phi(t) - \phi_0 \approx \frac{\sqrt{6}\beta z_c t}{\sqrt{1 - {z_c}^2}}, \qquad a(t) \approx a_0 e^{\frac{t z_c}{\sqrt{1 - {z_c}^2}}},$$

$$z_c = \pm \frac{\sqrt{V_0}}{\sqrt{3\sqrt{6}\alpha\beta + V_0}}.$$

This solution is free of Laplacian instabilities and ghosts in the region displayed in Fig. 5.

Furthermore, we have investigated the fixed points at infinity, and all of them are saddle points; thus, they have no relevance for the early- or late-time universe.

The Galileon modifications are particularly relevant for the radiation-dominated (early-time universe) as shown in [16] by investigating the regime where the coupling parameter satisfies $g_0 \gg 1$. On the other hand, for $g_0 \to 0$ the model is well suited for describing the late-time universe, and we have recovered the standard quintessence results found elsewhere, e.g., in the seminal work [48]. We have constructed asymptotic expansions of the power-law solution for large g_0 when the potential is turned on.

All the previous results were found for vacuum Galileon ($\rho_m = 0$). We have seen that for some points, $c_s^2 < 0$, and at some fixed points, $Q_S < 0$ also. Since we need to avoid Laplacian instabilities, we must have $c_s^2 \geq 0$, and if we demand the absence of ghosts, it is required that $Q_S > 0$. Moreover, we have investigated if the matter stabilizes the Galileon field, in the sense that it helps to restore the above conditions. When we include background matter in the form of dust, we recovered the fixed points for $\tilde{\Omega}_m = 0$, but some existence conditions change, especially for P_{10}, P_{11} and P_{12}, P_{13}. The stability conditions change accordingly. The eigenvalues and stability conditions for P_{10}, P_{11} remain unaffected when matter is included. The eigenvalues for $P_{12,13}$ change, but the point will still be a saddle. The results shown in Table II remain unaltered. We conclude, then, that the matter background has no imprint in the Galileon field concerning the avoidance of Laplacian instabilities and the absence of ghosts. The conditions $c_s^2 \geq 0$, $Q_S > 0$ are not restored for the points that violate them.

Finally, in order to get a cosmologically suitable scenario, we require $z > 0$ (i.e., $H > 0$) for accommodating a late-time accelerated expansion phase. For simplicity, we will restrict the analysis to the invariant set of the flow of the system (16) given by $z = 1$. In this invariant set the total equation-of-state parameter is $w_{\rm tot} = -y$. Now, for the special parameter choice $\alpha = \frac{3}{4\lambda(3-\lambda^2)}$, the fixed points $P_7(x_c)$ merge to one fixed point, denoted by dS, in the plane $(x, w_{\rm tot})$ with coordinates $(x, w_{\rm tot}) = (\sqrt{\frac{2}{3}}\lambda, -1)$ that mimic a de Sitter solution. It is an attractor for $0 < \lambda < \sqrt{\frac{3}{2}}$. For $0.72262 < \lambda < 1.22474$ the corresponding cosmological solution satisfies the physical conditions $c_s^2 > 0$, $Q_S \geq 0$. We also obtain a transient phase given by the fixed point $D_1 := (x, w_{\rm tot}) = (\frac{\sqrt{\frac{3}{2}}}{\lambda}, 0)$, which mimics a dustlike solution. Furthermore, we have a scaling solution $S := (x, w_{\rm tot}) = (\frac{\sqrt{\frac{3}{2}}}{\lambda}, -\frac{3}{2\lambda^2})$, which can be an attractor for $\lambda < -\frac{3}{\sqrt{2}}$ or $\sqrt{\frac{3}{2}} < \lambda < \sqrt{3}$, or a saddle otherwise. The scaling solution is accelerating for $-\frac{3}{\sqrt{2}} < \lambda < -\sqrt{3}$ or $0 < \lambda < \sqrt{3}$. The physical conditions $c_s^2 > 0$, $Q_S \geq 0$ are satisfied for $0.662827 < \lambda < 1.60021$. The scaling solution and the de Sitter are not attractors simultaneously. Besides, the model admits an additional dustlike solution $D_2 := (x, w_{\rm tot}) = (\frac{\sqrt{6}}{\lambda}, 0)$ which is unstable. In Fig. 6 we portray, in the plane $w_{\rm tot}$ vs x, the typical behavior of our model for $\alpha = \frac{3}{4\lambda(3-\lambda^2)}$ and different choices of λ. On the left panel, the attractor of the de Sitter point, dS, is such that the value $w_{\rm tot} = -1$ is approached asymptotically. However, we immediately see that there are trajectories connecting the dust point D_2 and the scaling solution S that crosses the phantom divide line (represented by a red dashed line), which eventually enter the $w = -1$ region. However, some solutions are trapped on the phantom regime. On the right

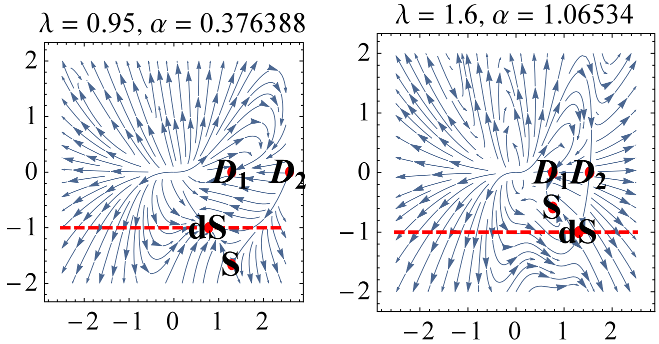

FIG. 6. Streamlines in the plane $(w_{\rm tot}, x)$, representing the typical behavior of our model for $\alpha = \frac{3}{4\lambda(3-\lambda^2)}$ and different choices of λ.

panel, the scaling point S is a stable spiral, and the de Sitter point is a transient one. There are trajectories connecting the dust point D_2 and the de Sitter solution dS crossing the phantom divide. As in the previous case, some orbits remain on the phantom region. As we have briefly shown, this model has quite interesting cosmological features, resembling the usual late-time dynamics found in conventional quintessence models for a nonzero value of α.

Different subjects of study are still open for the integrable model in which new critical points exist. Hence, the requirement of the existence of a conservation law has a physical interpretation in the evolution of the model. However, the exact nature of the physical interpretation for this model is still unknown; such an analysis is still in progress.

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APPENDIX A: NONHYPERBOLIC FIXED POINT AT INFINITY

For analyzing the stability of Q_1 we proceed as follows. Dividing by the positive quantity $\lambda^2\cos^2(\theta)$, we have that the flow of the system (25) is equivalent to the flow of

$$\frac{d\theta}{dT} = \hat{h}_1(\theta, z) \coloneqq z\sin(\theta)\cos(\theta), \tag{A1a}$$

$$\frac{dz}{dT} = \hat{h}_2(\theta, z) \coloneqq (z^2 - 1), \tag{A1b}$$

$$\frac{dr}{dT} = \hat{h}_3(r, \theta, z) \coloneqq r[z(\cos(2\theta) - 3)]. \tag{A1c}$$

To improve the range and accuracy, we calculate the diagonal first-order Padé approximants

$$[1/1]_{\hat{h}_1}(\theta), \qquad [1/1]_{\hat{h}_2}(\theta), \qquad [1/1]_{\hat{h}_3}(\theta),$$

around $\theta = \pi/2$, which yields the following approximate expressions:

$$\frac{d\theta}{dT} \approx \frac{1}{2}(\pi - 2\theta)z, \qquad \frac{dz}{dT} \approx -1 + z^2,$$
$$\frac{dr}{dT} \approx -4rz. \tag{A2}$$

Imposing the initial conditions

$$z(0) = -1 + \delta z, \qquad \theta(0) = \frac{\pi}{2} + \delta\theta,$$
$$r(0) = \delta r, \qquad \delta z \ll 1, \qquad \delta\theta \ll 1, \qquad \delta r \ll 1,$$

we obtain the solution

$$z = \delta z e^{-2T} - 1, \quad \theta = \delta\theta e^{T} + \frac{\pi}{2}, \quad r = \delta r e^{4T}. \tag{A3}$$

Observe that the small perturbations $\delta\theta$ and δr grow exponentially, and $z \to -1$ as $T \to +\infty$. At the cylinder at infinity ($\delta r = 0$), the orbits near Q_1 are attracted by the invariant set $z = -1$, but the angle departs from the equilibrium value $\theta = \frac{\pi}{2}$ as time goes forward. This means that Q_1 is the local past attractor at the circle $z = 1$, $X^2 + Y^2 = 1$, $Y \geq 0$, as is shown in Fig. 2. If we move to the interior of the phase space $\delta r > 0$, the orbits are repelled by the cylinder at infinity at an exponential rate, although $z \to -1$ as $T \to +\infty$. Hence, Q_1 is generically a saddle.

APPENDIX B: FIXED POINTS LYING ON SINGULAR SURFACES

The system (16) blows up when $f_4(x, y, z) = 0$, especially at the fixed points P_1, P_8, P_9. For this reason we had examined their stability numerically or by taking limits, since they lead to indeterminacy of the form 0/0.

For examining the stability of P_1, we introduce the new variables $u = z - \frac{\sqrt{\frac{3}{2}}x}{\lambda}$, $v = \frac{2z}{3}$, $w = y$, such that the system can be written in the canonical form

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \lambda u^2(\alpha(2\lambda^2-3)-\lambda) + \frac{3}{2}uv(2\lambda(\lambda-2\alpha(\lambda^2-3))-3) + \frac{9}{8}v^2(2\alpha(2\lambda^2-9)\lambda-2\lambda^2+3) \\ \frac{1}{12}(-4\alpha\lambda^3(2u-3v)^2 - 2\lambda^2(2u-3v)^2 - 27v^2) \\ \lambda^2 w(2u-3v) \end{pmatrix} + \mathcal{O}(|(u,v,w)^T|^3). \tag{B1}$$

Given H^2, the vector space of the homogeneous polynomials of second order in $\mathbf{u} = (u_1, u_2, u_3)$, let us consider the linear operator $\mathbf{L}_\mathbf{J}^{(2)}$ associated with

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which assigns to $\mathbf{h}(\mathbf{u}) \in H^2$ the Lie bracket of the vector fields $\mathbf{Ju}$ and $\mathbf{h}(\mathbf{u})$:

$$\mathbf{L}_\mathbf{J}^{(2)}\colon H^2 \to H^2 \qquad \mathbf{h} \to \mathbf{L}_\mathbf{J}\mathbf{h}(\mathbf{u}) = \mathbf{Dh}(\mathbf{u})\mathbf{Ju} - \mathbf{Jh}(\mathbf{u}), \tag{B2}$$

where H^2 are the real vector space of vector fields whose components are homogeneous polynomials of degree 2. The canonical basis for the real vector space of three-dimensional vector fields whose components are homogeneous polynomials of degree 2 is given by

$$H^2 = \mathrm{span}\left\{ \begin{pmatrix} u_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1u_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_3^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_3^2 \end{pmatrix} \right\}. \tag{B3}$$

By computing the action of $\mathbf{L}_\mathbf{J}^{(2)}$ on each basis element on H^2, we have

$$\mathbf{L}_\mathbf{J}^{(2)}(H^2) = \mathrm{span}\left\{ \begin{pmatrix} u_1u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_3^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_3^2 \end{pmatrix} \right\}. \tag{B4}$$

Thus, the second-order terms that are linear combinations of the six vectors in (B4) can be eliminated [49]. To determine the nature of the second-order terms that cannot be eliminated, we must compute the complementary space of (B4), which is

$$G^2 = \mathrm{span}\left\{ \begin{pmatrix} u_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1u_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2^2 \end{pmatrix} \right\}.$$

Following the above reasoning, we propose a transformation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} + \begin{pmatrix} h_2^{(1)}(\sigma_1,\sigma_2,\sigma_3) \\ h_2^{(2)}(\sigma_1,\sigma_2,\sigma_3) \\ h_2^{(3)}(\sigma_1,\sigma_2,\sigma_3) \end{pmatrix} \tag{B5}$$

where the $h_2^{(i)}, i = 1, 2, 3$, are homogeneous polynomials of degree 2 in $(\sigma_1, \sigma_2, \sigma_3)$. Up to this point $h_2^{(i)}, i = 1, 2, 3$, are completely arbitrary. Now we choose a specific form of them so as to simplify the $\mathcal{O}(2)$ terms as much as possible. We choose

$$\begin{pmatrix} h_2^{(1)}(\sigma_1,\sigma_2,\sigma_3) \\ h_2^{(2)}(\sigma_1,\sigma_2,\sigma_3) \\ h_2^{(3)}(\sigma_1,\sigma_2,\sigma_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}\lambda^2\sigma_1^2(2\alpha\lambda+1) - 2\lambda^2\sigma_1\sigma_2(2\alpha\lambda+1) + \frac{3}{4}\sigma_2^2(4\alpha\lambda^3+2\lambda^2+3) \end{pmatrix}. \tag{B6}$$

Hence, the normal form of the system (B1) is

$$\sigma_1' = \lambda\sigma_1^2(\alpha(2\lambda^2-3)-\lambda) + \sigma_1\sigma_2\left(3\lambda(\lambda-2\alpha(\lambda^2-3)) - \frac{9}{2}\right) + \frac{9}{8}\sigma_2^2(2\alpha(2\lambda^2-9)\lambda - 2\lambda^2+3) + \mathcal{O}(3), \tag{B7a}$$

$$\sigma_2' = \sigma_3 + \mathcal{O}(3), \tag{B7b}$$

$$\sigma_3'^2 = \sigma_1\sigma_3(\alpha\lambda+1) - \frac{3}{2}\sigma_2\sigma_3(4\lambda^2(\alpha\lambda+1)+3) + \mathcal{O}(3). \tag{B7c}$$

We propose the following ansatz:

$$\sigma_1 = \frac{a_1}{\tau} + \frac{a_2}{\tau^2} + \mathcal{O}(\tau^{-3}), \qquad \sigma_2 = \frac{b_1}{\tau} + \frac{b_2}{\tau^2} + \mathcal{O}(\tau^{-3}), \qquad \sigma_3 = \frac{c_1}{\tau} + \frac{c_2}{\tau^2} + \mathcal{O}(\tau^{-3}) \tag{B8}$$

as $\tau \to \infty$. Substituting back and comparing the coefficient of equal powers of τ, we find the solutions

$$a_1 = -\frac{\sqrt{3b_1(8\lambda(-2\alpha\lambda^2+6\alpha+\lambda)+9b_1(2\lambda(\alpha(2\lambda(3\alpha+\lambda)-9)-\lambda)+3)-12)+4} - 3b_1(4\alpha\lambda^3-12\alpha\lambda-2\lambda^2+3)+2}{4\lambda(\alpha(2\lambda^2-3)-\lambda)},$$
$$c_1 = 0, \quad c_2 = -b_1, \tag{B9a}$$

or

$$a_1 = -\frac{-\sqrt{3b_1(8\lambda(-2\alpha\lambda^2+6\alpha+\lambda)+9b_1(2\lambda(\alpha(2\lambda(3\alpha+\lambda)-9)-\lambda)+3)-12)+4} - 3b_1(4\alpha\lambda^3-12\alpha\lambda-2\lambda^2+3)+2}{4\lambda(\alpha(2\lambda^2-3)-\lambda)},$$
$$c_1 = 0, \quad c_2 = -b_1. \tag{B9b}$$

This approximated solution has the correct number of arbitrary constants: a_2, b_1, b_2. Then, we obtain

$$x = \frac{(3b_1-2a_1)\lambda}{\sqrt{6}\tau} + \frac{(3b_2-2a_2)\lambda}{\sqrt{6}\tau^2} + \mathcal{O}(\tau^{-3}), \tag{B10a}$$

$$y = \frac{4(2a_1-3b_1)^2\alpha\lambda^3 + 2(2a_1-3b_1)^2\lambda^2 + 3b_1(9b_1-4)}{12\tau^2} + \mathcal{O}(\tau^{-3}), \tag{B10b}$$

$$z = \frac{3b_1}{2\tau} + \frac{3b_2}{2\tau^2} + \mathcal{O}(\tau^{-3}). \tag{B10c}$$

From here it is clear that the origin attracts the orbits as $\tau \to \infty$.

On the other hand, we get

$$\dot{\phi}(t(\tau)) = \frac{(3b_1 - 2a_1)\lambda}{\tau} + \frac{(3b_2 - 2a_2)\lambda}{\tau^2} + \mathcal{O}(\tau^{-3}), \tag{B11a}$$

$$H(t(\tau)) = \frac{3b_1}{2\tau} + \frac{3b_2}{2\tau^2} + \mathcal{O}(\tau^{-3}), \tag{B11b}$$

$$\phi(t(\tau)) = \frac{\ln\left(\frac{4V_0}{4(2a_1-3b_1)^2\alpha\lambda^3 + 2(2a_1-3b_1)^2\lambda^2 + 3b_1(9b_1-4)}\right) + 2\ln\tau}{\lambda} + \mathcal{O}(\tau^{-1}). \tag{B11c}$$

Furthermore,

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{H^2+1}} = 1 - \frac{3b_1}{4\tau} + \frac{3(9b_1^2 - 8b_2)}{32\tau^2} + \mathcal{O}(\tau^{-3}) \tag{B12}$$

which implies

$$\Delta t = t(\tau) - t_1 = \tau - \frac{3}{4}b_1 \ln\tau + \frac{3(8b_2 - 9b_1^2)}{32\tau} + \mathcal{O}(\tau^{-2}). \tag{B13}$$

Inverting the above expression we have

$$\tau = \frac{3(9b_1^2 - 8b_2)}{32\Delta t} + \frac{9b_1^2 \ln\Delta t}{16\Delta t} + \frac{3}{4}b_1 \ln\Delta t + \Delta t + \mathcal{O}(\Delta t^{-2}). \tag{B14}$$

Finally,

$$\dot{\phi}(t) = -\frac{3b_1\lambda(3b_1 - 2a_1)\ln t}{4t^2} + \frac{\lambda(3b_1 - 2a_1)}{t} + \frac{\lambda(3b_2 - 2a_2)}{t^2} + \mathcal{O}(t^{-3}), \tag{B15a}$$

$$H(t) = -\frac{9b_1^2 \ln t}{8t^2} + \frac{3b_1}{2t} + \frac{3b_2}{2t^2} + \mathcal{O}(t^{-3}), \tag{B15b}$$

$$\phi(t) = \frac{\ln\left(\frac{4V_0}{4\alpha\lambda^3(2a_1-3b_1)^2 + 2\lambda^2(2a_1-3b_1)^2 + 3b_1(9b_1-4)}\right)}{\lambda} + \frac{2\ln t}{\lambda} + \mathcal{O}(t^{-1}), \tag{B15c}$$

$$a(t) = a_0 t^{\frac{3b_1}{2}}\left(1 + \frac{3(3{b_1}^2 \ln t + 3{b_1}^2 - 4b_2)}{8t} + \mathcal{O}(t^{-2})\right). \tag{B15d}$$

For simplicity we set $t_1 = 0$, $\Delta t = t$.

On the other hand, the new conservation law

$$\frac{g_0 a^3 e^{\lambda\phi}\dot{\phi}^2(6H - \lambda\dot{\phi})}{\lambda} + \frac{2a^3(\lambda H - \dot{\phi})}{\lambda} + I_1 = 0$$

has to be satisfied. Substituting the expression (B15), we obtain

$$\frac{I_1}{a_0^3} + \frac{1}{8}t^{\frac{9b_1}{2}-2}(9(4a_1 - 3b_1)(3b_1^2 - 4b_2) + 3b_1(9b_1 - 2)(4a_1 - 3b_1)\ln t + 32a_2 - 24b_2) + (4a_1 - 3b_1)t^{\frac{9b_1}{2}-1} + \mathcal{O}(t^{-3}) = 0,$$

where $a_1 = a_1(b_1, \lambda, \alpha)$. Thus, for eliminating the term $\propto t^{\frac{9b_1}{2}-1}$, we have the following choices:

(i) We set the exponent to zero; i.e., we set $b_1 = \frac{2}{9}$. Then a_1 is reduced to the two values

$$a_1^{\pm}(\lambda, \alpha) = -\frac{2\lambda(\lambda - 2\alpha(\lambda^2 - 3)) \pm \sqrt{6}\sqrt{\lambda(\alpha(6\alpha\lambda - 2\lambda^2 + 3) + \lambda)}}{6\lambda(\alpha(2\lambda^2 - 3) - \lambda)}.$$

Substituting $b_1 = \frac{2}{9}$, and specifying $a_1 = a_1^{\pm}(\lambda, \alpha)$, the constraint becomes

$$\frac{I_1}{{a_0}^3} + \frac{a_1(\frac{2}{3} - 18b_2) + 4a_2 - \frac{1}{9}}{t} + 4a_1 - \frac{2}{3} + \mathcal{O}(t^{-3}) = 0.$$

Now, in order to satisfy the above expressions up to the order $\mathcal{O}(t^{-3})$, we impose the conditions

$$a_1(6 - 162b_2) + 36a_2 - 1 = 0,$$
$$4a_1 - \frac{2}{3} + \frac{I_1}{{a_0}^3} = 0,$$

which have four arbitrary constants a_2, b_2, a_0, I_1, from which we cannot eliminate simultaneously a_2 and b_2, or I_1 and a_0. This leads to a two-parameter family of solutions. We solve for a_2 and I_1 to keep the parameters b_2 and a_0. Finally,

$$\dot{\phi}(t) = \frac{(\frac{2}{3} - 2a_1)\lambda}{t} + \frac{\lambda(-18a_2 + 27b_2 - (1 - 3a_1)\ln t)}{9t^2} + \mathcal{O}(t^{-3}), \tag{B16a}$$

$$H(t) = \frac{1}{3t} + \frac{27b_2 - \ln t}{18t^2} + \mathcal{O}(t^{-3}), \qquad \text{(B16b)}$$

$$\phi(t) = \frac{\ln(\frac{9V_0}{2(1-3a_1)^2\lambda^2(2\alpha\lambda+1)-3}) + 2\ln t}{\lambda} + \mathcal{O}(t^{-1}), \qquad \text{(B16c)}$$

$$a(t) = a_0\sqrt{3}t - \frac{1}{18}(a_0(27b_2 - \ln t - 1))t^{-\frac{2}{3}} + \mathcal{O}(t^{-\frac{5}{3}}). \qquad \text{(B16d)}$$

(ii) We can equate the coefficient of $t^{\frac{9b_1}{2}-1}$ to zero, that is, solve $(4a_1 - 3b_1) = 0$ for b_1. The next term in the expansion will be $(4a_2 - 3b_2)t^{\frac{9b_1}{2}-2}$. Setting $(4a_2 - 3b_2) = 0$, we obtain the constraints $I_1 = 0$ and $b_1 = \frac{4}{3}a_1$, $b_2 = \frac{4}{3}a_2$, $a_1(a_1\lambda(\alpha(2\lambda^2 - 15) - \lambda) + 1) = 0$.

(iii) Finally, we have the fine-tuned solutions $a_1 = \frac{1}{3}$ and $b_1 = \frac{4}{9}$, which lead to the constraints $2\alpha\lambda^3 - 15\alpha\lambda - \lambda^2 = 0$ and $\frac{I_1}{a_0{}^3} + 4a_2 - 3b_2 = 0$.

This point has not been obtained previously in [16] or [17] since in these works the authors used H-normalization, which obviously fails when $H = 0$.

For examining the stability of the fixed point P_8, we introduce the new variable $\tilde{z} = z + 1$. Then, the evolution equations (16) can be written as

$$x' = x\left(\frac{3}{2} - 6\alpha\lambda\right) + \sqrt{\frac{3}{2}}\lambda y + \mathcal{O}(2), \quad y' = -3y + \mathcal{O}(2),$$
$$\tilde{z}' = -3\tilde{z} + \mathcal{O}(2). \qquad \text{(B17)}$$

For examining the stability of the the fixed point P_9, we introduce the new variable $\tilde{z} = z - 1$. Then, the evolution equations (16) can be written as

$$x' = -x\left(\frac{3}{2} - 6\alpha\lambda\right) + \sqrt{\frac{3}{2}}\lambda y + \mathcal{O}(2),$$
$$y' = 3y + \mathcal{O}(2), \qquad \tilde{z}' = 3\tilde{z} + \mathcal{O}(2). \qquad \text{(B18)}$$

From these linearized equations we extract the eigenvalues for P_8 and P_9.

APPENDIX C: CENTER MANIFOLD FOR P_2

The point P_3 has the opposite stability behavior.

For investigating the stability of the center manifold of P_2, we introduce the new variables $w = y$, $u = z + 1$, $v = x - \frac{\sqrt{6}}{\lambda}z$. Hence, we have the evolution equations

$$w' = 12uw + \lambda\left(\sqrt{6}vw - \frac{w^2}{2\alpha}\right) + \frac{\lambda^3 w^2}{12\alpha} + \mathcal{O}(3), \qquad \text{(C1a)}$$

$$u' = -6u + 15u^2 + 2\sqrt{6}\lambda uv + \mathcal{O}(3), \qquad \text{(C1b)}$$

$$v' = -3v + 9uv + \frac{\lambda^2 w(u+w)}{4\sqrt{6}\alpha} - \frac{\sqrt{\frac{3}{2}}uw}{2\alpha} + \frac{\lambda(72\sqrt{6}\alpha^2 v^2 - 24\alpha vw + \sqrt{6}w^2)}{48\alpha^2} + - \frac{\lambda^3 w(\sqrt{6}w - 6\alpha v)}{72\alpha^2} + \frac{\lambda^5 w^2}{96\sqrt{6}\alpha^2} + \mathcal{O}(3). \qquad \text{(C1c)}$$

The center manifold of P_2 is then given locally by $\{(w,u,v): u = h_1(w), v = h_2(w), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |w| < \delta\}$, where δ is small enough, and the functions h_1, h_2 satisfy the equations

$$h_1'(w)\left(-12wh_1(w) - \sqrt{6}\lambda wh_2(w) - \frac{\lambda(\lambda^2-6)w^2}{12\alpha}\right) + h_1(w)(2\sqrt{6}\lambda h_2(w) - 6) + 15h_1(w)^2 = 0, \qquad \text{(C2a)}$$

$$h_2'(w)\left(-12wh_1(w) - \sqrt{6}\lambda wh_2(w) - \frac{\lambda(\lambda^2-6)w^2}{12\alpha}\right) + h_1(w)\left(9h_2(w) + \frac{(\lambda^2-6)w}{4\sqrt{6}\alpha}\right) + h_2(w)\left(\frac{\lambda(\lambda^2-6)w}{12\alpha} - 3\right) + 3\sqrt{\frac{3}{2}}\lambda h_2(w)^2 + \frac{\lambda w^2(24\alpha\lambda + \lambda^4 - 8\lambda^2 + 12)}{96\sqrt{6}\alpha^2} = 0. \qquad \text{(C2b)}$$

We have to solve these equations using Taylor series up to third order since Eqs. (C1) were truncated at third order. Assuming $h_1(w) = aw^2 + \mathcal{O}(w)^3$ and $h_2(w) = bw^2 + \mathcal{O}(w)^3$ [we start at second order in w since the conditions $h_1(0) = 0$, $h_2(0) = 0$, $h_1'(0) = 0$, $h_2'(0) = 0$ have to be satisfied], substituting back into (C2), and equating the coefficients with the same powers of w, we find $a = 0$, $b = \frac{\lambda(24\alpha\lambda+\lambda^4-8\lambda^2+12)}{288\sqrt{6}\alpha^2}$. The evolution on the center manifold is governed by $w' = \frac{\lambda(\lambda^2-6)w^2}{12\alpha} + \frac{\lambda^2 w^3(24\alpha\lambda+\lambda^4-8\lambda^2+12)}{288\alpha^2} + O(w^4)$, a "gradient" equation with potential $W(w) = -\frac{\lambda^2 w^4(24\alpha\lambda+\lambda^4-8\lambda^2+12)}{1152\alpha^2} - \frac{\lambda(\lambda^2-6)w^3}{36\alpha}$. Since the first nonzero derivative at $w = 0$ is the third one, then the point $w = 0$ is an inflection point of $W(u)$. Hence, for $\frac{\lambda(\lambda^2-6)}{12\alpha} > 0$, the solutions with $w(0) > 0$ leave the origin (and go off to ∞ in finite time), whereas the solutions with $w(0) < 0$ approach the equilibrium as time passes. For $\frac{\lambda(\lambda^2-6)}{12\alpha} < 0$, the solutions with $w(0) > 0$ approach the equilibrium as time passes, whereas the solutions with $w(0) < 0$ leave the origin (and go off to ∞ in finite time).

Such an equilibrium with one-sided stability is sometimes said to be semistable. However, since we have to restrict our attention to the region $w \geq 0$ (since $y \geq 0$), we have that P_2 is a saddle for $\frac{\lambda(\lambda^2-6)}{12\alpha} > 0$ and stable for $\frac{\lambda(\lambda^2-6)}{12\alpha} < 0$. But α is always non-negative; thus, P_2 is stable for $\lambda < -\sqrt{6}$ or $0 < \lambda < \sqrt{6}$, and a saddle for $-\sqrt{6} < \lambda < 0$ or $\lambda > \sqrt{6}$.

APPENDIX D: CENTER MANIFOLD FOR P_4

We discuss in more detail the stability of the fixed point P_4. The point P_5 has the opposite stability behavior.

For investigating the stability of the center manifold of P_4, we introduce the new variables $w = z$, $u = y - 2z + 1$, $v = x$. Hence, we have the evolution equations

$$w' = -3uw + \mathcal{O}(3), \tag{D1a}$$

$$u' = -3(u^2 + uw + u - v^2 - w^2) + \mathcal{O}(3), \tag{D1b}$$

$$v' = -\frac{3}{2}v(u + 2(\sqrt{6}\alpha v + w + 1)) + \mathcal{O}(3). \tag{D1c}$$

The center manifold of P_4 is then given locally by $\{(w,u,v): u = h_1(w), v = h_2(w), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |w| < \delta\}$, where δ is small enough, and the functions h_1, h_2 satisfy the equations

$$3wh_1(w)h_1'(w) - 3h_1(w)^2 - 3(w+1)h_1(w) + 3h_2(w)^2 + 3w^2 = 0, \tag{D2a}$$

$$3wh_1(w)h_2'(w) - \frac{3}{2}h_1(w)h_2(w) - 3\sqrt{6}\alpha h_2(w)^2 - 3(w+1)h_2(w) = 0. \tag{D2b}$$

We have to solve these equations using Taylor series up to third order since Eqs. (D1) were truncated at third order. Assuming $h_1(w) = aw^2 + \mathcal{O}(w)^3$ and $h_2(w) = bw^2 + \mathcal{O}(w)^3$ [we start at second order in w since the conditions $h_1(0) = 0$, $h_2(0) = 0$, $h_1'(0) = 0$, $h_2'(0) = 0$ have to be satisfied], substituting back into (D1), and equating the coefficients with the same powers of w, we find $a = 1$, $b = 0$. The evolution on the center manifold is governed by $w' = -3w^3$. The equilibrium $w = 0$ is then asymptotically stable. Furthermore, perturbations from the equilibrium grow or decay algebraically in time, not exponentially as in the usual linear stability analysis. This is the same result obtained previously in [16] and [17], for the de Sitter (constant potential) solution.

APPENDIX E: CENTER MANIFOLD FOR THE LINE OF FIXED POINTS $P_{14}(z_c)$

The line of fixed points $P_{14}(z_c): (0, z_c^2, z_c)$ exists for $\lambda = 0$. The eigenvalues are $(0, -3z_c, -3z_c)$. Thus, it is nonhyperbolic. For investigating the stability of the center manifold of $P_{14}(z_c)$, we introduce the new variables $w = \frac{-(y+1)z_c^2 + y + 2zz_c^3 - z_c^4}{2z_c}$, $u = \frac{(z_c^2-1)(y + z_c(z_c - 2z))}{2z_c}$, $v = x$. Hence, we have the evolution equations

$$w' = 3u^2 z_c^2 + \frac{3uw(z_c^2 - 3)z_c^2}{z_c^2 - 1} + \mathcal{O}(3), \tag{E1a}$$

$$u' = -\frac{3u^2(z_c^4 + 4z_c^2 - 1)}{2(z_c^2 - 1)} + u(-3w - 3z_c) + \frac{3}{2}v^2(z_c^2 - 1) + \frac{3}{2}w^2(z_c^2 - 1) + \mathcal{O}(3), \tag{E1b}$$

$$v' = \frac{uv(3 - 6z_c^2)}{z_c^2 - 1} - 3\sqrt{6}\alpha v^2 + v(-3w - 3z_c) + \mathcal{O}(3). \tag{E1c}$$

The center manifold of P_{14} is then given locally by $\{(w, u, v): u = h_1(w), v = h_2(w), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |w| < \delta\}$, where δ is small enough, and the functions h_1, h_2 satisfy the equations

$$\left(-3z_c^2 h_1(w)^2 - \frac{3w(z_c^2 - 3)z_c^2 h_1(w)}{z_c^2 - 1}\right)h_1'(w) - \frac{3(z_c^4 + 4z_c^2 - 1)h_1(w)^2}{2(z_c^2 - 1)} - 3h_1(w)(w + z_c) + \frac{3}{2}(z_c^2 - 1)h_2(w)^2 + \frac{3}{2}w^2(z_c^2 - 1) = 0, \tag{E2a}$$

$$\left(-3z_c^2 h_1(w)^2 - \frac{3w(z_c^2 - 3)z_c^2 h_1(w)}{z_c^2 - 1}\right)h_2'(w) + \frac{(3 - 6z_c^2)h_1(w)h_2(w)}{z_c^2 - 1} + - 3\sqrt{6}\alpha h_2(w)^2 - 3h_2(w)(w + z_c) = 0. \tag{E2b}$$

We have to solve these equations using Taylor series up to third order since Eqs. (E1) were truncated at third order. Assuming $h_1(w) = aw^2 + \mathcal{O}(w)^3$ and $h_2(w) = bw^2 + \mathcal{O}(w)^3$ [we start at second order in w since the conditions $h_1(0) = 0$, $h_2(0) = 0$, $h_1'(0) = 0$, $h_2'(0) = 0$ have to be satisfied], substituting back into (E1), and equating the coefficients with the same powers of w, we find $a = \frac{z_c^2 - 1}{2z_c}$, $b = 0$. The evolution on the center manifold is governed by $w' = -\frac{3}{2}w^3 z_c(3 - z_c^2)w^3$. The equilibrium $w = 0$ is then asymptotically stable for $0 < z_c \leq 1$ and asymptotically unstable for $-1 \leq z_c < 0$. As before, perturbations from the equilibrium grow or decay algebraically in time.

APPENDIX F: CENTER MANIFOLD FOR THE LINES OF FIXED POINTS $P_{15,16}(z_c)$

The lines of fixed points $P_{15,16}(z_c)\colon(\beta z_c, \sqrt{6}\alpha\beta {z_c}^2, z_c)$ exist for $\lambda=0$, $\beta=\beta_{1,2}$, where $\beta_{1,2}=\frac{1}{\sqrt{2}}(\sqrt{3}\alpha\mp\sqrt{3\alpha^2-2})$, respectively. The eigenvalues are $(0,-3z_c,-3z_c)$. Thus, they are nonhyperbolic. The two cases can be treated as one case under the choice $\alpha=\frac{\beta^2+1}{\sqrt{6}\beta}$ [this is equivalent to setting $\beta=\frac{1}{\sqrt{2}}(\sqrt{3}\alpha\mp\sqrt{3\alpha^2-2})$].

For investigating the stability of the center manifold of $P_{15,16}(z_c)$, we introduce the new variables $w=\frac{y-y{z_c}^2}{2\beta^2 z_c+2z_c}-\frac{1}{2}z_c(-2zz_c+{z_c}^2+1)$, $u=\frac{({z_c}^2-1)(y-(\beta^2+1)z_c(2z-z_c))}{2(\beta^2+1)z_c}$, $v=x+\frac{\beta(\frac{y({z_c}^2-1)}{\beta^2+1}+{z_c}^2(-2zz_c+{z_c}^2-1))}{2z_c}$. Hence, we have the evolution equations

$$w'=3u^2{z_c}^2+\frac{3uw({z_c}^2-3){z_c}^2}{{z_c}^2-1}+\mathcal{O}(3), \qquad \text{(F1a)}$$

$$u'=\frac{3u^2(2\beta^2+{z_c}^4+(4-6\beta^2){z_c}^2-1)}{2(\beta^2-1)({z_c}^2-1)}+u\left(-\frac{3\beta v({z_c}^2-1)}{\beta^2-1}-3w-3z_c\right)+\frac{3v^2({z_c}^2-1)}{2(\beta^2-1)}+\frac{3}{2}w^2({z_c}^2-1)+\mathcal{O}(3), \qquad \text{(F1b)}$$

$$v'=\frac{3\beta u^2({z_c}^2+1)({z_c}^4+2(\beta^2-2){z_c}^2+1)}{2(\beta^2-1)({z_c}^2-1)^2}+\frac{3uv({z_c}^2(4-\beta^2({z_c}^2+4))+1)}{(\beta^2-1)({z_c}^2-1)}+\frac{v^2(3\beta^2({z_c}^2+3)-6)}{2\beta(\beta^2-1)}+v(-3w-3z_c)+\frac{3}{2}\beta w^2({z_c}^2-1)+\mathcal{O}(3). \qquad \text{(F1c)}$$

The center manifold of P_{14} is then given locally by $\{(w,u,v)\colon u=h_1(w), v=h_2(w), h_1(0)=0, h_2(0)=0, h_1'(0)=0, h_2'(0)=0, |w|<\delta\}$, where δ is small enough, and the functions h_1, h_2 satisfy the equations

$$\left(-3{z_c}^2h_1(w)^2-\frac{3w({z_c}^2-3){z_c}^2h_1(w)}{{z_c}^2-1}\right)h_1'(w)+h_1(w)\left(-\frac{3\beta({z_c}^2-1)h_2(w)}{\beta^2-1}-3(w+z_c)\right)+\frac{3h_1(w)^2(2\beta^2+{z_c}^4+(4-6\beta^2){z_c}^2-1)}{2(\beta^2-1)({z_c}^2-1)}+\frac{3({z_c}^2-1)h_2(w)^2}{2(\beta^2-1)}+\frac{3}{2}w^2({z_c}^2-1)=0, \qquad \text{(F2a)}$$

$$\left(-3{z_c}^2h_1(w)^2-\frac{3w({z_c}^2-3){z_c}^2h_1(w)}{{z_c}^2-1}\right)h_2'(w)+\frac{3h_1(w)h_2(w)({z_c}^2(4-\beta^2({z_c}^2+4))+1)}{(\beta^2-1)({z_c}^2-1)}+\frac{3\beta({z_c}^2+1)h_1(w)^2({z_c}^4+2(\beta^2-2){z_c}^2+1)}{2(\beta^2-1)({z_c}^2-1)^2}+\frac{3h_2(w)^2(\beta^2({z_c}^2+3)-2)}{2\beta(\beta^2-1)}-3h_2(w)(w+z_c)+\frac{3}{2}\beta w^2({z_c}^2-1)=0. \qquad \text{(F2b)}$$

We have to solve these equations using Taylor series up to third order since Eqs. (F1) were truncated at third order. Assuming $h_1(w)=aw^2+\mathcal{O}(w)^3$ and $h_2(w)=bw^2+\mathcal{O}(w)^3$ [we start at second order in w since the conditions $h_1(0)=0$, $h_2(0)=0$, $h_1'(0)=0$, $h_2'(0)=0$ have to be satisfied], substituting back into (F1), and equating the coefficients with the same powers of w, we find $a=\frac{{z_c}^2-1}{2z_c}$, $b=\frac{\beta({z_c}^2-1)}{2z_c}$. The evolution on the center manifold is governed by $w'=-\frac{3}{2}w^3z_c(3-{z_c}^2)w^3$. The equilibrium $w=0$ is then asymptotically stable for $0<z_c\leq 1$ and asymptotically unstable for $-1\leq z_c<0$. As before, perturbations from the equilibrium grow or decay algebraically in time.

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