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Multipole calculation of gravitational forces

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In this paper we introduce a method to directly calculate the Newtonian gravitational forces using multipole moments. Gravitational torques for precision tests of Newtonian gravitation are regularly calculated with multipole expansions due to the elegance and efficiency of the calculations. Tests of Newtonian gravity which probe forces rather than torques often resort to less efficient numerical calculation of sextuple integrals. Unlike multipole expansions these cannot easily be adapted for numerous permutations of the system, and instead the calculation has to be repeated, often in full. The method derived in this paper calculates the forces from any $1/r$ potential given the outer multipoles of the system and the inner multipoles calculated at any arbitrary point. The result derived can be written as a simple recursion relation for efficient calculation.

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I. INTRODUCTION

Multipole moments are often used to express $1/r$ potentials. The formalism is particularly important for laboratory gravitational measurements [1] or tests of the weak equivalence principle [2]. Many of these instruments use torsion balances where torques can easily be calculated from the azimuthal symmetry of multipole moments. Multipole moment expansions are used due to a number of transformations [3,4] which allow very efficient calculation, and more importantly minimal recalculation for a perturbed system, allowing researchers to quickly investigate the effect of misalignments. Many other precision measurements of Newtonian gravity measure linear forces [5–7]. These measurements often rely on computationally expensive sextuple integrals, or specific solutions between an extended body and a point mass which still require triple integration. Efficient calculations of forces from Newtonian gravitation are also important for calculation of Newtonian noise in gravitational wave detectors [8]. Here we introduce a method to unlock the power of multipole expansions for the calculation of gravitational forces.

Considering an experiment where the interaction between two sets of masses is measured, we name these masses the inner masses and outer masses. Multipoles can be used only if a geometry is chosen such that the absolute length of all vector positions inside the inner masses are less than the absolute length of any vector position inside an outer mass. If this simple convergence condition is met, multipole moments provide a powerful framework for calculating gravitational interactions.

The gravitational potential energy of the system can be calculated as

$$V = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Q_{lm} q_{lm}, \tag{1}$$

where G is the Newtonian constant of gravitation, and q_{lm} and Q_{lm} are the inner and the outer multipoles, respectively. These are defined as

$$q_{lm} = \iiint_{V_i} \rho(\boldsymbol{r}) \mathcal{Y}^*_{lm}(\boldsymbol{r}) \mathrm{d}V_i, \tag{2}$$

where V_i is the volume of the inner mass. $\mathcal{Y}_{lm}$ is a regular solid harmonic defined as $\mathcal{Y}_{lm}(\boldsymbol{r}) = r^l Y_{lm}(\theta, \phi)$, where Y_{lm} is a spherical harmonic. Similarly, using an irregular solid harmonic

$$Q_{lm} = \iiint_{V_o} \rho(\boldsymbol{r}) \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) \mathrm{d}V_o, \tag{3}$$

where V_o is the volume of the outer mass. The convergence condition for Eq. (1) is that all values of r inside V_i must be less than all values of r inside V_o.

For torsion balance experiments one must calculate azimuthal torques, and as such differentiate Eq. (1) with respect to ϕ. Azimuthal rotations of multipole moments are simply calculated as $R_{\mathrm{Az}}(\phi) q_{lm} = q_{lm} e^{-im\phi}$, where $R_{\mathrm{Az}}(\phi)$ is an azimuthal rotation operator. Hence the torque can simply be written as [1]

$$\tau(\phi) = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} im e^{-im\phi} Q_{lm} q_{lm}. \tag{4}$$

The beauty of this method is that each multipole multiplication can be solved just once and the torque is then given for all angles. The torque can easily be written as a Fourier sum for easy comparison to experimental data,

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$$\tau(\phi) = \sum_{m=-\infty}^{\infty} e^{-im\phi}\left(-4\pi G i m \sum_{l=|m|}^{\infty} \frac{1}{2l+1} Q_{lm} q_{lm}\right). \quad (5)$$

Calculation of forces rather than torques is significantly more involved as the multipole symmetry cannot be exploited. The final result, however, is still highly efficient and can take advantage of numerous multipole and solid harmonic transformations. One can calculate a force by solving Eq. (1) for two positions and either perform an analytical limit of displacement tending to zero [9] or perform numerical differentiation. In this work we derive a concise analytic solution for the force on the inner multipole at any position (where the multipole convergence condition is valid), and this function can be efficiently solved by recursion.

II. THEORY

Starting with the inner multipole moment of an object, q_{lm} in a position which can be arbitrarily chosen to simplify calculation, we then define the notation $q_{lm}(x, y, z)$ as the inner multipole moments of the same object translated by a vector (x, y, z). To calculate the force of the inner moment at position (x_i, y_i, z_i) we take the gradient of Eq. (1),

$$\boldsymbol{F}(x_i, y_i, z_i) = 4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Q_{lm} \nabla_i q_{lm}(x_i, y_i, z_i). \quad (6)$$

Using the method described in D'Urso and Adelberger [3] we can expand the translated inner moment in terms of the original inner moments

$$\nabla_i q_{lm}(x_i, y_i, z_i) = \sum_{l_t, m_t, l_o, m_o} \delta_{l,(l_t+l_o)} \sqrt{\frac{4\pi(2l+1)!}{(2l_t+1)!(2l_o+1)!}} C^{lm}_{l_t m_t l_o m_o} q_{l_o m_o} \nabla_i \mathcal{Y}^*_{l_t m_t}(\boldsymbol{r}_i), \quad (7)$$

where the spherical polar vector (r_i, θ_i, ϕ_i) is equal to (x_i, y_i, z_i). Also $\delta_{l,(l_t+l_o)}$ is the Kronecker delta function and $C^{lm}_{l_t m_t l_o m_o}$ is a Clebsch-Gordan coefficient using the concise notation from Ref. [10].

Noting that Eq. (4.28) of Ref. [11] gives the regular solid harmonic in Cartesian coordinates, we take the gradient of its complex conjugate

$$\nabla_i \mathcal{Y}^*_{l_t m_t}(\boldsymbol{r}_i) = \sqrt{\frac{(2l_t+1)(l_t+m_t)!(l_t-m_t)!}{4\pi}} \boldsymbol{B}_{l_t m_t}(x_i, y_i, z_i), \quad (8)$$

where

$$\boldsymbol{B}_{lm}(x, y, z) \coloneqq (-1)^m \sum_k \left(\frac{\hat{\boldsymbol{i}}((m+2k)x + imy) + \hat{\boldsymbol{j}}((m+2k)y - imx)}{x^2+y^2} + \frac{\hat{\boldsymbol{k}}(l-2k-m)}{z}\right) \times \frac{(-1)^k (x-iy)^{k+m} (x+iy)^k z^{l-2k-m}}{2^{2k+m}(m+k)!k!(l-m-2k)!}. \quad (9)$$

Here k is summed over integers where all factorials are non-negative. We also note that $\boldsymbol{B}_{l_t m_t}(x_i, y_i, z_i)$ can be written as

$$\boldsymbol{B}_{lm}(x, y, z) = \nabla \tilde{\mathcal{Y}}^*_{lm}(\boldsymbol{r}), \quad (10)$$

where we define

$$\tilde{\mathcal{Y}}_{lm}(\boldsymbol{r}) \coloneqq \frac{1}{(l+m)!} r^l P_l^m(\cos\theta) e^{im\phi}, \quad (11)$$

an improperly normalized solid harmonic. Here, P_l^m is an associated Legendre polynomial which includes the Condon-Shortley phase factor.

Before combining Eqs. (7) and (8) we first consider the form of the Clebsch-Gordan coefficient. Because of the Kronecker delta in Eq. (7) and the selection rules for Clebsch-Gordan coefficients $l_t = (l - l_o)$ and $m_t = (m - m_o)$, inserting these conditions into the algebraic sum representation of the Clebsch-Gordan coefficients [10] leaves only one valid term, and this can then easily be simplified to just

$$C^{lm}_{l_t m_t l_o m_o} = \sqrt{\frac{(2l_t)!(2l_o)!(l+m)!(l-m)!}{(2l)!(l_t+m_t)!(l_t-m_t)!(l_o+m_o)!(l_o-m_o)!}} \quad \text{for } l_t = (l-l_o) \quad \text{and} \quad m_t = (m-m_o). \quad (12)$$

When combining Eqs. (6), (7), (8), and (12), most terms in the square roots cancel leaving

$$\boldsymbol{F}(x_i,y_i,z_i)=4\pi G\sum_{l_o=0}^{\infty}\sum_{m_o=-l_o}^{l_o}\sum_{l=l_o+1}^{\infty}\sum_{m=m_o-(l-l_o)}^{m_o+(l-l_o)}\frac{1}{2l+1}Q_{lm}q_{l_om_o}\sqrt{\frac{(2l+1)(l+m)!(l-m)!}{(2l_o+1)(l_o+m_o)!(l_o-m_o)!}}\times\boldsymbol{B}_{(l-l_o),(m-m_o)}(x_i,y_i,z_i). \tag{13}$$

For brevity this can be written as

$$\boldsymbol{F}(x_i,y_i,z_i)=4\pi G\sum_{l,m,l_o,m_o}\frac{1}{2l+1}Q_{lm}q_{l_om_o}\boldsymbol{S}_{lml_om_o}(x_i,y_i,z_i), \tag{14}$$

where the sums run over the same limits and we have defined

$$\boldsymbol{S}_{lml_om_o}(x_i,y_i,z_i)\coloneqq\sqrt{\frac{(2l+1)(l+m)!(l-m)!}{(2l_o+1)(l_o+m_o)!(l_o-m_o)!}}\boldsymbol{B}_{(l-l_o),(m-m_o)}(x_i,y_i,z_i). \tag{15}$$

III. PRACTICAL CALCULATION

The benefits of this result presented in Eq. (14) are significant. Once the inner multipoles have been calculated at an original position, only a simple expansion is needed to calculate force at any location. Also due to the algebraic cancellation of prefactors with the Clebsch-Gordan coefficients, the number of calculations to be performed is greatly reduced. Furthermore, it is possible to use closed form solutions for the inner multipoles which have been calculated at the origin for most elementary solids [12]. Thus, computationally all that is needed is an efficient calculation of $\boldsymbol{S}_{lml_om_o}$.

$\boldsymbol{S}_{lml_om_o}$ can be calculated with precalculated factors $\sqrt{(2l+1)(l+m)!(l-m)!}$ and precalculated $\boldsymbol{B}_{lm}$ for the translation vector. We note that the definition of $\boldsymbol{B}_{lm}$ given in Eq. (9), while useful for low degree calculations, is inefficient and numerically unstable for higher degree multipoles. Instead $\boldsymbol{B}_{lm}$ can be efficiently computed by recursion. Recursion relations for non-normalized solid harmonics (without Condon-Shortley phase) are give in Ref. [13], and we modify these to give recursions for our improperly normalized solid harmonics,

$$\tilde{\mathcal{Y}}_{00}(\boldsymbol{r})=1, \tag{16}$$

$$\tilde{\mathcal{Y}}_{ll}(\boldsymbol{r})=-\frac{x+iy}{2l}\tilde{\mathcal{Y}}_{l-1,l-1}(\boldsymbol{r}), \tag{17}$$

$$\tilde{\mathcal{Y}}_{lm}(\boldsymbol{r})=\frac{(2l-1)z\tilde{\mathcal{Y}}_{l-1,m}(\boldsymbol{r})-(x^2+y^2+z^2)\tilde{\mathcal{Y}}_{l-2,m}(\boldsymbol{r})}{l^2-m^2}, \tag{18}$$

where for the sake of recursion $\tilde{\mathcal{Y}}_{-1,m}=0$. These recursion conditions can be used together for all l and all positive m. For negative m we note that our improperly normalized solid harmonics still obey the relation

$$\tilde{\mathcal{Y}}_{l,-m}(\boldsymbol{r})=(-1)^m\tilde{\mathcal{Y}}^*_{lm}(\boldsymbol{r}). \tag{19}$$

Using the chain rule we arrive at recursion conditions for $\boldsymbol{B}_{lm}$,

$$\boldsymbol{B}_{00}(\boldsymbol{r})=0, \tag{20}$$

$$\boldsymbol{B}_{ll}(\boldsymbol{r})=-\frac{1}{2l}((\hat{\boldsymbol{\imath}}-i\hat{\boldsymbol{\jmath}})\tilde{\mathcal{Y}}^*_{l-1,l-1}(\boldsymbol{r})+(x-iy)\boldsymbol{B}_{l-1,l-1}(\boldsymbol{r})), \tag{21}$$

$$\boldsymbol{B}_{lm}(\boldsymbol{r})=\frac{(2l-1)(\hat{\boldsymbol{k}}\tilde{\mathcal{Y}}^*_{l-1,m}(\boldsymbol{r})+z\boldsymbol{B}_{l-1,m}(\boldsymbol{r}))-((2x\hat{\boldsymbol{\imath}}+2y\hat{\boldsymbol{\jmath}}+2z\hat{\boldsymbol{k}})\tilde{\mathcal{Y}}^*_{l-2,m}(\boldsymbol{r})+(x^2+y^2+z^2)\boldsymbol{B}_{l-2,m}(\boldsymbol{r}))}{l^2-m^2}, \tag{22}$$

again for the sake of recursion $\boldsymbol{B}_{-1,m}=0$ and $\boldsymbol{B}_{l,-m}(\boldsymbol{r})=(-1)^m\boldsymbol{B}^*_{lm}(\boldsymbol{r})$. The recursion relation for $\boldsymbol{B}_{lm}$ still depends on $\tilde{\mathcal{Y}}_{lm}$, which can be calculated with the above recursion relation, or most mathematical software packages have functions to efficiently calculate associated Legendre polynomials by recursion.

In the case where one is calculating the force for a multipole in its original position, $\boldsymbol{B}_{lm}(0)$, the recursion

relations in Eqs. (21) and (22) are significantly simplified as $\tilde{\mathcal{Y}}_{lm}(\mathbf{0}) = \delta_{l,0}\delta_{m,0}$. And hence the entire recursion series drops out, leaving only two terms with $m \geq 0$,

$$\boldsymbol{B}_{10}(\mathbf{0}) = \hat{\boldsymbol{k}}, \tag{23}$$

$$\boldsymbol{B}_{11}(\mathbf{0}) = -\frac{\hat{\boldsymbol{\imath}} - i\hat{\boldsymbol{\jmath}}}{2}. \tag{24}$$

IV. CONCLUSION

In this work a method to calculate the force between inner and outer multipole moments for any valid position of the inner moment is derived. For a practical gravity calculation one can calculate the outer moments of an experimental setup, and then calculate the inner moments at an initial position. One can then calculate the force at any position of the inner moment with a simple expansion. The terms in the expansion can be efficiently calculated with recursion relations. In the case where the force is only needed at the original location of the multipole, the recursion series simplifies to only two terms. As the new terms in the expansion are independent of the outer multipoles, the force can simply be recalculated for translated outer moments.

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