Canonical transformation path to gauge theories of gravity

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In this paper, the generic part of the gauge theory of gravity is derived, based merely on the action principle and on the general principle of relativity. We apply the canonical transformation framework to formulate geometrodynamics as a gauge theory. The starting point of our paper is constituted by the general De Donder-Weyl Hamiltonian of a system of scalar and vector fields, which is supposed to be form-invariant under (global) Lorentz transformations. Following the reasoning of gauge theories, the corresponding locally form-invariant system is worked out by means of canonical transformations. The canonical transformation approach ensures by construction that the form of the action functional is maintained. We thus encounter amended Hamiltonian systems which are form-invariant under arbitrary spacetime transformations. This amended system complies with the general principle of relativity and describes both, the dynamics of the given physical system's fields and their coupling to those quantities which describe the dynamics of the spacetime geometry. In this way, it is unambiguously determined how spin-0 and spin-1 fields couple to the dynamics of spacetime. A term that describes the dynamics of the "free" gauge fields must finally be added to the amended Hamiltonian, as common to all gauge theories, to allow for a dynamic spacetime geometry. The choice of this "dynamics" Hamiltonian is outside of the scope of gauge theory as presented in this paper. It accounts for the remaining indefiniteness of any gauge theory of gravity and must be chosen "by hand" on the basis of physical reasoning. The final Hamiltonian of the gauge theory of gravity is shown to be at least quadratic in the conjugate momenta of the gauge fields—this is beyond the Einstein-Hilbert theory of general relativity.

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I. INTRODUCTION

The theory of general relativity, as proposed by A. Einstein in 1915 [1]—in conjunction with the vacuum solution of K. Schwarzschild [2]—has provided a stunningly accurate description of the dynamics of celestial bodies. This fact becomes even more surprising as Einstein's approach was in fact an "educated guess," or—in H. Weyl's words—"a purely speculative theory" [3]. The modern comprehension of general relativity as a gauge theory arises from requiring a given Lorentz-invariant theory to be invariant as well under local Lorentz transformations. This approach was pioneered by Utiyama [4] in 1956. On the other hand, a rigorous derivation of the theory that describes the interaction of matter/energy with the spacetime fabric on the basis of the action principle and the requirement that the description of any system should be form-invariant under general spacetime transformations has not yet been delivered.

Moreover, Einstein's theory has severe limitations:

(i) The theory is not scale-invariant as the pertaining coupling constant is not dimensionless.

(ii) The underlying coarse-grained energy-momentum balance equation of the theory appears to be inaccessible to quantization. A more detailed and quantizable theory would describe the *direct* interaction of individual elementary particle fields with the gravitational field, the latter described by the (uncontracted) Riemann tensor—similarly to the form of the Maxwell equation.

(iii) The observed dynamics of clusters of galaxies and stars in galaxies led to postulates of the existence of "dark matter," in order to fit into the solutions of the Einstein equations.

In our previous attempts [5,6], we have advocated a strategy by which a formalism of extended canonical transformations is constructed in the realm of covariant Hamiltonian field theory [7,8], which enables a description of canonical transformations of fields under general mappings of the spacetime geometry. Any theory derived from an action principle must maintain the general form of the action principle under transformations of its dynamic quantities. Consequently, those mappings are most naturally formulated as canonical transformations, hence as transformations whose rules are derived from generating functions.

Any theory which conforms to the general principle of relativity—hence, which respects the requirement of

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form-invariance under general mappings of spacetime geometry—must then be formulated as a canonical transformation along the well-established procedures of gauge transformations, dating back to H. Weyl [9] and W. Fock [10]. The particular transformation rule for the Hamiltonian then "automatically" provides the structure of the "gauge Hamiltonian" which renders the original system locally form-invariant. The form-invariance of the action functional is achieved by simultaneously defining both, the appropriate transformation rules for the fields, the conjugate momentum fields, and the transformation rule for the Hamiltonian. In the present context, the particular gauge Hamiltonian is to be isolated which renders a given (globally) Lorentz-invariant Hamiltonian system form-invariant under local Lorentz transformations. The gauge Hamiltonian thus describes uniquely the coupling between the dynamic quantities of the given physical system with those describing the spacetime dynamics. This conforms to the procedure generally pursued in gauge theories of gravity [11] and is in stark contrast with a postulation of a particular Lagrangian—the latter procedure was first presented by D. Hilbert [12] with the appropriate Lagrangian that led to the postulated Einstein equation. For the reader's convenience, an outline of the gauge procedure in given in Sec. VII.

The canonical transformation approach is presented starting in Sec. II with a brief review of the extended canonical formalism in the realm of covariant Hamiltonian field theory. The theory of canonical transformations then prepares exactly those transformations of the dynamic variables which maintain the general form of the action principle—and hence the general form of the canonical field equations.

The formalism is then applied to a system of scalar and vector fields, in general curvilinear spacetime in Sec. III. The Hamiltonian is required to be form-invariant under general spacetime transformations. This enforces the introduction of the affine connections as the appropriate "gauge fields" which must obey their particular transformation rules. The affine gauge coefficients are not necessarily symmetric in their lower index pair. Hence, the torsion of spacetime is included in this theory explicitly. Introducing the affine connections as the gauge quantities promotes, in the language of gauge theories, the global Lorentz-symmetry of the given system into a local, general relativistic symmetry. This renders the action integral invariant under arbitrary mappings of the reference frame.

Now, the connection coefficients emerge as external "gauge fields" [4], and their dynamics is left open, at first. The connection coefficients are then converted into internal dynamic quantities: their transformation rules emerge from a particularly crafted generating function. The subsequent transformation rule for the Hamiltonian then yields the particular gauge Hamiltonian that amends the given Lorentz-invariant Hamiltonian, which thus becomes a generally covariant Hamiltonian. The gauge Hamiltonian is then inserted back into the action integral in Sec. V. It is shown that the integrand now constitutes a world scalar density—the amended action is thus form-invariant under general spacetime transformations. This constitutes the main result of our paper: the obtained generally covariant Hamiltonian represents the generic Hamiltonian that is common to any particular theory of gravity. Moreover, in order to encounter a closed set of field equations, we show that the final Hamiltonian with dynamic space-time must be at least quadratic in the conjugate momenta of the gauge fields. This contrasts with the Einstein-Hilbert theory of general relativity.

The Hamiltonian describing the dynamics of the "free" gauge fields is to be inserted "by hand," as is common to all gauge theories. This Hamiltonian is *not* determined by the gauge formalism—hence, it must be chosen on the basis of physical reasoning. The in depth discussion of this topic will be presented in a subsequent paper. Concluding remarks are given in Sec. IX.

II. CANONICAL TRANSFORMATION RULES UNDER A DYNAMIC SPACETIME GEOMETRY

The formalism of canonical transformations, in the realm of classical field theory under dynamic spacetime, was presented earlier [5,6]. Here it is reformulated and thereby simplified considerably. The extended canonical formalism of field theory involves the description how dynamic quantities transform under the transition from one reference frame to another, $x \mapsto X$. To achieve form-invariance of the action integral, the transformation of the volume form d^4x must be taken into account. $|\partial x/\partial X|$ denotes the determinant of the Jacobi matrix of the transformation $x \mapsto X$

$$\left|\frac{\partial x}{\partial X}\right| \equiv \frac{\partial(x^0, \ldots, x^3)}{\partial(X^0, \ldots, X^3)}. \tag{1}$$

The volume form d^4X transforms as a relative scalar of weight $w = -1$

$$\mathrm{d}^4X = \frac{\partial(X^0, \ldots, X^3)}{\partial(x^0, \ldots, x^3)}\mathrm{d}^4x = \left|\frac{\partial x}{\partial X}\right|^{-1}\mathrm{d}^4x. \tag{2}$$

A general covariant second rank tensor transforms as

$$G_{\mu\nu}(X) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu},$$

and hence its determinant transforms according to

$$(\det G_{\mu\nu})(X) = (\det g_{\mu\nu})(x)\left|\frac{\partial x}{\partial X}\right|^2.$$

In the following, $g_{\mu\nu}$ is supposed to denote the covariant representation of the metric tensor. Then $g_{\mu\nu}$ has maximum rank and $g \equiv \det g_{\mu\nu} < 0$. The transformation rule for the determinant of the covariant representation of the metric tensor, $g(x) \mapsto G(X)$, follows as:

$$\sqrt{-G} = \sqrt{-g}\left|\frac{\partial x}{\partial X}\right|. \tag{3}$$

$\sqrt{-g}$ thus represents a relative scalar of weight $w = 1$, i.e. a scalar density. The product $\sqrt{-g}\,\mathrm{d}^4x$ transforms as a scalar of weight $w = 0$, hence, in conjunction with Eq. (3), as an absolute scalar:

$$\sqrt{-G}\mathrm{d}^4X = \sqrt{-g}\,\mathrm{d}^4x. \tag{4}$$

$\sqrt{-g}\,\mathrm{d}^4x$ is thus referred to as the *invariant volume form*.

The variation of the action functional for a dynamical system of a scalar field ϕ and a vector field a_μ transforms in conjunction with their respective conjugate momentum field densities $\tilde{\pi}^\nu = \pi^\nu\sqrt{-g}$ and $\tilde{p}^{\mu\nu} = p^{\mu\nu}\sqrt{-g}$ as

$$\begin{aligned}\delta S_0 &\equiv \delta\int_R\left(\tilde{\pi}^\beta\frac{\partial\phi}{\partial x^\beta} + \tilde{p}^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta} - \tilde{\mathcal{H}}_0\right)\mathrm{d}^4x\\ &\overset{!}{=}\delta\int_{R'}\left(\tilde{\Pi}^\beta\frac{\partial\Phi}{\partial X^\beta} + \tilde{P}^{\alpha\beta}\frac{\partial A_\alpha}{\partial X^\beta} - \tilde{\mathcal{H}}'_0\right)\mathrm{d}^4X,\end{aligned} \tag{5}$$

where $\tilde{\mathcal{H}}_0 = \mathcal{H}_0\sqrt{-g}$ denotes the Hamiltonian scalar density pertaining to the scalar $\mathcal{H}_0(\phi, \tilde{\pi}^\nu, a_\mu, \tilde{p}^{\mu\nu}, g_{\mu\eta})$, while $\tilde{\Pi}^\nu = \Pi^\nu\sqrt{-G}$, $\tilde{P}^{\mu\nu} = P^{\mu\nu}\sqrt{-G}$, and $\tilde{\mathcal{H}}'_0 = \mathcal{H}'_0\sqrt{-G}$ denote the respective transformed quantities. The terms $\tilde{\pi}^\beta\partial\phi/\partial x^\beta$ and $\tilde{p}^{\alpha\beta}\partial a_\alpha/\partial x^\beta$ thus are *Lorentz* scalar densities. Additional gauge quantities must be introduced for general spacetime transformations, in order to ensure that the integrands in Eq. (5) are world scalar densities, thus maintaining their form under general local spacetime transformations. This corresponds to replacing the partial derivatives of the fields in (5) by covariant derivatives.

In other words, the differences of partial and covariant derivatives define the gauge quantities. This important result is worked out in Sec. III.

The action integral is to be varied, therefore Eq. (5) implies that the integrands may differ by the divergence of a set of functions $\tilde{\mathcal{F}}_1^\mu$, whose variation vanishes on the boundary ∂R of the integration region R in spacetime:

$$\delta\int_R\frac{\partial\tilde{\mathcal{F}}_1^\alpha}{\partial x^\alpha}\mathrm{d}^4x = \delta\oint_{\partial R}\tilde{\mathcal{F}}_1^\alpha\mathrm{d}S_\alpha \overset{!}{=} 0. \tag{6}$$

The variation of the action integral (5) is not modified by adding a term $\partial\tilde{\mathcal{F}}_1^\alpha/\partial x^\alpha$ to the integrand which can be converted into a surface integral according to Eq. (6)—commonly denoted briefly as a *surface term*: the integrand is only determined up to the divergence of the functions $\tilde{\mathcal{F}}_1^\mu(\Phi, \phi, A, a, x)$. The integrand condition for a canonical transformation writes

$$\begin{aligned}&\tilde{\pi}^\beta\frac{\partial\phi}{\partial x^\beta} + \tilde{p}^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta} - \tilde{\mathcal{H}}_0 - \left(\tilde{\Pi}^\beta\frac{\partial\Phi}{\partial X^\beta} + \tilde{P}^{\alpha\beta}\frac{\partial A_\alpha}{\partial X^\beta} - \tilde{\mathcal{H}}'_0\right)\left|\frac{\partial x}{\partial X}\right|^{-1}\\ &= \frac{\partial\tilde{\mathcal{F}}_1^\beta}{\partial\phi}\frac{\partial\phi}{\partial x^\beta} + \frac{\partial\tilde{\mathcal{F}}_1^\alpha}{\partial\Phi}\frac{\partial X^\beta}{\partial x^\alpha}\frac{\partial\Phi}{\partial X^\beta} + \frac{\partial\tilde{\mathcal{F}}_1^\beta}{\partial a_\alpha}\frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial\tilde{\mathcal{F}}_1^\xi}{\partial A_\alpha}\frac{\partial X^\beta}{\partial x^\xi}\frac{\partial A_\alpha}{\partial X^\beta}\\ &\quad + \left.\frac{\partial\tilde{\mathcal{F}}_1^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}},\end{aligned} \tag{7}$$

with the transformation rule of the volume form from Eq. (2) and $\tilde{\mathcal{F}}_1^\mu$ to be taken at x. The transformation rules are obtained by comparing the coefficients

$$\begin{aligned}\tilde{\pi}^\mu(x) &= \frac{\partial\tilde{\mathcal{F}}_1^\mu}{\partial\phi}\\ \tilde{\Pi}^\mu(X) &= -\frac{\partial\tilde{\mathcal{F}}_1^\beta}{\partial\Phi}\frac{\partial X^\mu}{\partial x^\beta}\left|\frac{\partial x}{\partial X}\right|\\ \tilde{p}^{\nu\mu}(x) &= \frac{\partial\tilde{\mathcal{F}}_1^\mu}{\partial a_\nu}\\ \tilde{P}^{\nu\mu}(X) &= -\frac{\partial\tilde{\mathcal{F}}_1^\beta}{\partial A_\nu}\frac{\partial X^\mu}{\partial x^\beta}\left|\frac{\partial x}{\partial X}\right|\\ \tilde{\mathcal{H}}'_0 &= \left(\tilde{\mathcal{H}}_0 + \left.\frac{\partial\tilde{\mathcal{F}}_1^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}}\right)\left|\frac{\partial x}{\partial X}\right|.\end{aligned} \tag{8}$$

Obviously, $\tilde{\mathcal{F}}_1^\mu$ can be devised to generate specific transformation rules of the involved fields and their conjugates—this is the reason that $\tilde{\mathcal{F}}_1^\mu$ is called a *generating function*. The generating function $\tilde{\mathcal{F}}_1^\mu(\Phi, \phi, A, a, x)$ can be Legendre-transformed into the equivalent generating function $\tilde{\mathcal{F}}_2^\mu(\tilde{\Pi}, \phi, \tilde{P}, a, x)$ according to

$$\tilde{\mathcal{F}}_2^\mu = \tilde{\mathcal{F}}_1^\mu + (\Phi\tilde{\Pi}^\beta + A_\alpha\tilde{P}^{\alpha\beta})\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}.$$

The transformation rules for $\tilde{\mathcal{F}}_2^\mu$ to be taken at the spacetime event x are

$$\begin{aligned}\tilde{\pi}^\mu(x) &= \frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial\phi}\\ \delta_\nu^\mu\Phi(X) &= \frac{\partial\tilde{\mathcal{F}}_2^\alpha}{\partial\tilde{\Pi}^\nu}\frac{\partial X^\mu}{\partial x^\alpha}\left|\frac{\partial x}{\partial X}\right|\\ \tilde{p}^{\nu\mu}(x) &= \frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial a_\nu}\\ \delta_\nu^\mu A_\alpha(X) &= \frac{\partial\tilde{\mathcal{F}}_2^\beta}{\partial\tilde{P}^{\alpha\nu}}\frac{\partial X^\mu}{\partial x^\beta}\left|\frac{\partial x}{\partial X}\right|\\ \tilde{\mathcal{H}}'_0 &= \left(\tilde{\mathcal{H}}_0 + \left.\frac{\partial\tilde{\mathcal{F}}_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}}\right)\left|\frac{\partial x}{\partial X}\right|.\end{aligned} \tag{9}$$

The total integrands in the action integrals (5) must be world scalars in order to keep their form under general spacetime transformations, while the Hamiltonians $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_0'$ do not necessarily represent scalar densities. This ensures that the canonical field equations emerge as tensor equations.

III. GENERAL SPACETIME TRANSFORMATION OF SYSTEMS OF SCALAR-, VECTOR-, AND TENSOR FIELDS

A Hamiltonian $\mathcal{H}_0(\phi, \pi^\nu, a_\mu, p^{\mu\nu}, g_{\mu\eta})$ is now considered which describes the dynamics of distinct classical fields, namely a scalar field ϕ, and a vector field a_μ. The quantity $g_{\mu\eta}$ in the argument list of the Hamiltonian is interpreted as the covariant representation of the (symmetric) metric tensor. The contravariant vector field π^ν denotes the canonical conjugate of ϕ in the context of covariant Hamiltonian field theory [13]. Hence, π^ν is dual quantity of the covariant vector of spacetime derivatives $\partial\phi/\partial x^\nu$ of the scalar field ϕ. Likewise, the (2,0) tensor $p^{\mu\nu}$ stands for the canonical conjugate of the covariant field vector a_μ, hence, for the dual quantity of the derivatives $\partial a_\mu/\partial x^\nu$. A (3,0) tensor $k^{\mu\eta\nu}$, accordingly, represents the canonical conjugate of the metric tensor $g_{\mu\eta}$ and hence is the dual quantity of the partial derivatives $\partial g_{\mu\eta}/\partial x^\nu$. The tensor $k^{\mu\eta\nu}$ will be introduced later, in the action functional, in order to describe the metric $g_{\mu\eta}$ as an internal dynamic quantity of an amended Hamiltonian system, rather than as the external field variable in $\mathcal{H}_0$. The Hamiltonian $\mathcal{H}_0$ is assumed to be form-invariant under global spacetime transformations, hence, $\mathcal{H}_0$ constitutes a Lorentz scalar. The scalar field, the vector field, and the metric tensor transform under local coordinate transitions, i.e., if the transformation of the spacetime event $x^\mu \mapsto X^\mu$ is applied, according to

$$\begin{aligned} \Phi(X) &= \phi(x) \\ A_\mu(X) &= a_\xi(x)\frac{\partial x^\xi}{\partial X^\mu} \\ G_{\mu\eta}(X) &= g_{\xi\zeta}(x)\frac{\partial x^\xi}{\partial X^\mu}\frac{\partial x^\zeta}{\partial X^\eta}. \end{aligned} \tag{10}$$

These transformations are generated, in the context of the extended canonical transformation formalism of covariant Hamiltonian field theory [14], by

$$\begin{aligned} \tilde{\mathcal{F}}_2^\mu(x) = \Bigg(&\tilde{\Pi}^\beta(X)\phi(x) + \tilde{P}^{\alpha\beta}(X)a_\xi(x)\frac{\partial x^\xi}{\partial X^\alpha} \\ &+ \tilde{K}^{\alpha\lambda\beta}(X)g_{\xi\zeta}(x)\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial x^\zeta}{\partial X^\lambda}\Bigg)\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}. \end{aligned} \tag{11}$$

Here x^μ and X^ν denote the independent variables in the two distinct reference frames. With Eqs. (1) and (3), the tensor density $\tilde{\Pi}^\beta(X) = \Pi^\beta(X)\sqrt{-G}$ denotes the canonical conjugate of the transformed scalar fields $\Phi(X)$.

In analogy, the $\tilde{P}^{\alpha\beta}(X) = P^{\alpha\beta}(X)\sqrt{-G}$ stand for the corresponding conjugates of the transformed vector fields, $A_\alpha(X)$, and $\tilde{K}^{\alpha\beta\xi}(X) = K^{\alpha\beta\xi}(X)\sqrt{-G}$ denote the momenta of the transformed tensor field $G_{\alpha\beta}(X)$.

The particular generating function (11) embodies a contravariant vector of weight $w = 1$ and, hence, a vector density. But this need not in general be the case: if transformations of nontensorial quantities are defined—such as the connection coefficients—then the corresponding $\tilde{\mathcal{F}}_2^\mu$ *cannot* represent a vector density.

The crucial requirement is that the total integrand of the action functional represents a world scalar density.

Equation (10) constitutes a global symmetry transformation if its coefficients $\partial x^\xi/\partial X^\mu$ do not depend on the spacetime event x and if $\mathcal{H}_0$ is form-invariant under this transformation. In contrast, the transformation is referred to as being local if the coefficients *do* depend on spacetime. The Hamiltonian $\mathcal{H}_0$ is then no longer form-invariant under the corresponding canonical transformation rule. Appropriate dynamic gauge quantities must then be introduced to restore the form-invariance of a then amended Hamiltonian system—as is usual for all gauge theories.

Explicitly, the new canonical transformation rules, which emerge from $\tilde{\mathcal{F}}_2^\mu(\phi, \tilde{\Pi}^\nu, a_\mu, \tilde{P}^{\mu\nu}, g_{\mu\eta}, \tilde{K}^{\mu\eta\nu})$ of Eq. (11), are

$$\begin{aligned} \tilde{\pi}^\mu &= \frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial\phi} = \tilde{\Pi}^\beta\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1} \\ \delta_\nu^\mu\Phi &= \frac{\partial\tilde{\mathcal{F}}_2^\kappa}{\partial\tilde{\Pi}^\nu}\frac{\partial X^\mu}{\partial x^\kappa}\left|\frac{\partial x}{\partial X}\right| = \delta_\nu^\beta\frac{\partial x^\kappa}{\partial X^\beta}\frac{\partial X^\mu}{\partial x^\kappa}\phi = \delta_\nu^\mu\phi \\ \tilde{p}^{\nu\mu} &= \frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial a_\nu} = \tilde{P}^{\alpha\beta}\delta_\xi^\nu\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1} \\ &= \tilde{P}^{\alpha\beta}\frac{\partial x^\nu}{\partial X^\alpha}\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1} \\ \delta_\beta^\mu A_\alpha &= \frac{\partial\tilde{\mathcal{F}}_2^\kappa}{\partial\tilde{P}^{\alpha\beta}}\frac{\partial X^\mu}{\partial x^\kappa}\left|\frac{\partial x}{\partial X}\right| = a_\xi\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial x^\kappa}{\partial X^\beta}\frac{\partial X^\mu}{\partial x^\kappa} \\ &= \delta_\beta^\mu a_\xi\frac{\partial x^\xi}{\partial X^\alpha} \\ \tilde{k}^{\xi\zeta\mu} &= \frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial g_{\xi\zeta}} = \tilde{K}^{\alpha\lambda\beta}\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial x^\zeta}{\partial X^\lambda}\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1} \\ \delta_\beta^\mu G_{\alpha\lambda} &= \frac{\partial\tilde{\mathcal{F}}_2^\kappa}{\partial\tilde{K}^{\alpha\lambda\beta}}\frac{\partial X^\mu}{\partial x^\kappa}\left|\frac{\partial x}{\partial X}\right| = g_{\xi\zeta}\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial x^\zeta}{\partial X^\lambda}\delta_\beta^\mu. \end{aligned}$$

Obviously, the required transformation rules (10) of the fields are reproduced. By virtue of Eq. (1), the momentum fields obey the rules required for relative vectors of weight $w = 1$. Hence, for vector densities

$$\tilde{\Pi}^{\mu}(X) = \tilde{\pi}^{\beta}(x)\frac{\partial X^{\mu}}{\partial x^{\beta}}\left|\frac{\partial x}{\partial X}\right|$$
$$\tilde{P}^{\nu\mu}(X) = \tilde{p}^{\alpha\beta}(x)\frac{\partial X^{\nu}}{\partial x^{\alpha}}\frac{\partial X^{\mu}}{\partial x^{\beta}}\left|\frac{\partial x}{\partial X}\right|$$
$$\tilde{K}^{\xi\zeta\mu}(X) = \tilde{k}^{\alpha\lambda\beta}(x)\frac{\partial X^{\xi}}{\partial x^{\alpha}}\frac{\partial X^{\zeta}}{\partial x^{\lambda}}\frac{\partial X^{\mu}}{\partial x^{\beta}}\left|\frac{\partial x}{\partial X}\right|. \tag{12}$$

The general transformation rule for the Hamiltonian densities is given by Eq. (9). For the actual generating function, Eq. (11), the divergence of the explicit x-dependent terms of $\tilde{\mathcal{F}}_2^{\mu}$ follows as (see Appendix A for the vanishing first term)

$$\left.\frac{\partial\tilde{\mathcal{F}}_2^{\alpha}}{\partial x^{\alpha}}\right|_{\mathrm{expl}} = \left(\tilde{\Pi}^{\beta}\phi + \tilde{P}^{\alpha\beta}a_{\xi}\frac{\partial x^{\xi}}{\partial X^{\alpha}} + \tilde{K}^{\alpha\lambda\beta}g_{\xi\zeta}\frac{\partial x^{\xi}}{\partial X^{\alpha}}\frac{\partial x^{\zeta}}{\partial X^{\lambda}}\right) \times \underbrace{\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\mu}}{\partial X^{\beta}}\left|\frac{\partial x}{\partial X}\right|^{-1}\right)}_{\equiv 0\ (\mathrm{Eq.(A3)})} + \left[\tilde{P}^{\alpha\beta}a_{\xi}\frac{\partial^2 x^{\xi}}{\partial X^{\alpha}\partial X^{\beta}} + \tilde{K}^{\alpha\lambda\beta}g_{\xi\zeta} \times \left(\frac{\partial^2 x^{\xi}}{\partial X^{\alpha}\partial X^{\beta}}\frac{\partial x^{\zeta}}{\partial X^{\lambda}} + \frac{\partial^2 x^{\zeta}}{\partial X^{\lambda}\partial X^{\beta}}\frac{\partial x^{\xi}}{\partial X^{\alpha}}\right)\right]\left|\frac{\partial x}{\partial X}\right|^{-1}. \tag{13}$$

The divergence (13) can be expressed completely in terms of the original dynamic quantities by inserting the transformation rules for the momentum fields from Eq. (12):

$$\left.\frac{\partial\tilde{\mathcal{F}}_2^{\alpha}}{\partial x^{\alpha}}\right|_{\mathrm{expl}} = (\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\frac{\partial^2 x^{\xi}}{\partial X^{\kappa}\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\eta}}{\partial x^{\beta}}. \tag{14}$$

The divergence of $\tilde{\mathcal{F}}_2^{\alpha}$ vanishes exactly only if the second derivatives of the $x^{\xi}(X)$ do all vanish:

$$\frac{\partial^2 x^{\xi}}{\partial X^{\kappa}\partial X^{\eta}} = 0 \Leftrightarrow \tilde{\mathcal{H}}_0' = \tilde{\mathcal{H}}_0\left|\frac{\partial x}{\partial X}\right|. \tag{15}$$

In this case, the Hamiltonian is kept unchanged. The transformation then does *not* depend on the spacetime location and is, therefore, referred to as *global*. Otherwise, for the case of a *local* transformation, the Hamiltonians are *not* form-invariant. A "gauge Hamiltonian" $\tilde{\mathcal{H}}_{\mathrm{G}}$ must be defined which matches in its dependencies on the fields those of Eq. (14). This finally yields the particular amended Hamiltonian which is form-invariant under the local transformation:

$$\tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\mathrm{G}}$$
$$\tilde{\mathcal{H}}_{\mathrm{G}} = (\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\gamma^{\xi}{}_{\alpha\beta}. \tag{16}$$

Herein, the $\gamma^{\xi}{}_{\alpha\beta}$ formally denote the gauge quantities, whose physical meaning will be clarified below setting up their transformation rule. Of course, the "gauge Hamiltonian" $\tilde{\mathcal{H}}_{\mathrm{G}}'$ of the transformed system must have the same form in order to work out the locally form-invariant amended Hamiltonian $\tilde{\mathcal{H}}_1$

$$\tilde{\mathcal{H}}_1' = \tilde{\mathcal{H}}_0' + \tilde{\mathcal{H}}_{\mathrm{G}}'$$
$$\tilde{\mathcal{H}}_{\mathrm{G}}' = (\tilde{P}^{\alpha\beta}A_{\xi} + \tilde{K}^{\alpha\lambda\beta}G_{\xi\lambda} + \tilde{K}^{\lambda\alpha\beta}G_{\lambda\xi})\Gamma^{\xi}{}_{\alpha\beta}. \tag{17}$$

The transformation rule of the gauge quantities $\gamma^{\xi}{}_{\alpha\beta}$ and $\Gamma^{\xi}{}_{\alpha\beta}$ is determined by expressing the transformed gauge Hamiltonian (17) in terms of the original fields according to the canonical transformation rules (10) and (12):

$$\tilde{\mathcal{H}}_{\mathrm{G}}' = (\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\frac{\partial x^{\xi}}{\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\tau}}{\partial x^{\beta}}\Gamma^{\eta}{}_{\kappa\tau}\left|\frac{\partial x}{\partial X}\right|. \tag{18}$$

The demanded correlation of the formally introduced gauge quantities $\gamma^{\xi}{}_{\alpha\beta}$ and $\Gamma^{\xi}{}_{\alpha\beta}$ is obtained by inserting the transformed gauge Hamiltonian in the representation of Eq. (18) and the original gauge Hamiltonian (16) with (14) into Eq. (9)

$$(\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\frac{\partial x^{\xi}}{\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\tau}}{\partial x^{\beta}}\Gamma^{\eta}{}_{\kappa\tau} = (\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\left(\gamma^{\xi}{}_{\alpha\beta} + \frac{\partial^2 x^{\xi}}{\partial X^{\kappa}\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\eta}}{\partial x^{\beta}}\right).$$

The coefficients are compared to yield the condition

$$\frac{\partial x^{\xi}}{\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\tau}}{\partial x^{\beta}}\Gamma^{\eta}{}_{\kappa\tau} = \gamma^{\xi}{}_{\alpha\beta} + \frac{\partial^2 x^{\xi}}{\partial X^{\kappa}\partial X^{\eta}}\frac{\partial X^{\kappa}}{\partial x^{\alpha}}\frac{\partial X^{\eta}}{\partial x^{\beta}}.$$

The transformation rule for the "gauge fields" follows, after solving for the $\Gamma^{\kappa}{}_{\alpha\beta}$, as

$$\Gamma^{\kappa}{}_{\alpha\beta}(X) = \gamma^{\xi}{}_{\eta\tau}(x)\frac{\partial x^{\eta}}{\partial X^{\alpha}}\frac{\partial x^{\tau}}{\partial X^{\beta}}\frac{\partial X^{\kappa}}{\partial x^{\xi}} + \frac{\partial^2 x^{\xi}}{\partial X^{\alpha}\partial X^{\beta}}\frac{\partial X^{\kappa}}{\partial x^{\xi}}. \tag{19}$$

This transformation rule corresponds to the transformation rule of the affine connection coefficients. In the following, we identify the gauge fields $\gamma^{\xi}{}_{\alpha\beta}$—formally introduced in Eq. (16)—with the affine connection coefficients. In this aspect, we follow the approach of Palatini [15], who first treated the metric and the connection coefficients as separate dynamic quantities, which entails an additional equation of motion providing their mutual correlation.

A Hamiltonian system, $\tilde{\mathcal{H}}_0 = \mathcal{H}_0(\phi, \pi^\nu, a_\mu, p^{\mu\nu}, g_{\mu\eta})\sqrt{-g}$, which is supposed to be invariant under Lorentz transformations as the global symmetry group, is then form-invariant under the local diffeomorphism group if and only if it is amended according to Eq. (16), provided that the gauge quantities $\gamma^\xi{}_{\alpha\beta}$ transform according to Eq. (19)

$$\left.\frac{\partial \tilde{\mathcal{F}}_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}} \left|\frac{\partial x}{\partial X}\right| = \tilde{\mathcal{H}}'_{\text{G}} - \tilde{\mathcal{H}}_{\text{G}} \left|\frac{\partial x}{\partial X}\right|$$
$$= (\tilde{P}^{\alpha\beta} A_\xi + \tilde{K}^{\alpha\lambda\beta} G_{\xi\lambda} + \tilde{K}^{\lambda\alpha\beta} G_{\lambda\xi})\Gamma^\xi{}_{\alpha\beta}$$
$$- (\tilde{p}^{\alpha\beta} a_\xi + \tilde{k}^{\alpha\lambda\beta} g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta} g_{\lambda\xi})\gamma^\xi{}_{\alpha\beta} \left|\frac{\partial x}{\partial X}\right|. \tag{20}$$

The requirement of form-invariance of the given Hamiltonian under local spacetime transformations thus induces a coupling term of the vector- and the tensor fields and their conjugates via the gauge coefficients $\gamma^\xi{}_{\alpha\beta}$. The $\gamma^\xi{}_{\alpha\beta}$—interpreted as *connection coefficients*—act in a way to convert the partial derivatives in the action functional (5) into covariant derivatives. Furthermore, the metric $g_{\mu\nu}$ is promoted by the gauge procedure from an external quantity in $\tilde{\mathcal{H}}_0$ to an internal spacetime-dependent quantity, whose coupling to the vector and tensor fields is described by $\tilde{\mathcal{H}}_1$.

Generally speaking, the form of the coupling term in gauge theories is uniquely determined by the particular global symmetry property of the system, which is rendered local. The $\gamma^\xi{}_{\alpha\beta}$ in Eq. (19) need not be symmetric in the lower indices α and β [16]. Yet, restricting the theory to only the symmetric part of $\gamma^\xi{}_{\alpha\beta}$—which is equivalent to postulating a vanishing torsion of spacetime—greatly simplifies the field equations for the spacetime dynamics.

Now, the connection coefficients $\gamma^\xi{}_{\alpha\beta}$ appear as external gauge fields whose dynamics are not determined by the amended Hamiltonian $\tilde{\mathcal{H}}_1$. The generating function (11) must be amended to define the transformation rule (19) in order to also include the dynamics of the gauge fields into the description of the dynamical system as provided by the final amended Hamiltonian. The set of canonical transformation rules then also yields the rules for the conjugates of the gauge fields and the rule for a second amended Hamiltonian $\tilde{\mathcal{H}}_2$. In other words, the gauge fields are now treated as internal fields, whose dynamics is described by a second amended Hamiltonian $\tilde{\mathcal{H}}_2$. As it comes out, the set of canonical equations then establishes a closed set of coupled field equations, hence, no further gauge quantities need to be introduced. This "miracle," as an important and welcome surprise, will be the topic of the next section.

IV. INCLUDE THE DYNAMICS OF THE GAUGE FIELDS

The extended generating function $\tilde{\mathcal{F}}_2^\mu$ from Eq. (11) will now be amended to define the transformation law (19), i.e., the canonical transformation which maps reference frame x to frame X

$$\tilde{\bar{\mathcal{F}}}{}^\mu_2(x) = \tilde{\mathcal{F}}^\mu_2(x) + \tilde{Q}_\eta{}^{\alpha\xi\beta}(X)\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}$$
$$\times \left(\gamma^k{}_{ij}(x)\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi} + \frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right). \tag{21}$$

The quantities $\tilde{Q}_\eta{}^{\alpha\xi\nu}(X) = Q_\eta{}^{\alpha\xi\nu}(X)\sqrt{-G}$ denote in this definition of an extended generating function of type $\tilde{\mathcal{F}}^\mu_2(x)$, formally the canonical conjugates of the $\Gamma^\eta{}_{\alpha\xi}(X)$ of the transformed system and, hence, the dual quantities to the X^ν-derivatives of the $\Gamma^\eta{}_{\alpha\xi}(X)$. As the $\gamma^\eta{}_{\alpha\xi}(x)$ stand for the gauge coefficients of the original system, the quantities $\tilde{q}_\eta{}^{\alpha\xi\nu}(x) \equiv q_\eta{}^{\alpha\xi\nu}(x)\sqrt{-g}$ denote, accordingly, the dual quantities of the x^ν-derivatives of the gauge coefficients $\gamma^\eta{}_{\alpha\xi}(x)$ of the original system. No prediction with respect to the physical meaning of $q_\eta{}^{\alpha\xi\nu}$ and $Q_\eta{}^{\alpha\xi\nu}$ is made, at this point. Rather, their physical meaning will be determined in a later paper by setting up the canonical field equations of the final, locally form-invariant Hamiltonian.

The amended generating function (21) entails the following additional transformation rules

$$\delta^\mu_\nu \Gamma^\eta{}_{\alpha\xi} = \frac{\partial \tilde{\bar{\mathcal{F}}}{}^\kappa_2(x)}{\partial \tilde{Q}_\eta{}^{\alpha\xi\nu}}\frac{\partial X^\mu}{\partial x^\kappa}\left|\frac{\partial x}{\partial X}\right|$$
$$= \delta^\lambda_\nu \frac{\partial x^\kappa}{\partial X^\lambda}\frac{\partial X^\mu}{\partial x^\kappa}\left(\gamma^k{}_{ij}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi} + \frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right)$$
$$= \delta^\mu_\nu\left(\gamma^k{}_{ij}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi} + \frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right)$$

and

$$\tilde{q}_k{}^{ij\mu} = \frac{\partial \tilde{\bar{\mathcal{F}}}{}^\mu_2}{\partial \gamma^k{}_{ij}} = \tilde{Q}_\eta{}^{\alpha\xi\lambda}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\frac{\partial x^\mu}{\partial X^\lambda}\left|\frac{\partial x}{\partial X}\right|^{-1},$$

hence, solved for $\tilde{Q}_\eta{}^{\alpha\xi\lambda}$,

$$\tilde{Q}_\eta{}^{\alpha\xi\lambda}(X) = \tilde{q}_n{}^{mrs}(x)\frac{\partial x^n}{\partial X^\eta}\frac{\partial X^\alpha}{\partial x^m}\frac{\partial X^\xi}{\partial x^r}\frac{\partial X^\lambda}{\partial x^s}\left|\frac{\partial x}{\partial X}\right|. \tag{22}$$

We observe that the *inhomogeneous transformation rule* for the gauge coefficients from Eq. (19) is recovered. Furthermore, the canonical conjugates of the gauge coefficients, introduced formally in the generating function (21), now transform as tensor densities.

The transformation rule for the Hamiltonians is again obtained by taking the divergence of $\bar{\tilde{\mathcal{F}}}_2^\mu$ from Eq. (21)

$$\frac{\partial \bar{\tilde{\mathcal{F}}}_2^\mu}{\partial x^\mu}\bigg|_{\mathrm{expl}} = \frac{\partial \tilde{\mathcal{F}}_2^\mu}{\partial x^\mu}\bigg|_{\mathrm{expl}} + \tilde{Q}_\eta{}^{\alpha\xi\beta}\Gamma^\eta{}_{\alpha\xi}\underbrace{\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\mu}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}\right)}_{\equiv 0 \text{ (Eq. (A3))}} + \frac{\tilde{Q}_\eta{}^{\alpha\xi\beta}}{\left|\frac{\partial x}{\partial X}\right|}\left[\gamma^k{}_{ij}\frac{\partial}{\partial X^\beta}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\right) + \frac{\partial}{\partial X^\beta}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right)\right]. \tag{23}$$

The first term on the right-hand side of Eq. (23) is given by Eq. (20). The transformation rule for the Hamiltonians shall be expressed completely in terms of the dynamic variables: All derivatives of the functions $x^\mu(X)$ and $X^\mu(x)$ in (23) are to be expressed in terms of the original and transformed gauge coefficients $\gamma^\eta{}_{\alpha\xi}(x)$ and $\Gamma^\eta{}_{\alpha\xi}(X)$, and their conjugates, $\tilde{q}_\eta{}^{\alpha\xi\mu}$ and $\tilde{Q}_\eta{}^{\alpha\xi\mu}$, by making use of the respective canonical transformation rules (19) and (22).

This calculation was worked out earlier [6] and is rewritten in Appendix A in a notation adapted to the actual context. Remarkably, the transformation rule (23) can indeed completely and symmetrically be expressed in terms of the canonical variables of the original and of the transformed system as

$$\begin{aligned}\frac{\partial \bar{\tilde{\mathcal{F}}}_2^\mu}{\partial x^\mu}\bigg|_{\mathrm{expl}}\left|\frac{\partial x}{\partial X}\right| &= (\tilde{P}^{\alpha\beta}A_\xi + \tilde{K}^{\alpha\lambda\beta}G_{\xi\lambda} + \tilde{K}^{\lambda\alpha\beta}G_{\lambda\xi})\Gamma^\xi{}_{\alpha\beta} \\ &\quad - (\tilde{p}^{\alpha\beta}a_\xi + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\gamma^\xi{}_{\alpha\beta}\left|\frac{\partial x}{\partial X}\right| \\ &\quad + \frac{1}{2}\tilde{Q}_\eta{}^{\alpha\xi\beta}\left(\frac{\partial \Gamma^\eta{}_{\alpha\xi}}{\partial X^\beta} + \frac{\partial \Gamma^\eta{}_{\alpha\beta}}{\partial X^\xi} + \Gamma^k{}_{\alpha\beta}\Gamma^\eta{}_{k\xi} - \Gamma^k{}_{\alpha\xi}\Gamma^\eta{}_{k\beta}\right) \\ &\quad - \frac{1}{2}\tilde{q}_\eta{}^{\alpha\xi\beta}\left(\frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} + \frac{\partial \gamma^\eta{}_{\alpha\beta}}{\partial x^\xi} + \gamma^k{}_{\alpha\beta}\gamma^\eta{}_{k\xi} - \gamma^k{}_{\alpha\xi}\gamma^\eta{}_{k\beta}\right)\left|\frac{\partial x}{\partial X}\right| \\ &= \tilde{\mathcal{H}}_2' - \tilde{\mathcal{H}}_2\left|\frac{\partial x}{\partial X}\right|. \end{aligned} \tag{24}$$

The terms on the right-hand side can be regarded as amendments to the given system Hamiltonians, $\tilde{\mathcal{H}}_0 = \mathcal{H}_0\sqrt{-g}$ and its transformed counterpart $\tilde{\mathcal{H}}_0' = \mathcal{H}_0'\sqrt{-G}$, which promote the given globally form-invariant Hamiltonians $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_0'$ to *locally* form-invariant Hamiltonians $\tilde{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_2'$. The Hamiltonians amended accordingly are form-invariant under the extended canonical transformation generated by Eq. (21).

Amending the Hamiltonian $\tilde{\mathcal{H}}_1$ from Eq. (16) further on according to (24) yields a second amended Hamiltonian $\tilde{\mathcal{H}}_2$

$$\tilde{\mathcal{H}}_2 = \tilde{\mathcal{H}}_0 + (\tilde{p}^{\alpha\beta}a_\xi + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\gamma^\xi{}_{\alpha\beta} + \frac{1}{2}\tilde{q}_\eta{}^{\alpha\xi\beta}\left(\frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} + \frac{\partial \gamma^\eta{}_{\alpha\beta}}{\partial x^\xi} + \gamma^\tau{}_{\alpha\beta}\gamma^\eta{}_{\tau\xi} - \gamma^\tau{}_{\alpha\xi}\gamma^\eta{}_{\tau\beta}\right). \tag{25}$$

In the first stage of Sec. III, the metric was rendered an internal system variable. The gauge coefficients $\gamma^\eta{}_{\alpha\xi}$ had to be introduced to restore form-invariance of the system. In contrast, no further gauge fields had to be introduced in the actual second stage, where the gauge coefficients $\gamma^\eta{}_{\alpha\xi}$ are promoted to internal dynamic quantities. Rather, the gauge fields $\gamma^\xi{}_{\alpha\beta}$ now interact with themselves, which induces the terms quadratic in γ to finally yield the locally form-invariant Hamiltonian $\tilde{\mathcal{H}}_2$.

Observe that the gauge terms occurring in Eq. (25)—hence the terms that must be added to the given Lorentz-invariant system Hamiltonian $\tilde{\mathcal{H}}_0$—have exactly the same structure as those of the SU(N) (Yang-Mills) gauge theory, in the canonical formalism [17]

$$\mathcal{H}_2 = \mathcal{H}_0(\pi,\phi) + \mathrm{i}g_{\mathrm{YM}}(\bar{\pi}_K^\alpha\phi_J - \bar{\phi}_K\pi_J^\alpha)a_{KJ\alpha} + \frac{1}{2}p_{JK}^{\alpha\beta}\left[\frac{\partial a_{KJ\alpha}}{\partial x^\beta} + \frac{\partial a_{KJ\beta}}{\partial x^\alpha} + \mathrm{i}g_{\mathrm{YM}}(a_{KI\beta}a_{IJ\alpha} - a_{KI\alpha}a_{IJ\beta})\right].$$

The set of fermionic fields ϕ_J for the Yang-Mills case corresponds to the vector field a_ξ of Eq. (25), whereas the $N\times N$ matrix of bosonic Yang-Mills 4-vector gauge fields $a_{KJ\mu}$ now reappear anew as the connection coefficients $\gamma^\eta{}_{\alpha\xi}$.

V. INSERT THE AMENDED HAMILTONIAN $\tilde{\mathcal{H}}_2$ INTO THE ACTION INTEGRAL

The derivative of the nontensorial dynamic quantity $\gamma^\eta{}_{\alpha\xi}$ now additionally appears in the amended action integral (for the notation see Sec. VII)

$$S_4 = \int_R\left(\tilde{\pi}^\beta\frac{\partial\phi}{\partial x^\beta} + \tilde{p}^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta} + \tilde{k}^{\alpha\lambda\beta}\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} + \tilde{q}_\eta{}^{\alpha\xi\beta}\frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} - \tilde{\mathcal{H}}_2\right)\mathrm{d}^4x, \tag{26}$$

as the affine connection $\gamma^\eta{}_{\alpha\xi}$, in conjunction with their conjugates, $\tilde{q}_\eta{}^{\alpha\xi\beta}$, are now internal dynamic variables of the system. In contrast to the partial derivatives of the tensors a_α and $g_{\alpha\lambda}$, the partial derivatives of nontensorial quantities $\gamma^\eta{}_{\alpha\xi}$ cannot simply be converted into covariant derivatives, in order to render the integrand into a world scalar.

Now it is obvious why the second amended $\tilde{\mathcal{H}}_2$ cannot depict a world scalar density just on its own: the terms in $\tilde{\mathcal{H}}_2$ must complement the nontensorial terms in (26), such that the total integrand is rendered as a world scalar density. Note that the terms proportional to $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$ sum up to the Riemann curvature tensor $R^{\eta}{}_{\alpha\xi\beta}$ after $\tilde{\mathcal{H}}_2$ from Eq. (25) is inserted into (26):

$$R^{\eta}{}_{\alpha\xi\beta} = \frac{\partial\gamma^{\eta}{}_{\alpha\beta}}{\partial x^{\xi}} - \frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\beta}} + \gamma^{\tau}{}_{\alpha\beta}\gamma^{\eta}{}_{\tau\xi} - \gamma^{\tau}{}_{\alpha\xi}\gamma^{\eta}{}_{\tau\beta}. \tag{27}$$

It is the total contraction of $R^{\eta}{}_{\alpha\xi\beta}$ with the tensor density $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$ which actually yields a world scalar density. With the abbreviation (27), the action integral (26) is equivalently expressed as

$$S_4 = \int_R \bigg(\tilde{\pi}^{\beta}\frac{\partial\phi}{\partial x^{\beta}} + \tilde{p}^{\alpha\beta}\frac{\partial a_{\alpha}}{\partial x^{\beta}} + \tilde{k}^{\alpha\lambda\beta}\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} - \frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}R^{\eta}{}_{\alpha\xi\beta} - \tilde{\mathcal{H}}_1\bigg)\mathrm{d}^4x. \tag{28}$$

As the Riemann tensor (27) is skew-symmetric in its last index pair, ξ and β, only the skew-symmetric part in ξ and β of the—as yet undetermined—conjugate field $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$ contributes to the action integral. Therefore, $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$ can be assumed to be skew-symmetric in its last index pair ξ and β as well

$$\tilde{q}_{\eta}{}^{\alpha\xi\beta} = -\tilde{q}_{\eta}{}^{\alpha\beta\xi}. \tag{29}$$

On this basis, the terms depending on γ as collected in the Riemann tensor $R^{\eta}{}_{\alpha\xi\beta}$ and the Hamiltonian $\tilde{\mathcal{H}}_1$ from Eq. (16) can be combined to recover the canonical form of the action integral from Eq. (26)

$$S_4 = \int_R \bigg(\tilde{\pi}^{\beta}\frac{\partial\phi}{\partial x^{\beta}} + \tilde{p}^{\alpha\beta}\frac{\partial a_{\alpha}}{\partial x^{\beta}} + \tilde{k}^{\alpha\lambda\beta}\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} + \tilde{q}_{\eta}{}^{\alpha\xi\beta}\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\beta}} - \tilde{\mathcal{H}}_{\mathrm{G}} - \tilde{\mathcal{H}}_0\bigg)\mathrm{d}^4x, \tag{30}$$

with $\tilde{\mathcal{H}}$ the redefined gauge Hamiltonian

$$\tilde{\mathcal{H}}_{\mathrm{G}} = (\tilde{p}^{\alpha\beta}a_{\xi} + \tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\gamma^{\xi}{}_{\alpha\beta} + \frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}(\gamma^{\tau}{}_{\alpha\beta}\gamma^{\eta}{}_{\tau\xi} - \gamma^{\tau}{}_{\alpha\xi}\gamma^{\eta}{}_{\tau\beta}). \tag{31}$$

The symmetry of the metric $g_{\alpha\lambda}$ induces the symmetry of its conjugate, $\tilde{k}^{\alpha\lambda\beta}$, in its first index pair. Therefore, the Hamiltonian (31) can be written equivalently as

$$\tilde{\mathcal{H}}_{\mathrm{G}} = (\tilde{p}^{\alpha\beta}a_{\xi} + 2\tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda} + \tilde{q}_{\eta}{}^{\alpha\lambda\beta}\gamma^{\eta}{}_{\xi\lambda})\gamma^{\xi}{}_{\alpha\beta}. \tag{32}$$

The gauge Hamiltonians (31) and (32) now directly reveal how the gauge procedure works. To set up a generally form-invariant action integral on the basis of a Lorentz-invariant one, the partial derivatives of the fields a_{μ} and $g_{\mu\nu}$ in the action integral (30) are amended by the linear terms in the connection coefficients γ to yield covariant derivatives. In contrast, the partial derivative of the connection coefficient γ—which cannot be converted into a covariant derivative due to its nontensorial character—is amended by the quadratic terms in γ to yield the Riemann tensor. In both cases, the gauge procedure provides tensor quantities in place of partial derivatives. This result of the canonical derivation of the gauge theory of gravity is new and is *not* encountered in a Lagrangian formalism.

The first amended Hamiltonian $\tilde{\mathcal{H}}_1$ from Eq. (16) is inserted into Eq. (28) in to order verify that the integrand of the action integral indeed depicts a world scalar. Then all partial derivatives of the tensors in Eq. (28) are promoted to covariant derivatives

$$S_4 = \int_R \bigg(\tilde{\pi}^{\beta}\phi_{;\beta} + \tilde{p}^{\alpha\beta}a_{\alpha;\beta} + \tilde{k}^{\alpha\lambda\beta}g_{\alpha\lambda;\beta} - \frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}R^{\eta}{}_{\alpha\xi\beta} - \tilde{\mathcal{H}}_0\bigg)\mathrm{d}^4x. \tag{33}$$

Equation (33) shows that it is the *Riemann tensor* $R^{\eta}{}_{\alpha\xi\beta}$ that does actually represent the dual of $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$. The given system Hamiltonian $\tilde{\mathcal{H}}_0$ is a scalar density by presupposition, hence, the entire integrand consists of contracted tensor quantities. It makes up a world scalar density, as required for a generally relativistic form-invariant action integral. The action (33) is *not* postulated, but emerges from the gauge principle, which here means to amend a given (globally) Lorentz-invariant system Hamiltonian $\tilde{\mathcal{H}}_0$ in a way to render it invariant under local, i.e., spacetime dependent, Lorentz transformations.

VI. ADDING THE "FREE-FIELD" HAMILTONIAN

As usual, the gauge formalism fixes the coupling of the gauge fields with the fields described by the given system Hamiltonian $\mathcal{H}_0$ but does *not* provide the dynamics of the "free" gauge fields, i.e., their dynamics in the absence of any coupling. If the respective gauge fields are considered dynamic (propagating) quantities, the obtained generally form-invariant Hamiltonian $\tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\mathrm{G}}$ must be further amended in order for the canonical equations to yield nonstatic solutions for the gauge fields. Hence, the above form-invariant Hamiltonians (24) must be further amended by "free field" Hamiltonians $\tilde{\mathcal{H}}'_{\mathrm{Dyn}}(G, \tilde{K}, \tilde{Q})$ and $\tilde{\mathcal{H}}_{\mathrm{Dyn}}(g, \tilde{k}, \tilde{q})$ that obey the transformation rule

$$\tilde{\mathcal{H}}'_{\mathrm{Dyn}}(G, \tilde{K}, \tilde{Q}) = \tilde{\mathcal{H}}_{\mathrm{Dyn}}(g, \tilde{k}, \tilde{q})\left|\frac{\partial x}{\partial X}\right| \tag{34}$$

in order for the final extended Hamiltonians to describe the dynamics of the gauge fields and to maintain the general invariance of the action integral (33).

In conjunction with the precondition (15) of a globally, i.e., Lorentz-invariant Hamiltonian $\tilde{\mathcal{H}}_0$, the final extended Hamiltonian $\tilde{\mathcal{H}}_3$, which is form-invariant under the corresponding local transformation generated by (21), now reads

$$\begin{aligned}\tilde{\mathcal{H}}_3(\phi,\tilde{\pi},a,\tilde{p},g,\tilde{k},\gamma,\tilde{q}) = \tilde{\mathcal{H}}_0(\phi,\tilde{\pi},a,\tilde{p},g)\\ +\tilde{\mathcal{H}}_{\mathrm{G}}(a,\tilde{p},g,\tilde{k},\gamma,\tilde{q})\\ +\tilde{\mathcal{H}}_{\mathrm{Dyn}}(g,\tilde{k},\tilde{q})\end{aligned} \tag{35}$$

with $\tilde{\mathcal{H}}_{\mathrm{G}}$ given by Eq. (32). The correlation of the derivatives of the fields with their momenta will be identified in Sec. VIII by setting up the respective canonical field equations.

VII. SUMMARY OF THE GAUGE PROCEDURE

The starting point is a classical Hamiltonian system of a real or complex scalar field ϕ and a vector field a_ν, with the metric $g_{\mu\nu}$ assumed to be of Minkowski type, $g_{\mu\nu}\equiv\eta_{\mu\nu}=\mathrm{const}$. The particular Hamiltonian $\mathcal{H}_0$ of this system needs not to be specified except for its property to be form-invariant under global coordinate transformations $x\mapsto X$ with $\partial X^\alpha/\partial x^\beta=\mathrm{const}$

$$S_0=\int_R\left(\pi^\beta\frac{\partial\phi}{\partial x^\beta}+p^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta}-\mathcal{H}_0(\pi,\phi,p,a,g\equiv\eta)\right)\mathrm{d}^4x. \tag{36}$$

The globally form-invariant action S_0 is to be amended to yield a locally form-invariant action that describes in addition to the scalar and vector fields the dynamics of the gauge quantities. In the first step, the metric in Eq. (36) is redefined as an arbitrary spacetime-dependent function, $g_{\mu\nu}=g_{\mu\nu}(x)$. Consequently, the invariant volume form is now given by $\sqrt{-g}\mathrm{d}^4x$. In the following, the factor $\sqrt{-g}$ is absorbed into defining the canonical momenta as tensor *densities* rather than absolute tensors, hence $\tilde{\pi}^\beta=\pi^\beta\sqrt{-g}$ and $\tilde{p}^{\alpha\beta}=p^{\alpha\beta}\sqrt{-g}$. Correspondingly, the scalar $\mathcal{H}_0$ in Eq. (36) is converted into a *scalar density* $\tilde{\mathcal{H}}_0=\mathcal{H}_0\sqrt{-g}$. Moreover, the conjugate momentum $\tilde{k}^{\mu\nu\lambda}$ as the dual of the derivative of the metric must be introduced into the action functional

$$\begin{aligned}S_1=\int_R\bigg(\tilde{\pi}^\beta\frac{\partial\phi}{\partial x^\beta}+\tilde{p}^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta}+\tilde{k}^{\mu\nu\beta}\frac{\partial g_{\mu\nu}}{\partial x^\beta}\\ -\tilde{\mathcal{H}}_0(\tilde{\pi},\phi,\tilde{p},a,g)\bigg)\mathrm{d}^4x.\end{aligned} \tag{37}$$

The metric $g_{\mu\nu}$ is now formally treated as an internal quantity in the action functional (37) of the system. As the partial derivatives of tensors do not transform as tensors, the integrand of (37) is now no longer a scalar. The scalar property of the integrand is restored by adding the gauge Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{G}}$ to the integrand

$$\begin{aligned}S_2=\int_R\bigg(\tilde{\pi}^\beta\frac{\partial\phi}{\partial x^\beta}+\tilde{p}^{\alpha\beta}\frac{\partial a_\alpha}{\partial x^\beta}+\tilde{k}^{\mu\nu\beta}\frac{\partial g_{\mu\nu}}{\partial x^\beta}-\tilde{\mathcal{H}}_0(\tilde{\pi},\phi,\tilde{p},a,g)\\ -\tilde{\mathcal{H}}_{\mathrm{G}}(\tilde{p},a,\tilde{k},g,\gamma)\bigg)\mathrm{d}^4x\end{aligned} \tag{38}$$

with

$$\tilde{\mathcal{H}}_{\mathrm{G}}=(\tilde{p}^{\alpha\beta}a_\xi+\tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda}+\tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi})\gamma^\xi{}_{\alpha\beta}.$$

This amounts to introducing the connection coefficients $\gamma^\xi{}_{\lambda\alpha}$ as external gauge quantities and then promoting the *partial* derivatives in (38) to covariant derivatives. The action (38) is thus equivalently expressed as

$$\begin{aligned}S_2=\int_R(\tilde{\pi}^\beta\phi_{;\beta}+\tilde{p}^{\alpha\beta}a_{\alpha;\beta}+\tilde{k}^{\mu\nu\beta}g_{\mu\nu;\beta}\\ -\tilde{\mathcal{H}}_0(\tilde{\pi},\phi,\tilde{p},a,g))\mathrm{d}^4x.\end{aligned} \tag{39}$$

In the next step, the so far external gauge fields $\gamma^\xi{}_{\alpha\beta}$ are treated as internal fields, which means that the description of their dynamics is to be included in a further amended action functional. This requires to define the canonical momentum $\tilde{q}_\xi{}^{\lambda\alpha\beta}$ as the dual of the partial x^β-derivative of the gauge field $\gamma^\xi{}_{\lambda\alpha}$ and to add the contraction of both terms to the action

$$\begin{aligned}S_3=\int_R\bigg(\tilde{\pi}^\beta\phi_{;\beta}+\tilde{p}^{\alpha\beta}a_{\alpha;\beta}+\tilde{k}^{\mu\nu\beta}g_{\mu\nu;\beta}+\tilde{q}_\xi{}^{\lambda\alpha\beta}\frac{\partial\gamma^\xi{}_{\lambda\alpha}}{\partial x^\beta}\\ -\tilde{\mathcal{H}}_0(\tilde{\pi},\phi,\tilde{p},a,g)\bigg)\mathrm{d}^4x.\end{aligned} \tag{40}$$

As the connection $\gamma^\xi{}_{\lambda\alpha}$ is no tensor, neither is its partial derivative. Again, the invariance property of the integrand (40) is restored by supplementing the appropriate term of the gauge formalism—hence the term quadratic in γ—to the integrand of Eq. (40)

$$\begin{aligned}S_4=\int_R\bigg(\tilde{\pi}^\beta\phi_{;\beta}+\tilde{p}^{\alpha\beta}a_{\alpha;\beta}+\tilde{k}^{\mu\nu\beta}g_{\mu\nu;\beta}+\tilde{q}_\xi{}^{\lambda\alpha\beta}\frac{\partial\gamma^\xi{}_{\lambda\alpha}}{\partial x^\beta}\\ -\tilde{q}_\xi{}^{\lambda\alpha\beta}\gamma^\tau{}_{\lambda\beta}\gamma^\xi{}_{\tau\alpha}-\tilde{\mathcal{H}}_0(\tilde{\pi},\phi,\tilde{p},a,g)\bigg)\mathrm{d}^4x.\end{aligned} \tag{41}$$

By virtue of the skew-symmetry of $\tilde{q}_\xi{}^{\lambda\alpha\beta}$ in its last index pair—which follows from the gauge formalism, as stated in Eq. (29)—the terms proportional to $\tilde{q}$ can be merged to yield the Riemann curvature tensor (27)

$$S_4 = \int_R \left(\tilde{\pi}^\beta \phi_{;\beta} + \tilde{p}^{\alpha\beta} a_{\alpha;\beta} + \tilde{k}^{\mu\nu\beta} g_{\mu\nu;\beta} - \frac{1}{2} \tilde{q}_\xi{}^{\lambda\alpha\beta} R^\xi{}_{\lambda\alpha\beta} - \tilde{\mathcal{H}}_0(\tilde{\pi}, \phi, \tilde{p}, a, g) \right) \mathrm{d}^4 x. \tag{42}$$

As a result of the gauge procedure, the action is obtained completely in terms of tensor quantities and thus represents a world scalar density. We observe that the gauge term for the partial derivative of the connection coefficients in Eq. (41) is given by the term quadratic in γ. This is a consequence of the fact that the coefficients in the transformation rule for the Hamiltonian from Eq. (A4) can completely be expressed in term of the connection coefficients. Therefore, no new gauge fields were needed—which is the reason why the procedure to promote the global Lorentz symmetry into a local one truncates here and does *not* produce an infinite hierarchy of gauge fields with their pertaining transformation conditions.

In the last step, a Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$ of the "free" momenta $\tilde{k}$ and $\tilde{q}$ must be introduced "by hand" in order for the corresponding fields g and γ to be dynamic

$$S_5 = \int_R \left(\tilde{\pi}^\beta \phi_{;\beta} + \tilde{p}^{\alpha\beta} a_{\alpha;\beta} + \tilde{k}^{\alpha\lambda\beta} g_{\alpha\lambda;\beta} - \frac{1}{2} \tilde{q}_\xi{}^{\lambda\alpha\beta} R^\xi{}_{\lambda\alpha\beta} - \tilde{\mathcal{H}}_0(\tilde{\pi}, \phi, \tilde{p}, a, g) - \tilde{\mathcal{H}}_{\mathrm{Dyn}}(\tilde{k}, g, \tilde{q}) \right) \mathrm{d}^4 x. \tag{43}$$

The options for defining $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$ will be discussed in a subsequent paper [18]. The canonical field equations, summarized in Sec. VIII E, follow from the action principle $\delta S_5 \stackrel{!}{=} 0$.

VIII. CANONICAL FIELD EQUATIONS: ARBITRARY $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$

A. Field equations for ϕ and $\tilde{\pi}^\nu$

As the locally form-invariant extended Hamiltonian (35) does not contain additional terms involving ϕ and $\tilde{\pi}^\nu$, the dynamics of these fields is determined by the globally form-invariant Hamiltonian $\mathcal{H}$ only. The respective field equations are

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{\pi}^\mu} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \tilde{\pi}^\mu}$$
$$\frac{\partial \tilde{\pi}^\alpha}{\partial x^\alpha} = -\frac{\partial \tilde{\mathcal{H}}_3}{\partial \phi} = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial \phi}.$$

For a vector density $\tilde{\pi}^\mu = \pi^\mu \sqrt{-g}$, the ordinary divergence $\partial \tilde{\pi}^\alpha / \partial x^\alpha$ of can be expressed in terms of the covariant divergence as

$$\tilde{\pi}^\alpha{}_{;\alpha} = \frac{\partial \tilde{\pi}^\alpha}{\partial x^\alpha} + \tilde{\pi}^\beta \gamma^\alpha{}_{\beta\alpha} - \tilde{\pi}^\alpha \gamma^\beta{}_{\beta\alpha} = \frac{\partial \tilde{\pi}^\alpha}{\partial x^\alpha} + 2\tilde{\pi}^\beta s^\alpha{}_{\beta\alpha},$$

wherein $s^\alpha{}_{\beta\alpha}$ denotes the contraction of the torsion tensor

$$s^\xi{}_{\beta\alpha} = \frac{1}{2}(\gamma^\xi{}_{\beta\alpha} - \gamma^\xi{}_{\alpha\beta}). \tag{44}$$

Thus, for connection coefficients that are symmetric in their lower index pair, the torsion tensor vanishes identically. The second field equation follows as the tensor equation

$$\tilde{\pi}^\alpha{}_{;\alpha} = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial \phi} + 2\tilde{\pi}^\beta s^\alpha{}_{\beta\alpha}.$$

Both field equations thus emerge as tensor equations.

B. Field equations for a_μ and $\tilde{p}^{\mu\nu}$

Due to the coupling term $\tilde{p}^{\alpha\beta} a_\eta \gamma^\eta{}_{\alpha\beta}$ in the extended Hamiltonian (35) the field equations for a_μ and $\tilde{p}^{\mu\nu}$ acquire an additional term. The respective field equations are

$$\frac{\partial a_\nu}{\partial x^\mu} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{p}^{\nu\mu}} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \tilde{p}^{\nu\mu}} + a_\xi \gamma^\xi{}_{\nu\mu}$$
$$\frac{\partial \tilde{p}^{\nu\beta}}{\partial x^\beta} = -\frac{\partial \tilde{\mathcal{H}}_3}{\partial a_\nu} = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial a_\nu} - \tilde{p}^{\alpha\beta} \gamma^\nu{}_{\alpha\beta}. \tag{45}$$

The partial derivatives of the fields and the terms proportional to the affine connections $\gamma^\nu{}_{\alpha\beta}$ can be combined to yield covariant derivatives

$$a_{\nu;\mu} = \frac{\partial a_\nu}{\partial x^\mu} - a_\eta \gamma^\eta{}_{\nu\mu}$$
$$\tilde{p}^{\nu\beta}{}_{;\beta} = \frac{\partial \tilde{p}^{\nu\beta}}{\partial x^\beta} + \tilde{p}^{\alpha\beta} \gamma^\nu{}_{\alpha\beta} + \tilde{p}^{\nu\alpha} \gamma^\beta{}_{\alpha\beta} - \tilde{p}^{\nu\beta} \gamma^\alpha{}_{\alpha\beta},$$

which yields the tensor equations

$$a_{\nu;\mu} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \tilde{p}^{\nu\mu}}, \qquad \tilde{p}^{\nu\beta}{}_{;\beta} = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial a_\nu} + 2\tilde{p}^{\nu\beta} s^\alpha{}_{\beta\alpha}. \tag{46}$$

The coupling term $\tilde{p}^{\alpha\beta} a_\xi \gamma^\xi{}_{\alpha\beta}$ in the extended Hamiltonian $\tilde{\mathcal{H}}_3$ thus converts the nontensor equations for a_μ and $\tilde{p}^{\mu\nu}$ which emerge from the system's Hamiltonian $\tilde{\mathcal{H}}_0$ into tensor equations which hold in any reference frame.

C. Field equations for $g_{\alpha\beta}$ and $\tilde{k}^{\alpha\beta\mu}$

The canonical equation for the metric $g_{\alpha\beta}$ is

$$\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{k}^{\alpha\lambda\beta}} = g_{\kappa\lambda} \gamma^\kappa{}_{\alpha\beta} + g_{\alpha\kappa} \gamma^\kappa{}_{\lambda\beta} + \frac{\partial \tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial \tilde{k}^{\alpha\lambda\beta}}, \tag{47}$$

hence

$$g_{\alpha\lambda;\beta} = \frac{\partial g_{\alpha\lambda}}{\partial x^\beta} - g_{\lambda\kappa}\gamma^\kappa{}_{\alpha\beta} - g_{\alpha\kappa}\gamma^\kappa{}_{\lambda\beta} = \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{k}^{\alpha\lambda\beta}}. \tag{48}$$

The field equation thus means that the covariant derivative of the metric is the dual of the *nonmetricity tensor*—which describes the length change of a vector under parallel transport.

The canonical equation for the conjugate of the metric follows as

$$\frac{\partial \tilde{k}^{\xi\lambda\beta}}{\partial x^\beta} = -\frac{\partial \tilde{\mathcal{H}}_3}{\partial g_{\xi\lambda}} = -\tilde{k}^{\alpha\lambda\beta}\gamma^\xi{}_{\alpha\beta} - \tilde{k}^{\xi\alpha\beta}\gamma^\lambda{}_{\alpha\beta} - \frac{\partial \tilde{\mathcal{H}}_0}{\partial g_{\xi\lambda}} - \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial g_{\xi\lambda}}, \tag{49}$$

hence

$$\tilde{k}^{\xi\lambda\beta}{}_{;\beta} = -\frac{\partial \tilde{\mathcal{H}}_0}{\partial g_{\xi\lambda}} - \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial g_{\xi\lambda}} + 2\tilde{k}^{\xi\lambda\beta}s^\alpha{}_{\beta\alpha}, \qquad \tilde{k}^{\xi\lambda\beta} = \tilde{k}^{\lambda\xi\beta}. \tag{50}$$

The coupling terms $\tilde{k}^{\alpha\lambda\beta}g_{\xi\lambda}\gamma^\xi{}_{\alpha\beta}$ and $\tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi}\gamma^\xi{}_{\alpha\beta}$ in the gauge-invariant extended Hamiltonian $\tilde{\mathcal{H}}_3$ from Eq. (35) thus convert the nontensor equations for $g_{\alpha\lambda}$ and $\tilde{k}^{\xi\lambda\beta}$ into tensor equations. As $g_{\xi\lambda}$ is symmetric, $\tilde{k}^{\xi\lambda\beta}$ is induced to be symmetric in its first index pair, ξ, λ.

The given Lorentz-invariant system Hamiltonian $\tilde{\mathcal{H}}_0$ describes the dynamics in a static spacetime background. For this reason, $\tilde{\mathcal{H}}_0$ is supposed not to depend on the conjugate of the metric, $\tilde{k}^{\xi\alpha\beta}$. The derivative of the Hamiltonian density $\tilde{\mathcal{H}}_0$ with respect to the metric $g_{\xi\lambda}$ then represents the symmetric energy-momentum tensor density $\tilde{T}^{\lambda\xi}$ of $\tilde{\mathcal{H}}_0$

$$\tilde{T}^{\lambda\xi} = 2\frac{\partial \tilde{\mathcal{H}}_0}{\partial g_{\xi\lambda}}. \tag{51}$$

Thus, $\tilde{T}^{\lambda\xi}$ does not describe the energy-momentum contributed by a gravitational field and a dynamic spacetime.

D. Field equations for $\gamma^\eta{}_{\alpha\beta}$ and $\tilde{q}_\eta{}^{\alpha\beta\nu}$

The canonical equation that provides the correlation of the x^β-derivative of the $\gamma^\eta{}_{\alpha\xi}$ with their duals, $\tilde{q}_\eta{}^{\alpha\xi\beta}$, follows as

$$\begin{aligned}\frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} &= \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} = \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} + \frac{\partial \tilde{\mathcal{H}}_{\rm G}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} \\ &= \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} + \gamma^\tau{}_{\alpha\beta}\gamma^\eta{}_{\tau\xi}.\end{aligned} \tag{52}$$

Solved for $\partial\tilde{\mathcal{H}}_{\rm Dyn}/\partial\tilde{q}_\eta{}^{\alpha\xi\beta}$, one finds

$$\frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} = \frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} - \gamma^\tau{}_{\alpha\beta}\gamma^\eta{}_{\tau\xi}.$$

Thus, by virtue of the skew-symmetry of $\tilde{q}_\eta{}^{\alpha\xi\beta}$ in ξ and β

$$\begin{aligned}2\frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} &= \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\xi\beta}} - \frac{\partial \tilde{\mathcal{H}}_{\rm Dyn}}{\partial \tilde{q}_\eta{}^{\alpha\beta\xi}} \\ &= \frac{\partial \gamma^\eta{}_{\alpha\xi}}{\partial x^\beta} - \frac{\partial \gamma^\eta{}_{\alpha\beta}}{\partial x^\xi} + \gamma^\tau{}_{\alpha\xi}\gamma^\eta{}_{\tau\beta} - \gamma^\tau{}_{\alpha\beta}\gamma^\eta{}_{\tau\xi} \\ &= -R^\eta{}_{\alpha\xi\beta}.\end{aligned} \tag{53}$$

On the right-hand side, the connection coefficients $\gamma^\eta{}_{\alpha\beta}$ and their derivatives sum up to the combination which represents the Riemann curvature tensor (27). The field Eq. (53) thus states that the Riemann tensor vanishes identically everywhere—and thus any curvature of spacetime—if there is no "free-field" Hamiltonian $\tilde{\mathcal{H}}_{\rm Dyn}$. Therefore, it must then be added "by hand" to the *derived* gauge Hamiltonian $\tilde{\mathcal{H}}_{\rm G}$ in order to allow for a consistent spacetime dynamics [19].

The divergence of $\tilde{q}_\xi{}^{\alpha\beta\lambda}$ is given by the derivative of the gauge Hamiltonian $\tilde{\mathcal{H}}_{\rm G}$ from Eq. (31) with respect to the $\gamma^\xi{}_{\alpha\beta}$

$$\frac{\partial \tilde{q}_\xi{}^{\alpha\beta\lambda}}{\partial x^\lambda} = -\frac{\partial \tilde{\mathcal{H}}_3}{\partial \gamma^\xi{}_{\alpha\beta}} = -\frac{\partial \tilde{\mathcal{H}}_{\rm G}}{\partial \gamma^\xi{}_{\alpha\beta}}.$$

This equation does not depend on the particular choice of $\tilde{\mathcal{H}}_{\rm Dyn}$ as the latter is supposed to not depend on the gauge fields $\gamma^\xi{}_{\alpha\beta}$. With the gauge Hamiltonian from Eq. (32), we find

$$\frac{\partial \tilde{q}_\xi{}^{\alpha\beta\lambda}}{\partial x^\lambda} = -\tilde{p}^{\alpha\beta}a_\xi - 2\tilde{k}^{\lambda\alpha\beta}g_{\lambda\xi} + \tilde{q}_\eta{}^{\alpha\beta\lambda}\gamma^\eta{}_{\xi\lambda} + \tilde{q}_\xi{}^{\eta\lambda\beta}\gamma^\alpha{}_{\eta\lambda}. \tag{54}$$

In order to express Eq. (54) manifestly as a tensor equation, we write the covariant divergence of the tensor density $\tilde{q}_\xi{}^{\alpha\beta\lambda}$

$$\begin{aligned}\tilde{q}_\xi{}^{\alpha\beta\lambda}{}_{;\lambda} = &\frac{\partial \tilde{q}_\xi{}^{\alpha\beta\lambda}}{\partial x^\lambda} - \tilde{q}_\eta{}^{\alpha\beta\lambda}\gamma^\eta{}_{\xi\lambda} + \tilde{q}_\xi{}^{\eta\beta\lambda}\gamma^\alpha{}_{\eta\lambda} \\ &+ \tilde{q}_\xi{}^{\alpha\eta\lambda}\gamma^\beta{}_{\eta\lambda} + \tilde{q}_\xi{}^{\alpha\beta\eta}\gamma^\lambda{}_{\eta\lambda} - \tilde{q}_\xi{}^{\alpha\beta\lambda}\gamma^\eta{}_{\eta\lambda}.\end{aligned}$$

As $\tilde{q}_\xi{}^{\alpha\eta\lambda}$ is skew-symmetric in η and λ, the first term in the second line can be expressed as well in terms of the torsion tensor

$$\begin{aligned}\tilde{q}_\xi{}^{\alpha\beta\lambda}{}_{;\lambda} = &\frac{\partial \tilde{q}_\xi{}^{\alpha\beta\lambda}}{\partial x^\lambda} - \tilde{q}_\eta{}^{\alpha\beta\lambda}\gamma^\eta{}_{\xi\lambda} - \tilde{q}_\xi{}^{\eta\lambda\beta}\gamma^\alpha{}_{\eta\lambda} + \tilde{q}_\xi{}^{\alpha\eta\lambda}s^\beta{}_{\eta\lambda} \\ &+ 2\tilde{q}_\xi{}^{\alpha\beta\eta}s^\lambda{}_{\eta\lambda}.\end{aligned}$$

The field Eq. (54) thus actually represents the tensor density equation

$$\tilde{q}_{\xi}^{\alpha\beta\lambda}{}_{;\lambda} + \tilde{p}^{\alpha\beta} a_{\xi} + 2\tilde{k}^{\lambda\alpha\beta} g_{\lambda\xi} - \tilde{q}_{\xi}{}^{\alpha\eta\lambda} s^{\beta}{}_{\eta\lambda} + 2\tilde{q}_{\xi}{}^{\alpha\eta\beta} s^{\lambda}{}_{\eta\lambda} = 0. \tag{55}$$

We thus found that *all* field equations emerging from $\mathcal{H}_3$ are tensor equations, hence, their forms are the same in any reference frame.

E. Summary of the coupled set of field equations

With the abbreviations (27) and (44) for particular combinations of the gauge fields $\gamma^{\eta}{}_{\xi\lambda}$ and their partial derivatives, the complete set of coupled field equations is summarized as

$$\begin{aligned}
\phi_{;\mu} &= \frac{\partial\tilde{\mathcal{H}}_0}{\partial\tilde{\pi}^{\mu}}, \qquad \tilde{\pi}^{\beta}{}_{;\beta} = -\frac{\partial\tilde{\mathcal{H}}_0}{\partial\phi} + 2\tilde{\pi}^{\beta} s^{\alpha}{}_{\beta\alpha} \\
a_{\nu;\mu} &= \frac{\partial\tilde{\mathcal{H}}_0}{\partial\tilde{p}^{\nu\mu}}, \qquad \tilde{p}^{\nu\beta}{}_{;\beta} = -\frac{\partial\tilde{\mathcal{H}}_0}{\partial a_{\nu}} + 2\tilde{p}^{\nu\beta} s^{\alpha}{}_{\beta\alpha} \\
g_{\xi\lambda;\mu} &= \frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial\tilde{k}^{\xi\lambda\mu}}, \qquad \tilde{k}^{\xi\lambda\beta}{}_{;\beta} = -\frac{\partial(\tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\text{Dyn}})}{\partial g_{\xi\lambda}} + 2\tilde{k}^{\xi\lambda\beta} s^{\alpha}{}_{\beta\alpha} \\
-\frac{R^{\eta}{}_{\xi\lambda\mu}}{2} &= \frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial\tilde{q}_{\eta}{}^{\xi\lambda\mu}}, \qquad \tilde{q}_{\eta}^{\xi\lambda\beta}{}_{;\beta} = -\tilde{p}^{\xi\lambda} a_{\eta} - 2\tilde{k}^{\beta\xi\lambda} g_{\beta\eta} + \tilde{q}_{\eta}{}^{\xi\beta\alpha} s^{\lambda}{}_{\beta\alpha} \\
&\qquad + 2\tilde{q}_{\eta}{}^{\xi\lambda\beta} s^{\alpha}{}_{\beta\alpha}
\end{aligned} \tag{56}$$

The surprising fact in the last line of Eqs. (56) is that the naïvely expected covariant derivatives of the gauge fields, i.e., the covariant derivatives of the connection coefficients $\gamma^{\eta}{}_{\xi\lambda}$—which do not exist—are replaced by the Riemann tensor as a result of the canonical gauge procedure—and thus again establish a tensor equation, as is required for a generally covariant theory.

Only with $\tilde{\mathcal{H}}_{\text{Dyn}}$ given, the entire set of eight canonical field equations for the fields $\phi, a_{\nu}, g_{\xi\lambda}$, and $\gamma^{\eta}{}_{\xi\lambda}$ and their respective conjugates $\tilde{\pi}^{\mu}$, $\tilde{p}^{\nu\mu}$, $\tilde{k}^{\xi\lambda\mu}$, and $\tilde{q}_{\eta}{}^{\xi\lambda\mu}$ is closed and can then be integrated to yield the combined dynamics of fields and spacetime geometry. As $\tilde{\mathcal{H}}_{\text{Dyn}}$ does *not* emerge from the gauge formalism, it must be chosen on the basis of physical reasoning. The field Eq. (56) thus depend on this choice. A particular choice of $\tilde{\mathcal{H}}_{\text{Dyn}}$ will be discussed in Sec. VIII G.

F. Consistency relation

Similar to U(1) and SU(N) gauge theories, the set of field equations brings about a *consistency condition*. Differentiating Eq. (54) with respect to x^{β}, the left-hand side vanishes due to the skew-symmetry of $\tilde{q}_{\xi}{}^{\alpha\beta\lambda}$ in its last index pair, as stated in Eq. (29). Accordingly, the right-hand side of (54) yields the condition

$$\frac{\partial}{\partial x^{\beta}}\left(\tilde{p}^{\alpha\beta} a_{\xi} + 2\tilde{k}^{\lambda\alpha\beta} g_{\lambda\xi} + \tilde{q}_{\eta}{}^{\alpha\lambda\beta}\gamma^{\eta}{}_{\xi\lambda} - \tilde{q}_{\xi}{}^{\eta\lambda\beta}\gamma^{\alpha}{}_{\eta\lambda}\right) = 0. \tag{57}$$

The partial derivative representations of the field Eqs. (45), (47), (49), (52), and (54) from Secs. VIII B, VIII C, and VIII D can now be inserted to yield the consistency condition (see Appendix B)

$$\begin{aligned}
&2\tilde{k}^{\lambda\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial\tilde{k}^{\lambda\xi\beta}} - 2g_{\xi\beta}\frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial g_{\alpha\beta}} + \tilde{q}_{\tau}{}^{\alpha\beta\lambda}\frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial\tilde{q}_{\tau}{}^{\xi\beta\lambda}} - \tilde{q}_{\xi}{}^{\tau\beta\lambda}\frac{\partial\tilde{\mathcal{H}}_{\text{Dyn}}}{\partial\tilde{q}_{\alpha}{}^{\tau\beta\lambda}} \\
&\quad = a_{\xi}\frac{\partial\tilde{\mathcal{H}}_0}{\partial a_{\alpha}} - \tilde{p}^{\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_0}{\partial\tilde{p}^{\xi\beta}} + 2g_{\xi\beta}\frac{\partial\tilde{\mathcal{H}}_0}{\partial g_{\alpha\beta}}.
\end{aligned} \tag{58}$$

This is a second rank tensor equation. In conjunction with Eq. (53), it relates the source terms of the right-hand side to the Riemann tensor terms on the left-hand side. Note that Eq. (58) holds as well for the cases nonvanishing torsion ($s^{\beta}{}_{\eta\lambda} \neq 0$) and nonmetricity ($g_{\xi\beta;\mu} \neq 0$). Equation (58) represents a generic Einstein equation that holds for any given system of scalar and vector fields described by $\tilde{\mathcal{H}}_0$ and the particular model for the dynamics of the free gravitational fields, as described by $\tilde{\mathcal{H}}_{\text{Dyn}}$.

At this point, it would be interesting for the reader to find out whether our theory makes observational predictions similar to general relativity. As general relativity predicts successfully observations made in the solar system (weak field), is the gauge theory given by Eqs. (56) and (58) capable of giving something like general relativity plus small corrections? We will address these issues briefly in the following section.

G. Sample $\tilde{\mathcal{H}}_{\text{Dyn}}$

As an example, we postulate $\tilde{\mathcal{H}}_{\text{Dyn}}$ as a linear combination of quadratic and linear terms in $\tilde{q}$

$$\tilde{\mathcal{H}}_{\text{Dyn}} = \frac{1}{4g_1}\tilde{q}_{\eta}{}^{\alpha\xi\beta}\tilde{q}_{\alpha}{}^{\eta\tau\lambda} g_{\xi\tau} g_{\beta\lambda}\frac{1}{\sqrt{-g}} - g_2\tilde{q}_{\eta}{}^{\alpha\eta\beta} g_{\alpha\beta}. \tag{59}$$

In contrast to the dimensionless coupling constant g_1, the coupling constant g_2 has the natural dimension Length^{-2}. Note that the sample Hamiltonian (59) does not depend on $\tilde{k}^{\xi\lambda\mu}$ and thus directly induces the metric compatibility condition $g_{\xi\lambda;\mu} = 0$ according to Eqs. (56). In a subsequent paper, we will discuss the more general case of a $\tilde{\mathcal{H}}_{\text{Dyn}}$ which also depends quadratically on $\tilde{k}^{\xi\lambda\mu}$.

The correlation of the canonical momentum q to the Riemann tensor then follows from Eq. (53) as

$$q_{\eta\alpha\xi\beta} = g_1\left(R_{\eta\alpha\xi\beta} - R_{\eta\alpha\xi\beta}|_{\max}\right), \tag{60}$$

with

$$R_{\eta\alpha\xi\beta}|_{\max} = g_2\left(g_{\eta\xi} g_{\alpha\beta} - g_{\eta\beta} g_{\alpha\xi}\right)$$

the Riemann tensor for a *maximally symmetric* 4-dimensional manifold with constant Ricci curvature $R = 12g_2$. The derivatives of $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$ with respect to $\tilde{q}$ in Eq. (58) then cancel.

The derivative of $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$ with respect to $g_{\alpha\beta}$ follows as

$$2g_{\xi\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial g_{\alpha\beta}} = \frac{1}{g_1}\left(q_{\beta\eta\lambda\xi}q^{\eta\beta\lambda\alpha} - \frac{1}{4}\delta_\xi^\alpha q_{\beta\eta\lambda\tau}q^{\eta\beta\lambda\tau}\right)\sqrt{-g} - g_2(q^{\alpha\eta}_{\eta\ \ \xi} + q_{\eta\xi}{}^{\eta\alpha})\sqrt{-g}.$$

Substituting the q-terms according to Eq. (60) and writing the derivative of $\tilde{\mathcal{H}}_0$ with respect to the metric $g_{\alpha\beta}$ as the given system's symmetric energy-momentum tensor according to Eq. (51)

$$2g_{\xi\beta}\frac{\partial\tilde{\mathcal{H}}_0}{\partial g_{\alpha\beta}} = T_\xi{}^\alpha\sqrt{-g},$$

the consistency relation (58) for the Hamiltonian (59) emerges as

$$g_1\left(R_{\eta\beta\lambda\xi}R^{\eta\beta\lambda\alpha} - \frac{1}{4}\delta_\xi^\alpha R_{\eta\beta\lambda\tau}R^{\eta\beta\lambda\tau}\right) + \frac{1}{8\pi G}\left(R_\xi{}^\alpha - \frac{1}{2}\delta_\xi^\alpha R + \Lambda\delta_\xi^\alpha\right) = a_\xi\frac{\partial\mathcal{H}_0}{\partial a_\alpha} - p^{\alpha\beta}\frac{\partial\mathcal{H}_0}{\partial p^{\xi\beta}} + T_\xi{}^\alpha \qquad (61)$$

and thus represents a generalized Einstein equation. The scalar (spin-0) field contributes to the source merely via its energy-momentum tensor terms, whereas the vector (spin-1) field contributes in addition with the first two terms on the right-hand side of Eq. (61). So, for systems with only scalar fields, the right-hand side of Eq. (61) reduces to that of the Einstein equation. Without the term proportional to the coupling constant g_1, the left-hand side of Eq. (61) reduces to the Einstein tensor. The coupling constants g_1 and g_2 contained in (59) can be expressed in terms of the cosmological constant Λ and the gravitational constant G as [20]

$$g_1 = -\frac{3}{16\pi G\Lambda}, \qquad g_2 = \frac{1}{3}\Lambda.$$

Solutions of the field equation for particular systems $\mathcal{H}_0$, namely for Klein-Gordon, Maxwell, and Proca systems, will be discussed in detail in our subsequent paper [18].

Whether or not our theory can provide new insights with respect to the dark matter issue remains to be clarified.

Remarkably, the metrics obtained from Eq. (61) for the exterior regions of nonrotating black holes or rotating black holes coincide with those solving the Einstein equation with cosmological constant. In other words, the vacuum field equation

$$R_{\eta\beta\lambda\xi}R^{\eta\beta\lambda\alpha} - \frac{1}{4}\delta_\xi^\alpha R_{\eta\beta\lambda\tau}R^{\eta\beta\lambda\tau} = 0 \qquad (62)$$

is likewise satisfied not only by the Schwarzschild metric [21], but also by the more general Schwarzschild-De Sitter and the Kerr-De Sitter metrics. Thus, for a vanishing right-hand side of Eq. (61), both parts of the left-hand side, the quadratic part (62) and the "Einstein part," are satisfied by the same metrics. As a consequence, the classical tests of general relativity, namely, the bending of light, the perihelion shift, and the Newtonian limit are equally passed by the solutions of the field Eq. (62). However, the metrics obtained from Eq. (61) for cases where matter/fields are present will be shown to be different from those emerging from the Einstein equation. This changes, for instance, the prediction of measurable observables of neutron stars.

IX. CONCLUSIONS

By means of the framework of canonical transformations, we have demonstrated that any (globally) Lorentz-invariant Lagrangian/Hamiltonian system can be converted into an amended Lagrangian/Hamiltonian which is form-invariant under a general local transformation of the reference frame, following the well-established lines of reasoning of gauge theories. No assumptions or postulates were incorporated into the theory. In particular, our approach includes a nonvanishing torsion of spacetime and is not restricted to the usual assumption of metric compatibility.

Thus, the description of the spacetime dynamics emerges from basic principles only, namely the action principle and the requirement of the form-invariance of the action integral under general spacetime transformations—which ensures the general principle of relativity to be satisfied. The ensuing coupling of spacetime dynamics with matter fields involve the affine connection coefficients, which thus act as gauge quantities. The derivation was worked out in the Hamiltonian framework making use of the canonical transformation formalism—which by construction ensures the action principle to be maintained in its form. The integrand (33) of the final action integral was shown to represent a world scalar density and thereby meets the requirement to be form-invariant under general spacetime transformations.

The reader might wonder about the constraints that arise in conventional Hamiltonian formulations of gauge theories. To address this issue, we must recall a general feature of the covariant (DeDonder-Weyl) Hamiltonian formalism. Generally, if a Hamiltonian in point dynamics does not depend on a dynamical quantity, then the canonical conjugate quantity is a constant of motion. The analogue applies in covariant Hamiltonian field theories. So, in our case of a diffeomorphism invariance, the divergence of the μth column (or row) of the total system's energy-momentum tensor vanishes if $\tilde{\mathcal{H}}_3$ from Eq. (35) does not explicitly depend on x^μ—which is the case for a background-independent system. In this regard, our formalism differs from the standard $3-1$-split Hamiltonian description (see, e.g., [22,23]).

For the closed description of the spacetime dynamics, a Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$ which describes the dynamics of the "free" gauge fields must be postulated. This is a common feature of all gauge theories and reflects here the residual indeterminacy of any gauge theory of gravity. In this sense, we have derived the generic part of the description of geometrodynamics which is common to all specific theories described by a Hamiltonian $\tilde{\mathcal{H}}_0$ that are based on a particular $\tilde{\mathcal{H}}_{\mathrm{Dyn}}$.

Most importantly, we found that in any case spin-1 fields contribute with additional source terms to the equation of motion for the metric—which do not occur for spin-0 fields. Work on extending the theory to half-integer fields is in progress [24]. Furthermore, the canonical formulation of the gauge theory of gravity requires a term quadratic in the canonical momenta $\tilde{q}$ of the gauge fields γ in order for the set of field equations to be closed. This contrasts with the Einstein approach, which is restricted—in its Hamiltonian formulation—to a linear momentum term.

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APPENDIX A: EXPLICIT CALCULATION OF THE TRANSFORMATION RULE (24)

First, we show that the second term on the right-hand side of Eq. (23) vanishes identically. According to the chain rule, we have

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}\right) = \left|\frac{\partial x}{\partial X}\right|^{-1}\left(\frac{\partial^2 x^\alpha}{\partial X^\beta \partial X^\xi}\frac{\partial X^\xi}{\partial x^\alpha} - \left|\frac{\partial x}{\partial X}\right|^{-1}\frac{\partial\left|\frac{\partial x}{\partial X}\right|}{\partial X^\beta}\right). \tag{A1}$$

By virtue of the general identity for the derivative of the determinant of a matrix with respect to a matrix element

$$\frac{\partial\left|\frac{\partial x}{\partial X}\right|}{\partial\left(\frac{\partial x^\alpha}{\partial X^\xi}\right)} = \frac{\partial X^\xi}{\partial x^\alpha}\left|\frac{\partial x}{\partial X}\right|,$$

the X^β-derivative of $\ln|\partial x/\partial X|$ in Eq. (A1) is converted into

$$\left|\frac{\partial x}{\partial X}\right|^{-1}\frac{\partial\left|\frac{\partial x}{\partial X}\right|}{\partial X^\beta} = \left|\frac{\partial x}{\partial X}\right|^{-1}\frac{\partial\left|\frac{\partial x}{\partial X}\right|}{\partial\left(\frac{\partial x^\alpha}{\partial X^\xi}\right)}\frac{\partial\left(\frac{\partial x^\alpha}{\partial X^\xi}\right)}{\partial X^\beta} = \frac{\partial X^\xi}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial X^\xi \partial X^\beta}. \tag{A2}$$

Inserting Eq. (A2) into (A1) then yields

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial X^\beta}\left|\frac{\partial x}{\partial X}\right|^{-1}\right) \equiv 0. \tag{A3}$$

In order to express the third term on the right-hand side of Eq. (23), the partial derivatives are first of all written in expanded form

$$\begin{aligned}\left.\frac{\partial \bar{\tilde{\mathcal{F}}}{}_2^\mu}{\partial x^\mu}\right|_{\mathrm{expl}} = \left.\frac{\partial \tilde{\mathcal{F}}_2^\mu}{\partial x^\mu}\right|_{\mathrm{expl}} + \tilde{Q}_\eta{}^{\alpha\xi\beta}\Bigg[&\gamma^k{}_{ij}\left(\frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial x^n}{\partial X^\beta}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi} + \frac{\partial^2 x^i}{\partial X^\alpha \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi} + \frac{\partial^2 x^j}{\partial X^\xi \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\right) \\ &+ \frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\frac{\partial x^n}{\partial X^\beta} + \frac{\partial^3 x^k}{\partial X^\alpha \partial X^\xi \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\Bigg]\left|\frac{\partial x}{\partial X}\right|^{-1}.\end{aligned} \tag{A4}$$

This expression is now split into a skew-symmetric and a symmetric part of $\tilde{Q}_\eta{}^{\alpha\xi\beta}$ in the indices ξ and β according to

$$\tilde{Q}_\eta{}^{\alpha\xi\beta} = \frac{1}{2}\left(\tilde{Q}_\eta{}^{\alpha\xi\beta} - \tilde{Q}_\eta{}^{\alpha\beta\xi}\right) + \frac{1}{2}\left(\tilde{Q}_\eta{}^{\alpha\xi\beta} + \tilde{Q}_\eta{}^{\alpha\beta\xi}\right) = \tilde{Q}_\eta{}^{\alpha[\xi\beta]} + \tilde{Q}_\eta{}^{\alpha(\xi\beta)}.$$

For the skew-symmetric part, $\tilde{Q}_\eta{}^{\alpha[\xi\beta]}$, the two terms in (A4) which are symmetric in ξ and β vanish, leaving

$$\begin{aligned}
&\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\left[\gamma^{k}{}_{ij}\left(\frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial x^n}{\partial X^\beta}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}+\frac{\partial^2 x^i}{\partial X^\alpha \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi}\right)+\frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial x^n}{\partial X^\beta}\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right]\\
&=\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\left[\frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial x^n}{\partial X^\beta}\left(\gamma^{k}{}_{ij}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}+\frac{\partial^2 x^k}{\partial X^\alpha \partial X^\xi}\right)+\gamma^{k}{}_{ij}\frac{\partial^2 x^i}{\partial X^\alpha \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi}\right]\\
&=\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\left[\Gamma^{j}{}_{\alpha\xi}\frac{\partial^2 X^\eta}{\partial x^k \partial x^n}\frac{\partial x^k}{\partial X^j}\frac{\partial x^n}{\partial X^\beta}+\gamma^{k}{}_{ij}\frac{\partial^2 x^i}{\partial X^\alpha \partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi}\right]\\
&=\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\left[\Gamma^{j}{}_{\alpha\xi}\left(\gamma^{i}{}_{kn}\frac{\partial X^\eta}{\partial x^i}\frac{\partial x^k}{\partial X^j}\frac{\partial x^n}{\partial X^\beta}-\Gamma^{\eta}{}_{j\beta}\right)+\gamma^{k}{}_{ij}\left(\Gamma^{a}{}_{\alpha\beta}\frac{\partial x^i}{\partial X^a}-\gamma^{i}{}_{ab}\frac{\partial x^a}{\partial X^\alpha}\frac{\partial x^b}{\partial X^\beta}\right)\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi}\right]\\
&=\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\left(-\Gamma^{i}{}_{\alpha\xi}\Gamma^{\eta}{}_{i\beta}-\gamma^{i}{}_{ab}\gamma^{k}{}_{ij}\frac{\partial x^a}{\partial X^\alpha}\frac{\partial x^b}{\partial X^\beta}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\xi}+\Gamma^{j}{}_{\alpha\xi}\gamma^{i}{}_{kn}\frac{\partial X^\eta}{\partial x^i}\frac{\partial x^k}{\partial X^j}\frac{\partial x^n}{\partial X^\beta}+\Gamma^{j}{}_{\alpha\beta}\gamma^{i}{}_{kn}\frac{\partial X^\eta}{\partial x^i}\frac{\partial x^k}{\partial X^j}\frac{\partial x^n}{\partial X^\xi}\right)\\
&=-\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\Gamma^{i}{}_{\alpha\xi}\Gamma^{\eta}{}_{i\beta}+\gamma^{i}{}_{ab}\gamma^{k}{}_{ij}\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\frac{\partial x^a}{\partial X^\alpha}\frac{\partial x^b}{\partial X^\xi}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^j}{\partial X^\beta}\\
&=-\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\Gamma^{i}{}_{\alpha\xi}\Gamma^{\eta}{}_{i\beta}+\tilde{q}_{k}{}^{a[bj]}\gamma^{i}{}_{ab}\gamma^{k}{}_{ij}\left|\frac{\partial x}{\partial X}\right|\\
&=-\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}\Gamma^{k}{}_{\alpha\xi}\Gamma^{\eta}{}_{k\beta}+\tilde{q}_{\eta}{}^{\alpha[\xi\beta]}\gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\beta}\left|\frac{\partial x}{\partial X}\right|\\
&=-\frac{1}{2}\tilde{Q}_{\eta}{}^{\alpha\xi\beta}(\Gamma^{k}{}_{\alpha\xi}\Gamma^{\eta}{}_{k\beta}-\Gamma^{k}{}_{\alpha\beta}\Gamma^{\eta}{}_{k\xi})+\frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}(\gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\beta}-\gamma^{k}{}_{\alpha\beta}\gamma^{\eta}{}_{k\xi})\left|\frac{\partial x}{\partial X}\right|.
\end{aligned}$$

The two mixed terms in Γ and γ cancel each other due to the skew-symmetry of $\tilde{Q}_{\eta}{}^{\alpha[\xi\beta]}$ in ξ and β.

The contribution of (23) emerging from the *symmetric* part $\tilde{Q}_{\eta}{}^{\alpha(\xi\beta)}$ can be expressed in terms of the derivatives of the connection coefficients, whose transformation rule is

$$\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\kappa}\frac{\partial X^\kappa}{\partial x^n}=\frac{\partial\gamma^{k}{}_{ij}}{\partial x^n}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}+\gamma^{k}{}_{ij}\frac{\partial}{\partial x^n}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\right)+\frac{\partial}{\partial x^n}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha\partial X^\xi}\right).$$

Thus

$$\begin{aligned}
&\tilde{Q}_{\eta}{}^{\alpha(\xi\beta)}\frac{\partial x^n}{\partial X^\beta}\left[\gamma^{k}{}_{ij}\frac{\partial}{\partial x^n}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\right)+\frac{\partial}{\partial x^n}\left(\frac{\partial X^\eta}{\partial x^k}\frac{\partial^2 x^k}{\partial X^\alpha\partial X^\xi}\right)\right]\\
&=\tilde{Q}_{\eta}{}^{\alpha(\xi\beta)}\frac{\partial x^n}{\partial X^\beta}\left(\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\kappa}\frac{\partial X^\kappa}{\partial x^n}-\frac{\partial\gamma^{k}{}_{ij}}{\partial x^n}\frac{\partial X^\eta}{\partial x^k}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\xi}\right)\\
&=\tilde{Q}_{\eta}{}^{\alpha(\xi\beta)}\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\beta}-\tilde{q}_{k}{}^{i(jn)}\frac{\partial\gamma^{k}{}_{ij}}{\partial x^n}\left|\frac{\partial x}{\partial X}\right|\\
&=\tilde{Q}_{\eta}{}^{\alpha(\xi\beta)}\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\beta}-\tilde{q}_{\eta}{}^{\alpha(\xi\beta)}\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^\beta}\left|\frac{\partial x}{\partial X}\right|\\
&=\frac{1}{2}\tilde{Q}_{\eta}{}^{\alpha\xi\beta}\left(\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\beta}+\frac{\partial\Gamma^{\eta}{}_{\alpha\beta}}{\partial X^\xi}\right)-\frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}\left(\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^\beta}+\frac{\partial\gamma^{\eta}{}_{\alpha\beta}}{\partial x^\xi}\right)\left|\frac{\partial x}{\partial X}\right|.
\end{aligned}$$

The total transformation rule (A4) expressed in terms of connection coefficients is then

$$\begin{aligned}
\left.\frac{\partial\bar{\tilde{\mathcal{F}}}_2^\mu}{\partial x^\mu}\right|_{\mathrm{expl}}\left|\frac{\partial x}{\partial X}\right|=\left.\frac{\partial\tilde{\mathcal{F}}_2^\mu}{\partial x^\mu}\right|_{\mathrm{expl}}\left|\frac{\partial x}{\partial X}\right|&+\frac{1}{2}\tilde{Q}_{\eta}{}^{\alpha\xi\beta}\left(\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^\beta}+\frac{\partial\Gamma^{\eta}{}_{\alpha\beta}}{\partial X^\xi}-\Gamma^{k}{}_{\alpha\xi}\Gamma^{\eta}{}_{k\beta}+\Gamma^{k}{}_{\alpha\beta}\Gamma^{\eta}{}_{k\xi}\right)\\
&-\frac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}\left(\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^\beta}+\frac{\partial\gamma^{\eta}{}_{\alpha\beta}}{\partial x^\xi}-\gamma^{k}{}_{\alpha\xi}\gamma^{\eta}{}_{k\beta}+\gamma^{k}{}_{\alpha\beta}\gamma^{\eta}{}_{k\xi}\right)\left|\frac{\partial x}{\partial X}\right|.
\end{aligned}$$

APPENDIX B: EXPLICIT CALCULATION OF THE CONSISTENCY EQUATION (58)

Equation (57) reads, in explicit form:

$$0=\frac{\partial\tilde{p}^{\alpha\beta}}{\partial x^{\beta}}a_{\xi}+\tilde{p}^{\alpha\beta}\frac{\partial a_{\xi}}{\partial x^{\beta}}+2\frac{\partial\tilde{k}^{\lambda\alpha\beta}}{\partial x^{\beta}}g_{\lambda\xi}+2\tilde{k}^{\lambda\alpha\beta}\frac{\partial g_{\lambda\xi}}{\partial x^{\beta}}+\frac{\partial\tilde{q}_{\eta}{}^{\alpha\lambda\beta}}{\partial x^{\beta}}\gamma^{\eta}{}_{\xi\lambda}+\tilde{q}_{\eta}{}^{\alpha\lambda\beta}\frac{\partial\gamma^{\eta}{}_{\xi\lambda}}{\partial x^{\beta}}-\frac{\partial\tilde{q}_{\xi}{}^{\eta\lambda\beta}}{\partial x^{\beta}}\gamma^{\alpha}{}_{\eta\lambda}-\tilde{q}_{\xi}{}^{\eta\lambda\beta}\frac{\partial\gamma^{\alpha}{}_{\eta\lambda}}{\partial x^{\beta}}.$$

The partial derivative representations (45), (47), (49), (52), and (54) of the canonical field equations can now be inserted to replace all derivatives with respect to x^{β}, which yields

$$\begin{aligned}0=&\left(-\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial a_{\alpha}}-\tilde{p}^{\eta\beta}\gamma^{\alpha}{}_{\eta\beta}\right)a_{\xi}+\tilde{p}^{\alpha\beta}\left(\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial\tilde{p}^{\xi\beta}}+a_{\eta}\gamma^{\eta}{}_{\xi\beta}\right)\\&-2\left(\tilde{k}^{\eta\alpha\beta}\gamma^{\lambda}{}_{\eta\beta}+\tilde{k}^{\lambda\eta\beta}\gamma^{\alpha}{}_{\eta\beta}+\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial g_{\lambda\alpha}}+\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial g_{\lambda\alpha}}\right)g_{\lambda\xi}+2\tilde{k}^{\lambda\alpha\beta}\left(g_{\eta\xi}\gamma^{\eta}{}_{\lambda\beta}+g_{\lambda\eta}\gamma^{\eta}{}_{\xi\beta}+\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{k}^{\lambda\xi\beta}}\right)\\&-(\tilde{p}^{\alpha\lambda}a_{\eta}+2\tilde{k}^{\beta\alpha\lambda}g_{\beta\eta}+\tilde{q}_{\beta}{}^{\alpha\tau\lambda}\gamma^{\beta}{}_{\eta\tau}-\tilde{q}_{\eta}{}^{\tau\beta\lambda}\gamma^{\alpha}{}_{\tau\beta})\gamma^{\eta}{}_{\xi\lambda}+\tilde{q}_{\eta}{}^{\alpha\lambda\beta}\left(\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\eta}{}^{\xi\lambda\beta}}+\gamma^{\tau}{}_{\xi\beta}\gamma^{\eta}{}_{\tau\lambda}\right)\\&+(\tilde{p}^{\eta\lambda}a_{\xi}+2\tilde{k}^{\tau\eta\lambda}g_{\tau\xi}-\tilde{q}_{\beta}{}^{\eta\lambda\tau}\gamma^{\beta}{}_{\xi\tau}+\tilde{q}_{\xi}{}^{\tau\lambda\beta}\gamma^{\eta}{}_{\tau\beta})\gamma^{\alpha}{}_{\eta\lambda}-\tilde{q}_{\xi}{}^{\eta\lambda\beta}\left(\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\alpha}{}^{\eta\lambda\beta}}+\gamma^{\tau}{}_{\eta\beta}\gamma^{\alpha}{}_{\tau\lambda}\right).\end{aligned}$$

All terms which do not depend on the Hamiltonians cancel, as can be seen after rearranging and relabeling some running indices

$$\begin{aligned}0=&-\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial a_{\alpha}}a_{\xi}+\tilde{p}^{\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial\tilde{p}^{\xi\beta}}-2\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial g_{\lambda\alpha}}g_{\lambda\xi}-2\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial g_{\lambda\alpha}}g_{\lambda\xi}+2\tilde{k}^{\lambda\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{k}^{\lambda\xi\beta}}+\tilde{q}_{\eta}{}^{\alpha\lambda\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\eta}{}^{\xi\lambda\beta}}-\tilde{q}_{\xi}{}^{\eta\lambda\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\alpha}{}^{\eta\lambda\beta}}\\&-\tilde{p}^{\eta\beta}a_{\xi}\gamma^{\alpha}{}_{\eta\beta}+\tilde{p}^{\eta\lambda}a_{\xi}\gamma^{\alpha}{}_{\eta\lambda}+\tilde{p}^{\alpha\beta}a_{\eta}\gamma^{\eta}{}_{\xi\beta}-\tilde{p}^{\alpha\lambda}a_{\eta}\gamma^{\eta}{}_{\xi\lambda}\\&-2\tilde{k}^{\eta\alpha\beta}g_{\lambda\xi}\gamma^{\lambda}{}_{\eta\beta}+2\tilde{k}^{\lambda\alpha\beta}g_{\eta\xi}\gamma^{\eta}{}_{\lambda\beta}-2\tilde{k}^{\lambda\eta\beta}g_{\lambda\xi}\gamma^{\alpha}{}_{\eta\beta}+2\tilde{k}^{\tau\eta\lambda}g_{\tau\xi}\gamma^{\alpha}{}_{\eta\lambda}+2\tilde{k}^{\lambda\alpha\beta}g_{\lambda\eta}\gamma^{\eta}{}_{\xi\beta}-2\tilde{k}^{\beta\alpha\lambda}g_{\beta\eta}\gamma^{\eta}{}_{\xi\lambda}\\&-\tilde{q}_{\beta}{}^{\alpha\tau\lambda}\gamma^{\eta}{}_{\xi\lambda}\gamma^{\beta}{}_{\eta\tau}+\tilde{q}_{\eta}{}^{\alpha\lambda\beta}\gamma^{\tau}{}_{\xi\beta}\gamma^{\eta}{}_{\tau\lambda}+\tilde{q}_{\eta}{}^{\tau\beta\lambda}\gamma^{\eta}{}_{\xi\lambda}\gamma^{\alpha}{}_{\tau\beta}-\tilde{q}_{\beta}{}^{\eta\lambda\tau}\gamma^{\beta}{}_{\xi\tau}\gamma^{\alpha}{}_{\eta\lambda}+\tilde{q}_{\xi}{}^{\tau\lambda\beta}\gamma^{\eta}{}_{\tau\beta}\gamma^{\alpha}{}_{\eta\lambda}-\tilde{q}_{\xi}{}^{\eta\lambda\beta}\gamma^{\tau}{}_{\eta\beta}\gamma^{\alpha}{}_{\tau\lambda}.\end{aligned}$$

The remaining terms constitute the second rank tensor equation

$$\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial a_{\alpha}}a_{\xi}-\tilde{p}^{\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial\tilde{p}^{\xi\beta}}+2\frac{\partial\tilde{\mathcal{H}}_{0}}{\partial g_{\lambda\alpha}}g_{\lambda\xi}=-2\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial g_{\lambda\alpha}}g_{\lambda\xi}+2\tilde{k}^{\lambda\alpha\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{k}^{\lambda\xi\beta}}+\tilde{q}_{\eta}{}^{\alpha\lambda\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\eta}{}^{\xi\lambda\beta}}-\tilde{q}_{\xi}{}^{\eta\lambda\beta}\frac{\partial\tilde{\mathcal{H}}_{\mathrm{Dyn}}}{\partial\tilde{q}_{\alpha}{}^{\eta\lambda\beta}}.$$

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