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Exact helicoidal and catenoidal solutions in five- and higher-dimensional Einstein-Maxwell theory

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We present several new exact solutions in five- and higher-dimensional Einstein-Maxwell theory by embedding the Nutku instanton. The metric functions for the five-dimensional solutions depend only on a radial coordinate and on two spatial coordinates for the six- and higher-dimensional solutions. The six- and higher-dimensional metric functions are convolutedlike integrals of two special functions. We find that the solutions are regular almost everywhere and some spatial sections of the solution describe wormhole handles. We also find a class of exact and nonstationary convolutedlike solutions to the Einstein-Maxwell theory with a cosmological constant.

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I. INTRODUCTION

Gravitational instantons are the regular and complete Euclidean signature solutions, with self-dual curvature 2-form, to the Einstein field equations in vacuum [1], or with a cosmological constant term [2]. There are several well-known solutions such as Taub-Newman, Tamburino, and Unti [3], Eguchi-Hanson [4], and Atiyah-Hitchin metrics [5]. The instanton solutions, in general, are the result of the reduction of the complex elliptic Monge-Ampère equation [6] on a complex manifold of dimension 2 to only one real variable. These solutions play an important role in the construction of higher-dimensional solutions to extended theories of gravity [7] and supergravity [8,9] as well as quantum properties of the black holes [10]. In fact, these self-dual geometries have been used in Ref. [11] to construct the M-theory realizations of the fully localized D2 branes in type-IIA string theory that intersect the D6 branes. The M-theory solutions involve the convolutedlike integrals of two special functions and upon compactification over a compact coordinate of the transverse self-dual geometry yield the supersymmetric solutions for the fully localized intersecting branes. One main feature of the solutions is that they are valid near and far from the core of D6 branes. Moreover, inspired by the convolutedlike structure of membrane solutions, some new convoluted solutions to the six- and higher-dimensional Einstein-Maxwell theory were constructed and studied in Ref. [12]. The convoluted solutions even can be generalized to include the cosmological constant in six- and higher-dimensional Einstein-Maxwell theory.

The gravitational instantons are also the dominant metrics in path-integral formulation of Euclidean quantum gravity and are closely related to minimal surfaces in Eulidean space [13]. In fact, the equations for any two-dimensional minimal surfaces provide a solution to the real elliptic Monge-Ampère equation on a real manifold of two dimensions. These solutions then lead to the Kähler metrics for some gravitational instantons besides the aforementioned well-known solutions.

In this article, we draw attention to gravitational instantons of novel geometries constructed from minimal surfaces by Nutku [13] and embed them to generate new classes of exact helicoidal and catenoidal solutions in five- and higher-dimensional Einstein-Maxwell theory. Inspired by the existence of such gravitational instantons in four dimensions and the well-known methods to construct the higher-dimensional convoluted solutions to the Einstein-Maxwell theory, in this article, we construct and study a new class of exact helicoidal and catenoidal solutions in Einstein-Maxwell theory. We note that, to our knowledge, these solutions which are generated from helicoid and catenoid minimal surfaces have not been studied before in Einstein-Maxwell theory.

The article is organized as follows. We begin with a brief overview of minimal surfaces and instantons and present the metric for the Nutku gravitational instantons in Sec. II. In Sec. III, we embed the Nutku metrics corresponding to helicoid and catenoid cases to get exact solutions in five dimensions. In Sec. IV, we construct convolutionlike general solutions for the Einstein-Maxwell theory in six dimensions and discuss the solutions. In Sec. V, we generalize the convolutedlike solutions for the Einstein-Maxwell theory in any dimensions greater than 6 and discuss the solutions. Finally, in Sec. VI, by using a very special separation of variables in the metric function, we find the most general cosmological solutions to the Einstein-Maxwell theory in the presence of a cosmological constant. We wrap up the article with concluding remarks.

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II. MINIMAL SURFACES AND THE FOUR-DIMENSIONAL NUTKU INSTANTONS

The study of minimal surfaces began with Lagrange's question about the existence of surfaces that minimize the area, subject to some boundary constraints. Physically, they represent soap films on wire frames. There are several equivalent mathematical definitions of minimal surfaces. We state two important ones [14]:

(1) A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature vanishes identically.
(2) A surface $M \subset \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported versions. The plane, the helicoid, the catenoid, the gyroid, the Scherk surface, the Enneper surface, and Costa's surface are some examples [15]. In $\mathbb{R}^3$, the helicoid and the plane are the only ruled 2-surfaces [14], i.e., surfaces generated by the rotation of a line. The helicoid and the catenoid are locally isometric [16] and are conjugates of each other [14].[3]

Observing that nearly all well-known solutions in Einstein-Maxwell theory have spherical symmetry, this article highlights the role of minimal surfaces and instantons in constructing new solutions of novel geometries. Instantons are pseudoparticles which first appeared in Yang-Mills theory as minimum-action, classical solutions in Euclidean[4] spacetime [17]. Their discovery inspired the notion of gravitational instantons, which soon found use in Schwarzschild black hole radiance calculations through Euclideanization [18]. In 1978, Comtet [19] showed that the multi-Belavin, Polyakov, Schwarz and Tyupkin-instanton solution of Witten [20] corresponds to minimal surfaces. In a similar vein, Nutku [13] proved that for every minimal surface in $\mathbb{R}^3$ there is a gravitational instanton with anti-self-dual curvature and gave some explicit metrics which we will embed in higher dimensions.

A class of gravitational instantons may be represented by the metric

$$ds^2 = \frac{1}{P(t,x)}\{(f_t^2+\kappa)(dt^2+dy^2)+(1+f_x^2)(dx^2+dz^2) + 2f_t f_x(dtdx+dydz)\}, \tag{2.1}$$

where $P(t,x)=\sqrt{1+\kappa f_t^2+f_x^2}$, $f_t=\frac{\partial f(t,x)}{\partial t}$ and $f_x=\frac{\partial f(t,x)}{\partial x}$, if the function $f(t,x)$ satisfies the quasilinear, elliptic-hyperbolic partial differential equation

$$(1+f_x^2)f_{tt}-2f_t f_x f_{tx}+(\kappa+f_t^2)f_{xx}=0. \tag{2.2}$$

The "Lagrange equation" (2.2) with $\kappa=1$ defines minimal surfaces in $\mathbb{R}^3$. For $\kappa=-1$, it reduces to the Born-Infeld equation [13], which arises in a nonlinear generalization of Maxwell electrodynamics. These two cases are also related through a Wick rotation ($t\to it$) [21]. Interestingly, the Born-Infeld equation is also related to the maximal surface equation in Lorentz-Minkowski space $\mathbb{L}^3$ by a Wick rotation [22]. Since the general metric above provides a large class of solutions, we will restrict our attention to the helicoid-catenoid solutions. Noting that the helicoid represented by $f(t,x)=a\tan^{-1}(\frac{x}{t})$ is a solution to Eq. (2.2), after substitution and subsequent coordinate transformations $x=r\cos\theta$ and $t=r\sin\theta$, we obtain the Nutku helicoid instanton as

$$ds^2=\frac{dr^2+(a^2+r^2)d\theta^2+(1+\frac{a^2(\sin(\theta))^2}{r^2})dy^2-\frac{a^2\sin(2\theta)dydz}{r^2}+(1+\frac{a^2(\cos(\theta))^2}{r^2})dz^2}{\sqrt{1+\frac{a^2}{r^2}}}, \tag{2.3}$$

where we consider $0<r<\infty$, $0\le\theta\le 2\pi$, and y and z could be considered the periodic coordinates on the 2-torus [23]. In Ref. [24], the authors studied the solutions of the Dirac equation in the background of the metric (2.3) and its singularities. The metric (2.3) is asymptotically Euclidean and would correspond to a catenoidal solution if a^2 were replaced with $-a^2$. From here on, we will use $\epsilon=\pm1$ to differentiate between the helicoid ($\epsilon=1$) and catenoid ($\epsilon=-1$) cases, respectively,

$$ds^2_{Nutku}=\frac{dr^2+(\epsilon a^2+r^2)d\theta^2+(1+\epsilon\frac{a^2(\sin(\theta))^2}{r^2})dy^2-\epsilon\frac{a^2\sin(2\theta)dydz}{r^2}+(1+\epsilon\frac{a^2(\cos(\theta))^2}{r^2})dz^2}{\sqrt{1+\epsilon\frac{a^2}{r^2}}}. \tag{2.4}$$

The Kretchmann invariant of the helicoid or catenoid instanton is given by

$$\mathcal{K}=\frac{72a^4}{r^4(r^2+\epsilon a^2)^2}+\frac{24a^8}{r^6(r^2+\epsilon a^2)^3}. \tag{2.5}$$

[3]Conjugate surfaces have the interesting property that a straight line in one surface can be mapped to a geodesic in the other and vice versa.

[4]Curiously, Wick rotation ($t\to it$) plays an important role in instanton physics in both quantum theory and general relativity.

The helicoid has only a curvature singularity at $r = 0$, while for the catenoid, there is another singularity at $r = a$.

III. HELICOID AND CATENOID SOLUTIONS IN FIVE-DIMENSIONAL EINSTEIN-MAXWELL THEORY

We consider the source-free Einstein-Maxwell equations with geometrized units in D dimensions that are given by

$$R_{\mu\nu} = F^{\lambda}_{\mu}F_{\nu\lambda} - \frac{1}{4+2(D-4)}g_{\mu\nu}F^2, \tag{3.1}$$

$$F^{\mu\nu}_{;\mu} = 0, \tag{3.2}$$

where the electromagnetic field tensor is given by

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \tag{3.3}$$

in terms of the electromagnetic potential A^{μ}. We will take its only nonzero component as

$$A_t = \sqrt{\frac{D-2}{D-3}}\frac{1}{H(r)}. \tag{3.4}$$

In five dimensions, we consider the following five-dimensional (5D) ansatz by adding a time coordinate to the Nutku helicoid or catenoid instanton:

$$\begin{aligned} ds^2 = &-\frac{dt^2}{(H(r))^2} + \frac{H(r)}{\sqrt{1+\frac{a^2}{r^2}}}\left(dr^2 + (a^2+r^2)d\theta^2\right. \\ &+ \left(1+\frac{a^2(\sin(\theta))^2}{r^2}\right)dy^2 - \frac{a^2\sin(2\theta)dydz}{r^2} \\ &\left.+ \left(1+\frac{a^2(\cos(\theta))^2}{r^2}\right)dz^2\right). \end{aligned} \tag{3.5}$$

We find that all Einstein and Maxwell equations are satisfied if the metric function $H(r)$ satisfies the differential equation

$$\epsilon\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}H(r)\right)a^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}H(r)\right)r^2 + \left(\frac{\mathrm{d}}{\mathrm{d}r}H(r)\right)r = 0. \tag{3.6}$$

The solutions to Eq. (3.6) are given by

$$H(r) = c_1 + C_2\ln\left(\frac{r+\sqrt{\epsilon a^2+r^2}}{a}\right), \tag{3.7}$$

where c_1 and C_2 are two constants.

The logarithmic solutions above correspond to $\sinh^{-1}(\frac{r}{a})$ and $\cosh^{-1}(\frac{r}{a})$ functions. Interestingly, these functions appear in the study of collapsing catenoidal soap films [25] and in the embedding of wormhole handles [26]. We note that a submanifold of the 5D helicoidal spacetime is conformally equivalent to a wormhole handle $[dr^2 + (a^2+r^2)d\theta^2]$ [26,27].

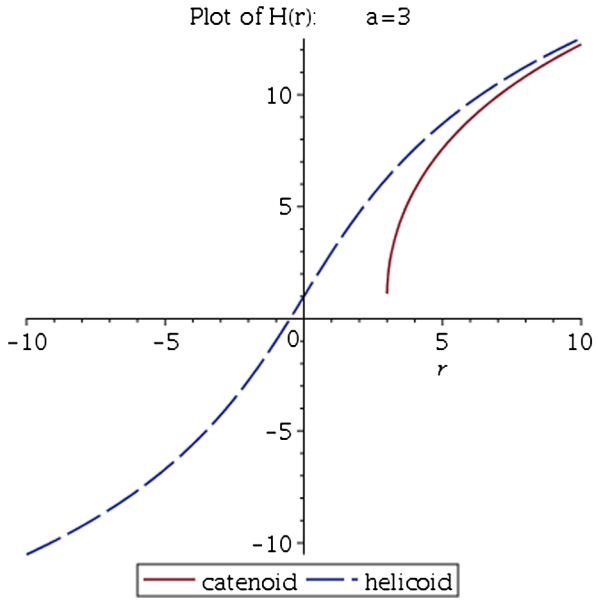

FIG. 1. $H(r)$, for the helicoid and catenoid solutions, with $a = 3$, $c_1 = 1$, $c_2 = 6$.

Inspired by the soap film solutions [25], we set $c_1 = 1$ and $C_2 = c_2 a^5$ so that in the limit $a \to 0$ the metric (3.5) becomes Minkowski spacetime. Thus, we have

$$H(r) = 1 + c_2 a\sinh^{-1}\left(\frac{r}{a}\right), \tag{3.8}$$

$$H(r) = 1 + c_2 a\cosh^{-1}\left(\frac{r}{a}\right), \tag{3.9}$$

for the helicoid and the catenoid solutions, respectively. Figure 1 shows the typical behavior of the metric functions $H(r)$ for helicoid and catenoid solutions. It is important to note here that in case of the helicoid the function $H(r)$ can become negative and hence change the metric signature from Lorentzian to Riemannian at some radial coordinate r. Although such signature changes are of interest in quantum cosmology [28], and may be dealt with through a Wick rotation, we wish to keep the metric well behaved and thus require $a \geq 0$ and $c_2 \geq 0$.

Figure 2 shows how the electric field may or may not diverge in the helicoid spacetime, depending on the choice of constants. The region $r < 0$ is included in the plots only to illustrate this behavior.

[5]In fact, we may even get rid of the constant parameter c_2 by setting it to be $c_2 = 2$ in order to match the height function for a catenoidal soap film. It is useful to keep c_2 for now to illustrate what it may do to the electric field.

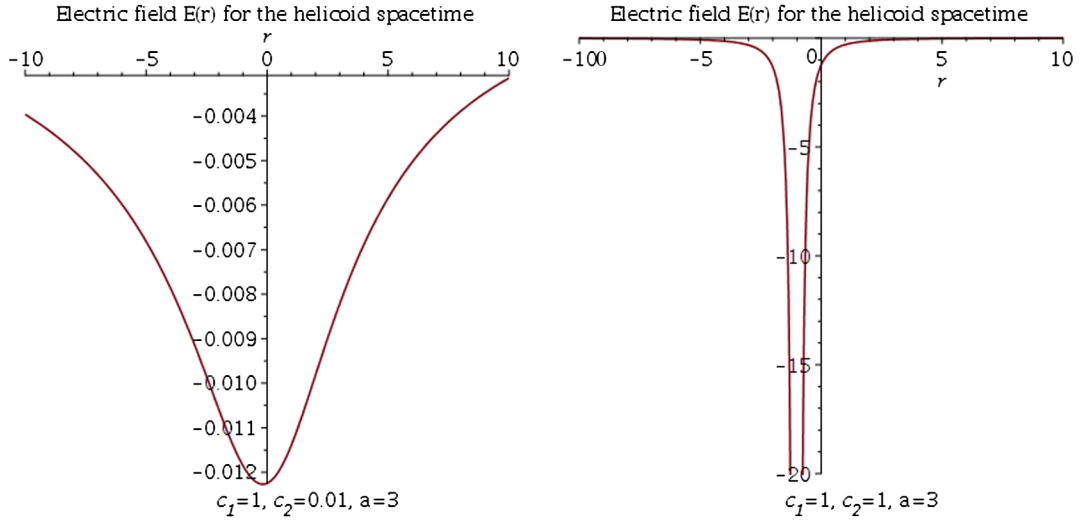

FIG. 2. The r component of the electric field as a function of r, for the helicoid solution with $a = 3$, $c_1 = 1$, $c_2 = 0.01$ (left) and $a = 2$, $c_1 = 1$, $c_2 = 1$ (right).

The electric field for the catenoid spacetime ($a < r < \infty$) is shown in Fig. 3.

If we replace $H(r)$ with $H(t, r)$ in the metric ansatz (3.5), we can get a dynamic solution to the Einstein-Maxwell equations with a cosmological constant Λ, which is given by

$$H(t,r) = 1 \pm \sqrt{\Lambda}t + c_2 a \ln\left(\frac{r + \sqrt{\epsilon a^2 + r^2}}{a}\right). \tag{3.10}$$

Again, if we are to avoid the metric signature change while keeping a and c_2 positive, we should only consider the $+\sqrt{\Lambda}t$ solution. Figure 4 illustrates the electric field as a function of r and t for the helicoid and catenoid spacetimes with a cosmological constant.

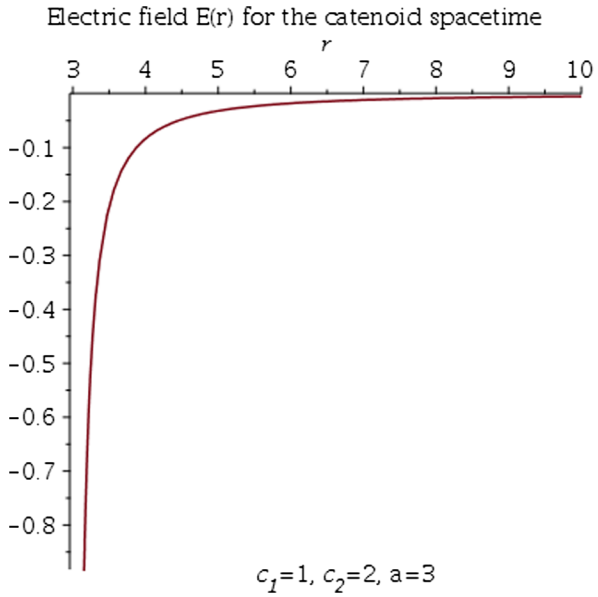

FIG. 3. $E_r(r)$ for the catenoid solution with $a = 3$, $c_1 = 1$, $c_2 = 2$.

IV. CONVOLUTED SOLUTIONS IN SIX-DIMENSIONAL EINSTEIN-MAXWEL THEORY BASED ON EMBEDDED NUTKU INSTANTONS

We consider the six-dimensional metric ansatz,

$$ds^2 = -H(r,x)^{-2}dt^2 + H(r,x)^{2/3}(dx^2 + ds^2_{\text{Nutku}}), \tag{4.1}$$

where ds^2_{Nutku} is the four-dimensional Nutku space with parameter a which is given by Eq. (2.4). We also consider the Maxwell gauge field as

$$A_t(r,x) = \frac{2\sqrt{3}}{3H(r,x)}. \tag{4.2}$$

The gravitational Einstein's field equations and the Maxwell equations provide that the metric function $H(r, x)$ must satisfy the partial differential equation

$$r^2\sqrt{\frac{a^2+r^2}{r^2}}\frac{\partial^2}{\partial x^2}H(r,x) + (r^2+a^2)\left(\frac{\partial^2}{\partial r^2}H(r,x)\right) + \left(\frac{\partial}{\partial r}H(r,x)\right)r = 0. \tag{4.3}$$

We can solve the partial differential equation (4.3) by separating the variables as $H(r, x) = 1 + R(r)X(x)$. We find two ordinary differential equations for $R(x)$ and $X(x)$ that are given by

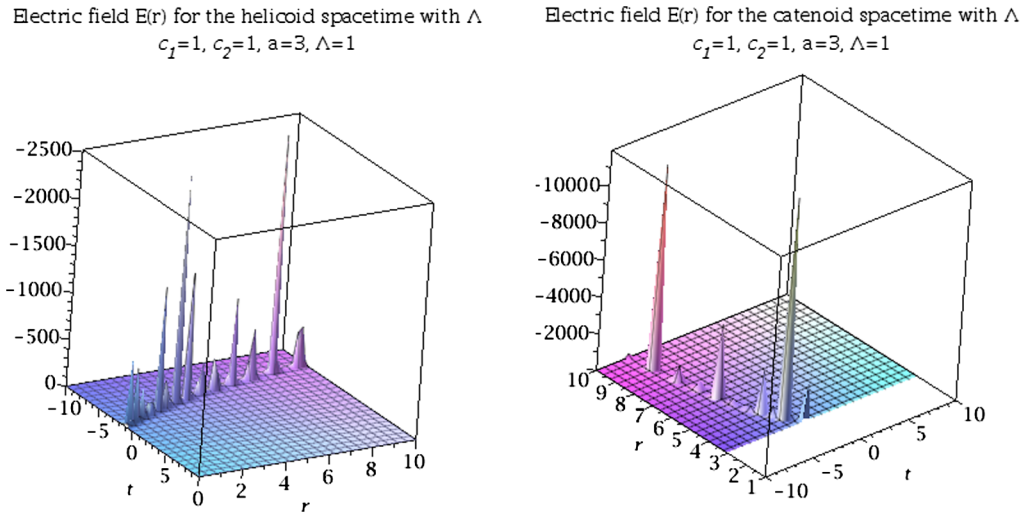

FIG. 4. $E_r(r,t)$ with $a=3$, $c_1=1$, $c_2=1$, $\Lambda=1$, for the helicoid solution (left) and at the catenoid solution (right).

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)+\epsilon c^2 R(r)\left(\sqrt{1+\frac{a^2}{r^2}}\right)^{-1}+\frac{(\frac{\mathrm{d}}{\mathrm{d}r}R(r))r}{a^2+r^2}=0, \tag{4.4}$$

and

$$\frac{d^2}{dx^2}X(x)-\epsilon c^2 X(x)=0, \tag{4.5}$$

respectively. We first consider $\epsilon=+1$, where the solutions to (4.4) are given by

$$\begin{aligned}R(r)=&A_1\mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\\&+A_2\mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\\&\times\int^{\sqrt{\frac{a^2+r^2}{r^2}}}\frac{1}{(f^2-1)(\mathcal{H}_D(0,0,a^2c^2,0,f))^2}df,\end{aligned} \tag{4.6}$$

in terms of Heun-D functions $\mathcal{H}_D$, and A_1 and A_2 are two constants. We note that the Heun-D functions $\mathcal{H}_D(\alpha,\beta,\gamma,\delta,z)$ are the solutions to the Heun double confluent equation

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}z^2}y(z)-\frac{(\alpha z^4-2z^5+4z^3-\alpha-2z)}{(z-1)^3(z+1)^3}\frac{\mathrm{d}}{\mathrm{d}z}y(z)\\&=\frac{(-z^2\beta+(-\gamma-2\alpha)z-\delta)}{(z-1)^3(z+1)^3}y(z),\end{aligned} \tag{4.7}$$

with the boundary conditions $y(0)=1$, $\frac{d}{dz}y(0)=0$. In Fig. 5, we show the typical behavior of the Heun-D function for a few different values of the separation constant c.

The solutions to Eq. (4.5) with $\epsilon=+1$ for $X(x)$ are given by

$$X(x)=B_1\mathrm{e}^{cx}+B_2\mathrm{e}^{-cx}, \tag{4.8}$$

where B_1 and B_2 are two constants. We can superimpose all the solutions (4.6) (where we choose $A_2=0$) and (4.8) with different values for the separation constant c, to construct the most general convoluted solutions to the partial differential equation (4.3), in the form

$$\begin{aligned}H(r,x)=1+\int_0^\infty&\mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\\&\times(f(c)e^{cx}+g(c)e^{-cx})dc,\end{aligned} \tag{4.9}$$

where $f(c)$ and $g(c)$ are two arbitrary functions in terms of separation constant c. To fix the arbitrary functions $f(c)$ and $g(c)$, we note that in the limit of $a\to 0$ the Nutku space describes a four-dimensional space $D^2\otimes T^2$ with the line element

$$ds^2_{a=0}=dr^2+r^2d\theta^2+dy^2+dz^2. \tag{4.10}$$

Quite interestingly, in this limit, we find an exact analytical solution to six-dimensional Einstein-Maxwell theory with the line element

$$d\mathcal{S}^2=-H_0(r,x)^{-2}dt^2+H_0(r,x)^{2/3}(dx^2+ds^2_{a=0}) \tag{4.11}$$

and the Maxwell gauge field

$$\mathcal{A}_t(r,x)=\frac{2\sqrt{3}}{3H_0(r,x)}, \tag{4.12}$$

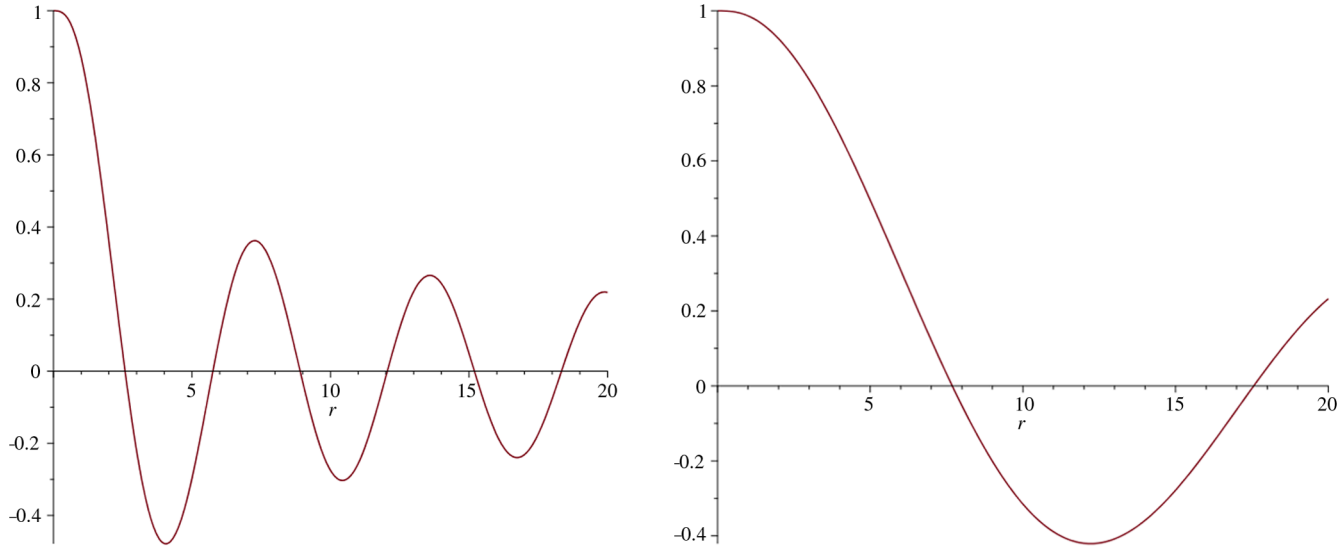

FIG. 5. The function $\mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)$ vs r, where we set $a=1$, $c^2=1$ (left) and $a=1$, $c^2=0.1$ (right).

where the exact analytic metric function $H_0(r,x)$ is

$$H_0(r,x)=1+\frac{h_0}{\sqrt{r^2+x^2}} \tag{4.13}$$

and h_0 is a constant. We can now fix the functions $f(c)$ and $g(c)$ by requiring that the metric (4.1) and the gauge field (4.2) must approach the exact analytical metric (4.11) and the gauge field (4.12), respectively, in the limit of $a\to 0$. These requirements imply that the convoluted metric function $H(r,x)$ in Eq. (4.9) must be equal to the exact analytic metric function $H_0(r,x)$ in Eq. (4.13), in the limit of $a\to 0$. The integrand of the convoluted metric function $H(r,x)$ in Eq. (4.9) contains the Heun-D function, which is the solution to the differential equation (4.4) with $\epsilon=+1$. This equation in the limit of $a\to 0$ reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)+c^2R(r)+\frac{\frac{\mathrm{d}}{\mathrm{d}r}R(r)}{r}=0, \tag{4.14}$$

for which the solutions are given by the Bessel functions $J_0(cr)$ and $Y_0(cr)$. The Bessel function $Y_0(cr)$ does not provide an oscillatory decaying behavior similar to that of Heun-D functions. However, the Bessel function $J_0(cr)$ provides such a desired behavior.

So, we find an integral equation for the functions $f(c)$ and $g(c)$, which is

$$\int_0^\infty J_0(cr)(f(c)e^{cx}+g(c)e^{-cx})dc=\frac{h_0}{\sqrt{r^2+x^2}}. \tag{4.15}$$

We find that the unique solutions to this integral equation for the functions $f(c)$ and $g(c)$ are given by

$$f(c)=\frac{h_0}{2}\xi_1, \qquad g(c)=\frac{h_0}{2}\xi_2, \tag{4.16}$$

where ξ_1 and ξ_2 are constants and $\xi_1+\xi_2=1$. Furnished by all the necessary results, we have the most general solution for the convoluted metric function $H(r,x)$, which is given by

$$H(r,x)=1+\frac{h_0}{2}\int_0^\infty \mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\times(\xi e^{cx}+(1-\xi)e^{-cx})dc, \tag{4.17}$$

where h_0 and ξ are two constants.

V. CONVOLUTED SOLUTIONS IN HIGHER-DIMENSIONAL EINSTEIN-MAXWELL THEORY BASED ON EMBEDDED NUTKU INSTANTONS

In this section, we find the general convoluted solutions for the embedding of Nutku geometry in higher-than-six-dimensional Einstein-Maxwell theory. We consider the metric in D dimensions as

$$ds^2=-H(r,x)^{-2}dt^2+H(r,x)^{\frac{2}{D-3}}(dx^2+x^2d\Omega^2_{D-6}+ds^2_{\mathrm{Nutku}}), \tag{5.1}$$

where $d\Omega^2_{D-6}$ shows the metric on a $(D-6)$-dimensional unit sphere. We consider the gauge field with the nonzero component

$$A_t=\sqrt{\frac{D-2}{D-3}}\frac{1}{H(r,x)}. \tag{5.2}$$

We find that all the D-dimensional Einstein and Maxwell equations are satisfied if the metric function

$H(r,x)$ provides a solution to the partial differential equation

$$r\sqrt{a^2+r^2}\left(\frac{\partial^2}{\partial x^2}H(r,x)+\frac{D-6}{x}\frac{\partial}{\partial x}H(r,x)\right)+(r^2+a^2)\left(\frac{\partial^2}{\partial r^2}H(r,x)\right)+\left(\frac{\partial}{\partial r}H(r,x)\right)r=0. \tag{5.3}$$

As in the six-dimensional case, we separate the coordinates in $H(r,x)$ by $H(r,x)=1+R(r)X(x)$. The partial differential equation (5.3) then separates into two ordinary differential equations for $R(r)$ and $X(x)$. The differential equation for the radial function $R(r)$ is given by (4.4) where the solutions are given in terms of Heun-D functions and the differential equation for $X(x)$ is

$$\frac{d^2}{dx^2}X(x)-\epsilon c^2X(x)+\frac{D-6}{x}\frac{dX(x)}{dx}=0. \tag{5.4}$$

The solutions to (5.4) with $\epsilon=1$ for $D>6$ are given by

$$X(x)=x^{\frac{7-D}{2}}I_{\frac{D-7}{2}}(cx)+x^{\frac{7-D}{2}}K_{\frac{D-7}{2}}(cx), \tag{5.5}$$

in terms of modified Bessel functions. As a result, we can write the most convoluted solution for the metric function in D dimensions $H_D(r,x)$ as

$$H_D(r,x)=1+\int_0^\infty \mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\times\left(f_D(c)I_{\frac{D-7}{2}}(cx)+g_D(c)K_{\frac{D-7}{2}}(cx)\right)x^{\frac{7-D}{2}}dc, \tag{5.6}$$

where $f_D(c)$ and $g_D(c)$ are two arbitrary functions in terms of separation constant c. We need to fix these two arbitrary functions to find a closed form for the metric function $H(r,x)$ in D dimensions. As in the case of six-dimensional theory, we consider the limit of $a\to 0$, where the Nutku space reduces to $D^2\otimes T^2$, with the line element (4.10). In this limit, we find an exact analytical solutions to D-dimensional Einstein-Maxwell theory with the line element

$$d\mathcal{S}^2=-H_{D0}(r,x)^{-2}dt^2+H_{D0}(r,x)^{\frac{2}{D-3}}\times\left(dx^2+x^2d\Omega^2_{D-6}+ds^2_{a=0}\right) \tag{5.7}$$

and the Maxwell gauge field

$$\mathcal{A}_t(r,x)=\sqrt{\frac{D-2}{D-3}}\frac{1}{H_{D0}(r,x)}, \tag{5.8}$$

where the exact analytic metric function $H_{D0}(r,x)$ is

$$H_{D0}(r,x)=1+\frac{h_{D0}}{(r^2+x^2)^{\frac{D-5}{2}}}, \tag{5.9}$$

where h_{D0} is a constant. To fix the functions $f_D(c)$ and $g_D(c)$, we demand that the metric (5.1) and the gauge field (5.2) must reduce to the exact analytical metric (5.7) and the gauge field (5.8), respectively, in the limit of $a\to 0$. In other words, these requirements imply that the convoluted metric function $H_D(r,x)$ in Eq. (5.6) must be equal to the exact analytic metric function $H_{D0}(r,x)$ in Eq. (5.9), in the limit of $a\to 0$. As in the case of six-dimensional solutions, the integrand of the convoluted metric function $H_D(r,x)$ in Eq. (5.6) contains the Heun-D function, which is the solution to the differential equation (4.4) with $\epsilon=+1$. This equation in the limit of $a\to 0$ reduces to Eq. (4.14), for which the solutions are given by the Bessel functions $J_0(cr)$ and $Y_0(cr)$.

As a result, we find an integral equation for the functions $f_D(c)$ and $g_D(c)$, which is

$$\int_0^\infty J_0(cr)\left(f_D(c)I_{\frac{D-7}{2}}(cx)+g_D(c)I_{\frac{D-7}{2}}(cx)\right)x^{\frac{7-D}{2}}dc=\frac{h_{D0}}{(r^2+x^2)^{\frac{D-5}{2}}}. \tag{5.10}$$

After lengthy calculations, we find that the unique solutions to this integral equation for the functions $f_D(c)$ and $g_D(c)$ are given by

$$f_D(c)=0,\qquad g_D(c)=\alpha_D h_{D0}c^{\frac{D-5}{2}}, \tag{5.11}$$

where α_D's are given by $\alpha_7=1,\alpha_8=\sqrt{\frac{2}{\pi}},\alpha_9=\frac{1}{2}$, $\alpha_{10}=\frac{1}{3}\sqrt{\frac{2}{\pi}},\cdots$. So, we can write the most general solution for the convoluted metric function $H_D(r,x)$, which is given by

$$H_D(r,x)=1+\alpha_D h_{D0}\int_0^\infty \mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right)\times K_{\frac{D-7}{2}}(cx)x^{\frac{7-D}{2}}c^{\frac{D-5}{2}}dc. \tag{5.12}$$

We also note that asymptotically a two-dimensional submanifold of our solutions (4.1) and (5.1) represents the handle of an Ellis wormhole [27].

VI. COSMOLOGICAL CONVOLUTED SOLUTIONS TO THE EINSTEIN-MAXWELL THEORY WITH A COSMOLOGICAL CONSTANT

We consider the Einstein-Maxwell theory with a cosmological constant in six dimensions, where the metric function H also depends explicitly on time coordinate

$$ds^2 = -H(t,r,x)^{-2}dt^2 + H(t,r,x)^{2/3}(dx^2 + ds^2_{\text{Nutku}}). \tag{6.1}$$

We also consider the Maxwell gauge field as

$$A_t(t,r,x) = \frac{2\sqrt{3}}{3H(t,r,x)}. \tag{6.2}$$

The Einstein equations and Maxwell equations in the presence of a cosmological constant lead to a second-order partial differential equation for $H(t,r,x)$, which is given by

$$r\left(\left(\frac{5}{3}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{3\Lambda}{2}\right)H^{5/3} + \left(\frac{\partial^2 H}{\partial t^2}\right)H^{8/3} - \frac{\partial^2 H}{\partial x^2}\right) \times \sqrt{a^2+r^2} - (a^2+r^2)\frac{\partial^2 H}{\partial r^2} - r\frac{\partial H}{\partial r} = 0. \tag{6.3}$$

The form of partial differential equation (6.3) leads us to separate the coordinates as

$$H(t,r,x) = 1 + T(t) + R(r)X(x). \tag{6.4}$$

We find three ordinary uncoupled differential equations for $R(r)$, $X(x)$, and $T(t)$ functions. The partial differential equations for $R(r)$ and $X(x)$ are given by Eqs. (4.4) and (4.5), and the solutions to the differential equation for $T(t)$ are

$$T(t) = \alpha t + \beta, \tag{6.5}$$

where $\alpha = 3\sqrt{\frac{\Lambda}{10}}$ and β is an arbitrary constant. Requiring in the limit of $\Lambda \to 0$ that the solution (6.4) for the metric function $H(t,r,x)$ approaches the metric function $H(r,x)$ in asymptotically flat spacetime yields $\beta = 0$. Moreover, we find the exact analytical nonstationary solutions to Einstein-Maxwell theory with a cosmological constant,

$$d\mathcal{S}^2 = -H_0(t,r,x)^{-2}dt^2 + H_0(t,r,x)^{2/3}(dx^2 + ds^2_{a=0}), \tag{6.6}$$

and the time-dependent Maxwell gauge field

$$\mathcal{A}_t(t,r,x) = \frac{2\sqrt{3}}{3H_0(t,r,x)}, \tag{6.7}$$

where the exact analytic metric function $H_0(t,r,x)$ is given by

$$H_0(t,r,x) = 1 + 3\sqrt{\frac{\Lambda}{10}}t + \frac{h_0}{\sqrt{r^2+x^2}} \tag{6.8}$$

and h_0 is a constant. We note that $ds^2_{a=0}$ in Eq. (6.6) is given by Eq. (4.10). We then can find the most general convoluted cosmological nonstationary solutions in six-dimensional Einstein-Maxwell theory with a cosmological constant where the metric function $H(t,r,x)$ approaches $H_0(t,r,x)$ in the limit of $a \to 0$. The metric function $H(t,r,x)$ is equal to

$$H(t,r,x) = 1 + 3\sqrt{\frac{\Lambda}{10}}t + \frac{h_0}{2}\int_0^\infty H_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right) \times (\xi e^{cx} + (1-\xi)e^{-cx})dc, \tag{6.9}$$

where h_0 and ξ are two constants. The result (6.9) for the metric function $H(t,r,x)$ in six dimensions can simply be generalized to solutions in D dimensions, where we get

$$H_D(t,r,x) = 1 + (D-3)\sqrt{\frac{2\Lambda}{(D-1)(D-2)}}t + \alpha_D h_{D0}\int_0^\infty \mathcal{H}_D\left(0,0,a^2c^2,0,\sqrt{\frac{a^2+r^2}{r^2}}\right) \times K_{\frac{D-7}{2}}(cx)x^{\frac{7-D}{2}}c^{\frac{D-5}{2}}dc. \tag{6.10}$$

As we notice, the general solutions for the metric functions (6.9) and (6.10) describe the asymptotically de Sitter (dS) spacetime. We can find the cosmological c function for the solutions in the context of dS/CFT correspondence. As is well known, for any D-dimensional asymptotically dS spacetime, one may define the c function [29]

$$c \sim (G_{\mu\nu}n^\mu n^\nu)^{1-D/2}, \tag{6.11}$$

where n^μ is the unit vector along the time coordinate. If we have an expanding patch of dS, the flow of the renormalization group is toward the high-energy region, and the c function is a monotonically increasing function in terms of the time coordinate. On the other hand, a contracting patch of dS, the flow of the renormalization group is toward the low-energy region, and the c function is then a monotonically decreasing function in terms of the time coordinate. For example, the c function for the D-dimensional solutions, given by the metric function (6.10), describes expanding patches of dS in different dimensions.

VII. CONCLUDING REMARKS

Inspired by the existence of helicoid-catenoid instantons in four dimensions, we constructed exact solutions to the five- and higher-dimensional Einstein-Maxwell theory with and without a cosmological constant. The solutions in five dimensions are given by the line element (3.5), the gauge

field (3.4), and the metric functions (3.7). We discussed the physical properties of the solution. We also found exact convolutedlike solutions to six- and higher-dimensional Einstein-Maxwell theory in which the metric functions are convoluted integrals of two special functions with some measure functions. We fixed the measure functions for all the solutions by considering the solutions in some appropriate limits and comparing them with some other exact solutions in D dimensions that are given by Eqs. (4.1) and (5.7) with the metric functions (4.17) and (5.9), respectively. We used a special separation of variables to construct the solutions to Einstein-Maxwell theory with a positive cosmological constant. In this case, the metric function depends on time and two spatial directions. The solutions are given by the metric (6.1) and gauge field (6.2) where the cosmological metric functions in six and higher-than-six dimensions are given by Eqs. (6.9) and (6.10), respectively. We constructed the c functions and noticed that for all the cosmological convoluted solutions the c function is a monotonically increasing function in agreement with the c theorem for asymptotically dS spacetimes. As noted in Ref. [23], the issue of singularities remains unresolved and needs further analysis.

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