

# Constant curvature black holes in Einstein-AdS gravity: Conserved quantities

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We study physical properties of constant curvature black holes in Einstein–anti-de Sitter (AdS) gravity. These objects, which are locally AdS throughout the space, are constructed from identifications of global AdS spacetime, in a similar fashion as the Banados-Teitelboim-Zanelli black hole in three dimensions. We find that, in dimensions equal to or greater than 4, constant curvature black holes have zero mass and angular momentum. Only in odd dimensions are we able to associate a nonvanishing conserved quantity to these solutions, which corresponds to the vacuum (Casimir) energy of the spacetime.

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## I. INTRODUCTION

The interest in three-dimensional gravity resides on the fact that black holes in this theory are able to capture essential features of the higher-dimensional counterparts.

It was more than two decades ago that Bañados, Teitelboim, and Zanelli (BTZ) found a black hole solution in 3D anti-de Sitter (AdS) gravity [1], characterized by mass and angular momentum, and with thermal properties analog to rotating black holes in  $D \geq 4$ .

However, a BTZ black hole possesses constant curvature, and therefore it is locally indistinguishable from global AdS space. It is only when the solution is obtained by identifications along isometries that one can understand that the global structure of the spacetime is modified [2]. In particular, mass and angular momentum appear in the holonomies computed for a flat connection of the AdS group [3]. They are also obtained as conserved quantities coming from surface integrals in a number of methods [4].

The striking properties of BTZ black holes led some authors to look for higher-dimensional generalizations, e.g., in the form of constant curvature black holes (CCBHs) [5].

In this paper, we study properties of CCBHs, as regards mass and angular momentum, using background-independent definitions of conserved quantities for asymptotically AdS spacetimes (AAdS).

## II. CONSTANT CURVATURE BLACK HOLES IN EINSTEIN AdS GRAVITY

We shall consider a pure gravity theory described by General Relativity in  $D$  dimensions, which is given in terms of Einstein-Hilbert action,

$$I_{\text{EH}} = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} (R - 2\Lambda). \quad (1)$$

The dynamic field is the metric  $g_{\mu\nu}$ , the cosmological constant is  $\Lambda$ , and  $R$  is the Ricci scalar, which comes from the double contraction in the indices of the Riemann tensor  $R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\gamma} \Gamma^\gamma_{\nu\mu} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\beta\mu}$ . Because we are interested in the case of a negative cosmological constant,  $\Lambda$  is given by the expression  $\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}$  in terms of the AdS radius  $\ell$ .

Arbitrary variations of the action (1) with respect to the metric give rise to the Einstein equations plus a surface term,

$$\delta I_{\text{EH}} = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} \mathcal{E}_\nu^\mu (g^{-1} \delta g)_\mu^\nu + \int_M d^D x \partial_\mu \Theta^\mu, \quad (2)$$

where  $\mathcal{E}_\nu^\mu$  stands for the Einstein tensor

$$\mathcal{E}_\nu^\mu = R_\nu^\mu - \frac{1}{2} R \delta_\nu^\mu + \Lambda \delta_\nu^\mu \quad (3)$$

and  $\Theta^\mu$  is a surface term that depends on the variation of the Christoffel symbol.

Casting the Einstein tensor in a more convenient form,

$$\mathcal{E}_\nu^\mu = -\frac{1}{4} \delta_{[\nu\sigma\lambda]}^{[\mu\alpha\beta]} \left( R_{\alpha\beta}^{\sigma\lambda} + \frac{1}{\ell^2} \delta_{[\alpha\beta]}^{[\sigma\lambda]} \right), \quad (4)$$

it is obvious that if the constant curvature condition

$$F_{\alpha\beta}^{\sigma\lambda} = R_{\alpha\beta}^{\sigma\lambda} + \frac{1}{\ell^2} \delta_{[\alpha\beta]}^{[\sigma\lambda]} = 0 \quad (5)$$

is satisfied everywhere the spacetime is a solution to AdS gravity.

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In a Riemannian gravity theory in  $D$  dimensions, the two-form  $F$  is the only nonvanishing part of the curvature associated to the AdS group  $SO(D-1, 2)$ .

While in three-dimensional AdS gravity, a global condition  $F = 0$  is equivalent to the equation of motion, imposing Eq. (5) in higher dimensions proves to be far more restrictive, as we shall see below.

### A. Construction of CCBHs

In what follows, we consider solutions of Einstein equations with a negative cosmological constant which are constant curvature black holes. This type of solution was originally constructed by Bañados in Ref. [5]. Here, we briefly review this construction.

Let us consider a  $D$ -dimensional AdS space as a hypersurface defined in  $(D+1)$ -dimensional pseudo-Euclidean spacetime, subjected to the constraint

$$-x_0^2 + x_1^2 + \dots + x_{D-2}^2 + x_{D-1}^2 - x_D^2 = -\ell^2. \quad (6)$$

In particular, the surface defined by the above relation possesses a Killing vector with the components  $\xi^\alpha = (r_+/\ell)(0, \dots, 0, x_D, x_{D-1})$ , which is a boost in the  $(x_D, x_{D-1})$  plane, with a norm  $\xi^2 = (r_+^2/\ell^2)(-x_{D-1}^2 + x_D^2)$ . Substituting  $\xi^2$  into Eq. (6) defines a  $D-1$ -dimensional hypersurface in AdS space (6):

$$x_0^2 = x_1^2 + \dots + x_{D-2}^2 + \ell^2(1 - \xi^2/r_+^2). \quad (7)$$

In the case in which  $\xi = r_+$ , the formula (7) leads to a null surface given by the relation

$$-x_0^2 + x_1^2 + \dots + x_{D-2}^2 = 0. \quad (8)$$

To construct a CCBH, one must identify points along the orbits of  $\xi^\alpha$ . In the region  $\xi^2 < 0$ , orbits of  $\xi^\alpha$  are timelike. However, after the identification, they will become closed. This means that the region  $\xi^2 < 0$  is not physical after the identification, and its boundary  $\xi^2 = 0$  is singular in this sense. Thus, the spacetime of the surface is divided into three regions:  $I := r_+^2 < \xi^2 < \infty$ ,  $II := 0 < \xi^2 < r_+^2$ , and  $III := -\infty < \xi^2 \leq 0$ .

To write down explicitly the identification along the orbits of  $\xi^\alpha$ , it is useful to introduce local coordinates of AdS space in the region  $\xi^2 > 0$ ,

$$\begin{aligned} x_{\tilde{\alpha}} &= \frac{2y_{\tilde{\alpha}}}{1-y^2}, & \tilde{\alpha} &= 0, \dots, D-2 \\ x_{D-1} &= \frac{\ell r}{r_+} \sinh\left(\frac{r_+\phi}{\ell}\right), \\ x_D &= \frac{\ell r}{r_+} \cosh\left(\frac{r_+\phi}{\ell}\right) \end{aligned} \quad (9)$$

where

$$\begin{aligned} r &= r_+(1+y^2)/(1-y^2), \\ y^2 &= \eta_{\tilde{\alpha}\tilde{\beta}} y^{\tilde{\alpha}} y^{\tilde{\beta}}, \\ \eta_{\tilde{\alpha}\tilde{\beta}} &= \text{diag}(-1, 1, \dots, 1). \end{aligned} \quad (10)$$

Because the range of coordinates is  $-\infty < x^\alpha < \infty$ , the new variables will be defined in the intervals  $-\infty < \phi < \infty$  and  $-\infty < y^{\tilde{\alpha}} < \infty$  with  $-1 < y^2 < 1$ . Then, the metric element for the surface (6) acquires the Kruskal form:

$$ds^2 = \frac{\ell^2(r+r_+)^2}{r_+^2} \eta_{\tilde{\alpha}\tilde{\beta}} dy^{\tilde{\alpha}} dy^{\tilde{\beta}} + r^2 d\phi^2. \quad (11)$$

In these coordinates, the boost Killing vector has only one component  $\xi^\phi = 1$ , and its norm is  $\xi^2 = r^2$ . Identifying  $\phi \sim \phi + 2\pi n$ , one obtains a compact gravitational object that is globally of constant curvature, and the hypercone (8) becomes the horizon. Let us follow the usual procedure that shows that the new object is a black hole. It turns out that one can construct a Schwarzschild-like metric in the outer  $I$  region. For this, one needs to introduce local *spherical* coordinates

$$\begin{aligned} y_0 &= f \cos \theta_1 \sinh \frac{r+t}{\ell}, \\ y_1 &= f \cos \theta_1 \cosh \frac{r+t}{\ell}, \\ y_2 &= f \sin \theta_1 \sin \theta_2 \dots \sin \theta_{D-4} \sin \theta_{D-3}, \\ y_3 &= f \sin \theta_1 \sin \theta_2 \dots \sin \theta_{D-4} \cos \theta_{D-3}, \\ y_4 &= f \sin \theta_1 \sin \theta_2 \dots \cos \theta_{D-4}, \\ &\dots = \dots, \\ y_{D-3} &= f \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ y_{D-2} &= f \sin \theta_1 \cos \theta_2, \end{aligned} \quad (12)$$

where  $f(r) = [(r-r_+)/(r+r_+)]^{1/2}$ . The new coordinates are defined in the patch  $r_+ < r$ , such that  $0 < \theta_1, \theta_2, \dots, \theta_{D-4} < \pi$ ,  $0 \leq \theta_{D-3} < 2\pi$  and  $-\infty < t < \infty$ . In the new coordinates, the metric (11) adopts the Schwarzschild-like form

$$\begin{aligned} ds^2 &= \ell^2 n^2(r) \left( -\cos^2 \theta_1 dt^2 + \frac{\ell^2}{r_+^2} d\Omega_{D-3}^2 \right) \\ &+ n^{-2}(r) dr^2 + r^2 d\phi^2, \end{aligned} \quad (13)$$

where  $n^2(r) = (r^2 - r_+^2)/\ell^2$  and, in a more explicit form, the line element of the  $(D-3)$ -dimensional sphere can be written as

$$d\Omega_{D-3}^2 = d\theta_1^2 + \sin^2 \theta_1 d\Omega_{D-4}^2.$$

It has been claimed that the geometry described by Eq. (13) represents a black hole with an event horizon at

$r = r_+$ . However, some of the properties of such a black hole are rather unusual with respect to the ones of the standard Schwarzschild solution. Indeed, the topology of CCBHs is  $R^{D-1} \times S^1$  instead of  $R^2 \times S^{D-2}$  (or locally flat or hyperbolic transversal sections, as in the case of topological black holes in AdS gravity). On the other hand, the horizon of a CCBH is degenerated into a one-dimensional circle, whereas the horizon in a Schwarzschild black hole is a  $(D-2)$ -dimensional lightlike closed surface.

Furthermore, the asymptotic form ( $r \rightarrow \infty$ ) of the metric (13) is not well defined in the limit of vanishing  $r_+$ . This is a reflection of the fact that is not possible to set  $r_+ = 0$  in the construction of CCBHs sketched in this section.

In what follows, we study physical properties of CCBHs in order to relate the solution parameters to conserved quantities, i.e., mass and angular momentum. Because the boundary of the region defined by  $r > r_+$  has a topology  $S^{D-3} \times S^1$ , which cannot be matched with the asymptotic region of global AdS, no clear background can be associated to this solution. This argument prevents the use of any background-subtraction technique [6–10] in order to define the energy of a CCBH.

### III. BACKGROUND-INDEPENDENT CHARGES IN AdS GRAVITY

In the previous section, we discussed on the fact that global AdS spacetime cannot be recovered from the metric (13) by simply switching off the parameter  $r_+$ . Therefore, we will resort to background-independent formulas for conserved quantities in AdS gravity to evaluate the mass and angular momentum of CCBHs.

#### A. Four dimensions

In any gravity theory of which the dynamics is described only by the metric field, the variation with respect to  $g_{\mu\nu}$  will give rise to the equations of motion. Then, the energy-momentum tensor of the system will be identically zero, unless we identify it with the one coming from the matter Lagrangian.

The alternative to introducing a matter source in the bulk is to consider a boundary stress tensor, an idea that was developed some time ago by Brown and York in Ref. [11]. In particular, in a Gauss-normal coordinate frame,

$$ds^2 = N^2(r)dr^2 + h_{ij}(x, r)dx^i dx^j, \quad (14)$$

this *quasilocal* stress tensor is obtained as the variation with respect to the metric  $h_{ij}$ , that is,

$$T^{ij} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ij}},$$

assuming that the field equations hold in the bulk.

In the above formula,  $I$  stands for the Einstein-Hilbert action supplemented by suitable boundary terms, such that it is stationary under arbitrary variations of the boundary metric.

As has been extensively discussed in the literature, the proper way to get rid of normal derivatives in the boundary metric in the surface term is to add a Gibbons-Hawking boundary term:

$$I = I_{\text{EH}} - \frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{-h} K. \quad (15)$$

The boundary  $\partial M$  is defined in terms of the foliation (14) as located at constant  $r$ . The term at the boundary is proportional to the trace of the extrinsic curvature, which is defined as

$$K_{ij} = -\frac{1}{2N} \partial_r h_{ij} \quad (16)$$

in the coordinate frame (14).

The action  $I$  produces a quasilocal stress tensor, which coincides with the definition of canonical momentum  $\pi^{ij}$  for the radial foliation of the spacetime (14), that is,

$$\pi^{ij} = \frac{1}{8\pi G} (K^{ij} - h^{ij} K). \quad (17)$$

Because the line element (14) describes the spacetime as an infinite series of concentric cylinders, the conservation of Eq. (17) for a given  $r$  is a consequence of Einstein equations in that frame.

When used for General Relativity with zero cosmological constant, this stress tensor provides a sensible definition of conserved quantities as surface integrals in the asymptotic region [11]. However, in asymptotically AdS gravity, this boundary energy-momentum tensor produces charges which are divergent even for three-dimensional AdS black holes.

In the context of anti-de Sitter gravity/conformal field theory (CFT) correspondence, there is a systematic way to construct finite conserved charges for asymptotically AdS spacetimes. To have a finite variation of the action, one needs to add a *counterterm* series  $\mathcal{L}_{ct}(h, \mathcal{R}, \nabla \mathcal{R})$ , with dependence on the boundary curvature  $\mathcal{R} = \mathcal{R}(h)$  and covariant derivatives of it, such that the total action is renormalized,

$$I_{\text{ren}} = I + \int_{\partial M} d^{D-1}x \mathcal{L}_{ct}(h, \mathcal{R}, \nabla \mathcal{R}). \quad (18)$$

With the addition of local counterterms, the Brown-York stress tensor adopts the general form

$$T^{ij} = \pi^{ij} + \frac{2}{\sqrt{-h}} \frac{\delta \mathcal{L}_{ct}}{\delta h_{ij}}. \quad (19)$$

In particular, in  $D = 4$ , the counterterms required for a proper regularization of AdS gravity action are [12,13]

$$\mathcal{L}_{ct} = \frac{\sqrt{-h}}{8\pi G} \left( \frac{2}{\ell} + \frac{\ell}{2} \mathcal{R}(h) \right), \quad (20)$$

such that the quasilocal stress tensor takes the form

$$T_i^j = \frac{1}{8\pi G} \left( K_i^j - \delta_i^j K + \frac{2}{\ell} \delta_i^j - \ell \left( \mathcal{R}_i^j(h) - \frac{1}{2} \delta_i^j \mathcal{R}(h) \right) \right). \quad (21)$$

Any asymptotically AdS spacetime can be described by a metric that has a divergence of order 2 in the radial coordinate, as one approaches the asymptotic region. In particular, holographic techniques employ a Fefferman-Graham frame [14] to realize the asymptotically locally AdS condition, i.e., the fact that the curvature tends to a constant at the boundary. In particular, this asymptotic behavior implies that the extrinsic properties of the boundary, given by  $K_{ij}$ , can be expressed as a series of intrinsic quantities,

$$K_i^j = \frac{1}{\ell} \delta_i^j + \ell S_i^j(h) + \mathcal{O}(\mathcal{R}^2) \quad (22)$$

where

$$S_i^j(h) = \mathcal{R}_i^j(h) - \frac{1}{4} \delta_i^j \mathcal{R}(h)$$

is the Schouten tensor of the boundary metric  $h_{ij}$ . It is not difficult to see that the linear terms in the extrinsic curvature can be regarded as a truncation of an expression that is quadratic in  $K$ , because the quasilocal stress tensor can be rewritten as

$$T_i^j = \frac{\ell}{16\pi G} \delta_{[inp]}^{[jkl]} \left( \frac{1}{2} \mathcal{R}_{kl}^{np}(h) - K_k^n K_l^p + \frac{1}{\ell^2} \delta_k^p \delta_l^n \right),$$

up to  $\mathcal{O}(\mathcal{R}^2)$  terms. Antisymmetrization of the indices leaves the last formula in the form

$$\begin{aligned} T_i^j &= \frac{\ell}{32\pi G} \delta_{[inp]}^{[jkl]} \left( \mathcal{R}_{kl}^{np}(h) - K_k^n K_l^p + K_k^p K_l^n + \frac{1}{\ell^2} \delta_{[kl]}^{[pn]} \right), \\ &= \frac{\ell}{32\pi G} \delta_{[inp]}^{[jkl]} \left( R_{kl}^{np} + \frac{1}{\ell^2} \delta_{[kl]}^{[pn]} \right), \end{aligned} \quad (23)$$

where we have used the Gauss-Codazzi relation (B4). The quantity in brackets corresponds to the boundary components of the AdS curvature, which is the tensor that appears in the left-hand side of Eq. (5), and that is identically vanishing for global AdS spacetime. Thus,  $T_i^j$  also vanish identically for a spacetime with constant curvature everywhere.

On the other hand, for Einstein gravity, where the Ricci tensor is  $R_{\mu\nu} = -\frac{3}{\ell^2} g_{\mu\nu}$ , the Weyl tensor

$$W_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} - \frac{1}{2} R_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} + \frac{1}{6} R \delta_{[\mu\nu]}^{[\alpha\beta]} \quad (24)$$

becomes

$$W_{\mu\nu}^{\alpha\beta} = F_{\mu\nu}^{\alpha\beta}, \quad (25)$$

when Einstein equations hold.

That means that the quasilocal stress tensor is nothing but a projection of the Weyl tensor

$$T_i^j = \frac{\ell}{32\pi G} \delta_{[inp]}^{[jkl]} W_{kl}^{np} = -\frac{\ell}{8\pi G} W_{jr}^{ir}, \quad (26)$$

where we have used the fact that the single and double traces of the Weyl are zero. In a more covariant form, the boundary stress tensor is the electric part of the Weyl tensor

$$T_i^j = -\frac{\ell}{8\pi G} W_{j\nu}^{i\beta} n^\nu n_\beta = E_i^j, \quad (27)$$

where  $n$  is a normal vector to the boundary. This implies that the conformal mass definition for AAdS spaces provided by Ashtekar and Magnon in Ref. [15] also gives a vanishing mass for CCBHs.

The electric part of the Weyl tensor in Eq. (23) is, in turn, a truncation (up to quadratic order in the boundary curvature) of a charge density obtained from the addition of a topological invariant (Gauss-Bonnet), which is cubic in the extrinsic curvature [16,17]

$$T_i^j = \frac{\ell}{32\pi G} \delta_{[inp]}^{[mkl]} K_m^j \left( \mathcal{R}_{kl}^{np}(h) - K_k^n K_l^p + K_k^p K_l^n + \frac{1}{\ell^2} \delta_{[kl]}^{[pn]} \right).$$

We stress the fact that the comparison is possible by using the standard asymptotic behavior of the metric and the curvature for AAdS spacetimes. Additional terms appearing in the different formulas for energy in AdS gravity may play a role in a modified asymptotic behavior or for a boundary located at a finite cutoff.

But what is of relevance here is the fact that the equivalence between different notions of conserved quantities shows that CCBHs always have zero mass and angular momentum.

## B. General even-dimensional case ( $D = 2n$ )

For the purpose of computation of the conserved quantities for the solution (13), we will employ the charges derived within the Kounterterm regularization scheme for AdS gravity [16,18].

The conservation of Noether charges, calculated as surface integrals, is a consequence of the existence of a conserved current  $J^\mu$ . Indeed, the quantity  $Q[\xi] = \int_{\partial M} \sqrt{-h} n_\mu J^\mu$  is a

constant of motion, where  $h$  is the determinant of the metric of the hypersurface orthogonal to the normal vector  $n_\mu$ . In terms of the radial foliation (14), this quantity gives rise to the energy and angular momentum (and, in principle, other conserved quantities) enclosed by the sphere at that radius.

For the line element on  $\partial M$ , we take the set of coordinates  $x^i = (t, x^m)$ , such that it adopts an Arnowitt-Deser-Misner form,

$$h_{ij}dx^i dx^j = -\tilde{N}^2(t)dt^2 + \sigma_{mn}(dx^m + \tilde{N}^m dt)(dx^n + \tilde{N}^n dt),$$

$$\sqrt{-h} = \tilde{N} \sqrt{\sigma}, \quad (28)$$

which is generated by the timelike unit vector  $u_i = (-\tilde{N}, \vec{0})$ . The metric  $\sigma_{mn}$  represents the geometry of the subspace  $\Sigma_\infty$ , which is the spatial section of the asymptotic region (at constant time).

Whenever the radial component of the current can be globally expressed on the boundary as a derivative of the

$$\sqrt{-g}J^r = \partial_j[\sqrt{-h}\xi^i(q_i^j + q_{(0)i}^j)], \quad (29)$$

the Noether theorem provides the conserved charges  $Q[\xi]$  of the theory,

$$Q[\xi] = q[\xi] + q_{(0)}[\xi], \quad (30)$$

where each term is expressed as surface integrals on  $\Sigma_\infty$  as

$$q[\xi] = \int_{\Sigma_\infty} d^{D-2}x \sqrt{\sigma} u_j \xi^i q_i^j, \quad (31)$$

$$q_{(0)}[\xi] = \int_{\Sigma_\infty} d^{D-2}x \sqrt{\sigma} u_j \xi^i q_{(0)i}^j \quad (32)$$

for a given set of asymptotic Killing vectors  $\{\xi\}$ . The splitting of the charge density in two parts is motivated by the fact that  $q_{(0)i}^j$  gives rise to the vacuum energy in odd-dimensional AAdS spacetimes. In turn, in even dimensions, it vanishes identically.

In the Kounterterm method, both  $q_i^j$  and  $q_{(0)i}^j$  are given as polynomials of the intrinsic and extrinsic curvatures.

The charge density in the  $2n$ -dimensional case is

$$q_i^j = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{n-2}(2n-2)!} \delta_{[i_1 i_2 \dots i_{2n-1}]^{[j j_2 \dots j_{2n-1}]}} \times K_i^{i_1} \left( R_{j_2 j_3}^{i_2 i_3} \dots R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - \frac{(-1)^{n-1}}{\ell^{2n-2}} \delta_{[j_2 j_3]^{[i_2 i_3]} \dots \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]}} \right). \quad (33)$$

It is possible to prove that the CCBH metric, for fixed  $r$  and  $\phi$  and after a proper time rescaling, represents a  $(D-2)$ -dimensional de Sitter space. This implies the absence of globally defined timelike Killing vector in this geometry

[19]. Employing a global coordinate chart to describe CCBHs leads to a time-dependent line element [20].

The alternative to the use of Killing vectors to define conserved quantities in a time-dependent geometry is the definition of Kodama vectors given in Ref. [21]. However, Kodama's construction provides an answer for simple metrics but cannot be straightforwardly extended to an arbitrary gravitational object.

However, the two terms under the bracket in the formula (33) can be always factorized by the AdS curvature

$$q_i^j = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{n-2}(2n-2)!} \delta_{[i_1 i_2 \dots i_{2n-1}]^{[j j_2 \dots j_{2n-1}]}} \times K_i^{i_1} \left( R_{j_2 j_3}^{i_2 i_3} + \frac{1}{\ell^2} \delta_{[j_2 j_3]^{[i_2 i_3]}} \right) P_{j_4 \dots j_{2n-1}}^{i_4 \dots i_{2n-1}}(R, \delta), \quad (34)$$

where  $P(R, \delta)$  is a polynomial of the spacetime Riemann tensor and the antisymmetric delta of rank 2, which is more conveniently written in a parametric integral form,

$$P_{j_4 \dots j_{2n-1}}^{i_4 \dots i_{2n-1}} = (n-1) \int_0^1 dt \left[ (1-t) R_{j_4 j_5}^{i_4 i_5} - \frac{t}{\ell^2} \delta_{[j_4 j_5]^{[i_4 i_5]}} \right] \dots \times \left[ (1-t) R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - \frac{t}{\ell^2} \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]}} \right]. \quad (35)$$

This fact implies that the integrand of the charge is identically zero for any spacetime which is globally of constant curvature. Thus, for CCBHs (13), the expression (34) gives zero identically, which is manifest even before evaluating the explicit metric.

### C. General odd-dimensional case ( $D = 2n + 1$ )

In odd dimensions, the expressions for the charge density  $q_i^j$  and  $q_{(0)i}^j$  are given by

$$q_i^j = -\frac{1}{2^{n-2}} \delta_{[k_1 \dots k_{2n-1}]^{[j j_1 \dots j_{2n-1}]}} \times K_i^{i_1} \delta_{j_1}^{i_1} \left[ \frac{1}{16\pi G (2n-1)!} \delta_{[j_2 j_3]^{[i_2 i_3]} \dots \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]}} \right. \\ \left. + nc_{2n} \int_0^1 dt \left( R_{j_2 j_3}^{i_2 i_3} + \frac{t^2}{\ell^2} \delta_{[j_2 j_3]^{[i_2 i_3]}} \right) \dots \right. \\ \left. \times \left( R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} + \frac{t^2}{\ell^2} \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]}} \right) \right], \quad (36)$$

where  $c_{2n}$  is a constant, given by

$$c_{2n} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2} n(n-1)!}. \quad (37)$$

The expression for the charge can be factorized as

$$q_i^j = -\frac{nc_{2n}}{2^{n-1}} \delta_{[k_1 \dots k_{2n-1}]^{[j j_1 \dots j_{2n-1}]}} K_i^{i_1} \delta_{j_1}^{i_1} \left( R_{j_2 j_3}^{i_2 i_3} + \frac{1}{\ell^2} \delta_{[j_2 j_3]^{[i_2 i_3]}} \right) \tilde{P}_{j_4 \dots j_{2n-1}}^{i_4 \dots i_{2n-1}}(R, \delta), \quad (38)$$

where  $\tilde{P}_{j_4 \dots j_{2n-1}}^{i_4 \dots i_{2n-1}}(R, \delta)$  is a Lovelock-type polynomial of degree  $(n-2)$  in the Riemann tensor, which takes the following form when written in terms of a double parametric integral:

$$\begin{aligned} \tilde{P}_{j_4 \dots j_{2n-1}}^{i_4 \dots i_{2n-1}} &= 2(n-1) \int_0^1 dt \int_0^1 ds \left[ s \left( R_{j_4 j_5}^{i_4 i_5} + \frac{1}{\ell^2} \delta_{[j_4 j_5]}^{[i_4 i_5]} \right) + \frac{t^2 - 1}{\ell^2} \delta_{[j_4 j_5]}^{[i_4 i_5]} \right] \times \dots \\ &\times \left[ s \left( R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} + \frac{1}{\ell^2} \delta_{[j_{2n-2} j_{2n-1}]}^{[i_{2n-2} i_{2n-1}]} \right) + \frac{t^2 - 1}{\ell^2} \delta_{[j_{2n-2} j_{2n-1}]}^{[i_{2n-2} i_{2n-1}]} \right]. \end{aligned} \quad (39)$$

Because  $q_i^j$  is factorized by  $(R_{j_2 j_3}^{i_2 i_3} + \frac{1}{\ell^2} \delta_{[j_2 j_3]}^{[i_2 i_3]})$ , then it is identically zero for a spacetime of constant curvature everywhere, as is the case for CCBHs.

The part (32) of the total charge density, which does not vanish for globally constant curvature spacetimes, has an explicit expression,

$$\begin{aligned} q_{(0)j}^i &= -\frac{nc_{2n}}{2^{n-2}} \delta_{[k i_1 \dots i_{2n-1}]}^{[j j_1 \dots j_{2n-1}]} (K_i^k \delta_{j_1}^{i_1} + K_{j_1}^k \delta_i^{i_1}) \int_0^1 duu \mathcal{F}_{j_2 j_3}^{i_2 i_3}(u) \\ &\times \dots \times \mathcal{F}_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}}(u), \end{aligned} \quad (40)$$

where

$$\mathcal{F}_{kl}^{ij}(u) = \mathcal{R}_{kl}^{ij} - u^2 (K_k^i K_l^j - K_l^i K_k^j) + \frac{u^2}{\ell^2} \delta_{[kl]}^{[ij]}. \quad (41)$$

Because of the lack of globally defined timelike Killing vectors, we compute the integrand  $q_{(0)t}^t$ , in order to compare with the results obtained for topological AdS black holes given in Appendix C. This quantity contains information on certain holographic modes of AAdS spaces which give rise to a nonzero vacuum energy. The main advantage is the fact that the expression for  $q_{(0)t}^t$  exists in any odd dimension, which allows us to perform a generic computation for  $D = 2n + 1$ .

As a warmup computation, let us first consider the five-dimensional case. The extrinsic and intrinsic curvatures expressions for CCBHs are given (see Appendix B). Once we set the components  $i$  and  $j$  as  $t$  in Eq. (40), we obtain

$$\begin{aligned} q_{(0)t}^t &= -\frac{\ell^2}{64\pi G} \delta_{[n_1 n_2 n_3]}^{[m_1 m_2 m_3]} (K_t^m \delta_{m_1}^{n_1} - K_{m_1}^n \delta_t^{n_1}) \\ &\times \int_0^1 duu \left[ \mathcal{R}_{m_2 m_3}^{n_2 n_3} - u^2 (K_{m_2}^{n_2} K_{m_3}^{n_3} - K_{m_3}^{n_2} K_{m_2}^{n_3}) \right. \\ &\left. + \frac{u^2}{\ell^2} \delta_{[m_2 m_3]}^{[n_2 n_3]} \right]. \end{aligned}$$

Because of the transversal symmetries of CCBHs, the indices  $m$  and  $n$  can be separated into the ones corresponding to the sphere  $S^2$  and an additional azimuthal angle  $\phi$ ,

$$\begin{aligned} q_{(0)t}^t &= -\frac{\ell^2}{128\pi G} \delta_{[b_1 b_2]}^{[a_1 a_2]} (K_t^a \delta_{a_1}^{b_1} - K_{a_1}^b \delta_t^{b_1}) \\ &\times \left[ \mathcal{R}_{a_1 a_2}^{b_1 b_2} - \frac{1}{2} (K_{a_1}^{b_1} K_{a_2}^{b_2} - K_{a_2}^{b_1} K_{a_1}^{b_2}) + \frac{1}{2\ell^2} \delta_{[a_1 a_2]}^{[b_1 b_2]} \right]. \end{aligned} \quad (42)$$

Because  $\mathcal{F}_{\phi a}^{\phi b}(u) = 0$ , all other contributions vanish. Using the explicit form for the intrinsic and extrinsic curvatures for this type of black holes, the above expression turns into

$$q_{(0)t}^t = \frac{1}{64\ell\pi G} \left( \frac{r_+^4}{r^4} \right) + \mathcal{O}\left(\frac{1}{r^6}\right), \quad (43)$$

where we used the formulas in Appendix A. This simple example already reflects an unusual dependence on the parameters  $r_+$  and  $\ell$  in the vacuum energy density, when compared to the one for five-dimensional Schwarzschild-AdS black holes (see Appendix C).

Computing the general case, the integrand takes the form

$$\begin{aligned} q_{(0)t}^t &= -\frac{nc_{2n}}{2^{n-2}} \int_0^1 duu \delta_{[b_1 \dots b_{2n-2}]}^{[a_1 \dots a_{2n-2}]} (K_t^a - K_{\phi}^a) \mathcal{F}_{a_1 a_2}^{b_1 b_2}(u) \times \dots \\ &\times \mathcal{F}_{a_{2n-3} a_{2n-2}}^{b_{2n-3} b_{2n-2}}(u). \end{aligned} \quad (44)$$

The values of  $\mathcal{F}$  are identical for all the  $a$  and  $b$  indices. Summing up all the contributions,

$$\begin{aligned} q_{(0)t}^t &= 2nc_{2n} (2n-2)! \left( \frac{r_+^2}{\ell r \sqrt{r^2 - r_+^2}} \right) \left( \frac{r_+^2}{\ell^2 (r^2 - r_+^2)} \right)^{n-1} \\ &\times \int_0^1 duu (1-u^2)^{n-1} \\ &= \frac{(-1)^n}{8\pi G \ell (2n-1)} \frac{r_+^{2n}}{r(r^2 - r_+^2)^{n-\frac{1}{2}}} \frac{(2n-1)!!^2}{(2n)!}. \end{aligned} \quad (45)$$

The leading order for the previous expression is

$$q_{(0)t}^t = \frac{(-1)^n}{8\pi G \ell (2n-1)} \frac{r_+^{2n}}{r^{2n}} \frac{(2n-1)!!^2}{(2n)!} + \mathcal{O}\left(\frac{1}{r^{2n+2}}\right). \quad (46)$$

A similar computation was carried for the five-dimensional CCBH by Cai in Ref. [20], using a quasilocal stress tensor, properly renormalized by the addition of local counterterms to the AdS gravity action. In this case, the metric is written in a manifestly time-dependent form, such that the boundary energy also exhibits a dependence on the parameter  $r_+$ . It is also claimed in that reference that proper rescalings of the coordinates in the CCBH solution can remove the dependence of  $r_+$  in the boundary stress tensor. What is left, however, contains powers of the AdS radius  $\ell$  over  $G$  such that the result cannot be understood as the

Casimir energy of a boundary CFT. Furthermore, the above result carries, in any odd dimension, the opposite sign with respect to formula (C10) for topological AdS black holes. The result in Eq. (46) suggests that the above conclusion is also valid in higher odd dimensions; it is not possible to attribute a well-defined zero-point energy to CCBHs.

#### D. Rotating CCBHs

The construction of spinning CCBHs relies on the identification along isometries which are different than the ones of the static case. This procedure is equivalent to boosting the static metric along the  $t - \phi$  plane [5],

$$\begin{aligned} t &\rightarrow \beta t \frac{r_+}{\ell^2} + (\phi - \Omega\beta t) \frac{r_-}{\ell}, \\ \phi &\rightarrow \beta t \frac{r_-}{\ell^2} + (\phi - \Omega\beta t) \frac{r_+}{\ell}, \end{aligned} \quad (47)$$

which introduces  $r_+$  and  $r_-$  as two arbitrary constants ( $r_+ > r_-$ ) in the metric. This is motivated by a similar transformation acting on the static BTZ black hole in three dimensions, which indeed generates angular momentum. In Ref. [5], it is stated that a different choice of the parameters  $\beta$  and  $\Omega$  does not modify the conserved charges of this type of locally AdS solutions.

In a way, one could understand this statement along the line of reasoning of the previous section. Indeed, no matter what particular values  $\beta$  and  $\Omega$  take, the mass and angular momentum of the solution are identically zero, which is evident from Eq. (38). To complete the present discussion, we can compute the vacuum charge density (40) in any odd dimension. The integrand in Eq. (40) for the boosted metric can be expanded as

$$\begin{aligned} q'_{(0)t} &= -\frac{nc_{2n}}{2^{n-2}} \delta_{[b_1 \dots b_{2n-2}]^{[a_1 \dots a_{2n-2}]} \int_0^1 duu [(K_t^t - K_\phi^\phi) \mathcal{F}_{a_1 a_2}^{b_1 b_2}(u) \\ &\quad + (K_t^t \delta_{a_1}^{b_1} - K_{a_1}^{b_1}) \mathcal{F}_{\phi a_2}^{\phi b_2}(u) - K_t^\phi \delta_{a_1}^{b_1} \mathcal{F}_{\phi a_2}^{t b_2}(u)] \mathcal{F}_{a_3 a_4}^{b_3 b_4}(u) \\ &\quad \times \dots \times \mathcal{F}_{a_{2n-3} a_{2n-2}}^{b_{2n-3} b_{2n-2}}(u). \end{aligned} \quad (48)$$

Summing up all the contributions,

$$\begin{aligned} q'_{(0)t} &= 2nc_{2n}(2n-2)! \int_0^1 duu (1-u^2)^{n-1} \\ &\quad \times \frac{r_+^4 (r_+^2 - r_-^2 + 2nr_-(r_+ \Omega \ell - r_-))}{r \ell^3 (r_+^2 - r_-^2) (r^2 - r_+^2)^{\frac{3}{2}}} \left( \frac{r_+^2}{\ell^2 (r^2 - r_+^2)} \right)^{n-2} \\ &= \frac{(-1)^n (2n-1)!!^2}{8\pi G \ell (2n-1) (2n)!} \\ &\quad \times \frac{r_+^{2n} (r_+^2 - r_-^2 + 2nr_-(r_+ \Omega \ell - r_-))}{r (r_+^2 - r_-^2) (r^2 - r_+^2)^{n-\frac{1}{2}}}. \end{aligned} \quad (49)$$

The leading order for this expression is

$$\begin{aligned} q'_{(0)t} &= \frac{(-1)^n (2n-1)!!^2}{8\pi G \ell (2n-1) (2n)!} \\ &\quad \times \frac{r_+^{2n} (r_+^2 - r_-^2 + 2nr_-(r_+ \Omega \ell - r_-))}{r^{2n} (r_+^2 - r_-^2)} + \mathcal{O}\left(\frac{1}{r^{2n+2}}\right). \end{aligned} \quad (50)$$

In Ref. [22] a particular choice of the parameters  $\Omega = \frac{r_-}{r_+ \ell}$  and  $\beta = r_+$  is taken in order to define a five-dimensional Lorentzian rotational CCBH. Plugging these values into Eq. (50), we obtain

$$q'_{(0)t} = \frac{(-1)^n (2n-1)!!^2 r_+^{2n}}{8\pi G \ell (2n-1) (2n)! r^{2n}} + \mathcal{O}\left(\frac{1}{r^{2n+2}}\right), \quad (51)$$

which is the same result as in the static case.

Therefore, the presence of terms of the type  $dt d\phi$  in the line element does not bring in rotation in the solution. That means that the falloff of these crossed terms is such that they do not contribute to surface integrals defined at radial infinity.

In sum, the boost (47) does not generate a new solution, as it is unable to change the physical parameters of CCBHs.

#### IV. DISCUSSION AND CONCLUSIONS

We have shown that, in even-dimensional AdS gravity, mass and angular momentum for CCBHs is always zero. This is a consequence of the fact that the conserved charge formulas for AdS gravity can be always factorized by the AdS curvature or, equivalently, the Weyl tensor.

This result is not surprising in the light of a recent work [23], where it is shown that, for AAdS spaces, conformal mass definition is linked to the addition of Kounterterms. The fact that two energy definitions in AdS gravity give the same zero result for CCBH mass is then reassuring.

In principle, one could always go against this reasoning, claiming that the mass for a black hole may also be obtained from the integration of the matter stress tensor on the right-hand side of the Einstein equation. However, delta-type singularities are hidden inside the horizon and not needed by methods that compute conserved charges as surface integrals in the asymptotic region.

But what looks undeniable is the fact that conserved charges computed within Kounterterm regularization provide the correct answer for a large class of AAdS solutions regardless of a particular form of the matter energy-momentum tensor. This is confirmed by the relation between Kounterterm charges and conformal mass definition in every dimension [15,23,24].

In odd dimensions, a nonvanishing value for  $q'_{(0)t}$  does not make the situation more promising for CCBHs. Indeed, the formulas that usually give the mass and angular momentum for AAdS solutions also vanish identically in this case. For static CCBHs, the vacuum energy contains a

rather unusual dependence on the parameter  $r_+$ . Even though one can eliminate this dependence by coordinate rescalings, the result is not proportional to  $\ell^{2n-1}/G$ , which is the only sensible value one can interpret as a Casimir energy for the CFT on the boundary of AdS gravity in  $D = 2n + 1$  dimensions.

In the rotating case, the second parameter in the solution,  $r_-$ , does not appear anywhere in the conserved quantities nor in the expression for the vacuum energy. This result contrasts with the interpretation of  $r_-$  as related to angular momentum in the original reference [5] because it is a well-known fact that, in  $D = 2n + 1$  dimensions, the vacuum energy for Kerr-AdS black holes, depends on the rotation parameters  $\{a_i\}_{i=1..n}$  [25–28]. Even in the simplest case (Myers-Perry–AdS), with a single rotation parameter  $a$ , the vacuum energy depends on  $r_-$ .

If one attempts to consider CCBHs as solutions of Einstein-Gauss-Bonnet (EGB) or a more general Lovelock gravity theory with AdS asymptotics, the main conclusions of this paper would remain the same. In fact, for EGB AdS gravity, Kounterterm regularization will give rise to conserved charges of which the formulas are factorizable by the corresponding AdS curvature [29,30],

$$\tilde{F}_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} + \frac{1}{\ell_{\text{eff}}^2} \delta_{[\mu\nu]}^{[\alpha\beta]}, \quad (52)$$

in terms of an effective AdS radius,

$$\frac{1}{\ell_{\text{eff}}^2} = \frac{1}{2(D-3)(D-4)\alpha} \left( 1 \pm \sqrt{1 - \frac{4(D-3)(D-4)\alpha}{\ell^2}} \right), \quad (53)$$

where  $\alpha$  is the Gauss-Bonnet coupling.

A black hole which is globally of constant curvature can indeed have mass in three-dimensional AdS gravity, as is the case of BTZ solution [1,2]. This is understood in the light of the general formulas for the conserved charges in odd dimensions, Eqs. (36) and (40). In three dimensions, there are not enough indices in Eq. (36) to produce an expression proportional to the AdS curvature (or, equivalently, the Weyl tensor), as happens in higher dimensions. As a consequence, the equivalence to formula (36) vanishes identically in three dimensions. Then, Eq. (40) is the formula responsible for the mass and angular momentum for a BTZ black hole [31].

As AdS gravity in three dimensions is derived from a Chern-Simons action for the  $SO(2, 2)$  group, the previous result can be extended to higher odd dimensions. Indeed, in Lovelock-Chern-Simons gravity, a formula proportional to Eq. (40) produces the mass of black holes in that theory [32]. As a matter of fact, this is the only case in which one can attribute a nonvanishing mass to CCBHs [33].

An independent approach that might shed some light on the problem of vanishing charges for CCBHs is the computation of holonomies, motivated by the result in 3D AdS gravity [3]. However, this computation would require the existence of noncontractible curves which enclose the *horizon* in higher dimensions. On the other hand, in the context of supersymmetry, arbitrary values of the parameters in the CCBH solution should produce an obstruction to the existence of globally defined Killing spinors [34,35].

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### APPENDIX A: KRONECKER DELTA OF RANK $p$

The totally antisymmetric Kronecker delta of rank  $p$  is defined as the determinant

$$\delta_{[\mu_1 \dots \mu_p]}^{[\nu_1 \dots \nu_p]} := \begin{vmatrix} \delta_{\mu_1}^{\nu_1} & \delta_{\mu_1}^{\nu_2} & \dots & \delta_{\mu_1}^{\nu_p} \\ \delta_{\mu_2}^{\nu_1} & \delta_{\mu_2}^{\nu_2} & & \delta_{\mu_2}^{\nu_p} \\ \vdots & & \ddots & \\ \delta_{\mu_p}^{\nu_1} & \delta_{\mu_p}^{\nu_2} & \dots & \delta_{\mu_p}^{\nu_p} \end{vmatrix}. \quad (A1)$$

A contraction of  $k \leq p$  indices in the Kronecker delta of rank  $p$  produces a delta of rank  $p - k$ ,

$$\delta_{[\mu_1 \dots \mu_k \dots \mu_p]}^{[\nu_1 \dots \nu_k \dots \nu_p]} \delta_{\nu_1}^{\mu_1} \dots \delta_{\nu_k}^{\mu_k} = \frac{(N - p + k)!}{(N - p)!} \delta_{[\mu_{k+1} \dots \mu_p]}^{[\nu_{k+1} \dots \nu_p]}, \quad (A2)$$

where  $N$  is the range of indices.

### APPENDIX B: EXTRINSIC AND INTRINSIC CURVATURES FOR STATIC CCBHs

The radial foliation of the spacetime

$$ds^2 = N^2(r)dr^2 + h_{ij}(x, r)dx^i dx^j \quad (B1)$$

implies the Gauss-Codazzi relations for the spacetime Riemann tensor

$$R_{kl}^{ir} = \frac{1}{N} (\nabla_l K_k^i - \nabla_k K_l^i), \quad (B2)$$

$$R_{kr}^{ir} = \frac{1}{N} (K_k^i)' - K_l^i K_l^k, \quad (B3)$$

$$R_{kl}^{ij} = \mathcal{R}_{kl}^{ij}(h) - K_k^i K_l^j + K_l^i K_k^j, \quad (B4)$$



where  $\nabla_l = \nabla_l(\Gamma_{ij}^k)$  is the covariant derivative defined in the Christoffel symbol of the boundary  $\Gamma_{ij}^k(g) = \Gamma_{ij}^k(h)$ .

On the other hand, the boundary metric for CCBHs takes the block-diagonal form

$$h_{ij} = \begin{pmatrix} -(r^2 - r_+^2) \cos^2 \theta_1 & 0 & 0 \\ 0 & (r^2 - r_+^2) \frac{\ell^2}{r_+^2} \gamma_{ab} & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad (\text{B5})$$

where the boundary indices split as  $i = (t, a, \phi)$  and the metric  $\gamma_{ab}$  is the metric of the unit sphere  $S^{D-3}$ .

From the line element on the boundary metric

$$h_{ij} dx^i dx^j = (r^2 - r_+^2) \left( -\cos^2 \theta_1 dt^2 + \frac{\ell^2}{r_+^2} d\Omega_{D-3} \right) + r^2 d\phi^2, \quad (\text{B6})$$

it is easy to read off the lapse function and the induced metric in an Arnowitt-Deser-Misner foliation with vanishing shift functions,

$$\begin{aligned} h_{ij} dx^i dx^j &= -\tilde{N}^2 dt^2 + \sigma_{mn} dx^m dx^n \\ &= -\tilde{N}^2 dt^2 + r^2 (\gamma_{ab} dy^a dy^b + d\phi^2), \end{aligned} \quad (\text{B7})$$

such that

$$\tilde{N}^2 = (r^2 - r_+^2) \cos^2 \theta_1, \quad (\text{B8})$$

where  $y^a$  and  $\gamma_{ab}$  (with  $a, b = \{1, \dots, D-3\}$ ) are the angles and the metric of the  $(D-3)$ -dimensional sphere, respectively, and  $\phi$  is an additional azimuthal angle. The lapse functions set the only nonvanishing component of the timelike normal vector as  $u_t = -(r^2 - r_+^2)^{1/2} |\cos \theta_1|$ , and the determinant  $\sigma$  of the spatial metric is given by

$$\sqrt{\sigma} = r^{D-2} \left( \frac{\ell}{r_+} \right)^{D-3} \left( 1 - \frac{r_+^2}{r^2} \right)^{\frac{D-3}{2}} \sqrt{\gamma_{D-3}}. \quad (\text{B9})$$

The components of the extrinsic curvature for the metric (13) are

$$\begin{aligned} K_t^t &= -\frac{r}{\ell (r^2 - r_+^2)^{1/2}}, \\ K_b^a &= -\frac{r}{\ell (r^2 - r_+^2)^{1/2}} \delta_b^a, \\ K_\phi^\phi &= -\frac{(r^2 - r_+^2)^{1/2}}{r\ell}. \end{aligned} \quad (\text{B10})$$

Recalling the fact that the solution (13) is a constant curvature spacetime, the intrinsic curvature  $\mathcal{R}_{kl}^{ij}$  can be directly obtained by Gauss-Codazzi relations. In doing so, the only nonvanishing components of the boundary Riemann tensor are

$$\begin{aligned} \mathcal{R}_{tb}^{ta}(h) &= \frac{r_+^2}{\ell^2 r^2 - r_+^2} \delta_b^a, \\ \mathcal{R}_{cd}^{ab}(h) &= \frac{r_+^2}{\ell^2 r^2 - r_+^2} \delta_{[cd]}^{[ab]}. \end{aligned} \quad (\text{B11})$$

In the case of rotating CCBHs, the components of the extrinsic curvatures are

$$K_t^t = -\frac{r^2(r_+^2 - r_-^2) + r_+^2 r_- (r_- - r_+ \Omega \ell)}{\ell r (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}}, \quad (\text{B12})$$

$$K_\phi^\phi = -\frac{r^2(r_+^2 - r_-^2) + r_+^3 (\Omega \ell r_- - r_+)}{\ell r (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}}, \quad (\text{B13})$$

$$K_t^\phi = -\frac{r_+^2 \beta (\Omega \ell (r_+^2 - r_-^2) - r_+ r_- (1 + \Omega^2 \ell^2))}{\ell^2 r (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}}, \quad (\text{B14})$$

$$K_\phi^t = -\frac{r_+^3 r_-}{r \beta (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}}, \quad (\text{B15})$$

$$K_a^b = -\frac{r}{\ell \sqrt{r^2 - r_+^2}} \delta_b^a; \quad (\text{B16})$$

the boundary Riemann tensors

$$\mathcal{R}_{ab}^{cd} = \frac{r_+^2}{\ell^2 (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}} \delta_{[ab]}^{[cd]}, \quad (\text{B17})$$

$$\mathcal{R}_{\phi a}^{\phi b} = \frac{r_+^2 r_- (\Omega \ell r_+ - r_-)}{\ell^2 (r_+^2 - r_-^2) (r^2 - r_+^2)} \delta_b^a, \quad (\text{B18})$$

$$\mathcal{R}_{ta}^{\phi b} = \frac{r_+^3 r_-}{\beta \ell (r_+^2 - r_-^2) (r^2 - r_+^2)} \delta_b^a; \quad (\text{B19})$$

and the values of  $\mathcal{F}_{\alpha\beta}^{\mu\nu}(u)$

$$\mathcal{F}_{b_1 b_2}^{a_1 a_2}(u) = (1 - u^2) \frac{r_+^2}{\ell^2 (r_+^2 - r_-^2) \sqrt{r^2 - r_+^2}} \delta_{[b_1 b_2]}^{[a_1 a_2]}, \quad (\text{B20})$$

$$\mathcal{F}_{\phi b}^{\phi a}(u) = (1 - u^2) \frac{r_+^2 r_- (\Omega \ell r_+ - r_-)}{\ell^2 (r_+^2 - r_-^2) (r^2 - r_+^2)} \delta_b^a, \quad (\text{B21})$$

$$\mathcal{F}_{tb}^{\phi a}(u) = (1 - u^2) \frac{r_+^3 r_-}{\beta \ell (r_+^2 - r_-^2) (r^2 - r_+^2)} \delta_b^a. \quad (\text{B22})$$

### APPENDIX C: VACUUM ENERGY IN TOPOLOGICAL BLACK HOLES IN AdS GRAVITY

To compare to the results in Sec. III, we review here the computation of the zero-point energy for globally AdS spacetime.

Static topological black holes are solutions of the Einstein equations with a negative cosmological constant, which are described by the metric

$$ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2 d\Sigma_{D-2}^2, \quad (\text{C1})$$

where the line element of the transversal section  $\Sigma^{(k)}$  is

$$d\Sigma_{D-2}^2 = \gamma_{mn}^{(k)} dy^m dy^n, \quad (\text{C2})$$

and

$$f^2(r) = k - \frac{2G\mu}{r^{D-3}} + \frac{r^2}{\ell^2}, \quad (\text{C3})$$

where the parameter  $\mu$  is the mass density. The topological parameter  $k = +1, 0, -1$  denotes the curvature of the transversal section, which can be a sphere, a locally flat surface, or a hyperboloid, respectively.

Using the general formula for the vacuum energy for AAdS spacetimes (40) for the particular choice of the timelike vector  $u_j$  and a timelike Killing vector  $\xi^i$  leads to the expression

$$q_{(0)t}^t = \frac{nc_{2n}}{2^{n-2}} \delta_{[m_1 \dots m_{2n-1}]}^{[n_1 \dots n_{2n-1}]} (K_t^i \delta_{n_1}^{m_1} - K_{n_1}^{m_1} \delta_t^i) \times \int_0^1 duu \mathcal{F}_{n_2 n_3}^{m_2 m_3}(u) \times \dots \times \mathcal{F}_{n_{2n-2} n_{2n-1}}^{m_{2n-2} m_{2n-1}}(u). \quad (\text{C4})$$

The components of the tensorial quantities defined at the boundary—which are relevant for the evaluation of the above formula—are given by

$$K_t^i = -f'(r), \quad K_{n_1}^{m_1} = -\frac{f(r)}{r} \delta_{n_1}^{m_1} \quad (\text{C5})$$

for the extrinsic curvature and

$$\mathcal{F}_{n_2 n_3}^{m_2 m_3}(u) = \frac{1}{r^2} \left[ k - u^2 \left( f^2(r) - \frac{r^2}{\ell^2} \right) \right] \delta_{[n_2 n_3]}^{[m_2 m_3]} \quad (\text{C6})$$

for the parametric curvature (41).

In the static black hole ansatz (C1), the vacuum energy integrand adopts the form

$$q_{(0)t}^t = 2nc_{2n}(2n-1)! \left( f'(r) - \frac{f(r)}{r} \right) \left( \frac{1}{r^{2n-2}} \right) \times \int_0^1 duu \left[ k - u^2 \left( f^2(r) - \frac{r^2}{\ell^2} \right) \right]^{n-1}, \quad (\text{C7})$$

which, taking the value of the metric function in Eq. (C3), adopts a much simpler form,

$$q_{(0)t}^t = 2nc_{2n}(2n-1)! \left( -\frac{\ell k}{r^2} + \mathcal{O}(r^{1-D}) \right) \left( \frac{1}{r^{2n-2}} \right) \times \int_0^1 duu \left[ k - u^2 \left( k - \frac{2G\mu}{r^{D-3}} \right) \right]^{n-1}, \quad (\text{C8})$$

in the limit  $r \rightarrow \infty$  for the leading order,

$$q_{(0)t}^t = -c_{2n}(2n)! \frac{k^n \ell}{r^{2n}} \int_0^1 duu (1-u^2)^{n-1} + \dots \quad (\text{C9})$$

When performing the trivial integration in the parameter  $u$ , the standard result of the vacuum energy integrand for topological AdS black holes is recovered:

$$q_{(0)t}^t = \frac{(-1)^{n-1} \ell^{2n-1} (2n-1)!!^2 k^n}{8\pi G (2n)! r^{2n}}.$$

The standard result of the vacuum energy for topological AdS black holes is recovered when integrated in the transversal section considering  $\sqrt{\sigma} = r^{2n-1} \sqrt{\gamma^{(k)}}$ ,

$$E_0 = \int_{\Sigma_\infty} d^{2n-1}x (-f(r)) q_{(0)t}^t \sqrt{\sigma} = \frac{\ell^{2n-2}}{8\pi G} (-k)^n \frac{(2n-1)!!^2}{(2n)!} \text{Vol}(\Sigma^{(k)}), \quad (\text{C10})$$

where  $\gamma^{(k)}$  is the determinant of the surface  $\Sigma^{(k)}$ .

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