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Partially massless graviton on beyond Einstein spacetimes

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We show that a partially massless graviton can propagate on a large set of spacetimes which are not Einstein spacetimes. Starting from a recently constructed theory for a massive graviton that propagates the correct number of degrees of freedom on an arbitrary spacetime, we first give the full explicit form of the scalar constraint responsible for the absence of a sixth degree of freedom. We then spell out generic conditions for the constraint to be identically satisfied, so that there is a scalar gauge symmetry which makes the graviton partially massless. These simplify if one assumes that spacetime is Ricci symmetric. Under this assumption, we find explicit non-Einstein spacetimes (some, but not all, with vanishing Bach tensors) allowing for the propagation of a partially massless graviton. These include in particular the Einstein static Universe.

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I. INTRODUCTION

A natural physical question to ask is whether the graviton has a mass. General relativity predicts that the graviton is a self-interacting massless spin-2 particle, but it is possible that small mass corrections to general relativity are present. This question has received renewed interest since the experimental discovery of the acceleration of the universe [1,2], the possibility that this acceleration may be explained by a large distance modification of gravity related to the one appearing when the graviton has a mass [3–5], the better understanding of the so-called Vainshtein mechanism of massive gravity [6–8], and the recent theoretical discovery of nonlinear ghost free theories of massive spin-2 [9,10] (see [11–13] for reviews).

From the point of view of naturalness, an explanation for the observed acceleration using a mechanism driven by a graviton mass would be an improvement over the standard explanation in terms of a cosmological constant. This is because a small graviton mass is technically natural [14,15] whereas a small cosmological constant is not. But such a mechanism would still leave open the question of why various large contributions to the cosmological constant expected from such sources as phase transitions and heavy particle states do not gravitate (see [16] for a recent summary of the cosmological status of massive gravity and its extensions).

When spacetime is not flat, the division of particles into massless or massive no longer covers all the possibilities. On de Sitter (dS) space, there exists the mathematical possibility of gravitons which are neither massive nor massless, but instead propagate a number of degrees of freedom greater than that of a massless graviton but less than that of a massive graviton. These are called "partially massless" (PM) gravitons, and they enjoy a scalar gauge invariance responsible for removing one of the degrees of freedom of the fully massive graviton [17–27].

This extra scalar gauge symmetry fixes the mass of the graviton relative to the background curvature, and hence is a candidate symmetry which could fix the cosmological constant relative to the already small graviton mass, thus explaining why the cosmological constant itself is small. It is primarily for this reason why partially massless fields are of interest cosmologically[1] (see the review [42]). For this mechanism to be nontrivial, we need to have it realized in a fully nonlinear theory whose graviton is partially massless and can be coupled to massive matter. This has led to many studies of the properties of the linear theory and possible nonlinear extensions [32,43–64]. As a result of these studies, various obstructions and no-go results to an interacting theory have been found, and at this point there is no known four dimensional example of a nonlinear ghost-free theory with a finite number of fields

[1]Quite apart from cosmology, partially massless fields appear in holographic duals to conformal field theories describing so-called multicritical points, see e.g. [28–35] for some recent work. These duals are theories with infinite towers of partially massless fields of all spins [36–41].

in which a partially massless graviton mode propagates fully nonlinearly with a gauge symmetry persisting to all orders.

If such a fully nonlinear theory exists through some loophole in the various no-go results, then the partially massless graviton would be the fluctuation of some fully nonlinear field around a dS background solution. We would then expect there to be other solutions which are more general than dS, and expanding around these other solutions we would expect to see a PM graviton propagating around these non-dS backgrounds. Thus, to get more insight into a putative nonlinear theory, we can start by asking about what kinds of backgrounds a PM graviton can propagate on. For example, in the fully massless case, a massless graviton is only known to propagate on an Einstein space [65,66]. This is a clue pointing to the fact that the only two-derivative fully nonlinear theory of a massless spin 2 is Einstein gravity [67,68], whose vacuum solutions are precisely Einstein spaces.

Up until now, the only known backgrounds upon which a PM graviton can propagate are Einstein spaces, and obstructions to propagating on more general spacetimes, under certain assumptions about the possible gravitational couplings, have been presented [69,70]. Here, we will relax one of these assumptions and show that there are more general non-Einstein spaces upon which a PM graviton can propagate.

We do this by starting with the construction of [71–73], which shows how to couple a fully massive graviton to an arbitrary background metric in such a way that only the correct five degrees of freedom propagate. We give the full explicit form of the scalar constraint responsible for eliminating a possible sixth degree of freedom. We then proceed to ask under what conditions on the background and parameters of the theory this constraint becomes identically satisfied, indicating an extra scalar gauge symmetry that emerges and makes the graviton partially massless. While we will not be able to solve for the most general such conditions, we will be able to find several classes of examples which are not Einstein spaces, and thus go beyond the backgrounds previously thought possible. This gives a hint that there may be some leeway for a fully interacting partially massless theory of some kind, and raises the question of what the most general background in such a theory will be.

This paper is organized as follows. In the following section we review some technical properties of a massive and partially massless graviton on curved background spacetimes. In the next section we construct explicitly examples of non Einstein spacetimes on which a partially massless graviton can propagate.

Conventions: We work in four spacetime dimensions and use the mostly plus metric signature $(-,+,+,+)$. The curvature conventions are those of [74]. We (anti) symmetrize tensors with unit weight, e.g. $S_{(\mu\nu)} = \frac{1}{2}(S_{\mu\nu} + S_{\nu\mu})$.

II. MASSIVE AND PARTIALLY MASSLESS GRAVITONS ON CURVED BACKGROUND SPACETIMES

A. Massive graviton on an Einstein background

Before discussing the theory for a massive graviton on a general curved background, as introduced in [71–73], we first review the well known properties of a massive graviton propagating on a generic Einstein spacetime.[2] An Einstein spacetime is one whose metric $g_{\mu\nu}$ obeys

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor and Λ is a constant, the cosmological constant.

A massive graviton propagating on this background has field equations reading (here and henceforth the symbol $\simeq$ denotes an on-shell equality)

$$E_{\mu\nu} \simeq 0, \tag{2}$$

where $E_{\mu\nu}$ is the field equation operator defined by

$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\right) + \frac{m^2}{2}(h_{\mu\nu} - g_{\mu\nu} h). \tag{3}$$

Here $h_{\mu\nu}$ is a symmetric rank two covariant tensor representing the graviton field, indices are raised and lowered and traced with the background metric, e.g. $h = g^{\mu\nu} h_{\mu\nu}$, and the linear kinetic operator is given by

$$\begin{aligned}\mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\frac{1}{2}[&\delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu \nabla^\sigma\nabla_\nu - \delta^\rho_\nu \nabla^\sigma \nabla_\mu \\ &- g_{\mu\nu} g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma] h_{\rho\sigma},\end{aligned} \tag{4}$$

where ∇ denotes the covariant derivative associated with the background metric $g_{\mu\nu}$. These field equations derive from an action given by

$$\begin{aligned}S[h] = -M_{\rm Pl}^2 \int d^4x \sqrt{|g|}\bigg[&h_{\mu\nu}\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \Lambda\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) \\ &+ \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2)\bigg].\end{aligned} \tag{5}$$

Note that the kinetic and Λ dependent terms appearing on the right-hand side of (3) come from the linearization around an Einstein spacetime of the usual vacuum Einstein equations with a cosmological constant.

[2] Such a theory was first formulated and studied for a Minkowski background spacetime by Fierz and Pauli [75]. It was then further generalized to a maximally symmetric and later to a generic Einstein spacetime [18,19,76,77]. See also [78–81].

We now review how to count the degrees of freedom in this model in a covariant way [77], since we will follow the same pattern in the more complicated case of a general background. One first notices that, due to the Bianchi identities identically satisfied by the kinetic operator,

$$\nabla^\mu\left[\mathcal{D}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} - \Lambda\left(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h\right)\right] = 0, \qquad (6)$$

one has from the definition (3)

$$\nabla^\mu E_{\mu\nu} = \frac{m^2}{2}(\nabla^\mu h_{\mu\nu} - g^{\rho\sigma}\nabla_\nu h_{\rho\sigma}), \qquad (7)$$

resulting in the on-shell relation (assuming $m^2 \neq 0$)

$$\nabla^\mu h_{\mu\nu} - \nabla_\nu h \simeq 0. \qquad (8)$$

This relation provides four constraint equations (because they contain at most first order derivatives) for $h_{\mu\nu}$, which are referred to as the vector constraints. These eliminate four degrees of freedom from the original 10 components of the symmetric tensor $h_{\mu\nu}$, leaving six.

To get down to five, the correct count for a massive spin-2 field, we need to find an additional scalar constraint. A second covariant divergence of the field equations gives the identity

$$\nabla^\mu\nabla^\nu E_{\mu\nu} = \frac{m^2}{2}(\nabla^\mu\nabla^\nu h_{\mu\nu} - \nabla^2 h). \qquad (9)$$

If, on the other hand, we trace the field operator (3) with the metric we get,

$$g^{\mu\nu}E_{\mu\nu} = \nabla^2 h - \nabla^\mu\nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2}\right)h. \qquad (10)$$

By comparing (9) and (10) one sees that the linear combination,

$$2\nabla^\mu\nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu}E_{\mu\nu} = \frac{m^2}{2}(2\Lambda - 3m^2)h, \qquad (11)$$

does not contain any second order derivatives (and in fact no first order derivatives either), and hence constitutes a scalar constraint reading (assuming $m^2 \neq 2\Lambda/3$)

$$(2\Lambda - 3m^2)h \simeq 0. \qquad (12)$$

For general parameters this constraint implies $h \simeq 0$ which together with (8) simplifies the vector constraint into $\nabla^\mu h_{\mu\nu} \simeq 0$ and shows that $h_{\mu\nu}$ is transverse-traceless in vacuum. By enforcing these constraints the equations of motion (3) are reduced to the following system,

$$\begin{aligned}&(\nabla^2 - m^2)h_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma h_{\rho\sigma} \simeq 0,\\ &\nabla^\mu h_{\mu\nu} \simeq 0, \qquad h \simeq 0.\end{aligned} \qquad (13)$$

Hence, on generic Einstein spacetime the above theory describes a massive graviton with five degrees of freedom.

An exception to the above arises in two cases: $m^2 = 0$ and $2\Lambda = 3m^2$. When $m^2 = 0$, the case of a massless graviton, we do not have any direct constraint (8) but instead a Noether identity which signals a vector gauge symmetry, which is nothing but the linearized diffeomorphism symmetry of general relativity. The case $2\Lambda = 3m^2$ is when the mass saturates the so-called Higuchi bound [19]. In this case, the linear combination (11) vanishes off-shell, and (11) becomes a Noether identity signaling the presence of a new scalar gauge symmetry [21,22]; the field equations are invariant under $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$, where $\Delta h_{\mu\nu}$ is given by

$$\begin{aligned}\Delta h_{\mu\nu} &= \left(\nabla_\mu\nabla_\nu + \frac{m^2}{2}g_{\mu\nu}\right)\xi(x)\\ &= \left(\nabla_\mu\nabla_\nu + \frac{\Lambda}{3}g_{\mu\nu}\right)\xi(x),\end{aligned} \qquad (14)$$

for an arbitrary scalar gauge function $\xi(x)$. This is a symmetry of the quadratic action (5); for an arbitrary variation $\delta h_{\mu\nu}$ of $h_{\mu\nu}$, the variation of the action (5) is given by

$$\delta S = -2M_{\rm Pl}^2 \int d^4x\sqrt{|g|}E^{\mu\nu}\delta h_{\mu\nu}. \qquad (15)$$

Inserting (14) for the variation of $h_{\mu\nu}$, $\delta h_{\mu\nu} = \Delta h_{\mu\nu}$, integrating by parts, and using the off-shell vanishing of the right hand side of (11) whenever the Higuchi bound is saturated (i.e. $2\Lambda = 3m^2$), we see that δS vanishes for the variation $\Delta h_{\mu\nu}$. A gauge symmetry removes two degrees of freedom (in contrast to a constraint which removes only one), and so we are left with a total of $10 - 4 - 2 = 4$ degrees of freedom. This can be interpreted as a scalar mode becoming a non-propagating pure gauge mode at the enhanced symmetry point, leaving just the four helicity modes ($\pm 2, \pm 1$). This is a partially massless graviton.

B. Massive graviton on an arbitrary background

Going beyond Einstein backgrounds, a theory for a massive graviton $h_{\mu\nu}$ on an arbitrary background spacetime with metric $g_{\mu\nu}$ has been obtained in Refs. [71–73]. It is written using a tensor $S_{\mu\nu}$ which is defined in terms of the Ricci curvature $R_{\mu\nu}$ of the metric by the relation[3]

[3]If we define another rank two tensor, $f_{\mu\nu}$, out of the metric $g_{\mu\nu}$ and the tensor $S_{\mu\nu}$ by the following relation

$$f_{\mu\nu} = g_{\mu\sigma}S^\sigma{}_\rho S^\rho{}_\nu, \qquad (16)$$

then this tensor is what is usually called the "reference metric" in the framework of the de Rham, Gabadadze and Tolley (dRGT) theory [9,10,82–86]. However, we stress that we use here a point of view where this reference metric plays no role since we only consider a linear theory defined on a spacetime endowed with a single metric $g_{\mu\nu}$ and all quantities of interest concerning the background spacetimes (e.g. the curvature tensor, the covariant derivatives...) are defined with respect to $g_{\mu\nu}$.

$$R^\mu{}_\nu = m^2\left[\left(\beta_0+\frac{1}{2}e_1\beta_1\right)\delta^\mu_\nu + (\beta_1+\beta_2 e_1)S^\mu{}_\nu - \beta_2(S^2)^\mu{}_\nu\right], \tag{17}$$

where β_0, β_1 and β_2 are dimensionless parameters, and $(S^2)^\mu{}_\nu$ denotes the square of the tensor $S^\mu{}_\nu$ considered as a matrix, $(S^2)^\mu{}_\nu = S^\mu{}_\rho S^\rho{}_\nu$. Note that the tensor $S_{\mu\nu}$ is symmetric. This can be seen in several ways, one being to notice that (17) can be formally solved for $S_{\mu\nu}$ and obtain an infinite series consisting of powers of $R_{\mu\nu}$. Since $R_{\mu\nu}$ is symmetric and powers of a symmetric matrix are symmetric, $S_{\mu\nu}$ is also symmetric, as well as algebraic in $R_{\mu\nu}$. Also, since the inverse of a symmetric matrix is symmetric, $(S^{-1})^{\mu\nu}$, when it exists, is also symmetric. The theory is expressed using the elementary symmetric polynomials $e_n(S)$, $n = 0, \ldots, 4$

$$e_0 = 1, \tag{18}$$

$$e_1 = S^\rho{}_\rho, \tag{19}$$

$$e_2 = \frac{1}{2}(S^\rho{}_\rho S^\nu{}_\nu - S^\rho{}_\nu S^\nu{}_\rho), \tag{20}$$

$$e_3 = \frac{1}{6}(S^\rho{}_\rho S^\nu{}_\nu S^\mu{}_\mu - 3S^\mu{}_\mu S^\rho{}_\nu S^\nu{}_\rho + 2S^\rho{}_\nu S^\nu{}_\mu S^\mu{}_\rho), \tag{21}$$

$$e_4 = \det(S). \tag{22}$$

The theory obtained in [71–73] for a massive graviton $h_{\mu\nu}$ on an arbitrary background has the vacuum field equations

$$\begin{aligned} E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} + \frac{m^2}{2}[&2(\beta_0+\beta_1 e_1+\beta_2 e_2)h_{\mu\nu} \\ &- (\beta_1+\beta_2 e_1)(h_{\mu\rho}S^\rho{}_\nu + h_{\nu\rho}S^\rho{}_\mu) \\ &- (\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu})h_{\rho\sigma}S^{\rho\sigma} + \beta_2 g_{\mu\nu}h_{\rho\sigma}(S^2)^{\rho\sigma} \\ &- (\beta_1+\beta_2 e_1)(g_{\mu\rho}\delta S^\rho{}_\nu + g_{\nu\rho}\delta S^\rho{}_\mu)] \simeq 0, \end{aligned} \tag{23}$$

where we have defined the linearized Einstein operator

$$\begin{aligned} \mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} \equiv -\frac{1}{2}[&\delta^\rho_\mu\delta^\sigma_\nu\nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu\nabla^\sigma\nabla_\nu - \delta^\rho_\nu\nabla^\sigma\nabla_\mu \\ &- g_{\mu\nu}g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma + \delta^\rho_\mu\delta^\sigma_\nu R - g_{\mu\nu}R^{\rho\sigma}]h_{\rho\sigma}, \end{aligned} \tag{24}$$

and the tensor δS has components given by

$$\begin{aligned} \delta S^\lambda{}_\mu = \frac{1}{2}g^{\nu\lambda}[&e_4c_1(\delta^\rho_\nu\delta^\sigma_\mu + \delta^\sigma_\nu\delta^\rho_\mu - g_{\mu\nu}g^{\rho\sigma}) + e_4c_2(S^\rho_\nu\delta^\sigma_\mu + S^\sigma_\nu\delta^\rho_\mu - S_{\mu\nu}g^{\rho\sigma} - g_{\mu\nu}S^{\rho\sigma}) - e_3c_1(\delta^\rho_\nu S^\sigma_\mu + \delta^\sigma_\nu S^\rho_\mu) \\ &+ (e_2c_1 - e_4c_3 + e_3c_2)S_{\mu\nu}S^{\rho\sigma} + e_4c_3[\delta^\sigma_\mu[S^2]^\rho_\nu + \delta^\rho_\mu[S^2]^\sigma_\nu - g^{\rho\sigma}[S^2]_{\mu\nu} + \delta^\rho_\nu[S^2]^\sigma_\mu + \delta^\sigma_\nu[S^2]^\rho_\mu - g_{\mu\nu}[S^2]^{\rho\sigma}] \\ &- e_3c_2(S^\rho_\nu S^\sigma_\mu + S^\sigma_\nu S^\rho_\mu) - e_3c_3(S^\sigma_\mu[S^2]^\rho_\nu + S^\rho_\mu[S^2]^\sigma_\nu + S^\rho_\nu[S^2]^\sigma_\mu + S^\sigma_\nu[S^2]^\rho_\mu) + (e_3c_3 - e_1c_1)(S^{\rho\sigma}[S^2]_{\mu\nu} + S_{\mu\nu}[S^2]^{\rho\sigma}) \\ &- (c_1 - e_2c_3)([S^2]^\rho_\nu[S^2]^\sigma_\mu + [S^2]^\sigma_\nu[S^2]^\rho_\mu) + c_4[S^2]_{\mu\nu}[S^2]^{\rho\sigma} + c_1([S^3]_{\mu\nu}S^{\rho\sigma} + S_{\mu\nu}[S^3]^{\rho\sigma}) \\ &+ c_2([S^3]_{\mu\nu}[S^2]^{\rho\sigma} + [S^2]_{\mu\nu}[S^3]^{\rho\sigma}) + c_3[S^3]_{\mu\nu}[S^3]^{\rho\sigma}]h_{\rho\sigma}, \end{aligned} \tag{25}$$

with the coefficients c_i are given by

$$c_1 = \frac{e_3 - e_1e_2}{-e_1e_2e_3 + e_3^2 + e_1^2e_4}, \qquad c_2 = \frac{e_1^2}{-e_1e_2e_3 + e_3^2 + e_1^2e_4},$$
$$c_3 = \frac{-e_1}{-e_1e_2e_3 + e_3^2 + e_1^2e_4}, \qquad c_4 = \frac{e_3 - e_1^3}{-e_1e_2e_3 + e_3^2 + e_1^2e_4}. \tag{26}$$

This theory can only be formulated if the tensor S is such that the spectrum of eigenvalues of S, $\sigma(S)$, and the spectrum of its negative, $\sigma(-S)$, do not intersect, i.e. we should have $\sigma(S) \cap \sigma(-S) = \emptyset$. This is generically the case and will be checked for the solutions we present further down. This additionally ensures that the denominators in Eqs. (26) do not vanish. It also implies that the tensor $S^\lambda{}_\mu$, considered as a matrix, is invertible which will be used later. Since the equations are linear in $h_{\mu\nu}$, the theory has a quadratic action given by

$$S = -M_{\mathrm{Pl}}^2\int \mathrm{d}^4x\sqrt{|g|}h^{\mu\nu}E_{\mu\nu}. \tag{27}$$

Note that the above theory was derived in Refs. [71–73] by linearizing the nonlinear ghost-free massive gravity theory of de Rham, Gabadadze and Tolley [10] (dRGT) on an arbitrary background [87]. However, the details of this derivation are inessential for our present purposes and will not be used here. Rather, we will just use the linear theory as defined here as our starting point.

C. Covariant constraint counting on a general background

A key feature of this theory is that, as shown using purely covariant constraints in [71–73], it always propagates at most 5 degrees of freedom on generic backgrounds and hence it is free from the linearized version of the "Boulware-Deser" ghost [88] that was present in previous constructions of nonlinear massive gravity or non Fierz-Pauli linear theories [75]. This covariant constraint analysis is crucial for us, as it will allow us to formulate generic covariant conditions for partial masslessness.

The analysis parallels the analysis we reviewed in Sec. II A for the case of Einstein spacetimes. One first notes that, because $\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}$ is the linearization of the Einstein tensor $\mathcal{G}_{\mu\nu}$, the Bianchi identity $\nabla^\mu \mathcal{G}_{\mu\nu} = 0$ implies that

$$\nabla^\mu E_{\mu\nu} \simeq 0 \tag{28}$$

does not involve second order derivatives of $h_{\mu\nu}$ and thus gives vector constraints which are the generalization of Eq. (8). These generalized vector constraints eliminate four degrees of freedom of the graviton.

An extra (scalar) constraint can be found [71–73] and reads

$$\mathcal{C} \equiv (S^{-1})^\nu{}_\rho \nabla^\rho \nabla^\mu E_{\mu\nu} + \frac{m^2\beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0, \tag{29}$$

where we emphasize that the equality on the right-hand side only holds on-shell, while the fact that this is a constraint relies on the crucial property that the left-hand side of this equality has been shown to depend off-shell only on undifferentiated or once differentiated $h_{\mu\nu}$ but not on second or higher order derivatives of $h_{\mu\nu}$.

We now give the full off-shell expression of this constraint. It is most easily expressed using new field variables $\tilde{h}_{\mu\nu}$ defined through the following equation

$$h_{\mu\nu} = (S^\lambda{}_\mu \delta^\beta_\nu + S^\lambda{}_\nu \delta^\beta_\mu)\tilde{h}_{\beta\lambda}.$$

This, for a given $h_{\mu\nu}$, leads to a unique $\tilde{h}_{\mu\nu}$ provided that S obeys the same condition that allows the theory to be well defined, i.e. that $\sigma(S) \cap \sigma(-S) = \emptyset$ [73]. In terms of the new variable $\tilde{h}_{\mu\nu}$, the scalar constraint reads

$$\mathcal{C} = m^2[(A^{\beta\lambda} + \tilde{A}^{\beta\lambda})\tilde{h}_{\beta\lambda} + B^{\beta\lambda}_\rho \nabla^\rho \tilde{h}_{\beta\lambda}], \tag{30}$$

where,

$$A^{\beta\lambda} \equiv m^2 S^\beta{}_\rho \left[\left(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1\right) g^{\rho\lambda} + \left(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2\right) S^{\rho\lambda} - \beta_2^2 e_1 [S^2]^{\rho\lambda}\right], \tag{31}$$

$$\begin{aligned}
\tilde{A}^{\beta\lambda} \equiv\ & \frac{1}{2}(\beta_1 + \beta_2 e_1)[S^{-1}]^\nu_\gamma[-\nabla^\gamma S^{\rho\lambda}\nabla_\nu S^\beta_\rho + \nabla^\gamma S^\beta_\rho \nabla^\lambda S^\rho_\nu + \nabla^\gamma S^\rho_\nu \nabla^\lambda S^\beta_\rho - \nabla^\gamma S_{\rho\nu}\nabla^\rho S^{\beta\lambda} - S^{\rho\lambda}\nabla^\gamma\nabla_\nu S^\beta_\rho + S^\beta_\rho \nabla^\gamma \nabla^\lambda S^\rho_\nu] \\
& + \beta_2 [S^{-1}]^\nu{}_\gamma [S^\beta_\rho \nabla^\lambda S^\rho_\nu \nabla^\gamma e_1 - S^\beta_\rho \nabla_\nu S^{\rho\lambda}\nabla^\gamma e_1 + S^\lambda_\rho \nabla^\gamma S^\beta_\mu \nabla_\nu S^{\rho\mu} + S^\beta_\mu \nabla^\gamma S^\lambda_\rho \nabla_\nu S^{\rho\mu} + S^\lambda_\mu \nabla^\gamma S^{\mu\rho}\nabla_\nu S^\beta_\rho + S^{\mu\rho}\nabla^\gamma S^\lambda_\mu \nabla_\nu S^\beta_\rho \\
& - 2S^\beta_\mu \nabla^\gamma S^{\mu\lambda}\nabla_\nu e_1 - S^\rho_\mu \nabla^\gamma S^\mu_\nu \nabla^\beta S^\lambda_\rho - S^\beta_\mu \nabla^\gamma S^\mu_\rho \nabla^\lambda S^\rho_\nu - S^\mu_\rho \nabla^\gamma S^\beta_\mu \nabla^\lambda S^\rho_\nu - S^\beta_\rho \nabla^\gamma S^\mu_\nu \nabla^\lambda S^\rho_\mu + S^\beta_\mu \nabla^\gamma S^\mu_\nu \nabla^\lambda e_1 + S^\mu_\rho \nabla^\gamma S_{\mu\nu}\nabla^\rho S^{\beta\lambda} \\
& - S^\lambda_\mu \nabla^\gamma S^\mu_\nu \nabla^\rho S^\beta_\rho - S^\lambda_\mu \nabla^\gamma S^\beta_\rho \nabla^\mu S^\rho_\nu - S^\beta_\rho \nabla^\gamma S^\lambda_\mu \nabla^\mu S^\rho_\nu - S^\lambda_\mu \nabla^\gamma S^\rho_\nu \nabla^\mu S^\beta_\rho + 2S^\beta_\mu \nabla^\gamma S^{\mu\lambda}\nabla^\rho S_{\rho\nu} + 2S^\beta_\rho \nabla^\gamma S_{\mu\nu}\nabla^\mu S^{\rho\lambda} \\
& + S^\lambda_\rho S^\beta_\mu \nabla^\gamma \nabla_\nu S^{\rho\mu} + [S^2]^{\lambda\rho}\nabla^\gamma\nabla_\nu S^\beta_\rho - [S^2]^{\beta\lambda}\nabla^\gamma\nabla_\nu e_1 - [S^2]^\beta_\rho \nabla^\gamma\nabla^\lambda S^\rho_\nu - S^\lambda_\mu S^\beta_\rho \nabla^\gamma\nabla^\mu S^\rho_\nu + [S^2]^{\beta\lambda}\nabla^\gamma\nabla^\rho S_{\rho\nu}] \\
& + \beta_2[+\nabla^\beta S^\lambda_\gamma \nabla^\gamma e_1 - \nabla_\gamma S^{\beta\lambda}\nabla^\gamma e_1 - \nabla^\mu S^\rho_\mu \nabla^\beta S^\lambda_\rho - \nabla^\mu S^\beta_\rho \nabla^\lambda S^\rho_\mu + \nabla^\mu S^\beta_\mu \nabla^\lambda e_1 + \nabla_\mu S^\mu_\rho \nabla^\rho S^{\beta\lambda} - \nabla^\mu S^\lambda_\mu \nabla^\rho S^\beta_\rho \\
& - \nabla^\rho S^\lambda_\mu \nabla^\mu S^\beta_\rho + 2\nabla_\mu S^\beta_\rho \nabla^\mu S^{\rho\lambda} - S^\beta_\rho \nabla^\lambda\nabla^\mu S^\rho_\mu + S^\beta_\gamma \nabla^\gamma\nabla^\lambda e_1 - S^\lambda_\gamma \nabla^\gamma \nabla^\rho S^\beta_\rho + S^\beta_\rho \nabla^\gamma\nabla_\gamma S^{\rho\lambda}] + (\beta \leftrightarrow \lambda),
\end{aligned} \tag{32}$$

$$\begin{aligned}
B^{\beta\lambda}_\rho \equiv\ & \frac{1}{2}(\beta_1 + \beta_2 e_1)[S^{-1}]^\nu_\gamma[-S^{\sigma\lambda}\delta^\gamma_\rho \nabla_\nu S^\beta_\sigma + \delta^\gamma_\rho S^\beta_\sigma \nabla^\lambda S^\sigma_\nu + \delta^\lambda_\rho S^\beta_\sigma \nabla^\gamma S^\sigma_\nu - S^{\beta\lambda}\nabla^\gamma S_{\nu\rho}] \\
& + \beta_2[S^{-1}]^\nu{}_\gamma[\delta^\gamma_\rho S^\lambda_\delta S^\beta_\mu \nabla_\nu S^{\delta\mu} + \delta^\gamma_\rho [S^2]^{\lambda\mu}\nabla_\nu S^\beta_\mu - \delta^\gamma_\rho [S^2]^{\beta\lambda}\nabla_\nu e_1 - \delta^\gamma_\rho [S^2]^\beta_\mu \nabla^\lambda S^\mu_\nu - \delta^\gamma_\rho S^\lambda_\mu S^\beta_\delta \nabla^\mu S^\delta_\nu + \delta^\gamma_\rho [S^2]^{\beta\lambda}\nabla^\mu S_{\mu\nu} \\
& + S^{\beta\lambda}S^\mu_\rho \nabla^\gamma S_{\mu\nu} + [S^2]^{\beta\lambda}\nabla^\gamma S_{\rho\nu} - \delta^\beta_\rho [S^2]^\lambda_\mu \nabla^\gamma S^\mu_\nu - S^\beta_\rho S^\lambda_\mu \nabla^\gamma S^\mu_\nu] \\
& + \beta_2[-S^\beta_\delta \nabla^\lambda S^\delta_\rho + S^\beta_\rho \nabla^\lambda e_1 - S^\lambda_\mu \nabla^\mu S^\beta_\rho + 2S^\beta_\delta \nabla_\rho S^{\delta\lambda} + \delta^\beta_\rho S^\lambda_\gamma \nabla^\gamma e_1 - S^{\beta\lambda}\nabla_\rho e_1 + S^{\beta\lambda}\nabla_\mu S^\mu_\rho - \delta^\beta_\rho S^\lambda_\delta \nabla^\mu S^\delta_\mu - S^\beta_\rho \nabla^\mu S^\lambda_\mu] \\
& + (\beta \leftrightarrow \lambda).
\end{aligned} \tag{33}$$

These tensors are all symmetric in β, λ.

Given these constraints (vector plus scalar), the theory propagates at most $10 - 4 - 1 = 5$ degrees of freedom on any background spacetime, appropriate for a massive graviton. One can check that the above constraint (30) degenerates to the form (12) on Einstein spacetimes.

D. Condition for partial masslessness on a generic background spacetime

Following the discussion in Sec. II A for Einstein spacetimes, we now ask whether the analog of partial masslessness can be found on spacetimes more generic than Einstein. We will not be able to find here the most general spacetimes for which this happens, but we will be able to exhibit for the first time explicit examples of non Einstein background spacetimes allowing partial masslessness.

This will happen when the expression for the constraint $\mathcal{C}$ in Eq. (30) vanishes identically off-shell. The off-shell vanishing of $\mathcal{C}$ automatically implies the existence of a Noether identity and hence a new scalar gauge symmetry. Indeed, in this case the explicit form of $\mathcal{C}$ as written in Eq. (29) shows that the action and the equations of motion are invariant under the transformation $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$, where $\Delta h_{\mu\nu}$ is now given by

$$\Delta h_{\mu\nu} = [(S^{-1})_\mu{}^\rho \nabla_\rho \nabla_\nu + (S^{-1})_\nu{}^\rho \nabla_\rho \nabla_\mu + m^2\beta_1 g_{\mu\nu} + 2m^2\beta_2 S_{\mu\nu}]\xi(x), \tag{34}$$

with $\xi(x)$ an arbitrary scalar gauge function. The action is quadratic,

$$S[h] = -M_{\rm Pl}^2 \int \mathrm{d}^4x \sqrt{|g|} h^{\mu\nu} E_{\mu\nu}, \tag{35}$$

so upon varying and using (34), the invariance of the action follows from the vanishing of $\mathcal{C}$. As before, a gauge symmetry removes two degrees of freedom, and so we are left with at most a total of $10 - 4 - 2 = 4$ degrees of freedom.

III. BACKGROUND SPACETIMES FOR PARTIAL MASSLESSNESS

We now proceed to find several classes of background spacetimes for which the scalar constraint vanishes identically so that we have partial masslessness.

A. General solution for models with vanishing β_2

Let us first consider the case of models with vanishing β_2. In this case, we will find that the only possible backgrounds are Einstein spaces.

The vanishing of β_2 in turn implies that β_1 must be non vanishing for a sensible massive graviton theory to be formulated (otherwise, the Eq. (17) does not define a proper tensor $S_{\mu\nu}$). For such a theory, $A^{\beta\lambda}$, $\tilde{A}^{\beta\lambda}$ and $B_\rho^{\beta\lambda}$ are given by

$$A^{\beta\lambda} = \beta_1 \frac{m^2}{2}[(2\beta_0 + \beta_1 e_1)S^{\beta\lambda} - \beta_1 [S^2]^{\beta\lambda}], \tag{36a}$$

$$\tilde{A}^{\beta\lambda} = \frac{1}{2}\beta_1 [S^{-1}]^\nu_\gamma [-\nabla^\gamma S^{\rho\lambda}\nabla_\nu S^\beta_\rho + \nabla^\gamma S^\beta_\rho \nabla^\lambda S^\rho_\nu + \nabla^\gamma S^\rho_\nu \nabla^\lambda S^\beta_\rho - \nabla^\gamma S_{\rho\nu}\nabla^\rho S^{\beta\lambda} - S^{\rho\lambda}\nabla^\gamma\nabla_\nu S^\beta_\rho + S^\beta_\rho \nabla^\gamma \nabla^\lambda S^\rho_\nu] + (\beta \leftrightarrow \lambda), \tag{36b}$$

$$B_\rho^{\beta\lambda} = \frac{1}{2}\beta_1 [S^{-1}]^\nu_\gamma [-S^{\sigma\lambda}\delta^\gamma_\rho \nabla_\nu S^\beta_\sigma + \delta^\gamma_\rho S^\beta_\sigma \nabla^\lambda S^\sigma_\nu + \delta^\lambda_\rho S^\beta_\sigma \nabla^\gamma S^\sigma_\nu - S^{\beta\lambda}\nabla^\gamma S_{\nu\rho}] + (\beta \leftrightarrow \lambda), \tag{36c}$$

and the relation (17) reads

$$S_{\mu\nu} = \frac{1}{m^2\beta_1}\left(R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R - \frac{m^2\beta_0}{3} g_{\mu\nu}\right). \tag{37}$$

The identical vanishing of the constraint (30) implies that we must have

$$B_\rho^{\beta\lambda} = 0. \tag{38}$$

Now, using the definitions (36) we find that

$$B_\rho^{\beta\lambda}(S^{-1})_{\beta\lambda} = \beta_1 [S^{-1}]^\nu_\gamma [-\delta^\gamma_\rho \nabla_\nu S^\sigma_\sigma + \delta^\gamma_\rho \nabla_\sigma S^\sigma_\nu - 3\nabla^\gamma S_{\nu\rho}], \tag{39}$$

while the relation (37) implies that

$$\nabla_\sigma S^\sigma_\nu = \frac{1}{m^2\beta_1}\frac{1}{3}\nabla_\nu R = \nabla_\nu S^\sigma_\sigma. \tag{40}$$

This shows that the first two terms in the bracket of the left-hand side of (39) cancel each other and one is left with

$$B_\rho^{\beta\lambda}(S^{-1})_{\beta\lambda} = -3\beta_1 [S^{-1}]^\nu_\gamma [\nabla^\gamma S_{\nu\rho}]. \tag{41}$$

Thus a necessary condition to get partial masslessness is

$$[S^{-1}]^\nu_\gamma [\nabla^\gamma S_{\nu\rho}] = 0. \tag{42}$$

Now, using $[S^{-1}]^\nu_\gamma S_{\nu\rho} = g_{\gamma\rho}$, we get

$$[S^{-1}]^\nu_\gamma [\nabla^\gamma S_{\nu\rho}] = -S_{\nu\rho}(\nabla^\gamma [S^{-1}]^\nu_\gamma) = 0. \tag{43}$$

The last equality means that $(\nabla^\gamma [S^{-1}]^\nu_\gamma)$ is a vector in the kernel of the (invertible) matrix S^ρ_ν and hence it must vanish. So we must have

$$(\nabla^\gamma [S^{-1}]^\nu_\gamma) = 0, \tag{44}$$

which will be used later. We first note that (42) implies that the last two terms entering in the definition of $B_\rho^{\beta\lambda}$ in Eq. (33) with $\beta_2 = 0$ vanish, hence we are left with the expression

$$B_\rho^{\beta\lambda} = \frac{1}{2}\beta_1 [S^{-1}]^\nu_\rho [-S^{\sigma\lambda}\nabla_\nu S^\beta_\sigma + S^\beta_\sigma \nabla^\lambda S^\sigma_\nu] + (\beta \leftrightarrow \lambda), \tag{45}$$

which must vanish when symmetrized over β and λ. Similarly, after using the condition (42) we are left with the expression for $\tilde{A}^{\beta\lambda}$ given by

$$\tilde{A}^{\beta\lambda} = \frac{1}{2}\beta_1 [S^{-1}]^\nu_\gamma [-\nabla^\gamma S^{\rho\lambda}\nabla_\nu S^\beta_\rho + \nabla^\gamma S^\beta_\rho \nabla^\lambda S^\rho_\nu - S^{\rho\lambda}\nabla^\gamma \nabla_\nu S^\beta_\rho + S^\beta_\rho \nabla^\gamma \nabla^\lambda S^\rho_\nu] + (\beta \leftrightarrow \lambda). \tag{46}$$

Let us then compute $\nabla^\rho B_\rho^{(\beta\lambda)}$. The condition (44) implies that the operator ∇^ρ "goes through" the prefactor of S^{-1} on the r.h.s. of the above Eq. (45), and we find that when the condition (44) [and the equivalent (42)] is obeyed, one has

$$\nabla^\rho B_\rho^{\beta\lambda} = \frac{1}{2}\beta_1[S^{-1}]^\nu_\gamma[-\nabla^\gamma S^{\rho\lambda}\nabla_\nu S^\beta_\rho + \nabla^\gamma S^\beta_\rho \nabla^\lambda S^\rho_\nu - S^{\rho\lambda}\nabla^\gamma\nabla_\nu S^\beta_\rho + S^\beta_\rho \nabla^\gamma\nabla^\lambda S^\rho_\nu] + (\beta \leftrightarrow \lambda). \quad (47)$$

The right-hand side of the above is found to be identical to the r.h.s. of (46) which means that the vanishing of $B_\rho^{\beta\lambda}$ implies the vanishing of $\tilde{A}^{\beta\lambda}$.

After all this, the only remaining condition for partial masslessness is the vanishing of $A^{\beta\lambda}$. Factoring out one power of S and using that S is invertible in the expression for $A^{\beta\lambda}$, this implies that $S_{\mu\nu}$ is proportional to the metric, which in turns shows, using (37), that the spacetime must be an Einstein spacetime satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for some appropriate value of Λ. We show next that this conclusion does not extend to the more general cases with nonvanishing β_2.

B. Sufficient conditions for partial masslessness in the general case

We now return to the general case with nonvanishing β_2, and attempt to find classes of partially massless spacetimes beyond Einstein. Finding the most general spacetimes for which the right-hand side of (30) vanishes identically is a difficult task given in particular the involved form of the derivative expressions in (32) and (33). However, one sees that the form of the constraint $\mathcal{C}$ becomes much simpler if one assumes that $S_{\mu\nu}$ is covariantly constant, i.e. that it obeys

$$\nabla_\rho S_{\mu\nu} = 0. \quad (48)$$

The background spacetime then admits a covariantly constant tensor, namely $S_{\mu\nu}$. We will make this assumption in the rest of this section, stressing that this assumption severely restricts the spacetime, and is not necessarily a property of the most general solution. But, as we will see, it allows us to find spacetimes more generic than the Einstein spacetime where the phenomenon of partial masslessness is known to exist.

There are several important properties that hold whenever S obeys Eq. (48). First, it does not imply that $S_{\mu\nu} \propto g_{\mu\nu}$, and second, it obviously implies that all the scalar invariants made out of the tensor $S_{\mu\nu}$ are constant, and hence in particular, we must have that all of e_1, e_2, e_3 and e_4 are constant. This together with (48) used in the relation (17) implies that one must also have

$$\nabla_\rho R_{\mu\nu} = 0, \quad (49)$$

i.e. that the Ricci tensor is covariantly constant. This last condition defines what is known as a *Ricci symmetric spacetime*. Such spacetimes, which obviously form a subclass of spacetimes with a covariantly constant tensor, have been studied and classified in various contexts (see e.g. [89–93]). In the next subsection, we introduce explicitly some properties of the Ricci-symmetric background spacetimes which will be used later to define our partially massless theory.

C. Some properties of suitable Ricci symmetric backgrounds

The geometry of a spacetime admitting a covariantly constant tensor, say $H_{\mu\nu}$, is constrained by the integrability conditions that derive from the vanishing of the covariant derivatives of the covariantly constant tensor, i.e.

$$\nabla_\rho H_{\mu\nu} = 0. \quad (50)$$

For instance, using that

$$\nabla_{[\mu}\nabla_{\nu]} H_{\rho\lambda} = R_{\mu\nu\rho}{}^\sigma H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^\sigma H_{\rho\sigma}, \quad (51)$$

and inserting (50), we find that such a spacetime must obey the following primary integrability conditions

$$R_{\mu\nu\rho}{}^\sigma H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^\sigma H_{\rho\sigma} = 0. \quad (52)$$

For a Ricci symmetric space (49), when the Ricci tensor is covariantly conserved, the above relation becomes a nontrivial relation involving only the curvature tensor,

$$R_{\mu\nu\rho}{}^\sigma R_{\sigma\lambda} + R_{\mu\nu\lambda}{}^\sigma R_{\rho\sigma} = 0. \quad (53)$$

Furthermore, 4-dimensional (simply connected) spacetimes admitting a covariantly constant symmetric tensor field $H_{\mu\nu}$ can be classified as follows. If $H_{\mu\nu}$ is not a constant multiple of the metric then such spacetimes can be shown to fall in one of the following two categories [89] (see also e.g. [90,93])

(1) The spacetime is $2 \otimes 2$ decomposable, meaning that the metric can be written in the form

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}(x^c)\mathrm{d}x^a\mathrm{d}x^b + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j, \quad (54)$$

where here $a, b, c = 0, 1$ and $i, j, k = 2, 3$. In this case, a symmetric and idempotent covariantly constant tensor $H_{\mu\nu}$ can be found, i.e. a symmetric covariantly constant tensor which satisfies in addition $H_{\mu\rho}H^\rho{}_\nu = H_{\mu\nu}$ (that such a covariantly constant idempotent and symmetric tensor exists is true also for $1 \otimes 3$ decomposable spacetimes but then falls under point 2. below). For spacetimes (54), the Ricci tensor is found to obey

$$R^0{}_0 = R^1{}_1, \qquad R^2{}_2 = R^3{}_3, \tag{55}$$

with all other components vanishing. This follows from the fact that the metric (54) is a direct product of two dimensional metrics, and that the Einstein tensor vanishes identically in $d = 2$ dimensions so that the $2d$ Ricci tensors are proportional to their metrics.

An explicit example of such a spacetime which is also Ricci symmetric and that will be used later is

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{2\mathrm{d}x^0\mathrm{d}x^1}{(1+(R-E)x^0x^1/8)^2} - \frac{2\mathrm{d}x^2\mathrm{d}x^3}{(1-(R+E)x^2x^3/8)^2}, \tag{56}$$

where $(R-E)/4$ is the scalar curvature of the (x^0, x^1)-space and $(R+E)/4$ is the scalar curvature of the (x^2, x^3)-space, where both R and E are constant. As the notation suggests, the constant R is just the Ricci scalar, $R^\rho_\rho = R$, as can be seen explicitly from the expression for the Ricci tensor

$$R^\rho{}_\nu = \frac{1}{4}\mathrm{diag}[(R-E),(R-E),(R+E),(R+E)]. \tag{57}$$

It can be verified that the Ricci tensor satisfies $\nabla_\rho R_{\mu\nu} = 0$. Furthermore one can show, using Eq. (55), that the Ricci tensor obeys the interesting relation

$$(R^2)^\rho{}_\nu = \frac{R}{2}R^\rho{}_\nu - \frac{1}{16}(R^2 - E^2)\delta^\rho_\nu, \tag{58}$$

which can also be deduced directly from the relation (53). The Weyl ($W_{\mu\nu\rho\sigma}$) and Bach ($B_{\mu\nu}$) tensors for this spacetime can also easily be computed and are both nonvanishing; the explicit form of the Bach tensor is

$$B^\rho{}_\nu = \frac{\mathrm{ER}}{24}\mathrm{diag}[-1,-1,1,1]. \tag{59}$$

We see that in order for the Bach tensor to be nonzero we need $ER \neq 0$. In contrast, the case $R = 0$ and $E \neq 0$ provide examples of Bach flat non Einstein spacetimes, which will play a role below. Note further that the spacetimes (56) all belong to the Petrov type D spacetimes[4] and will be referred to in that way in the next section.

[4]The Petrov classification classifies spacetimes into 6 classes depending on the nature of principal null directions, see e.g. [93].

(2) There exists a covariantly constant vector (CCV), N^μ, in terms of which a covariantly constant tensor can be constructed as $H_{\mu\nu} = N_\mu N_\nu$ (to which one can also add an arbitrary constant times the metric). The CCV can either be spacelike, timelike or null and its existence implies an integrability condition similar to (52), given by

$$R_{\mu\nu\rho}{}^\sigma N_\sigma = 0. \tag{60}$$

This implies that

$$R_\mu{}^\nu N_\nu = 0, \tag{61}$$

which in turn implies that N_μ is in the kernel of $R_\mu{}^\nu$. Ricci symmetric spacetimes of interest here and belonging to this class have their line element given by

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 - \epsilon \mathrm{d}w^2}{[1 + R(x^2 + y^2 - \epsilon w^2)/24]^2} + \epsilon \mathrm{d}z^2, \tag{62}$$

where $\epsilon = \pm 1$ depending on whether there is a spacelike $(+)$ or timelike $(-)$ CCV. The Ricci tensor is given by

$$R^\rho{}_\nu = \frac{R}{3}\mathrm{diag}[1,1,1,0]. \tag{63}$$

This satisfies $R^n_{\mu\nu} = (R/3)^n R_{\mu\nu}$. In fact, for these spacetimes we have that

$$R_{\mu\nu} = \frac{R}{3}(g_{\mu\nu} - \epsilon N_\mu N_\nu), \tag{64}$$

where N^μ is a CCV of square norm equal to ϵ. In the case where N^μ is timelike, i.e. $\epsilon = -1$, the above spacetime is an Einstein static spacetime. These spacetime are both Bach flat and conformally flat, i.e. have vanishing Bach and Weyl tensors. The spacetimes (62) all belong to the Petrov type O class and will be referred to in that way in the next section. We note that other Ricci symmetric spacetimes admitting a CCV can be found, in particular for a null CCV certain pp-wave spacetimes of Petrov type N (see e.g. [93]). However, considering some specific examples of this type, we did not find that those allowed for partial masslessness.

To summarize, the Ricci symmetric spacetimes which will be shown below to admit partial masslessness will be either the Petrov type D spacetimes (56) or the Petrov type O spacetimes (62). The later type includes the well known Einstein static Universe. These spacetimes all obey an interesting idempotency-like relation for the Ricci tensor, reading

$$(R^2)^\rho{}_\nu = r_1 R^\rho{}_\nu + r_2 \delta^\rho_\nu, \tag{65}$$

where the constant r_1 and r_2 are given in the different cases by

$$\text{type D}: \ r_1 = \frac{R}{2}, \qquad r_2 = -\frac{1}{16}(R^2 - E^2),$$
$$\text{type O}: \ r_1 = \frac{R}{3}, \qquad r_2 = 0. \tag{66}$$

D. Ricci symmetric spacetimes admitting partial masslessness

1. Conditions for partial masslessness on Ricci symmetric spacetimes

In the case where (48) [and hence also (49)] is obeyed, and for generic values of β_0, β_1 and β_2, the scalar constraint (30) reads simply $\mathcal{C} = m^2 A^{\beta\lambda}\tilde{h}_{\beta\lambda}$, so that the condition to get a partially massless graviton is the vanishing of $A^{\beta\lambda}$. Assuming that S is invertible and factoring out one power of S we get that this vanishing reads

$$\delta^\beta_\lambda\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2}e_1\right) + S^\beta_\lambda\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^\beta_\lambda(e_1\beta_2^2) = 0. \tag{67}$$

This equation together with the relation between $S_{\mu\nu}$ and $R_{\mu\nu}$ given by (17) form the system of equation we have to solve. Note that when $\beta_1 = 0$, taking the trace of Eq. (67) implies that $\beta_0\beta_2 e_1 = 0$, which further implies $\beta_0 e_1 = 0$ as we have seen that $\beta_2 = 0$ is not admissible. Then $e_1 = 0$ or $\beta_0 = 0$ only yields Einstein spacetime solutions, as can be seen from Eq. (67). Note that when $e_1 = 0$ the previous conclusion still hold, even when β_1 is not vanishing. Thus we now assume that $\beta_1, \beta_2, e_1 \neq 0$.

When (67) is obeyed, one can in general obtain $(S^2)_{\mu\nu}$ as a linear combination of $S_{\mu\nu}$ and the metric. Using this in turn in Eq. (17), one obtains a linear relation between the tensor $S_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the metric $g_{\mu\nu}$,

$$R^\rho{}_\nu = \frac{m^2}{\beta_2 e_1}\left[\left(-\beta_0\beta_1 + \frac{1}{2}\beta_1\beta_2 e_1^2 - \frac{1}{2}\beta_1^2 e_1\right)\delta^\rho_\nu + \left(\beta_1\beta_2 e_1 + 2\beta_0\beta_2 + \frac{1}{2}\beta_1^2 + 2\beta_2^2 e_2\right)S^\rho{}_\nu\right]. \tag{68}$$

This can be used back in Eq. (67) to obtain a nontrivial relation between the metric tensor, the Ricci tensor and the square of the Ricci tensor. This relation is precisely of the type (65) found above and explains why Ricci symmetric spacetimes of the kind introduced previously are suitable backgrounds allowing the propagation of a partially massless graviton.

We introduce the following notations that will be used later; we define the parameters u, v and c by

$$u \equiv \frac{\beta_0\beta_2}{\beta_1^2}, \qquad v \equiv m^2\beta_0\frac{16u^2 - 24u - 3}{u(4u+3)}, \qquad \text{and} \quad c \equiv \beta_1^2(4u+3). \tag{69}$$

Note that u and c are dimensionless, while v has mass dimension two. The relation (68) between the Ricci tensor and the tensor $S_{\mu\nu}$ can then be written as

$$R^\rho{}_\nu = \frac{1}{8}\left[v\delta^\rho_\nu + \frac{c}{2\beta_0\beta_1}(v - 2R)S^\rho{}_\nu\right]. \tag{70}$$

From this expression it is obvious that we must demand $(v - 2R) \neq 0$ in order not to get an Einstein spacetime (i.e. $R^\rho{}_\nu \propto \delta^\rho_\nu$). From this linear relation we can also obtain $S^\rho{}_\nu$ in terms of $R^\rho{}_\nu$

$$S^\rho{}_\nu = \frac{2\beta_0\beta_1}{c(v-2R)}[8R^\rho{}_\nu - v\delta^\rho_\nu], \tag{71}$$

and also its square

$$(S^2)^\rho{}_\nu = \frac{4\beta_0^2\beta_1^2}{c^2(v-2R)^2}[64(R^2)^\rho{}_\nu - 16vR^\rho{}_\nu + v^2\delta^\rho_\nu]. \tag{72}$$

This allows us to compute any quantity that depends on $S^\rho{}_\nu$ in terms of $R^\rho{}_\nu$. In particular, Eq. (67) can be rewritten as a condition on the curvature, which reads

$$u[v^2(16u^2 + 12u - 9) - 8vR(8u^2 + 6u - 3) + 4R^2\beta_2(16u^2 + 8u - 3)]g^{\beta\lambda} + \left[v(-64u^3 + 48u^2 + 132u + 9) + 4R(64u^3 - 16u^2 - 108u - 9) - \frac{4c^2R^2\beta_2}{m^2\beta_1^2}\right]R^{\beta\lambda} - 256u^2[R^2]^{\beta\lambda} = 0. \tag{73}$$

The two Eqs. (67), (17) are then equivalent to the pair (70), (73) whenever the conditions

$$\beta_1 \neq 0, \tag{74a}$$

$$\beta_2 e_1 \neq 0, \tag{74b}$$

$$(v - 2R) \neq 0, \tag{74c}$$

$$c \neq 0, \tag{74d}$$

are obeyed. We have checked that whenever one of these conditions is not fulfilled, the only possible solutions are Einstein spacetimes.A theory admitting partial masslessness on a Ricci symmetric background of the kind introduced in Sec. III C can thus be searched for, first solving for the β_k and m parameters such that (73) holds [considering that the background spacetime curvature also obeys (65)], and then defining the tensor $S_{\mu\nu}$ from the curvature from the Eq. (70). Specifically, inserting (65) into (73) we obtain the following two equations to be solved for u and v

$$\begin{cases} v^2(16u^2+12u-9)-8vR(8u^2+6u-3)+4[(16u^2+8u-3)R^2-64ur_2]=0, \\ v^2(-64u^3+48u^2+132u+9)+4v[(64u^3-16u^2-108u-9)R-64u^2r_1]-4R^2(64u^3-48u^2-84u-9)=0. \end{cases} \tag{75}$$

In order to solve these equations we now restrict to the different Petrov-type spacetimes introduced in Sec. III C.

2. Examples of Ricci symmetric spacetimes with partial masslessness

Petrov type O.—In this case, for which we use the line element (62), we have $r_1=\frac{R}{3}$ and $r_2=0$, and the system of Eqs. (75) reduces to:

$$\begin{cases} v^2(16u^2+12u-9)-8vR(8u^2+6u-3)+4(16u^2R^2+8uR^2-3R^2)=0, \\ v^2(-64u^3+48u^2+132u+9)+4vR\left(64u^3-\frac{112}{3}u^2-108u-9\right)-4(64u^3-48u^2-84u-9)R^2=0. \end{cases} \tag{76}$$

First we investigate the case when $v=0$. The only solution in this case is flat spacetime, $R^{\mu}{}_{\nu}=0$, so we now assume that $v\neq 0$. In order to have a solution, both factors of v^2 are nonzero, and combining these two equations we obtain

$$\left(\frac{2048}{3}u^4+1280u^3+1344u^2+720u+108\right)Rv \\ -(1024u^4+768u^3+1344u^2+1296u+216)R^2=0. \tag{77}$$

As when $R=0$ the system (76) does not admit any solution, we now assume that $R\neq 0$. If the two expressions in parenthesis simultaneously vanish, the solution to the system is given by

$$u=-\frac{3}{4} \quad \text{and} \quad v=\frac{8}{3}R. \tag{78}$$

It further implies that $c=0$, and thus that the spacetime is of Einstein-type. Thus we insist that the two parenthesis do not vanish simultaneously, yielding an expression for v,

$$v=\frac{6(32u^3+42u+9)}{128u^3+144u^2+144u+27}R. \tag{79}$$

Inserting this back into the system (76), we obtain the following equation for u

$$256u^4+1280u^3+4320u^2+144u-135=0. \tag{80}$$

This quartic equation admits two real solutions $u\simeq(-0.2006, 0.1575)$. To each one of these values corresponds a value for v, namely $v\simeq(0.6622R, 1.757R)$.

In addition to the conditions (74), we also have to check that the matrix S is invertible and that its spectrum does not intersect the spectrum of $-S$. For these solutions the corresponding matrix S is given by

$$S^{\rho}{}_{\nu}=\frac{2\beta_0\beta_1}{c(v-2R)}\mathrm{Diag}\left[\frac{8}{3}R-v,\frac{8}{3}R-v,\frac{8}{3}R-v,-v\right]. \tag{81}$$

Then all the conditions on S reduce to

$$v\neq 2R, \qquad v\neq\frac{4R}{3}, \quad \text{and} \quad v\neq\frac{8R}{3}, \tag{82}$$

which are verified by our set of solutions. To summarize, for Petrov-type O spacetimes, we have two distinct sets of partially massless solutions whose parameters are given by

$$\begin{cases} \beta_0\beta_2\simeq-0.2006\beta_1^2 \quad \text{and} \quad m^2\beta_0\simeq-0.119R, \\ \beta_0\beta_2\simeq 0.1575\beta_1^2 \quad \text{and} \quad m^2\beta_0\simeq-0.157R. \end{cases} \tag{83}$$

Note in the above that β_0 can be scaled to one, as it is nonzero and is degenerate with the value of the mass scale m. Hence, for a fixed value of R we obtain two one parameter sets of theories, each parametrized by β_1.

Petrov type D.—In this case we have $r_1=\frac{R}{2}$ and $r_2=-\frac{1}{16}(R^2-E^2)$. The system of equations then reduces to

$$\begin{cases} v^2(16u^2+12u-9)-8vR(8u^2+6u-3)+4[(16u^2+12u-3)R^2-4uE^2]=0, \\ v^2(-64u^3+48u^2+132u+9)+4vR(64u^3-48u^2-108u-9)-4(64u^3-48u^2-84u-9)R^2=0. \end{cases} \quad (84)$$

As for type O we can combine these equations to obtain the following equation (provided that both factors in front of v^2 are not zero)

$$(768u^3+1728u^2+720u+108)Rv - (1536u^3+3456u^2+1440u+216)R^2 + (-1024u^4+768u^3+2112u^2+144u)E^2=0. \quad (85)$$

As we want to avoid having an Einstein spacetime solution, we assume that $E \neq 0$. First we study the particular case when $R=0$. The system of Eqs. (84) then admits six real solutions, given by

$$\begin{cases} u \simeq 1.884 \quad \text{and} \quad v \simeq \pm 0.654E, \\ u \simeq -0.070 \quad \text{and} \quad v \simeq \pm 0.339E, \\ u \simeq -1.604 \quad \text{and} \quad v \simeq \pm 2.159E. \end{cases} \quad (86)$$

We stress that, as can be seen from Eq. (59), the above six solutions provide background solutions which are non Einstein but Bach flat. That such solutions existed was argued impossible in [69].

We now assume also that $R \neq 0$, Eqs. (85) can be solved for v, and we obtain

$$v = 2R + \frac{4u(-9-132u-48u^2+64u^3)}{3R(9+60u+144u^2+64u^3)}E^2. \quad (87)$$

Inserting this into Eqs. (84), we obtain the following equation

$$R^2(-648-4320u-10368u^2-4608u^3) + E^2(81+2376u+18288u^2+11520u^3 - 14592u^5+4096u^6)=0. \quad (88)$$

We can solve this equation for E to obtain

$$E^2 = \frac{72(9+60u+144u^2+64u^3)}{(-9-132u-48u^2+64u^3)^2}R^2, \quad (89)$$

which when inserted into Eq. (87) gives

$$v = \left(2 + \frac{96u}{-9-132u-48u^2+64u^3}\right)R. \quad (90)$$

For given values of R and E, v and u are determined by the above two Eqs. (89)–(90). Note that one might question the existence of real solutions to these equations. It is however easy to see that it is always possible to find such solutions, at least by setting the value of the ratio $r = R^2/E^2$ to be sufficiently large. Indeed, the only worrisome equation is the Eq. (89) obeyed by u [once u is found, it determines a value of v through Eq. (90)], which can be rewritten as

$$4096u^6-61440u^5-14592u^4+(11520-4608r)u^3 + (18288-10368r)u^2+(2376-4300r)u + 81-648r=0. \quad (91)$$

A large enough value of r shows that the sextic polynomial to solve is negative at the origin and diverges to $+\infty$ for large values of u and hence the sextic equation must have in this case at least two real solutions.

We now look after the conditions that the matrix S should fulfill. For the solutions given by (89)–(90), as well as for the solutions (86), the matrix S takes the form

$$S^\rho{}_\nu = \frac{2\beta_0\beta_1}{c(v-2R)}\mathrm{Diag}[2(R-E)-v, 2(R-E)-v, 2(R+E)-v, 2(R+E)-v]. \quad (92)$$

In order for S to be invertible and not to have common eigenvalues with $-S$, the following conditions need to be verified,

$$v \neq 2(R-E), \qquad v \neq 2(R+E), \quad \text{and} \quad v \neq 2R. \quad (93)$$

These are obviously true for the solutions (86) (which have vanishing R), but are nontrivial for the solutions (89)–(90). In this latter case these conditions translate into some forbidden values for u, that in turn correspond to some forbidden values for E and R. More precisely we obtain the following prohibited set of parameters:

$$u \neq 0, \quad \text{corresponding to} \quad E^2 \neq 8R^2, \quad (94a)$$

$$u \neq -\frac{3}{4}, \quad \text{corresponding to} \quad E^2 \neq R^2, \quad (94b)$$

$$u \neq -\frac{1}{4}, \quad \text{corresponding to} \quad E^2 \neq \frac{9}{25}R^2, \quad (94c)$$

$$u \neq -1.764, \quad \text{corresponding to} \quad E^2 \neq 0. \quad (94d)$$

To summarize, for Petrov-type D spacetimes, we have an infinite number of solutions. More precisely, for any given value of the set (R, E) [except for the forbidden values of Eq. (94)], we have up to six different solutions given by Eqs. (89) and (90), or for the case $R=0$ by (86).

E. Example of a non-Einstein PM theory

We can now write explicitly the field equations for the solutions we have obtained. For simplicity's sake, we only give here the expression for one specific example of the type O solution, the Einstein static universe. We consider the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2 \mathrm{d}\Sigma^2, \tag{95}$$

where the metric on the 3-dimensional space is given in spherical coordinates by $\mathrm{d}\Sigma^2 = \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$, with $k = 0, \pm 1$ the curvature. For this metric we have

$$R^t_t = 0, \quad \text{and} \quad R^r_r = R^\theta_\theta = R^\phi_\phi = \frac{2k}{a^2}, \tag{96}$$

$$R = \frac{6k}{a^2}. \tag{97}$$

We can then write the field equations for a partially massless graviton propagating on an Einstein static universe, and get

$$E_{tt} = \mathcal{D}_{tt}{}^{\rho\sigma}h_{\rho\sigma} + \frac{1}{6(3+4u)(-3-24u+16u^2)(v-2R)^2}\Big[(-18uv^3 + 24u^2v^3 + 96u^3v^3 + 27v^2R + 468uv^2R - 48u^2v^2R - 576u^3v^2R - 108vR^2 - 1368uvR^2 - 256u^2vR^2 + 1152u^3vR^2 + 108R^3 + 1008uR^3 + 576u^2R^3 - 768u^3R^3)\tilde{h} - \frac{1}{3}(54uv^3 - 27v^2R - 504uv^2R - 96u^2v^2R + 192u^3v^2R + 108vR^2 + 1296uvR^2 + 448u^2vR^2 - 768u^3vR^2 - 108R^3 - 1008uR^3 - 576u^2R^3 + 768u^3R^3)h_{tt}\Big], \tag{98a}$$

$$E_{ti} = \mathcal{D}_{ti}{}^{\rho\sigma}h_{\rho\sigma} + \frac{1}{6(3+4u)(-3-24u+16u^2)(3v-4R)(v-2R)^2}[-108uv^4 + 72u^2v^4 + 288u^3v^4 + 81v^3R + 1440uv^3R - 96u^2v^3R - 2112u^3v^3R - 432v^2R^2 - 5304uv^2R^2 - 1472u^2v^2R^2 + 5760u^3v^2R^2 + 756vR^3 + 7728uvR^3 + 3776u^2vR^3 - 6912u^3vR^3 - 432R^4 - 4032uR^4 - 2304u^2R^4 + 3072u^3R^4]h_{ti}, \tag{98b}$$

$$E_{ij} = \mathcal{D}_{ij}{}^{\rho\sigma}h_{\rho\sigma} + \frac{1}{3(3+4u)(-3-24u+16u^2)(v-2R)^2}\Big[\frac{1}{2}(-36uv^3 + 24u^2v^3 + 96u^3v^3 + 27v^2R + 360uv^2R + 96u^2v^2R - 576u^3v^2R - 108vR^2 - 1080uvR^2 - 640u^2vR^2 + 1152u^3vR^2 + 108R^3 + 1008uR^3 + 576u^2R^3 - 768u^3R^3)h_{ij} + (uv^2(9v - 18R + 8uR))a^2\gamma_{ij}h_{tt} + \frac{1}{6}(54uv^3 - 27v^2R - 504uv^2R - 192u^2v^2R + 192u^3v^2R + 108vR^2 + 1296uvR^2 + 704u^2vR^2 - 768u^3vR^2 - 108R^3 - 1008uR^3 - 576u^2R^3 + 768u^3R^3)a^2\gamma_{ij}\tilde{h}\Big], \tag{98c}$$

where we have defined $\tilde{h} \equiv \delta^i_j h^j_i$. The parameters (u, v) are given by the solutions (83). Looking, for example, at the first set of solutions, namely $(u, v) = (-0.2006, 0.6622R)$, we obtain

$$E_{tt} = \mathcal{D}_{tt}{}^{\rho\sigma}h_{\rho\sigma} - 0.0414R\tilde{h} + 0.0926Rh_{tt}, \tag{99a}$$

$$E_{ti} = \mathcal{D}_{ti}{}^{\rho\sigma}h_{\rho\sigma} - 0.4233Rh_{ti}, \tag{99b}$$

$$E_{ij} = \mathcal{D}_{ij}{}^{\rho\sigma}h_{\rho\sigma} + 0.0709Ra^2\gamma_{ij}\tilde{h} - 0.0414Ra^2\gamma_{ij}h_{tt} - 0.5177Rh_{ij}. \tag{99c}$$

These equations of motion can be derived from the quadratic action

$$S = -M_{\mathrm{Pl}}^2 \int \mathrm{d}^4x\sqrt{|g|}h^{\mu\nu}E_{\mu\nu}. \tag{100}$$

Finally the scalar gauge symmetry under which the action is invariant is given by

$$\Delta h_{tt} = \left[-\frac{(4u+3)(v-2R)}{v}\nabla_t\partial_t + \frac{u(6R+8Ru-3v)v}{(-3-24u+16u^2)(-2R+vv)}\right]\xi(x),$$
$$\Delta h_{ti} = -\frac{(4u+3)(v-2R)(\frac{8R}{3}-2v)}{2v(\frac{8R}{3}-v)}\nabla_{(t}\partial_{i)}\xi(x),$$
$$\Delta h_{ij} = \left[-\frac{(4u+3)(v-2R)}{\frac{8R}{3}-v}\nabla_i\partial_j + \frac{uv(-18R+8Ru+9v)}{3(-3-24u+16u^2)(-2R+v)}a^2\gamma_{ij}\right]\xi(x), \quad (101)$$

or, when (u, v) are evaluated at their partially massless value,

$$\Delta h_{tt} = [4.4397\nabla_t\partial_t - 0.0973R]\xi(x),$$
$$\Delta h_{ti} = 1.4865\nabla_{(t}\partial_{i)}\xi(x),$$
$$\Delta h_{ij} = [-1.4668\nabla_i\partial_j - 0.1838Ra^2\gamma_{ij}]\xi(x). \quad (102)$$

IV. CONCLUSIONS

We have shown that there are non-Einstein backgrounds on which a partially massless graviton can propagate. We have done this by taking the construction in [71–73] of a fully massive graviton propagating on an arbitrary background, and finding backgrounds and values of the parameter for which an additional scalar gauge symmetry emerges.

Before this, the only known backgrounds on which a partially massless graviton could propagate were Einstein spaces. Indeed, it was argued in [69] that only Einstein backgrounds can propagate a PM graviton. There were certain assumptions going into this argument, and one of these assumptions was that the mass term for the graviton is at most linear in the curvature of the background spacetime. Our examples here violate this assumption by having mass terms with arbitrarily high powers of the curvature of the background metric (if one expands the mass term in a power series of the curvature). However, there are still at most two derivatives acting on the dynamical field as well as on the background metric. In other words, there are highly nonminimal curvature couplings that allow for PM on non Einstein backgrounds. Note that this allows not only a PM graviton to propagate on some non-Einstein backgrounds, but also on such backgrounds [in the case of solutions (86)] with a vanishing Bach tensor, which was also argued impossible in [69].

There are several questions that this raises. One is the nature of the most general backgrounds allowing for a partially massless mode. We have not attempted to find the most general possible background within the class of models [71–73] which propagate a partially massless mode. We have only found a few restricted classes of backgrounds beyond Einstein spaces, and there may very well be more examples. It would be interesting to find and characterize the most general possible background, and this would give a clue as to the nature of the backgrounds which may emerge from the equations of motion of a putative nonlinear theory.

Another open question is the nature of propagation of the partially massless graviton about these non-Einstein backgrounds. The partially massless graviton on dS space is known to propagate exactly luminally [17]. It would be interesting to see whether this continues to be true for these more general backgrounds.

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