Horizon instability of extremal Reissner-Nordström black holes to charged perturbations

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We investigate the stability of highly charged Reissner-Nordström black holes to charged scalar perturbations. We show that the near-horizon region exhibits a transient instability which becomes the Aretakis instability in the extremal limit. The rates we obtain match the enhanced rates for non-axisymmetric perturbations of the near-extremal and extremal Kerr solutions. The agreement is shown to arise from a shared near-horizon symmetry of the two scenarios.

I. INTRODUCTION

The importance of black holes in modern theoretical physics cannot be understated. At large scales, numerous astrophysical phenomena feature black holes as major players: stellar mass black holes are the end states of "high-mass" stellar collapse [1], supermassive black holes have the capacity to source high-energy jets seen in the centers of galaxies [2], and binary black holes provide a source of gravitational waves, recently detected for the first time [3]. At small scales, black holes provide a testing ground for theories of quantum gravity, which must reproduce the area law of entropy [4,5] at the semiclassical level. Mathematically, the nonlinear stability of black holes is far from understood, though the linear stability problem of black hole solutions has seen tremendous progress in recent years [6–8].

Solutions describing black holes near their extremal limits have attracted additional interest. This may be due in part to the fact that in the extremal limit many problems become analytically tractable. For instance, microstate counting simplifies near the Bogomol'nyi-Prasad-Sommerfield (extremal) bound where certain fields behave classically. Hence, extremal solutions have been important for string theory calculations of black hole entropy [9]. Apart from their analytical tractability [10–20], extremal black holes have uniquely interesting physical features. These include turbulentlike dynamics [21], unique telltale observational features [22–27], and enhanced symmetries shared by certain conformal field theories [28].

A new development came in 2010 when Aretakis proved that extremal horizons are linearly unstable [29–31]. For perturbations of nonextremal black holes the redshift of outgoing radiation at the horizon controls transverse derivatives occurring in energy estimates [32]. However, the redshift factor degenerates in the extremal limit. The lack of a redshift, combined with the existence of conserved quantities on the future horizon [33], generically gives rise

unbounded polynomial growth of sufficiently high-order derivatives at late times [34–36]. The polynomial growth was shown to arise in a mode expansion from a branch point in the Laplace transform at the super-radiant frequency [37].

Physically, extremal solutions occupy a set of measure zero and can never be realized thermodymically in finite time [38]. However, given the continuous dependence of the Kerr solution on the spin parameter, the Aretakis "instability" is exhibited in a tamer form for near-extremal solutions as well [39,40].

Here we consider charged (denoted q) massless scalar field perturbations ψ of a Reissner-Nordström (RN) black hole, the unique static electrovacuum solution in four dimensions.

There are many reasons to study charged perturbations of the RN spacetime. For one, the highly charged RN scenario provides a simplified setting which captures many features of the rapidly spinning astrophysical Kerr black hole. In this paper we show that the linearized near-horizon dynamics of RN, including the horizon instability, share many salient features with Kerr and can be understood within background spherical symmetry. These shared features include a set of long-lived modes and an Aretakis rate enhancement relative to neutral scalar perturbations that is identical to the non-axisymmetric rate enhancement found in [37] for perturbations of Kerr. In fact, many results carry over *exactly* to the Kerr analysis by simply mapping $q \to m$, where m is the azimuthal mode number of a Kerr perturbation. Moreover, the (near-) extremal RN geometry offers an ideal setup to study nonlinear interactions between long-lived modes. Such a study may unearth more evidence of gravitational turbulence in asymptotically flat spaces [21], pushing the limits of the established fluid gravity correspondence [41,42] wherein anti–de Sitter (AdS) black branes have been shown to exhibit turbulent features [43,44]. When allowing for a negative cosmological constant, yet another reason to study charged fields in (near-) extremal Reissner-Nordström comes from the AdS/CMT correspondence [45], as it may inform our understanding of novel low-temperature strongly coupled states of matter [46,47].

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The scalar field ψ is decomposed into a set of discrete spherical harmonic modes (ℓ, m) on $\mathbb{S}^2$. It is also meromorphically extended into the complex frequency plane through a Laplace transform, which we invert to resolve the time dependence. The near-horizon field modes are shown to organize into spaces (modules) imaged by the action of the near-horizon symmetry group. The representations are labeled in part by the so-called "conformal weight" h. Through the field equations, we find that the conformal weight is related to the (eigenvalues of the) $SO(3)$ and $U(1)$ Casimirs, $K_\ell := \ell(\ell + 1)$ and q^2, by

$$h = \frac{1}{2} + \sqrt{\frac{1}{4} + K_\ell - q^2}. \tag{1}$$

Our main results apply to modes for which $h \in \mathbb{C}$. In the language of representation theory, these are the "principal series" representations [48]. The principal representations were shown to dominate the near-horizon response of the rapidly spinning Kerr black hole [40], where the azimuthal mode number m plays the role of q in (1). In the charged case studied here, we again find that the principal modes run the show, giving rise to horizon dynamics replicating extremal Kerr.

One explanation of the similarities with the Kerr scenario lies in the shared presence of a critical frequency at which the horizon flux is zero—the super-radiant bound. The perturbation spectra of both (near-extremal) spacetimes feature a family of weakly damped quasinormal modes clustered around the super-radiant bound frequency [24,25,49–58]. The collective excitation of these slowly decaying quasinormal modes gives rise to a steeply graded field with large local energy density measured near the horizon. In the extremal limit, a confluence of these resonant modes forms a branch point precisely at the super-radiant bound frequency. In this paper we analytically compute the field arising from the long-lived quasinormal modes (near-extremal case) and branch cut integral (extremal case), presenting decay rates for each.

We follow the conventions of [59] and use geometric units $G = c = 1$.

II. FIELD EQUATIONS

We consider the charged scalar field

$$D_\alpha D^\alpha \psi = 0, \tag{2}$$

where $D_\alpha = \nabla_\alpha - iqA_\alpha$, as a linear perturbation of the Reissner-Nordström spacetime. The field ψ is constructed via the Green function governing the linear response to suitable initial data or a prescribed compact source. The Green function is defined as the distributional solution of the adjoint equation

$$D_\alpha^* D^{\alpha*} G(X, X') = \delta_4(X, X'), \tag{3}$$

where $\delta_4(X, X')$ is the covariant Dirac distribution and "$*$" signifies taking the complex conjugate. We choose $G(X, X')$ to vanish when the spacetime point X is not in the chronological future of X'. With this choice, the Kirchhoff representation of the "forward" solution in the absence of sources is

$$\psi = \int_\Sigma (\psi_\Sigma D_\alpha^* G^* - G^* D_\alpha \psi_\Sigma) n^\alpha \sqrt{h} d^3 y \tag{4}$$

for given compact support initial data ψ_Σ and $n^\alpha D_\alpha \psi_\Sigma$ on the spacelike surface Σ with future directed normal n^α and intrinsic metric h_{jk}. We restrict to initial data supported away from the future outer horizon.

In ingoing coordinates $x^\mu = (v, r, \theta, \phi)$, the background metric and gauge field are given by

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2, \tag{5}$$

$$A = -\frac{Q}{r}dv, \tag{6}$$

where $f(r) = 1 - 2M/r + Q^2/r^2 = (r - r_-)(r - r_+)/r^2$ with r_+ and r_- denoting the locations of the inner and outer horizons, $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The quantities Q and M are the ADM charge and mass of the black hole, respectively.

To separate variables[1] we adopt a mode decomposition

$$G^* = \frac{1}{2\pi} \sum_{\ell m} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi')$$
$$\times \int_{ic-\infty}^{ic+\infty} e^{-i\omega v} \tilde{g}(r, r') \mu(r') d\omega, \tag{7}$$

where $Y_{\ell m}$ are spherical harmonics satisfying $\Delta_{\mathbb{S}^2} Y_{\ell m} = -K_\ell Y_{\ell m}$ and $\tilde{g}(r, r')$ is the "transfer function" obeying

$$fr^2 \partial_r^2 \tilde{g}(r, r') + 2(r - M - ir(\omega r - qQ)) \partial_r \tilde{g}(r, r')$$
$$+ (-K_\ell + i(qQ - 2\omega r)) \tilde{g}(r, r') = \mu^{-1}(r) \delta(r - r'). \tag{8}$$

We have also introduced a "weight factor" given by

$$\mu(r) = \frac{1}{r^2 f} \exp\left(\int 2(r - M - ir(\omega r - qQ))/(r^2 f) dr\right) \tag{9}$$

$$= e^{-2i\omega r_*} \exp\left(2iqQ \int \frac{dr}{rf}\right) \tag{10}$$

$$\sim e^{-2i\omega r_*}(r/r_+ - 1)^{2iqQ}, \qquad r \to \infty, \tag{11}$$

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[1]Here we are separating the adjoint of (3).

where $r_* = \int dr/f$.[2] The quantity c is a small positive constant chosen to put the integration contour in a strip where $\tilde{g}$ is holomorphic.

A suitable transfer function consistent with the causal conditions imposed on G may be constructed from homogeneous solutions R_{in} and R_{up}, which are defined as having no incoming radiation from the past horizon (in) and past null infinity (up). To construct this quantity explicitly, we employ the "variation of parameters" formula

$$\tilde{g}(r, r') = \frac{R_{\text{in}}(r_<) R_{\text{up}}(r_>)}{\mathcal{W}}, \qquad (12)$$

making use of the notation $r_< := \min(r, r')$, $r_> := \max(r, r')$, and introducing the scaled Wronskian $\mathcal{W} = r^2 f \mu(r)(R^{\text{in}} \partial_r R^{\text{up}} - R^{\text{up}} \partial_r R^{\text{in}})$ such that $\partial_r \mathcal{W} = 0$. As defined, the in solution is regular on the future horizon and the up solution is regular at infinity.

Following [10], homogeneous radial solutions are obtained in two asymptotic regions, the near and far zones, and then matched in a region of common overlap: the "buffer." The existence of a buffer region relies on the presence of an additional parameter deemed small. When the black hole is extremal, a sufficiently small parameter arises in our analysis as the deviation of the frequency of the perturbation from the super-radiant bound frequency. This is a simple but profound consequence of the fact that waves of all frequencies defined with respect to the near-horizon geometry (with frequency conjugate to the near-horizon time coordinate) limit to the super-radiant bound when viewed in coordinates adapted to the asymptotically flat region. We thus pattern our analysis on [11], obtaining asymptotic solutions by solving the radial equation directly in the limiting spacetimes [limiting sections on the $U(1)$ bundle]. In doing so we make use of the near-horizon symmetries to streamline our derivation of the Aretakis instability.

III. HOMOGENEOUS SOLUTIONS

A. Near-zone solutions

1. Near-extremal near-zone solution

We now derive the ingoing near-zone solution by solving the radial equation near the horizon and imposing no incoming radiation. For near-extremal solutions

$$\sigma := \frac{r_+ - r_-}{r_+} \ll 1. \qquad (13)$$

We further introduce a shifted radial coordinate

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[2]The function μ is a weight in the following sense. Define $E := f r^2 \partial_r^2 + 2(r - M - ir(\omega r - qQ))\partial_r + (-K_\ell + i(qQ - 2\omega r))$. Then E is symmetric in the bilinear $(a, b) = \int a(r)b(r)\mu(r)dr$ in the sense that $(a, Eb) = (Ea, b)$.

$$x := \frac{r - r_+}{r_+}, \qquad (14)$$

which puts the outer horizon at 0, forming the inner boundary of our working domain.

To begin, we construct the near-horizon geometry by introducing a scaling parameter λ. For the near-extremal RN black hole, the near-horizon geometry of interest is the end point ($\lambda \to 0$) of a flow along a one-parameter family of spacetimes having $\sigma = \lambda\bar{\sigma}$. The flow is taken at fixed "scaled" coordinates $\bar{v} = \lambda v/r_+$, $\bar{x} = x/\lambda$, and $\bar{\sigma}$. The limiting metric forms a patch of the Robertson-Bertotti universe, $\text{AdS}_2 \times \mathbb{S}^2$, with metric [60–63]

$$r_+^{-2} ds^2 = -\bar{x}(\bar{x} + \bar{\sigma})d\bar{v}^2 + 2d\bar{v}d\bar{x} + d\Omega^2. \qquad (15)$$

The gauge field in the form (6) is singular in this limit. We choose a section on the $U(1)$ bundle compatible with the near-horizon limit by shifting $A \to A + d\chi$, where $\chi = \frac{Q}{r_+} v$, such that $A(r_+) = 0$. The limit $\lambda \to 0$ gives

$$\bar{A} = Q\bar{x}d\bar{v}. \qquad (16)$$

The near-zone scalar field frequency in this gauge is then given by

$$\bar{\omega} := \frac{\bar{\sigma} r_+ (\omega - q\Phi_+)}{\sigma}, \qquad \Phi_+ := \frac{Q}{r_+}. \qquad (17)$$

Upon separating variables using $e^{-i\bar{\omega}\bar{v}} R(\bar{x}) Y_{\ell m}(\theta, \phi)$, the homogeneous radial equation is found to be an ordinary hypergeometric equation

$$\bar{x}(\bar{x} + \bar{\sigma})R'' + (\bar{\sigma} - 2i(\bar{\omega} + \mathsf{q}\bar{x} + i\bar{x}))R' - (K_\ell + i\mathsf{q})R = 0, \qquad (18)$$

where $\mathsf{q} := r_+ q$. Of the two linearly independent solutions we choose the ingoing one

$$R_{\text{near}}^{\text{in}} = {}_2F_1\left(1 - h - i\mathsf{q}, h - i\mathsf{q}, 1 - i\bar{k}, -\frac{\bar{x}}{\bar{\sigma}}\right), \qquad (19)$$

where we have introduced

$$\bar{k} := 2r_+(\omega - q\Phi_+)/\sigma.$$

Using Eq. (15.8.2) of [64], we find that the ingoing solution has the large-$\bar{x}$ (buffer-zone) asymptotics

$$R_{\text{near}}^{\text{in}} \sim \bar{A}(\bar{x}/\bar{\sigma})^{h-1+i\mathsf{q}} + \bar{B}(\bar{x}/\bar{\sigma})^{-h+i\mathsf{q}}, \qquad \bar{x} \to \infty, \qquad (20)$$

where

$$\bar{A} = \frac{\Gamma(2h-1)\Gamma(1-i\bar{k})}{\Gamma(h-iq)\Gamma(h-i\bar{k}+iq)}, \qquad (21a)$$

$$\bar{B} = \bar{A}(h \leftrightarrow 1-h). \qquad (21b)$$

The operational meaning of $h \leftrightarrow 1-h$ in (21b) is to interchange h with $1-h$ everywhere in $\bar{A}$.

2. Extremal near-zone solution

We are also interested in seeing what happens in the extremal limit. To study this regime, we require the near-horizon radial functions. These functions are solutions to the radial differential equation derived from the field equation in the extremal near-horizon geometry. The extremal near-horizon limit is obtained by the same limiting procedure used in the previous section, but now we instead fix the scaled coordinates

$$\hat{x} = \frac{x}{\lambda^p}, \qquad \hat{v} = \frac{\lambda^p v}{r_+} \qquad (22)$$

with $p \in (0,1)$ and take $\lambda \to 0$ while keeping $\bar{\sigma}$ fixed. The extremal limiting geometry forms a different patch of $\text{AdS}_2 \times \mathbb{S}^2$ with metric

$$r_+^{-2} ds^2 = -\hat{x}^2 d\hat{v}^2 + 2d\hat{v}d\hat{x} + d\Omega^2, \qquad (23)$$

which is diffeomorphic to the near-extremal patch (15) [62,65].

In the frequency domain, one also must hold

$$\hat{\omega} = \lambda^{-p}k, \quad \text{where } k := r_+(\omega - q\Phi_+),$$

fixed to have a well-defined Laplace transform. Then, the mode functions $e^{-i\hat{\omega}\hat{x}}R(\hat{x})Y_{\ell m}(\theta, \phi)$ reduce the extremal near-horizon field equation to

$$\partial_{\hat{x}}(\hat{x}^2 \partial_{\hat{x}} R) - 2i(\hat{\omega} + q\hat{x})\partial_{\hat{x}}R - (K_\ell + iq)R = 0. \quad (24)$$

Notice that the regular singular points at the inner and outer horizon which determined the solution of the near-extremal near-horizon radial equation (18) have "come together" to form an irregular singular point at the degenerate horizon $\hat{x} = 0$. The confluence of singular points may be exploited to generate the ingoing solution of (24) from the previously obtained ingoing (near-extremal) near-zone solution by taking the scaling limit

$$\lim_{\lambda \to 0} {}_2F_1(1-h-iq, h-iq, 1-2i\lambda^{p-1}\hat{\omega}/\bar{\sigma}, -\lambda^{p-1}\hat{x}/\bar{\sigma}) \quad (25)$$

and using the confluence identity [17] $W_{\nu,\mu}(z) = \lim_{c\to\infty} {}_2F_1(\mu-\nu-1/2, 1/2-\mu-\nu, c; 1-c/z)e^{-z/2}z^\nu$, where $W_{\nu,\mu}(z)$ is the Whittaker confluent hypergeometric function. It is easily checked that the result

$$R_{\text{near}}^{\text{in}}(\hat{x}) = \left(-\frac{2i\hat{\omega}}{\hat{x}}\right)^{-iq} \exp\left(-\frac{i\hat{\omega}}{\hat{x}}\right) W_{iq,h-1/2}\left(-\frac{2i\hat{\omega}}{\hat{x}}\right) \qquad (26)$$

satisfies (24). In the neighborhood of the horizon, the asymptotic ingoing solution is given by [64]

$$R_{\text{near}}^{\text{in}} \sim \sum_{j=0}^{\infty} \frac{(h-iq)_j(1-h-iq)_j}{j!}\left(\frac{\hat{x}}{2i\hat{\omega}}\right)^j, \quad \hat{x} \to 0, \qquad (27)$$

where $(a)_j := \Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol. To find the buffer-zone asymptotics, one may either take the scaling limit $\lambda \to 0$ of Eq. (20) at fixed hatted coordinates and with fixed finite $\hat{\omega}$, or equivalently, employ the large-$\hat{x}$ asymptotics of (27) directly via Eq. (13.14.18) of [64]. Taking the former route, and applying Eq. (5.11.12) of [64], one finds

$$R_{\text{near}}^{\text{in}} \sim \hat{A}\hat{x}^{h-1+iq} + \hat{B}\hat{x}^{-h+iq}, \qquad \hat{x} \to \infty, \qquad (28)$$

where

$$\hat{A} = \frac{\Gamma(2h-1)}{\Gamma(h-iq)}(-2i\hat{\omega})^{1-h-iq}, \qquad (29a)$$

$$\hat{B} = \hat{A}(h \leftrightarrow 1-h). \qquad (29b)$$

Notice that the extremal buffer-zone coefficients exhibit branch points at the super-radiant bound $\hat{\omega} = 0$.

B. Far-zone solutions

To obtain the far-horizon limit, we fix the far (asymptotic) coordinates $x^\mu = (v, r, \theta, \phi)$ and take $\sigma \to 0$ and $k = r_+(\omega - q\Phi_+) \to 0$. This limit corresponds to perturbations of the extremal geometry at the critical frequency $k = 0$.

In this limit, the inner and outer horizons (singular points of the ODE) "converge" and the radial equation is

$$(x^2 R')' - 2iqx(1+x)R' - (K_\ell + iq(1+2x))R = 0. \quad (30)$$

Equation (30) is a confluent hypergeometric differential equation with solution

$$R_{\text{far}} = Px^{h-1+iq}{}_1F_1(h+iq, 2h, 2iqx) + x^{-h+iq}{}_1F_1(1-h+iq, 2-2h, 2iqx). \qquad (31)$$

Again, we have used that $h(h-1) = K_\ell - q^2$. The far solution is regular for the choice

$$P = -\frac{(-2iq)^{2h-1}\Gamma(2-2h)\Gamma(h-iq)}{\Gamma(2h)\Gamma(1-h-iq)}. \qquad (32)$$

The buffer-zone asymptotics are obtained using $_1F_1(a, b, 0) = 1$ for any a and $b \notin \mathbb{Z}^- \cup \{0\}$. One may verify that the large-$\bar{x}$ asymptotic near-extremal solution, as given in Eq. (20), matches onto the far-horizon solutions just obtained. Moreover, the far-zone solution matches onto the hatted solution Eq. (28), as can be seen by reintroducing $k := \lambda^p \hat{\omega}$ in the near-horizon solution and replacing $\hat{A}\hat{x}^{h-1+iq}$ by $A(x/k)^{h-1+iq}$ (likewise for the $\hat{B}$ term).

IV. LINEAR RESPONSE: CHARGED FIELD INSTABILITY

The time dependence of the Green function has an intricate relationship with the analytic structure of the transfer function $\tilde{g}$ and the contour of the inversion integral (3) [66]. Subtleties aside, the qualitative picture goes as follows. At "early" times, direct propagation on the future light cone derives from the large-$|\omega|$ arc. At very late times, the field exhibits a power-law time dependence deriving from the branch point(s) located at $\omega = 0$ for the nonextremal black hole, and $\omega = q\Phi_+$ and $\omega = 0$ at extremality. At "intermediate" times, the field takes the form of a decaying sinusoid coming from the poles of the transfer function, the quasinormal modes.

A. Near-extremal case

1. Quasinormal mode spectrum

The spectrum of long-lived quasinormal modes which characterize the dominant near-horizon ringing is now obtained. From Eq. (12) we see that for holomorphic homogeneous solutions, the poles of the transfer function are given by the 0's of the Wronskian $\mathcal{W}$ alone. Matching the near and far solutions and demanding linear dependence ($\mathcal{W} = 0$) gives the quasinormal mode condition $\bar{A} = P\bar{B}$, or

$$\frac{\Gamma^2(1-2h)\Gamma^2(h-iq)\Gamma(h+iq-i\bar{k})}{\Gamma^2(2h-1)\Gamma^2(1-h-iq)\Gamma(1-h+iq-i\bar{k})} = (-2i\sigma q)^{1-2h}.$$

(33)

To find the quasinormal modes, we adopt an ansatz originating with Hod [67], which approximates the zeros of $\mathcal{W}$ by the negative integer poles of certain gamma functions containing $\bar{k}$ in (33). The deviation of the quasinormal mode pole position from the pole of the gamma function is parametrized by a complex number η, which weakly depends on the integer overtone index n. When $h \in \mathbb{R}$ the approximation is validated by the perturbative smallness of σ and $\eta = O(\sigma^h)$, while for complex h the Hod approximation has been found to be numerically accurate to within $|\eta| \approx 10^{-3}$ in typical cases [55,68]. We find that the near-horizon quasinormal modes fall into two categories,

$$\bar{k}_n = q + i(h - 1 - n) + \eta \quad \text{(principal)}, \quad (34a)$$

$$\bar{k}_n = q + i(-h - n) \quad \text{(supplementary)}, \quad (34b)$$

where $\bar{k}_n := 2r_+(\omega_n - q\Phi_+)/\sigma$ at the frequency ω_n. The terminology principal and supplementary has its roots in the classification of $SO(2,1)$ representations to be discussed in some detail in Sec. V.

In addition to the weakly decaying near-horizon modes, there are also damped far-horizon quasinormal modes distributed in $\mathbb{C}$ away from the super-radiant frequency [56]. As these modes have non-σ-suppressed exponential decay, they have a negligible contribution relative to the near-horizon modes after an inverse-σ time scale.

2. Overtone sum

For simplicity, let us assume that the perturbation develops from compactly supported initial data with amplitude peaked deep within the far zone where $x' \gg 1$. In this case, we approximate $R_{\text{up}}(x')$ by its asymptotics

$$R_{\text{up}} \sim C_\infty e^{2iqx'}(x')^{2iq-1} \qquad x' \to \infty. \quad (35)$$

At large x' the weight is given by

$$\mu \sim e^{-2iqx'}(x')^{-2iq}, \qquad x' \to \infty \quad (36)$$

such that complete x' dependence is asymptotically given by $\mu(x')R_{\text{up}}(x') \sim C_\infty/x'$ as $x' \to \infty$. The transmission amplitude C_∞ is straightforwardly computed from the asymptotics of (31) subject to the outgoing condition (32).

As in [40], we define a near-horizon-mode (NHM) Green function due to poles near the super-radiant bound,

$$G_{\text{NHM}} = -\frac{i\sigma}{2r_+x'}\sum_{\ell m}C_\infty Y_{\ell m}(\theta, \phi)Y_{\ell m}^*(\theta', \phi')$$

$$\times \sum_{n=0}^\infty e^{-i\bar{k}_n\bar{v}}\frac{R_{\text{in}}(\bar{X}; \bar{k}_n)}{d\mathcal{W}/d\bar{k}|_{\bar{k}=\bar{k}_n}}, \quad (37)$$

where

$$\bar{V} := \sigma(v - v')/(2r_+), \qquad \bar{X} := x/\sigma. \quad (38)$$

The values of $\bar{k}_n := 2\bar{\omega}_n/\bar{\sigma}$ are given by (34).

The residues at the quasinormal mode frequencies are derived from the zero's of the scaled Wronskian

$$\mathcal{W} := \Gamma(1 - i\bar{k})\left(\frac{a\sigma^{1-h-iq}}{\Gamma(h - i\bar{k} + iq)} + \frac{b\sigma^{h-iq}}{\Gamma(1 - h - i\bar{k} + iq)}\right),$$

(39)

where $a = -\Gamma(2h)/\Gamma(h - iq)$, $b = \Gamma(2h - 2)/\Gamma(1 - h - iq) \times P$, with P given in Eq. (32). To find the approximate zero's of $\mathcal{W}$ we substitute the Hod ansatz (34), differentiate

with respect to η, and drop $O(\eta)$ error terms. For the dominant principal modes, we find

$$\frac{d\mathcal{W}}{d\bar{k}}\bigg|_{\bar{k}=\bar{k}_n} = b\sigma^{h-iq}(i(-n)^{n+1}n!)\Gamma(h - iq - n), \qquad h \in \mathbb{C},$$

(40)

where we have used $1/\Gamma(-n - i\eta) = -i\eta(-1)^n n! + O(\eta^2)$. The supplementary modes are obtained from (40) by swapping $b \leftrightarrow a$ and $h \leftrightarrow 1 - h$.

To compute the overtone sum we again follow [40]. However, here we restrict to the principal series modes for which

$$G_{\mathrm{NHM}} = \frac{-i}{2r_+ x'} \sum_{\ell m} \sigma^{1-h+iq} C_\infty b^{-1} \cdot Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta', \phi')$$

$$\times e^{-i(q+i(h-1))\bar{V}} \mathcal{S}_P.$$

(41)

The overtone sum

$$\mathcal{S}_P := \sum_{n=0}^{\infty} \frac{(-1)^n e^{-n\bar{V}}}{n!\Gamma(h - iq - n)}$$

$$\times {}_2F_1(1 - h - iq, h - iq, h - iq - n, -\bar{X}) \quad (42)$$

is evaluated by invoking the series representation of the hypergeometric function and interchanging the two sums. The result of both summations is

$$\mathcal{S}_P = \frac{(1 - e^{-\bar{V}})^{h-iq-1}}{\Gamma(h - iq)}[1 + \bar{X}(1 - e^{-\bar{V}})]^{h-1+iq} \quad (43)$$

At late times $\bar{V} \to \infty$, $\mathcal{S}_P \to 1/\Gamma(h - iq)$, the response decays exponentially according to (41). At semiearly times—after a light-crossing time but before the exponential decay "takes over"—the overtone sum takes the simple form

$$\mathcal{S}_P \approx \frac{\bar{V}^{h-iq-1}}{\Gamma(h - iq)}(1 + \bar{X}\bar{V})^{h-1+iq}, \qquad \bar{V} \ll 1, \quad (44)$$

where $\approx$ signifies we have expanded $e^{-\bar{V}}$ and kept first order in $\bar{V}$ only. We further overload the $\approx$ symbol by restricting to values of V large enough that the contribution from the large-$|\omega|$ arc may be ignored, ensuring that the quasinormal mode sum is convergent. Finally, taking n radial derivatives of (44) one finds

$$\mathcal{S}_P^{(n)} \approx \frac{\bar{V}^{h-1-iq+n}}{\Gamma(h - iq)}(h - 1 + iq)_n(1 + \bar{X}\bar{V})^{h-1+iq-n}. \quad (45)$$

Therefore we have demonstrated that n derivatives of a principal harmonic grow at a rate $\bar{V}^{-1/2+n}$ for $\bar{V} \ll 1$. The same transitory growth, including the rate, was found for

nonaxisymmetric perturbations of the near-extremal rotating Kerr black hole [40].

B. Extremal case

The analysis of the previous section suggests unbounded growth of radial derivatives on the horizon as $\sigma \to 0$. We now show how the Aretakis instability arises at the extremal horizon as a steep-gradient tail by examining the Laplace transform near the branch point at the super-radiant bound frequency.

Near the horizon, the asymptotic transfer function for far-zone initial data ($x' \gg 1$) is given by

$$\tilde{g} \sim \frac{R_{\mathrm{up}}(x')(-2ik)^{iq}}{a(-2ik)^{1-h} + b(-2ik)^h} \sum_{j=0}^{\infty} \mathcal{R}_j \left(\frac{x}{2ik}\right)^j, \quad x \to 0, \quad (46)$$

where $a = -\Gamma(2h)/\Gamma(h - iq)$, $b = \Gamma(2 - 2h)/\Gamma(1 - h - iq) \times P$ and $\mathcal{R}_j := (h - iq)_j(1 - h - iq)_j/j!$. The k-independent quantity P appearing above in b was previously given in Eq. (32).

In general, the late-time asymptotics of the Green function are determined by the singular points of $\tilde{g}$ in $\mathbb{C}$ with the largest real part [69]. For perturbations of four-dimensional stationary (asymptotically flat) extremal spacetimes, the known uppermost singular points of $\tilde{g}$ are the static ($\omega = 0$) and super-radiant ($k = 0$) frequencies, both existing as branch points on the real axis. These branch points arise because the extremal radial equation has *irregular* singular points at the horizon and infinity. We have found that the power-law contribution from the branch point at the static frequency is subleading in the case of extremal Kerr [70] and we assume the same to hold for extremal Reissner-Nordström.

First consider modes having supplemental representations, where $h > 1/2$. In this case the a-term in the denominator of $\tilde{g}$ in (46) is more singular near $k = 0$. Then, the nth radial derivative of the transfer function at the horizon to leading order in the late-time "$1/k$" expansion [37] is given by

$$\tilde{g}_S^{(n)}|_{\mathcal{H}}(k \to 0) \sim f_n(x')(-2ik)^{h-1-n+iq}, \quad h > 1/2, \quad (47)$$

where $f_n(x') := n!(-1)^n \mathcal{R}_n R_{\mathrm{up}}(x')$ and the subscript S means supplementary. Using Theorem 37.1 of [69], the results of which are succinctly summarized in [37], we find the inverse transform to be

$$g_S^{(n)}|_{\mathcal{H}}(v \to \infty) \sim \frac{f_n(x')e^{-iqv}}{2\Gamma(1 - h + n - iq)}\left(\frac{v}{2}\right)^{n-h-iq}. \quad (48)$$

For principal representation modes, both the a and b terms in the Wronksian [denominator of (46)] are equally singular [37] at $k = 0$. The nth derivative of the horizon transfer function in this case is

$$\tilde{g}_P^{(n)}|_{\mathcal{H}}(k \to 0) \sim \frac{\mathfrak{f}_n(x')(-2ik)^\alpha}{(-2ik)^{-2i\lambda} + \zeta}, \qquad h \in \mathbb{C}, \quad (49)$$

where we have introduced $\mathfrak{f}_n(x') := (-1)^n n! \mathcal{R}_n \mathcal{R}_{\text{up}}(x')/a$, $\zeta := b/a$, and $\alpha := -1/2 - n + i(\text{q} - \lambda)$ as simplifying factors. Despite considerable effort, we are unable to analytically invert (49). Instead, we resort to numerical evaluation with the MATHEMATICA package NumericalLaplaceInversion [71], which computes the inverse Laplace transform to arbitrary numerical precision. Through experience with the same form of integral in the extremal Kerr study [37], we find it may be fitted to

$$g_P^{(n)}|_{\mathcal{H}}(v \to \infty) \sim D_n e^{-iqv} v^{n-1/2+ip} \quad (50)$$

for complex D_n and $p \in \mathbb{R}$. Comparison with (48) reveals that the principal modes are dominant at late times.

V. SYMMETRY INTERPRETATION

We have shown that a collective ringing of overtones at the critical frequency gives rise to transient power-law behavior with duration $1/\sigma$. Though the field itself decays, the nth transverse derivative grows with scaling $\sigma^{1/2-n}$. Therefore, the energy density $T_{\alpha\beta} u^\alpha u^\beta \sim \sigma^{-1}$ measured by infalling observers at the horizon becomes infinite in the extremal limit.

The same phenomena were found for nonaxisymmetric perturbations of the near-extremal and extremal Kerr black holes [26,37]. In hindsight, the similarities may have been predicted from their mutual symmetries. Both scenarios contain a $U(1)$ symmetry in addition to the near-horizon $2 + 1$ Lorentz symmetry $SO(2, 1)$. For Kerr the $U(1)$ manifests as axisymmetry of spacetime, while for charged scalars it is a gauge symmetry on the principal bundle with q playing the role of ∂_ϕ angular momentum m. The triviality of the $U(1)$ bundle—existence of a global section—enabled us to globally utilize a smooth gauge adapted to the $\lambda \to 0$ limit wherein the connection between m and q was furnished explicitly through the near-horizon frequency $\bar{\omega} = r_+(\omega - q\Phi_+)/\lambda$. In this way, the nonaxisymmetric modes of rapidly spinning Kerr may be thought of as a tower of charged scalar perturbations with integer separated charges.

We now sketch how the near-horizon charged perturbations of extremal Reissner-Nordström fall into modules of the symmetry group. Symmetry in the background is defined as

$$\mathcal{L}_{X,\chi} g_{\alpha\beta} = \mathcal{L}_X g_{\alpha\beta} = 0 \quad (51a)$$

$$\mathcal{L}_{X,\chi} A = \mathcal{L}_X A - d\chi = 0 \quad (51b)$$

where X is the set of left-invariant vector fields generating spacetime isometries and χ is the set of $U(1)$ transition functions, which, for smooth A, can be uniquely paired

with each X [72]. The generators act on the charged scalar by $\mathcal{L}_{X,\chi}\psi = \mathcal{L}_X\psi + iq\chi\psi$. In the extremal limit the $\mathfrak{so}(2, 1)$ Cartan-Weyl generators carry simple expressions in ingoing coordinates

$$H_0 = v\partial_v - x\partial_x, \quad (52a)$$

$$H_+ = \partial_v, \quad (52b)$$

$$H_- = v^2\partial_v - 2(xv + 1)\partial_x, \quad (52c)$$

and satisfy the commutation relations $[H_+, H_-] = 2H_0$ and $[H_\pm, H_0] = H_\pm$. These vector fields Lie derive the metric (23). To Lie derive the vector potential $A = vdx$ we pair H_- with the transition function $\chi_- = -2v$. We also pair H_0 with $\chi_0 = 1$. With this pairing all generators commute with the D^2 operator. Then, by Schur's lemma, D^2 is proportional to the identity. Therefore we have

$$D^2\psi = (h(h - 1) - K_\ell + C)\psi \quad (53)$$

where K_ℓ labels $\mathfrak{so}(3)$ as before, $C = \text{q}^2$ trivially labels $\mathfrak{u}(1)$, and $h(h - 1)$ is the $\mathfrak{so}(2, 1)$ Casimir [73]. The wave equation then implies $h = \frac{1}{2} \pm \sqrt{1/4 + K_\ell - C}$ for nontrivial solutions. Thus for a given ℓ and q, the near-horizon solutions ψ may be partially classified according to h. The Casimir h provides only a partial classification because $\mathfrak{so}(2, 1)$ is noncompact. In general, simple noncompact Lie algebras require not only the Casimir $h(h - 1)$, but also a set of roots—simultaneous eigenvalues of the diagonal generators. For $\mathfrak{so}(2, 1)$, the Cartan subalgebra is rank 1 (single eigenvalue), so the choice of generator we diagonalize is arbitrary without restricting to unitary representations. Our choice is to classify by the eigenvalue E_0 of H_0, which is already diagonal in the ingoing coordinate basis,

$$\mathcal{L}_{H_0}\psi = E_0\psi.$$

One solution of this equation is $\psi = v^{E_0} f(vx)$, while another is $\psi = x^{-E_0} f(vx)$. The physical eigensolution is determined by fixing boundary conditions such as regularity at $x = 0$.

General finite representations of $SO(2, 1)$ are classified into four types, three of which have unbounded weight spectra $E_0 + n$, where $n \in \mathbb{Z}$, and the other is finite [48,74–77]. The infinite representations are further divided into principal/supplementary series which have a bilateral unbounded weight spectrum, and highest/lowest weight series which are unbounded from above/below $n = 0$, respectively. For the principal/supplementary series modes, h is complex/real, whereas the highest and lowest weight modes have real $h = \pm E_0$. For the finite representation, which occurs for $\text{q} = 0$, h is an integer. The only unitary finite representation is trivial ($\ell = 0$). Axisymmetric linear

perturbations of Kerr transform as a finite representation also.

We have been primarily interested in the principal series representations, which have the dominant contribution to the horizon instability. As K_ℓ is positive semidefinite, the field is a principal series solution of (53) when

$$C > 1/4 + K_\ell. \tag{54}$$

For charged fields, C is an arbitrary parameter that may be chosen such that Eq. (54) is satisfied for some ℓ. In the case of generic mode perturbations of Kerr,[3] principal representations correspond to $1/4 \lesssim m/\ell$ [78]. In near-horizon regions of both RN and Kerr, the principal harmonics [modes on which the generators act such that (53) and (54) are satisfied] which respect *causal* boundary conditions, and continuously match to the far region radiation, realize nonunitary representations. With respect to the global near-horizon AdS$_2$ geometry, these modes lack a positive definite Hermitian form [79] and violate the Breitenlohner-Freedman bound [80]. Stability results supporting these observations were found for the near-horizon perturbations of extremal Kerr in [81,82].

Lastly we remark on the explicit action of $\mathfrak{so}(2,1) \in \mathfrak{g}$ on the physical solutions. With the v and x dependence derived in (44), let us write $\psi = v^{h-1-iq} f(vx)$, ignoring overall constants and also the angular dependence. As $f(vx)$ is trivially

Lie derived along H_0, $\mathcal{L}_{H_0}\psi = (h-1)\psi$ follows.[4] This weight spectrum is unbounded, $(\mathcal{L}_{H_\pm})^j\psi \neq 0 \ \forall j \in \mathbb{N}$, and has neither a highest nor a lowest weight module. Starting with ψ one can generate descendant solutions of (53) in both stable and unstable directions by acting on ψ with either H_+ or H_- at the horizon: H_- raises E_0 by an integer and enhances the Aretakis rate by an integer power of v, while H_+ lowers it in the same way in the opposite direction. Though we can map solutions into solutions by the action of $\mathfrak{so}(2,1)$, to maintain consistency with the boundary conditions, one must also act with the Cartan-Weyl generators of $\mathfrak{so}(3)$, changing the multipole number, *and* adjust q suitably to ensure uniqueness of the solution. This "mixing" limits the practical application of the construction based on symmetry alone.

ACKNOWLEDGMENTS

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[3]For scalar fields in Kerr $K = A_{\ell m \omega} + a^2\omega^2$ where $A_{\ell m \omega}$ is the spheroidal eigenvalue. It can be shown that $K \geq |m|(|m| + 1)$.

[4]The $-iq$ in E_0 was removed by automorphically shifting H_0 by the $\mathfrak{u}(1)$ generator.

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