

Rotation of polarization by a moving gravitational lens

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We present a simple prescription for the rotation of polarization produced by a relativistically moving gravitational lens, applicable to arbitrary deflection angle and arbitrary velocity of the lens. When geometric optics is applicable, two independent components contribute to the total rotation of polarization: (i) in the frame of the lens the polarization vector experiences minimal rotation defined by the deflection angle (as measured by a set of remote observers, or no rotation if defined in terms of a parallel-propagated tetrad); (ii) the effect of the motion of the lens on the polarization can be taken into account exactly using special-relativistic Lorentz transformation of polarization. The effects of the gravitational lensing are thus parametrized by the deflection angle of the null geodesics (not necessarily small) and the motion of the lens (not necessarily with velocities much smaller than that of light).

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I. INTRODUCTION

In general relativity (GR) the polarization vector is parallel-transported along the light ray [1]. In a nonrotating spherically symmetric metric the final direction of polarization is uniquely determined by the initial polarization and the total deflection angle of the lens (Sec. II). (In a rotating metric there is an additional contribution from the spin parameter [2,3].)

A moving gravitational lens does lead to the rotation of polarization (see, e.g., Ref. [4]). In Ref. [5] the GR principles behind this observer-dependent rotation of polarization were discussed, including the analysis of the related controversies. In this paper we point out that the rotation of polarization due to the relative velocity of the lens and the observer is a special-relativistic effect [6–8] and thus can be taken into account exactly, to any value of the velocity (and not only much smaller than the speed of light; see Sec. III).

II. POLARIZATION ROTATION DUE TO A STATIONARY SCHWARZSCHILDIAN LENS

For completeness, let us first discuss the transformation of polarization by a nonrotating (Schwarzschildian) lens. The discussion below outlines the concept of parallel-transport of polarization. The method we use to calculate polarization is somewhat different from what is usually done. For example, in Ref. [4] the polarization transfer was done using an invariant relativistic formulation and then projecting the four-dimensional results onto the three-dimensional (3D) space of an observer. Instead, we define

the observers from the beginning; calculations of the polarization vector are done in an absolute 3D space. This method is simpler for the problem of moving lenses, when both the emitter and the observer are located far away from the source of gravity.

Let the initial direction of propagation and polarization in the frame of the lens be \mathbf{n}'_1 and \mathbf{p}'_1 , $\mathbf{n}'_1 \cdot \mathbf{p}'_1 = 0$, and the final states are \mathbf{n}'_2 and \mathbf{p}'_2 , $\mathbf{n}'_2 \cdot \mathbf{p}'_2 = 0$ (primes denote quantities measured in the frame of the lens). The relation between \mathbf{n}'_1 and \mathbf{n}'_2 is given by the GR bending of light [1]. We would like to find the final polarization \mathbf{p}'_2 in terms of \mathbf{n}'_1 , \mathbf{n}'_2 and \mathbf{p}'_1 .

Introducing the unit vector

$$\mathbf{k}_r = \frac{\mathbf{n}'_1 \times \mathbf{n}'_2}{|\mathbf{n}'_1 \times \mathbf{n}'_2|}, \quad (1)$$

the polarization will be rotated from \mathbf{p}'_1 around the axis \mathbf{k}_r by the angle $\sin \theta_{\text{rot}} = |\mathbf{n}'_1 \times \mathbf{n}'_2|$. The rotation matrix around a direction \mathbf{k}_r is

$$T_r = (1 - \cos \theta_{\text{rot}})\mathbf{k}_r \times \mathbf{k}_r + \cos \theta_{\text{rot}}\mathbf{I} + \sin_{\text{rot}}\theta \mathbf{k}_{r \times} \quad (2)$$

where $\mathbf{k}_{r \times}$ is the tensor product.

For example, if the initial direction of propagation is $\mathbf{n}'_1 = \{0, 0, 1\}$, the polarization is $\mathbf{p}'_1 = \{1, 0, 0\}$ and the final direction is $\mathbf{n}'_2 = \{\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta\}$ (θ is the deflection angle that can be calculated, in the case when both the source and the observer are at null infinities, using GR for any photon trajectory), we find

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$$\begin{aligned}
\mathbf{k}_r &= \{-\sin \phi, \cos \phi, 0\}, \\
\theta_{\text{rot}} &= \theta, \\
T_r &= \begin{pmatrix} (1 - \cos \theta) \sin^2 \phi + \cos \theta & -(1 - \cos \theta) \sin \phi \cos \phi & \sin \theta \cos \phi \\ -(1 - \cos \theta) \sin \phi \cos \phi & (1 - \cos \theta) \cos^2 \phi + \cos \theta & \sin \theta \sin \phi \\ -\sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \end{pmatrix}, \\
\mathbf{p}'_2 &= \{\cos \theta \cos^2 \phi + \sin^2 \phi, -(1 - \cos \theta) \sin \phi \cos \phi, -\sin \theta \cos \phi\}.
\end{aligned} \tag{3}$$

It may be verified that $\mathbf{p}'_2 \cdot \mathbf{n}'_2 = 0$. (The statement that the polarization is not rotated during lensing by a stationary nonrotating lens, implies that the polarization is just rotated by the above amount, and nothing else.)

For example, for the photon's motion in the $x-z$ plane, $\phi = 0$, the new direction of polarization is $\mathbf{p}'_2 = \{\cos \theta, 0, -\sin \theta\}$, while for the photon's motion in the $y-z$ plane, $\phi = \pi/2$, the polarization remains along the x direction, $\mathbf{p}'_2 = \{1, 0, 0\}$. For small deviation angles $\theta \ll 1$, $\mathbf{p}'_2 \approx \{1, 0, -\theta \cos \phi\}$.

III. RELATIVISTIC TRANSFORMATION OF POLARIZATION

Let us next derive explicit relations for the Lorentz transformation of polarization. Let a photon in some frame be defined by a triad of unit vectors along the electric field, magnetic field and the direction of propagation \mathbf{e}' , \mathbf{b}' , \mathbf{n}' , so that $\mathbf{n}' = \mathbf{e}' \times \mathbf{b}'$. The easiest way to derive the relativistic transformation of polarization is using relations for the aberration of light,

$$\mathbf{n} = \frac{\mathbf{n}' + \mathbf{v}\gamma(1 + \frac{\gamma}{1+\gamma}(\mathbf{n}' \cdot \mathbf{v}))}{\gamma(1 + (\mathbf{n}' \cdot \mathbf{v}))} \tag{4}$$

and the transformation of the electromagnetic fields,

$$\begin{aligned}
\mathbf{n}_{(y)} &= \frac{\beta \sin \theta_v (1 + \gamma(1 + \beta \cos \theta_v))}{(\gamma + 1)(1 + \beta \cos \theta_v)} \begin{pmatrix} \cos \phi_v \\ \sin \phi_v \end{pmatrix}, \\
n_z &= \frac{\beta \gamma \cos \theta_v + (\gamma - 1) \cos^2 \theta_v + 1}{\beta \gamma \cos \theta_v + \gamma}, \\
\tan \chi &= \frac{\mathbf{e} \cdot \hat{y}}{\mathbf{e} \cdot \hat{x}} = -\frac{\sin^2 \theta_v \sin \phi_v \cos \phi_v (1 + \beta \cos \theta_v)}{\frac{\beta(\gamma+1)\cos(\theta_v)+1}{\gamma-1} + \sin^2 \theta_v \sin^2 \phi_v (1 + \beta \cos \theta_v) + \beta \cos^3 \theta_v + \frac{(2\gamma+1)\cos^2 \theta_v}{\gamma}}.
\end{aligned} \tag{8}$$

In the limit $\gamma \rightarrow \infty$,

$$\tan \chi = -\frac{\sin^2 \theta_v \sin \phi_v \cos \phi_v}{\sin^2 \theta_v \sin^2 \phi_v + \cos^2 \theta_v + \cos \theta_v}. \tag{9}$$

$$\begin{aligned}
\mathbf{B} &= \gamma \mathbf{B}' - \frac{\gamma^2}{\gamma + 1} (\mathbf{B}' \cdot \mathbf{v}) \mathbf{v} - \gamma \mathbf{v} \times \mathbf{E}', \\
\mathbf{E} &= \gamma \mathbf{E}' - \frac{\gamma^2}{\gamma + 1} (\mathbf{B}' \cdot \mathbf{v}) \mathbf{v} + \gamma \mathbf{v} \times \mathbf{B}'
\end{aligned} \tag{5}$$

($\beta = v/c$ is γ , the corresponding Lorentz factor). Using $\mathbf{E}' = -\mathbf{n}' \times \mathbf{B}'$, for the unit vector along the magnetic field we find

$$\begin{aligned}
\mathbf{b} &= \frac{\mathbf{q}'}{(1 + \gamma)(1 + (\mathbf{n}' \cdot \mathbf{v}))}, \\
\mathbf{q}' &= \mathbf{b}' + \mathbf{v} \times (\mathbf{b}' \times \mathbf{n}') - \frac{\gamma}{\gamma + 1} (\mathbf{b}' \cdot \mathbf{v}) \mathbf{v}.
\end{aligned} \tag{6}$$

Thus, the new direction of polarization (unit vector along \mathbf{e}) is

$$\mathbf{e} = \frac{\mathbf{n} \times \mathbf{q}'}{\sqrt{q'^2 - (\mathbf{n} \cdot \mathbf{q}')^2}}, \tag{7}$$

(cf. Ref. [8]). This solves in general the polarization transformation. (In the Appendix we discuss how general relations for polarization transformation agree with the relativistic transformation of acceleration for dipolar-type emission.)

For example, if initially the photon propagates along the \hat{z} direction and is polarized along the \hat{x} direction, and the relative velocity is $\mathbf{v} = \{\sin \theta_v \cos \phi_v, \sin \theta_v \sin \phi_v, \cos \theta_v\} \beta$, the new direction of propagation and the polarization angle are (Fig. 1)

Thus, there is no rotation of polarization for special cases of motion along the photon direction, $\theta_v = 0$, in the polarization plane, $\phi_v = 0$, or perpendicular to the polarization plane, $\phi_v = \pi/2$.

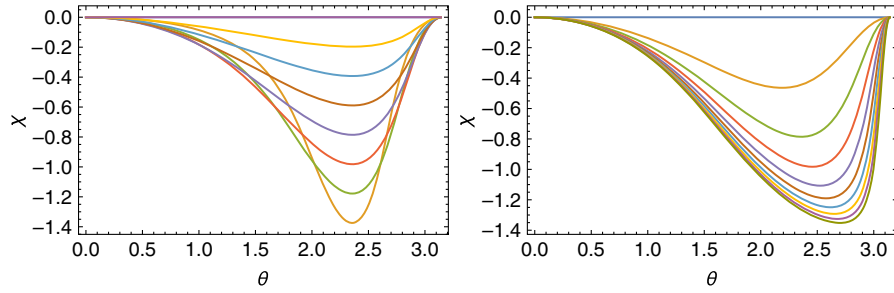


FIG. 1. Rotation of the polarization angle χ as a function of the direction of the velocity θ with respect to the photon propagation in the rest frame. Left panel: $\gamma = 2$, for different position angles $\phi = 0, \pi/16, \dots, \pi/2$. Right panel: $\phi = \pi/4$, for different Lorentz factor $\gamma = 1, 2, \dots, 10$. (Only a fraction of the curve is revealed to the observer through the generally narrow radio beam.)

IV. POLARIZATION ROTATION DUE TO A MOVING LENS

In Sec. II we derived the polarization rotation due to a stationary lens and in Sec. III we derived the Lorentz transformation of the polarization. Combining the two expressions we can find the polarization rotation due to a (relativistically) moving lens. For practical purposes, it is easiest to start in the lens frame, and then (i) do a boost to the lab frame to find the initial polarization in the lab frame, and (ii) do a GR deflection and the corresponding rotation of polarization in the lens frame and do a boost to the lab frame to find the final polarization in the lab frame.

Thus, the general procedure of polarization transfer by a moving lens involves a Lorentz boost of polarization vectors $\mathbf{p}'_1 = \{1, 0, 0\}$ and \mathbf{p}'_2 [Eq. (3)] using the transformation (7). This procedure formally solves the problem of polarization rotation by a moving lens. (Alternatively, starting with the wave vector and polarization in the laboratory frame, $\mathbf{n}_1, \mathbf{p}_1$, do a boost to the lens' frame

and find directions and polarization, $\mathbf{n}'_1, \mathbf{p}'_1$, do a radiative transfer in the frame of the lens to find $\mathbf{n}'_2, \mathbf{p}'_2$, and then do a Lorentz boost back to the observer's frame to find $\mathbf{n}_2, \mathbf{p}_2$.) Importantly, each step can be done exactly, without the limitations of small angle deflection or small relative velocity.

As practical applications are mostly limited to small lens velocities, next we find explicit relations for $\beta \ll 1$. If the velocity of the lens in the lab frame is

$$\mathbf{v} = \{\cos \phi_v \sin \theta_v, \sin \phi_v \sin \theta_v, \cos \theta_v\} \beta \quad (10)$$

then, keeping only linear terms in β , we find the initial polarization vector

$$\mathbf{p}_0 = \{1, 0, 0\} - \beta \{0, 0, \cos \phi_v \sin \theta_v\} \quad (11)$$

[cf. Ref. [9], Eq. (1)]. There is no rotation if the lens is moving along the direction of propagation of the photon, $\theta_v = 0$.

The final polarization vector is

$$\begin{aligned} \mathbf{p}_2 &= \mathbf{p}'_2 + \beta \cos \mu \hat{\mathbf{r}}, \\ \hat{\mathbf{r}} &= \{\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}, \\ \cos \mu &= \sin \theta \cos \phi \cos \theta_v - \sin \theta_v (\cos \theta \cos \phi \cos (\phi - \phi_v) + \sin \phi \sin (\phi - \phi_v)). \end{aligned} \quad (12)$$

For example, consider a lens moving in a direction perpendicular to the direction of the photon propagation, $\theta_v = \pi/2$. We find then

	\mathbf{p}_0	\mathbf{p}_2
$\phi = 0, \phi_v = 0$	$\{1, 0, -\beta\}$	$\{\cos \theta (\beta \sin \theta + 1), 0, \beta \cos^2 \theta - \sin(\theta)\} \approx \{\beta \theta + 1, 0, \beta - \theta\}$
$\phi = \pi/2, \phi_v = 0$	$\{1, 0, -\beta\}$	$\{1, \beta \sin \theta, \beta \cos \theta\} \approx \{1, \beta \theta, \beta\}$
$\phi = \pi/2, \phi_v = \pi/2$	$\{1, 0, 0\}$	$\{1, 0, 0\}$
$\phi = 0, \phi_v = \pi/2$	$\{1, 0, 0\}$	$\{\cos \theta, 0, -\sin \theta\} \approx \{1, 0, -\sin \theta\}$

If the lens moves along the initial propagation of the photon, $\theta_v = 0$, then $\mathbf{p}_0 = \{1, 0, 0\}$ and

$$\mathbf{p}_2 = \begin{cases} \{\cos \theta - \beta \sin^2 \theta, 0, \sin \theta (-\beta \cos(\theta) - 1)\} \approx \{1, 0, -(1 + \beta)\theta\} \approx \{1, 0, -\theta\}, & \text{for } \phi = \pi/2, \\ \{1, 0, 0\}, & \text{for } \phi = 0. \end{cases} \quad (13)$$

V. DISCUSSION

In this paper we solved exactly the polarization transfer by a moving gravitational lens by separating the motion of the lens and gravitational light bending in the frame of the lens. In the frame of the spherical (nonrotating) lens the final polarization direction is determined only by the total deflection angle in the frame of the lens (not necessarily small), while the motion of the lens can be taken into account exactly using the Lorentz transformation of polarization. A typical value of the rotation angle is small $\sim\theta\beta$, where $\theta \approx 4GM/(rc^2)$ is the deflection angle (M is the mass of the lens and r is the impact parameter) and β is the velocity of the lens [10].

Our method of separating the GR effects in the frame of the lens and special-relativistic effects due to the motion of the lens—both of which can be strong—can be applied to a more general case of rotating lens as well. The relations of Sec. II then need to be changed to include the effects of the lens' rotation (see e.g., Ref. [4]).

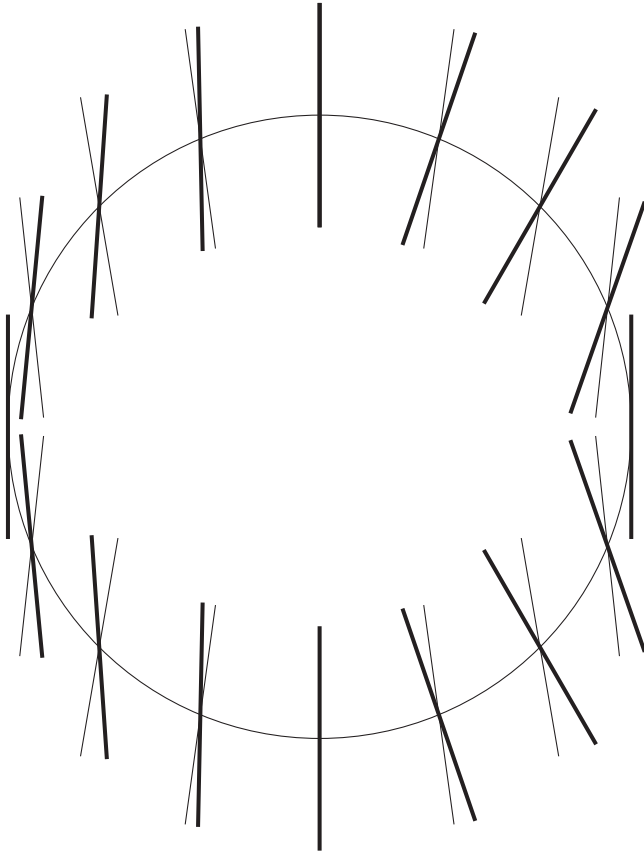


FIG. 2. Polarization pattern produced by a moving strong lens as viewed by an observer. The source is polarized along the vertical direction. Thin lines indicate the polarization direction for a stationary lens; thick lines indicate the polarization direction for a lens moving horizontally, in a direction transverse to the initial photon propagation. The circle represents the Einstein ring. The effects of motion are exaggerated for clarity. The lens is moving from left to right, as seen by the observer.

Possible applications of the effect include the double pulsar [11], where the line of sight passes $\sim 5 \times 10^8$ cm from the pulsar B [12], so the deflection angle is $\approx 1.5 \times 10^{-3}$. The pulsar B moves with a velocity ~ 600 km s^{-1} , $\beta = 2 \times 10^{-3}$, so that the polarization rotation angle is $\approx 3 \times 10^{-6}$. (Another case of nearly in-the-orbital-plane line of sight [13] produces a much smaller effect.)

In the case of cosmological gravitational lensing, the deflection angle for strong lensing is typically a few arcseconds, while the random velocity of the lens can be estimated as a virial velocity in a large cluster of galaxies, $\beta \leq 10^{-2}$. Thus, the expected rotation of polarization is in the range of milliarcseconds (see also Ref. [14]). The corresponding pattern for a lens propagating in a direction $\theta_v = \pi/2$, $\phi_v = \pi/2$ is given in Fig. 2.

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APPENDIX: DIPOLAR EMISSION: POLARIZATION TRANSFORMATION FROM TRANSFORMATION OF ACCELERATION

In the case of dipolar-type emission, the electric field in the wave is perpendicular to the projection of the particle acceleration on the plane of the sky [Ref. [15], Eq. (67.6)]:

$$\begin{aligned} \mathbf{e} &\propto (\mathbf{a} \times \mathbf{n}) \times \mathbf{n} \propto \mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\mathbf{n}, \\ \mathbf{b} &\propto (\mathbf{a} \times \mathbf{n}). \end{aligned} \quad (\text{A1})$$

The Lorentz transformation of acceleration reads [16]

$$\begin{aligned} \mathbf{a} &= \frac{\mathbf{q}_a}{\gamma^2(1 - \mathbf{v}' \cdot \mathbf{v})^3}, \\ \mathbf{q}_a &= \mathbf{a}' - \frac{\gamma}{1 + \gamma} (\mathbf{a}' \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v}' \times \mathbf{a}'), \end{aligned} \quad (\text{A2})$$

where \mathbf{a}' is an acceleration of a particle moving with velocity \mathbf{v}' in the primed frame, \mathbf{v} is the relative velocity of the two frames, and the Lorentz factor refers to the relative motion, $\gamma = 1/\sqrt{1 - v^2}$.

The similarity between the expressions for \mathbf{q}_a and \mathbf{q}' , Eqs (A2) and (6), is highly revealing. For example, consider emission by a highly relativistic particle moving along a curved trajectory. In this case $\mathbf{v}' \times \mathbf{a}'$ is directed along the normal to the orbital plane. The particle's velocity in the primed frame is both along the trajectory and along the direction of emission, $\mathbf{n}' \parallel \mathbf{v}'$. Thus,

$$\mathbf{q}_a = \mathbf{a}' - \frac{\gamma}{1 + \gamma} (\mathbf{a}' \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{n}' \times \mathbf{a}'), \quad (\text{A3})$$

so that the direction of acceleration (actually, the component perpendicular to the direction of wave propagation) transforms exactly as the electric field in the wave.

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