Gluon orbital angular momentum at small *x*

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We present a general analysis of the orbital angular momentum (OAM) distribution of gluons $L_g(x)$ inside the nucleon with particular emphasis on the small-x region. We derive a novel operator representation of $L_g(x)$ in terms of Wilson lines and argue that it is approximately proportional to the gluon helicity distribution $L_g(x) \approx -2\Delta G(x)$ at small x. We also compute longitudinal single-spin asymmetry in exclusive diffractive dijet production in lepton-nucleon scattering in the next-to-eikonal approximation and show that the asymmetry is a direct probe of the gluon helicity/OAM distribution as well as the QCD odderon exchange.

I. INTRODUCTION

After nearly thirty years since the discovery of "spin crisis" by the EMC Collaboration [1], the partonic decomposition of the nucleon spin continues to be a fascinating research area. Among the four terms in the Jaffe-Manohar decomposition formula [2],

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g, \qquad (1)$$

the quark helicity contribution $\Delta\Sigma$ is reasonably well constrained by the experimental data. The currently accepted value is $\Delta\Sigma \sim 0.30$. Over the past decade or so, there have been worldwide experimental efforts to determine the gluon helicity contribution ΔG as the integral of the polarized gluon distribution function $\Delta G = \int_0^1 dx\Delta G(x)$. The most recent NLO global QCD analysis has found a nonvanishing gluon polarization in the moderate-x region $\int_{0.05}^1 dx\Delta G(x) \approx 0.2^{+0.06}_{-0.07}$ [3]. However, uncertainties from the small-x region $x < 0.05$ are quite large, of order unity. Future experimental data from RHIC at $\sqrt{s} = 510$ GeV [4] and the planned electron-ion collider (EIC) [5] are expected to drastically reduce these uncertainties.

In contrast to these achievements in the helicity sector, it is quite frustrating that very little is known about the orbital angular momentum (OAM) of quarks L_q and gluons L_g. In fact, even the proper, gauge-invariant definitions of $L_{q,g}$ have long remained obscure (see, however, Ref. [6]). Thanks to recent theoretical developments, it is now understood that $L_{q,g}$ can be defined in a manifestly gauge-invariant (albeit nonlocal) way [7,8]. Moreover, this construction naturally allows one to define, and also gauge invariantly, the associated partonic distributions [9,10]

$$L_{q,g} = \int_0^1 dx L_{q,g}(x). \qquad (2)$$

A detailed analysis shows that $L_{q,g}(x)$ is sensitive to the twist-3 correlations in the longitudinally polarized nucleon.

Introducing the x distributions, $L_{q,g}(x)$ is essential for the experimental measurement of OAMs. Just like $\Delta\Sigma$, which is the integral of the polarized quark distribution $\Delta\Sigma = \int_0^1 dx\Delta q(x)$, $L_{q,g}$ can only be determined through a global analysis of the "OAM parton distributions" $L_{q,g}(x)$ extracted from various observables. However, accessing $L_{q,g}(x)$ experimentally is quite challenging, and there has been some recent debate over whether they can be in principle related to observables in the first place [11–13].

In this paper, we propose a method to experimentally measure the *gluon* OAM distribution $L_g(x)$ for small values of x. This is practically important in view of the above mentioned large uncertainties in ΔG from the small-x region, as well as a strong coupling analysis [14] which suggests that a significant fraction of spin comes from OAM at small x. Together with a related proposal which focuses on the moderate-x region [15], our work represents a major step forward towards understanding the spin sum rule (1).[1] We shall make a crucial use of the relation [8,18,19] between $L_{q,g}$ and the QCD Wigner distribution [20], or its Fourier transform, the generalized transverse-momentum-dependent distribution (GTMD) [8,21,22], which actually holds at the density level $L_{q,g}(x)$. Since

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[1]Very recently, a different observable related to the quark OAM distribution $L_q(x)$ for generic values of x [16] has been suggested. Moreover, the first direct computation of L_q in lattice QCD simulations [17] has appeared.

the gluon Wigner distribution is measurable at small x [23], $L_g(x)$ should also be measurable through this relation.

In Sec. II, we review the gauge-invariant gluon OAM L_g and its x distribution $L_g(x)$. In Sec. III, we discuss the said relation between $L_g(x)$ and the gluon Wigner distribution, and we prove some nontrivial identities. From Sec. IV on, we focus on the small-x regime. We derive a novel operator representation of $L_g(x)$ in terms of lightlike Wilson lines. The operator is unusual (for those who are familiar with nonlinear small-x evolution equations), as it is composed of half-infinite Wilson lines and covariant derivatives. We observe that exactly the same operator is relevant to the polarized gluon distribution $\Delta G(x)$ at small x. This, together with the arguments in Appendix B, has led us to advocate the relation

$$L_g(x) \approx -2\Delta G(x), \qquad (x \ll 1), \qquad (3)$$

which puts strong constraints on the small-x behavior of $L_g(x)$ and $\Delta G(x)$ and their uncertainties. It also suggests that the measurement of $L_g(x)$ at small x is closely related to that of $\Delta G(x)$. Based on this expectation, in Sec. V we compute longitudinal single-spin asymmetry $d\Delta\sigma = d\sigma^\rightarrow - d\sigma^\leftarrow$ in diffractive dijet production in lepton-nucleon scattering. It turns out that the asymmetry vanishes in the leading eikonal approximation, and the first non-vanishing contributions come from the next-to-eikonal corrections. This involves precisely the OAM operator found in Sec. IV, and as a result, the asymmetry is directly proportional to $L_g(x)$ in certain kinematic regimes. Interestingly, the asymmetry is also proportional to the odderon amplitude in QCD. Finally, we comment on the small-x evolution of $L_g(x)$ and $\Delta G(x)$ in Sec. VI and conclude in Sec. VII.

II. GLUON ORBITAL ANGULAR MOMENTUM

In this section, we review the gluon OAM L_g and its associated parton distribution $L_g(x)$ following Refs. [7–9]. The precise gauge-invariant definition of L_g is given by the nonperturbative proton matrix element

$$\lim_{\Delta\to 0}\langle P'S|F^{+\alpha}\overset{\leftrightarrow}{D}{}^i_{\text{pure}}A^{\text{phys}}_\alpha|PS\rangle = -i\epsilon^{ij}\Delta_{\perp j}S^+L_g, \quad (4)$$

where $P^\mu \approx \delta^\mu_+ P^+$ is the proton momentum, and the spin vector is longitudinally polarized: $S^\mu \approx \delta^\mu_+ S^+$. On the right-hand side, we keep only the linear term in the transverse momentum transfer $\Delta_\perp = P'_\perp - P_\perp$, which is assumed to be small. We use the notations $\overset{\leftrightarrow}{D}{}^\mu \equiv \frac{\overset{\rightarrow}{\partial}{}^\mu - \overset{\leftarrow}{\partial}{}^\mu}{2} + igA_\mu$ and $D^\mu_{\text{pure}} \equiv D^\mu - igA^\mu_{\text{phys}}$. A^μ_{phys} is a nonlocal operator defined by [7]

$$A^\mu_{\pm\text{phys}}(y) = \mp \int dz^- \theta(\pm(z^- - y^-))$$
$$\times \tilde{U}_{y^-,z^-}(y_\perp)F^{+\mu}(z^-, y_\perp), \qquad (5)$$

where $\tilde{U}$ is the lightlike Wilson line segment in the adjoint representation. L_g does not depend on the choice of the $\pm$ sign in Eq. (5) due to PT symmetry [8]. In the light-cone gauge $A^+ = 0$, $A^\mu_{\text{phys}} = A^\mu$, and Eq. (4) reduces to the canonical gluon OAM originally introduced by Jaffe and Manohar [2]. The operator structure (4) was first written down in Ref. [24], but the authors proposed a different A^μ_{phys}. We emphasize that the choice (5) is unique if one identifies ΔG in (1) with the usual gluon helicity ΔG that has been measured at RHIC and other experimental facilities.

Next, we discuss the gluon OAM distributions $L_g(x)$ with the property[2]

$$L_g = \int_0^1 dx L_g(x) = \frac{1}{2}\int_{-1}^1 dx L_g(x). \qquad (6)$$

The x distributions for the quark and gluon OAMs $L_{q,g}(x)$ have been previously introduced in Refs. [25,26], and their DGLAP evolution equation has been derived to one loop. However, the definition in Refs. [25,26] is not gauge invariant, and the computation of the anomalous dimensions has been performed in the light-cone gauge $A^+ = 0$. The gauge-invariant canonical OAM distributions $L_{q,g}(x)$ have been first introduced in Ref. [9]. They reduce to the previous definitions [25,26] if one takes the light-cone gauge.[3] While the notion of OAM parton distributions is not yet widely known, we emphasize that they are crucial for the measurability of OAMs. Just as one has to measure the polarized quark and gluon distributions $\Delta q(x), \Delta G(x)$ in order to extract $\Delta\Sigma = \int_0^1 dx\Delta q(x)$ and $\Delta G = \int_0^1 dx\Delta G(x)$, any attempt to experimentally determine $L_{q,g}$ must start by measuring its x distribution, $L_{q,g}(x)$.

For the gauge-invariant gluon OAM (4) with A^μ_{phys} given by Eq. (5), the distribution $L_g(x)$ is also gauge invariant and is defined through the relation [9]

$$\delta(x - x')\frac{L_g(x)}{2} = \frac{M_F(x, x')}{x(x - x')} - \frac{M_D(x, x')}{x}, \qquad (7)$$

where M_F and M_D are the "F-type" and "D-type" three-gluon collinear correlators

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[2]The normalization of $L_g(x)$ in Eqs. (6) and (7) differs by a factor of 2 from that in Ref. [9], where $L_g(x)$ was defined as $L_g = \int_{-1}^1 dx L_g(x) = 2\int_0^1 dx L_g(x)$. The present choice is in parallel with the definition of $\Delta G(x)$: $\int_0^1 dx\Delta G(x) = \Delta G$.

[3]There is an alternative gauge-invariant definition in Ref. [27], but this is different from the one [9] we discuss in the following.

$$\int \frac{dy^- dz^-}{(2\pi)^2} e^{ixP^+ y^- + i(x'-x)P^+ z^-} \langle P'S|F^{+\alpha}(0)gF^{+i}(z^-)F^+_\alpha(y^-)|PS\rangle$$

$$= -ixP^+ \int \frac{dy^- dz^-}{(2\pi)^2} e^{ixP^+ y^- + i(x'-x)P^+ z^-} \langle P'S|F^{+\alpha}(0)gF^{+i}(z^-)A^{\pm\text{phys}}_\alpha(y^-)|PS\rangle$$

$$= \epsilon^{ij} \Delta_{\perp j} S^+ M_F(x, x') + \cdots, \tag{8}$$

$$\int \frac{dy^- dz^-}{(2\pi)^2} e^{ixP^+ y^- + i(x'-x)P^+ z^-} \langle P'S|F^{+\alpha}(0)\overleftrightarrow{D}^i(z^-)F^+_\alpha(y^-)|PS\rangle$$

$$= -ixP^+ \int \frac{dy^- dz^-}{(2\pi)^2} e^{ixP^+ y^- + i(x'-x)P^+ z^-} \langle P'S|F^{+\alpha}(0)\overleftrightarrow{D}^i(z^-)A^{\pm\text{phys}}_\alpha(y^-)|PS\rangle$$

$$= \epsilon^{ij} \Delta_{\perp j} S^+ M_D(x, x') + \cdots. \tag{9}$$

(In the above, we omitted Wilson lines $\tilde{U}$ for simplicity.) The quark OAM distribution $L_q(x)$ can be similarly defined through the collinear quark-gluon-quark operators. Interestingly, although $L_{q,g}(x)$ are related to three-parton correlators which are twist-3, a partonic interpretation is possible because one of the three partons has a vanishing longitudinal momentum fraction $x - x' = 0$ due to the delta function constraint in Eq. (7). After using the QCD equations of motion, one can reveal the precise twist structure of $L_g(x)$: It can be written as the sum of the "Wandzura-Wilczek" part and the genuine twist-3 part [9]:

$$\frac{1}{2} L_g(x) = \frac{x}{2} \int_x^1 \frac{dx'}{x'^2} (H_g(x') + E_g(x')) - x \int_x^1 \frac{dx'}{x'^2} \Delta G(x')$$

$$+ 2x \int_x^1 \frac{dx'}{x'^3} \int dX \Phi_F(X, x')$$

$$+ 2x \int_x^1 dx_1 \int_{-1}^1 dx_2 \tilde{M}_F(x_1, x_2) \mathcal{P} \frac{1}{x_1^3(x_1 - x_2)}$$

$$+ 2x \int_x^1 dx_1 \int_{-1}^1 dx_2 M_F(x_1, x_2) \mathcal{P} \frac{2x_1 - x_2}{x_1^3(x_1 - x_2)^2}, \tag{10}$$

where $H_g = xG(x)$ and E_g are the gluon generalized parton distributions (GPDs) at vanishing skewness. Φ_F and $\tilde{M}_F$ are the quark-gluon-quark and three-gluon correlators defined similarly to Eq. (8) (see Ref. [9] for the details). Equation (10) shows that $L_g(x)$ and $\Delta G(x)$ are related, albeit in a complicated way. Later, we shall find a more direct relation between the two distributions special to the small-x region.

Before leaving this section, we show the DGLAP equations for $L_{q,g}(x)$. They can be extracted from the results of the anomalous dimensions in Refs. [25,26] (see also Ref. [28]).

$$\frac{d}{d \ln Q^2} \begin{pmatrix} L_q(x) \\ L_g(x) \end{pmatrix}$$

$$= \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \begin{pmatrix} \hat{P}_{qq}(z) & \hat{P}_{qg}(z) & \Delta\hat{P}_{qq}(z) & \Delta\hat{P}_{qg}(z) \\ \hat{P}_{gq}(z) & \hat{P}_{gg}(z) & \Delta\hat{P}_{gq}(z) & \Delta\hat{P}_{gg}(z) \end{pmatrix}$$

$$\times \begin{pmatrix} L_q(x/z) \\ L_g(x/z) \\ \Delta q(x/z) \\ \Delta G(x/z) \end{pmatrix}, \tag{11}$$

$$\hat{P}_{qq}(z) = C_F \left(\frac{z(1+z^2)}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right), \tag{12}$$

$$\hat{P}_{qg}(z) = n_f z(z^2 + (1-z)^2), \tag{13}$$

$$\hat{P}_{gq}(z) = C_F(1 + (1-z)^2), \tag{14}$$

$$\hat{P}_{gg}(z) = 6\frac{(z^2 - z + 1)^2}{(1-z)_+} + \frac{\beta_0}{2}\delta(z-1), \tag{15}$$

$$\Delta\hat{P}_{qq}(z) = C_F(z^2 - 1), \tag{16}$$

$$\Delta\hat{P}_{qg}(z) = n_f(1 - 3z + 4z^2 - 2z^3), \tag{17}$$

$$\Delta\hat{P}_{gq}(z) = C_F(-z^2 + 3z - 2), \tag{18}$$

$$\Delta\hat{P}_{gg}(z) = 6(z-1)(z^2 - z + 2), \tag{19}$$

where $C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$, n_f is the number of flavors, and $\beta_0 = 11 - \frac{2n_f}{3}$. For completeness and a later use, we also note the DGLAP equation for the helicity distributions:

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\Delta q(x)\\ \Delta G(x)\end{pmatrix} = \frac{\alpha_s}{2\pi}\int_x^1\frac{dz}{z}\begin{pmatrix}\Delta P_{qq}(z) & \Delta P_{qg}(z)\\ \Delta P_{gq}(z) & \Delta P_{gg}(z)\end{pmatrix}$$
$$\times\begin{pmatrix}\Delta q(x/z)\\ \Delta G(x/z)\end{pmatrix}, \qquad (20)$$

$$\Delta P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right), \qquad (21)$$

$$\Delta P_{qg}(z) = \frac{n_f}{2}(2z-1), \qquad (22)$$

$$\Delta P_{gq}(z) = C_F(2-z), \qquad (23)$$

$$\Delta P_{gg}(z) = 6\left(\frac{1}{(1-z)_+}-2z+1\right)+\frac{\beta_0}{2}\delta(z-1). \qquad (24)$$

III. OAM AND THE WIGNER DISTRIBUTION

The original definition (7) is technical and does not immediately invoke its physical meaning as the OAM. Fortunately, there exists an equivalent and very intuitive definition of $L_g(x)$ in terms of the Wigner distribution. The gluon Wigner distribution is defined as

$$xW(x,q_\perp,b_\perp,S)$$
$$= 2\int\frac{dz^-d^2z_\perp}{(2\pi)^3P^+}\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-ixP^+z^-+iq_\perp\cdot z_\perp}$$
$$\times\left\langle P+\frac{\Delta_\perp}{2},S\left|\mathrm{Tr}F^{+i}\left(b_\perp+\frac{z}{2}\right)F^{+i}\left(b_\perp-\frac{z}{2}\right)\right|P-\frac{\Delta_\perp}{2},S\right\rangle, \qquad (25)$$

where the trace is in the fundamental representation. It is convenient to also consider the Fourier transform of the Wigner distribution with respect to $b_\perp$—namely, the generalized transverse-momentum-dependent distribution (GTMD) [8,21,22]

$$xW(x,q_\perp,\Delta_\perp,S)$$
$$= \int d^2b_\perp W_g(x,q_\perp,b_\perp,S)e^{i\Delta_\perp\cdot b_\perp}$$
$$= 4\int\frac{d^3xd^3y}{(2\pi)^3}e^{-ixP^+(x^--y^-)+iq_\perp\cdot(x_\perp-y_\perp)+i\frac{\Delta_\perp}{2}\cdot(x_\perp+y_\perp)}$$
$$\times\langle\mathrm{Tr}F^{+i}(x)F^{+i}(y)\rangle, \qquad (26)$$

where $\langle\cdots\rangle\equiv\frac{\langle P+\frac{\Delta_\perp}{2},S|\cdots|P-\frac{\Delta_\perp}{2},S\rangle}{\langle P,S|P,S\rangle}$. In Eqs. (25) and (26), we have to specify the configuration of Wilson lines to make the nonlocal operator $F(x)F(y)$ gauge invariant. There are two interesting choices for this [29,30]: One is the Weizsäcker-Williams (WW) type

$$\langle\mathrm{Tr}F^{+i}(x)F^{+i}(y)\rangle$$
$$\to\langle\mathrm{Tr}F^{+i}(x)U_\pm(x,y)F^{+i}(y)U_\pm(y,x)\rangle, \qquad (27)$$

and the other is the dipole type

$$\langle\mathrm{Tr}F^{+i}(x)F^{+i}(y)\rangle$$
$$\to\langle\mathrm{Tr}F^{+i}(x)U_-(x,y)F^{+i}(y)U_+(y,x)\rangle, \qquad (28)$$

where $U_\pm(x,y)\equiv U_{x^-,\pm\infty}(x_\perp)U_{x_\perp,y_\perp}(\pm\infty)U_{\pm\infty,y^-}(y_\perp)$ is a staple-shaped Wilson line in the fundamental representation. We denote the corresponding distributions as $W_\pm$ and W_{dip}, respectively.

The Wigner distribution describes the phase distribution of gluons with transverse momentum $q_\perp$ and impact parameter $b_\perp$. Their cross product $b_\perp\times q_\perp$ classically represents the orbital angular momentum. It is thus natural to define L_g as [8]

$$L_g \equiv \int_{-1}^1 dx\int d^2b_\perp d^2q_\perp\epsilon_{ij}b_\perp^i q_\perp^j W_\pm(x,q_\perp,b_\perp)$$
$$= -i\int_{-1}^1 dx\int d^2q_\perp\epsilon^{ij}q_\perp^j\lim_{\Delta_\perp\to0}\frac{\partial}{\partial\Delta_\perp^i}W_\pm(x,q_\perp,\Delta_\perp), \qquad (29)$$

where our default choice is the WW-type Wigner distribution, because it is consistent with a partonic interpretation. One can check that (29) agrees with (4), with the $\pm$ sign taken over to that in (5). W has the following spin-dependent structure:

$$W(x,q_\perp,\Delta_\perp,S) = i\frac{S^+}{P^+}\epsilon^{ij}\Delta_\perp^i q_\perp^j(f(x,|q_\perp|)$$
$$+ i\Delta_\perp\cdot q_\perp h(x,|q_\perp|))+\cdots. \qquad (30)$$

Substituting this into Eq. (29), one finds

$$L_g = \lambda\int_{-1}^1 dx\int d^2q_\perp q_\perp^2 f(x,|q_\perp|), \qquad (31)$$

where $\lambda = \frac{S^+}{P^+} = \pm1$ is the helicity of the proton.

The result (31), together with a similar relation for the quark OAM, is by now well established [8,18,19]. We now discuss this relation at the level of the x distribution. Since (29) involves an integration over x, it is tempting to identify the integrand with $L_g(x)$:

$$L_g(x) = 2\int d^2b_\perp d^2q_\perp\epsilon_{ij}b_\perp^i q_\perp^j W_\pm(x,q_\perp,b_\perp)$$
$$= -2i\int d^2q_\perp\epsilon^{ij}q_\perp^j\lim_{\Delta_\perp\to0}\frac{\partial}{\partial\Delta_\perp^i}W_\pm(x,q_\perp,\Delta_\perp). \qquad (32)$$

(The factor of 2 is because $\int_{-1}^1 dx = 2\int_0^1 dx$.) It turns out that this exactly agrees with $L_g(x)$ defined in Eq. (7). The proof was essentially given in Ref. [9] for the quark OAM distribution $L_q(x)$. The generalization to the gluon case is straightforward, and this is outlined in Appendix A. Here we prove another nontrivial fact—that $L_g(x)$'s defined through the WW and dipole Wigner distribution are identical for all values of x. Namely,

$$\int d^2b_\perp d^2q_\perp \epsilon^{ij} b^i_\perp q^j_\perp W_\pm(x, b_\perp, q_\perp) = \int d^2b_\perp d^2q_\perp \epsilon^{ij} b^i_\perp q^j_\perp W_{\mathrm{dip}}(x, b_\perp, q_\perp). \tag{33}$$

The proof goes as follows: Consider the part that involves $q_\perp$; $\int d^2q_\perp q^j_\perp W$. For the WW-type Wigner, this is evaluated as

$$
\begin{aligned}
\int d^2q_\perp q^j_\perp \int \frac{d^2z_\perp}{(2\pi)^2} e^{iq_\perp \cdot z_\perp} \mathrm{Tr} F^{+i}\left(\frac{z}{2}\right) U_\pm F^{+i}\left(-\frac{z}{2}\right) U^\dagger_\pm &= i \lim_{z_\perp \to 0} \frac{\partial}{\partial z^j_\perp}\left(\mathrm{Tr} F^{+i}\left(\frac{z}{2}\right) U_\pm F^{+i}\left(-\frac{z}{2}\right) U^\dagger_\pm\right) \\
&= \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)(i\bar{D}_j U - iUD_j)F^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&\quad + \frac{1}{2}\mathrm{Tr}\left[[F^{+i}, gA^j_{\pm\mathrm{phys}}]\left(\frac{z^-}{2}\right)UF^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&\quad - \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)U[F^{+i}, gA^j_{\pm\mathrm{phys}}]\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)(i\bar{D}^{\mathrm{pure}}_j U - iUD^{\mathrm{pure}}_j)F^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right], \tag{34}
\end{aligned}
$$

where we only show the relevant operator structure and suppress the arguments of Wilson lines U which should be obvious from gauge invariance. The same type of calculation for the dipole Wigner distribution gives

$$
\begin{aligned}
\int d^2q_\perp q^j_\perp \int \frac{d^2z_\perp}{(2\pi)^2} e^{iq_\perp \cdot z_\perp} \mathrm{Tr} F^{+i}\left(\frac{z}{2}\right) U_- F^{+i}\left(-\frac{z}{2}\right) U^\dagger_+ &= \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)(i\bar{D}_j U - iUD_j)F^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&\quad + \frac{1}{2}\mathrm{Tr}\left[g(F^{+i}A^j_{-\mathrm{phys}} - A^j_{+\mathrm{phys}}F^{+i})\left(\frac{z^-}{2}\right)UF^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&\quad - \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)Ug(F^{+i}A^j_{+\mathrm{phys}} - A^j_{-\mathrm{phys}}F^{+i})\left(-\frac{z^-}{2}\right)U^\dagger\right]. \tag{35}
\end{aligned}
$$

Taking the plus sign in Eq. (34) (the minus sign leads to the same conclusion) and subtracting (35), we obtain

$$
\begin{aligned}
i\lim_{z_\perp \to 0}\frac{\partial}{\partial z^j_\perp}&\left(\mathrm{Tr} F^{+i}\left(\frac{z}{2}\right) U_+ F^{+i}\left(-\frac{z}{2}\right) U^\dagger_+\right) - i\lim_{z_\perp \to 0}\frac{\partial}{\partial z^j_\perp}\left(\mathrm{Tr} F^{+i}\left(\frac{z}{2}\right) U_- F^{+i}\left(-\frac{z}{2}\right) U^\dagger_+\right) \\
&= \frac{1}{2}\mathrm{Tr}\left[F^{+i}(A_{+\mathrm{phys}} - A_{-\mathrm{phys}})\left(\frac{z^-}{2}\right)UF^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] + \frac{1}{2}\mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)U(A_{+\mathrm{phys}} - A_{-\mathrm{phys}})F^{+i}\left(-\frac{z^-}{2}\right)U^\dagger\right] \\
&= -\int dy^- \mathrm{Tr}\left[F^{+i}\left(\frac{z^-}{2}\right)U_{\frac{z^-}{2}, y^-}F^{+i}(y^-)U_{y^-, -\frac{z^-}{2}}F^{+i}\left(-\frac{z^-}{2}\right)U_{-\frac{z^-}{2}, \frac{z^-}{2}}\right]. \tag{36}
\end{aligned}
$$

The question is whether the nonforward matrix element $\langle\ldots\rangle$ of the operator (36) contains the structure $i\frac{S^+}{P^+}\epsilon^{ij}\Delta^i_\perp \delta L(x)$. If so, the function δL would contribute to the difference $L^{WW}_g(x) - L^{\mathrm{dip}}_g(x)$. However, this is impossible, as one can easily see by applying the PT transformation to the matrix element. Under PT, $F^{\mu\nu} \to -F^{\mu\nu}$; and one obtains an identity

$$i\frac{S^+}{P^+}\epsilon^{ij}\Delta^i_\perp \delta L(x) = -i\frac{-S^+}{P^+}\epsilon^{ij}(-\Delta^i_\perp)\delta L(x), \tag{37}$$

which immediately gives $\delta L(x) = 0$.

The above proof is crucial for the measurability of $L_g(x)$. While $L_g(x)$ is naturally defined by the WW-type Wigner distribution, the dipole Wigner distribution has a better chance to be measured in experiments [23]. Below, we only consider W_{dip} and omit the subscript.

IV. SMALL-x REGIME

Our discussion so far has been general and valid for any value of x. From now on, we focus on the small-x regime. In this section we derive a novel operator representation of $L_g(x)$ and point out its unexpected relation to the polarized gluon distribution $\Delta G(x)$.

A. Leading order

In order to study the properties of the (dipole) Wigner distribution at small x, as a first step we approximate $e^{-ixP^+(x^- - y^-)} \approx 1$ in Eq. (26). We shall refer to this as the eikonal approximation. We then use the identity

$$
\partial_i U(x_\perp) = -ig \int_{-\infty}^{\infty} dx^- U_{\infty, x^-} F^{+i}(x) U_{x^-, -\infty}
$$
$$
- igA^i(\infty, x_\perp) U(x_\perp)
$$
$$
+ igU(x_\perp) A^i(-\infty, x_\perp), \tag{38}
$$

where $U(x_\perp) \equiv U_{\infty, -\infty}(x_\perp)$, and perform integration by parts. This leads us to [23]

$$
W(x, \Delta_\perp, q_\perp, S) \approx W_0(x, \Delta_\perp, q_\perp)
$$
$$
= \frac{4N_c}{xg^2(2\pi)^3} \left(q_\perp^2 - \frac{\Delta_\perp^2}{4} \right) F(x, \Delta_\perp, q_\perp),
\tag{39}
$$

where F is the Fourier transform of the so-called dipole S-matrix:

$$
F(x, \Delta_\perp, q_\perp) \equiv \int d^2x_\perp d^2y_\perp e^{iq_\perp \cdot (x_\perp - y_\perp) + i(x_\perp + y_\perp) \cdot \frac{\Delta_\perp}{2}}
$$
$$
\times \left\langle \frac{1}{N_c} \text{Tr}[U(x_\perp) U^\dagger(y_\perp)] \right\rangle. \tag{40}
$$

The last two terms in Eq. (38) have been canceled against the terms which come from the derivative of the transverse gauge links connecting $x_\perp$ and $y_\perp$ at $x^- = \pm\infty$ [not shown

in Eq. (40) for simplicity]. The x dependence of F arises from the quantum evolution of the dipole operator $\text{Tr}U(x_\perp)U^\dagger(y_\perp)$. To linear order in $\Delta_\perp$, we can parametrize F as

$$
F(x, \Delta_\perp, q_\perp) = P(x, \Delta_\perp, q_\perp) + iq_\perp \cdot \Delta_\perp O(x, |q_\perp|). \tag{41}
$$

The imaginary part O comes from the so-called odderon operator [31,32]. It is important to notice that F cannot depend on the longitudinal spin S^+, and therefore W_0 cannot have the structure in Eq. (30). This follows from PT symmetry, which dictates that

$$
\left\langle P + \frac{\Delta}{2}, S \middle| \text{Tr}[U(x_\perp)U^\dagger(y_\perp)] \middle| P - \frac{\Delta}{2}, S \right\rangle
$$
$$
= \left\langle P - \frac{\Delta}{2}, -S \middle| \text{Tr}[U(-x_\perp)U^\dagger(-y_\perp)] \middle| P + \frac{\Delta}{2}, -S \right\rangle,
$$

so that $W_0(x, q_\perp, \Delta_\perp, S) = W_0(x, -q_\perp, -\Delta_\perp, -S)$. Therefore, it is impossible to access any information about spin and OAM in the eikonal approximation. This is actually expected on physical grounds. At high energy, spin effects are suppressed by a factor of x (or inverse energy) compared to the "Pomeron" contribution as represented by the first term P in Eq. (41).[4]

B. First subleading correction

In order to be sensitive to the spin and OAM effects, we have to go beyond the eikonal approximation. By taking into account the second term in the expansion $e^{-ixP^+(x^- - y^-)} = 1 - ixP^+(x^- - y^-) + \cdots$ and writing $W = W_0 + \delta W$ accordingly, we find

$$
\delta W(x, \Delta_\perp, q_\perp, S) = -\frac{4P^+}{g(2\pi)^3} \int d^2x_\perp d^2y_\perp e^{iq_\perp \cdot (x_\perp - y_\perp) + i(x_\perp + y_\perp) \cdot \frac{\Delta_\perp}{2}}
$$
$$
\times \left\{ \int_{-T}^{T} dx^- (x^- + T) \frac{\partial}{\partial y_\perp^i} \langle \text{Tr}[U_{T,x} F^{+i}(x) U_{x,-T} U^\dagger(y_\perp)] \rangle \right.
$$
$$
\left. + \int_{-T}^{T} dy^- (y^- + T) \frac{\partial}{\partial x_\perp^i} \langle \text{Tr}[U(x_\perp) U_{-T,y} F^{+i}(y) U_{y,T}] \rangle \right\}
$$
$$
= \frac{4P^+}{g^2(2\pi)^3} \int d^2x_\perp d^2y_\perp e^{i(q_\perp + \frac{\Delta_\perp}{2}) \cdot x_\perp + i(-q_\perp + \frac{\Delta_\perp}{2}) \cdot y_\perp}
$$
$$
\times \left\{ \int_{-T}^{T} dz^- \left(q_\perp^i - \frac{\Delta_\perp^i}{2} \right) \langle \text{Tr}[U_{Tz^-}(x_\perp) \overleftarrow{D}_i U_{z^- - T}(x_\perp) U^\dagger(y_\perp)] \rangle \right.
$$
$$
\left. + \int_{-T}^{T} dz^- \left(q_\perp^i + \frac{\Delta_\perp^i}{2} \right) \langle \text{Tr}[U(x_\perp) U_{-Tz^-}(y_\perp) D_i U_{z^- T}(y_\perp)] \rangle \right\}. \tag{42}
$$

---

[4]The situation is different when the spin is transversely polarized. In this case, F can have the structure $\epsilon_{ij} S_\perp^i q_\perp^j$, and the corresponding amplitude has been dubbed the "spin-dependent odderon" [33]. While this is subleading compared to the leading Pomeron term P, it is suppressed only by a fractional power x^α with $\alpha \sim 0.3$.

The first equality is obtained by splitting $x^- - y^- = x^- + T - (y^- + T)$, where T is eventually sent to infinity. In the second equality, we write $x^- + T = \int_{-T}^{x^-} dz^-$ and switch the order of integration between $\int dx^-$ and $\int dz^-$.

In contrast to W_0, δW can have the structure (30): From PT symmetry, one can show that $\delta W(x, q_\perp, \Delta_\perp, S) = -\delta W(x, -q_\perp, -\Delta_\perp, -S)$.[5] The most general parameterization of the near-forward matrix element in (42) is, to linear order in $\Delta_\perp$ and S^+,

$$\frac{4P^+}{g^2(2\pi)^3} \int d^2x_\perp d^2y_\perp e^{i(q_\perp + \frac{\Delta_\perp}{2})\cdot x_\perp + i(-q_\perp + \frac{\Delta_\perp}{2})\cdot y_\perp} \int dz^- \langle \mathrm{Tr}[U_{\infty, z^-}(x_\perp)\bar{D}_i U_{z^-, -\infty}(x_\perp)U^\dagger(y_\perp)]\rangle$$

$$= -i\frac{S^+}{2P^+}\epsilon_{ij}\left\{ \left(q_\perp^j + \frac{\Delta_\perp^j}{2}\right)f(x,|q_\perp|) + \left(q_\perp^j - \frac{\Delta_\perp^j}{2}\right)g(x,|q_\perp|) + q_\perp^j \Delta_\perp \cdot q_\perp A(x,|q_\perp|)\right\}$$

$$- \frac{S^+}{2P^+}\epsilon_{ij}\left\{ \left(q_\perp^j + \frac{\Delta_\perp^j}{2}\right)B(x,|q_\perp|) + \left(q_\perp^j - \frac{\Delta_\perp^j}{2}\right)C(x,|q_\perp|) - 2q_\perp^j \Delta_\perp \cdot q_\perp h(x,|q_\perp|)\right\} + \cdots, \qquad (43)$$

$$\frac{4P^+}{g^2(2\pi)^3} \int d^2x_\perp d^2y_\perp e^{i(q_\perp + \frac{\Delta_\perp}{2})\cdot x_\perp + i(-q_\perp + \frac{\Delta_\perp}{2})\cdot y_\perp} \int dz^- \langle \mathrm{Tr}[U(x_\perp)U_{-\infty, z^-}(y_\perp)D_i U_{z^-, \infty}(y_\perp)]\rangle$$

$$= i\frac{S^+}{2P^+}\epsilon_{ij}\left\{ \left(q_\perp^j - \frac{\Delta_\perp^j}{2}\right)f(x,|q_\perp|) + \left(q_\perp^j + \frac{\Delta_\perp^j}{2}\right)g(x,|q_\perp|) - q_\perp^j \Delta_\perp \cdot q_\perp A(x,|q_\perp|)\right\}$$

$$- \frac{S^+}{2P^+}\epsilon_{ij}\left\{ \left(q_\perp^j - \frac{\Delta_\perp^j}{2}\right)B(x,|q_\perp|) + \left(q_\perp^j + \frac{\Delta_\perp^j}{2}\right)C(x,|q_\perp|) + 2q_\perp^j \Delta_\perp \cdot q_\perp h(x,|q_\perp|)\right\} + \cdots. \qquad (44)$$

We recognize the functions f and h that appear in Eq. (30); the former is related to the OAM as in (31). The other real-valued functions g, A, B, C do not contribute to the Wigner distribution. Integrating both sides over $q_\perp$, we obtain the following sum rules:

$$\int d^2q_\perp (f - g + q_\perp^2 A) = 0,$$

$$\int d^2q_\perp (B - C - 2q_\perp^2 h) = 0. \qquad (45)$$

Equation (43) uncovers a novel representation of the OAM distribution at small x in terms of an unusual Wilson line operator in which the covariant derivative D_i is inserted at an intermediate time z^-. Such operators do not usually appear in the context of high-energy evolution. In the next section, we shall see that this structure is related to the next-to-eikonal approximation. Here we point out that the same operator is relevant to the polarized gluon distribution $\Delta G(x)$. This elucidates an unexpected relation between $\Delta G(x)$ and $L_g(x)$.

Let us define the "unintegrated" (transverse-momentum-dependent) polarized gluon distribution $\Delta G(x, q_\perp)$ as

$$ix\Delta G(x, q_\perp)\frac{S^+}{P^+} \equiv 2\int \frac{d^2z_\perp dz^-}{(2\pi)^3 P^+} e^{-ixP^+ z^- + iq_\perp \cdot z_\perp}$$

$$\times \langle PS|\epsilon_{ij}\mathrm{Tr}F^{+i}\left(\frac{z}{2}\right)U_- F^{+j}\left(-\frac{z}{2}\right)U_+|PS\rangle$$

$$= 4\int \frac{d^3x d^3y}{(2\pi)^3} e^{-ixP^+(x^- - y^-) + iq_\perp \cdot (x_\perp - y_\perp)}$$

$$\times \frac{\langle PS|\epsilon_{ij}\mathrm{Tr}[F^{+i}(x)U_- F^{+j}(y)U_+]|PS\rangle}{\langle PS|PS\rangle}, \qquad (46)$$

such that $\int d^2q_\perp \Delta G(x, q_\perp) = \Delta G(x)$ and $\int_0^1 dx \Delta G(x) = \Delta G$. Note that (46) is a forward matrix element $\Delta_\perp = 0$. Using the same approximation as above, we obtain the following representation at small x:

$$i\Delta G(x, q_\perp)\frac{S^+}{P^+}$$

$$= \frac{4P^+}{g^2(2\pi)^3}\int d^2x_\perp d^2y_\perp e^{iq_\perp \cdot (x_\perp - y_\perp)}$$

$$\times \epsilon_{ij}\left\{ q_\perp^j \int_{-\infty}^\infty dz^- \langle \mathrm{Tr}[U_{\infty z^-}(x_\perp)\bar{D}_i U_{z^- -\infty}(x_\perp)U^\dagger(y_\perp)]\rangle \right.$$

$$\left. + q_\perp^i \int_{-\infty}^\infty dz^- \langle \mathrm{Tr}[U(x_\perp)U_{-\infty z^-}(y_\perp)D_j U_{z^- \infty}(y_\perp)]\rangle \right\}, \qquad (47)$$

---

[5]More generally, in the Taylor expansion of the phase factor $e^{-ixP^+(x^- - y^-)}$, the odd terms in x can contribute to the OAM.

or equivalently,

$$
\Delta G(x, q_\perp) \frac{S^+}{P^+}
$$
$$
= \frac{8P^+}{g^2 (2\pi)^3} \epsilon_{ij} q_\perp^j \, \mathfrak{Im} \left[\int d^2 x_\perp d^2 y_\perp e^{iq_\perp \cdot (x_\perp - y_\perp)} \right.
$$
$$
\left. \times \int_{-\infty}^{\infty} dz^- \langle \mathrm{Tr}[U_{\infty z^-}(x_\perp) \bar{D}_i U_{z^- -\infty}(x_\perp) U^\dagger(y_\perp)] \rangle \right].
$$
(48)

Substituting Eq. (43), we find

$$
\Delta G(x) = - \int d^2 q_\perp q_\perp^2 (f(x, |q_\perp|) + g(x, |q_\perp|))
$$
$$
= - \frac{1}{2} L_g(x) - \int d^2 q_\perp q_\perp^2 g(x, |q_\perp|).
$$
(49)

This is a rather surprising result. From Eq. (10), one can argue that if $\Delta G(x)$ shows a power-law behavior at small x, $\Delta G(x) \sim x^{-\alpha}$, the OAM distribution grows with the same exponent, $L_g(x) \sim x^{-\alpha}$. Equation (49) imposes a strong constraint on the respective prefactors, and the relation is preserved by the small-x evolution because both $L_g(x)$ and $\Delta G(x)$ are governed by the same operator. Moreover, in Appendix B, we present three different arguments which indicate that $|f| \gg |g|$. If this is true, a very intriguing relation emerges:

$$
L_g(x) \approx -2 \Delta G(x).
$$
(50)

As mentioned in the Introduction, reducing the huge uncertainty in ΔG from the small-x region $x < 0.05$ [3] is a pressing issue in QCD spin physics. Equation (50) suggests that, if the integral $\int_0^{0.05} dx \Delta G(x)$ turns out to be sizable in the future, one should expect an even larger contribution from the gluon OAM in the same x region, which reverses the sign of the net gluon angular momentum:

$$
\int_0^{0.05} dx \Delta G(x) + \int_0^{0.05} dx L_g(x) \approx - \int_0^{0.05} dx \Delta G(x). \quad (51)
$$

This has profound implications on the spin sum rule (1). In particular, it challenges the idea that $\Delta \Sigma$ and ΔG alone can saturate the sum rule. There must be OAM contributions.

Equation (50) is reminiscent of a similar relation observed in the large-Q^2 asymptotic scaling behavior of the components in the spin decomposition formula, Eq. (1) [28]. To one-loop order,

$$
\Delta \Sigma(t) = \mathrm{const}, \tag{52}
$$

$$
L_q(t) = -\frac{1}{2} \Delta \Sigma + \frac{1}{2} \frac{3 n_f}{16 + 3 n_f}, \tag{53}
$$

$$
\Delta G(t) = -\frac{4 \Delta \Sigma}{\beta_0} + \frac{t}{t_0} \left(\Delta G_0 + \frac{4 \Delta \Sigma}{\beta_0} \right), \tag{54}
$$

$$
L_g(t) = -\Delta G(t) + \frac{1}{2} \frac{16}{16 + 3 n_f}, \tag{55}
$$

where $t = \ln(Q^2 / \Lambda_{\mathrm{QCD}}^2)$ and we have neglected the subleading terms at large Q^2. ΔG_0 represents the gluon helicity contribution at some initial scale t_0. From these equations, we find that the large negative gluon orbital angular momentum would cancel out the gluon helicity contribution if the latter is large and positive. It is interesting to see how this behavior imposes a constraint on the small-x contribution to ΔG and L_g when we apply Eq. (51) as the initial condition. The scale evolution of $L_g(x)$ and $\Delta G(x)$ can be an important agenda for the future electron-ion collider [5], where one of the primary goals is to investigate the sum rule (1).

V. SINGLE-SPIN ASYMMETRY IN DIFFRACTIVE DIJET PRODUCTION

In this section, we calculate longitudinal single-spin asymmetry in forward dijet production in exclusive diffractive lepton-nucleon scattering. As observed recently [23], in this process one can probe the gluon Wigner distribution at small x (see also Ref. [34]) and its characteristic angular correlations. Here we show that the same process, with the proton being longitudinally polarized, is directly sensitive to the function $f(x, q_\perp)$.

A. Next-to-eikonal approximation

Exclusive diffractive forward dijet production in ep collisions has been extensively studied in the literature, mostly in the BFKL framework [35–39], and more recently in the color glass condensate framework [23,34]. We work in the so-called dipole frame, where the left-moving virtual photon with virtuality Q^2 splits into a $q\bar{q}$ pair and scatters off the right-moving proton. The proton emerges elastically with momentum transfer $\Delta_\perp$. The $q\bar{q}$ pair is detected in the forward region (i.e., at large negative rapidity) as two jets with the total transverse momentum $k_{1\perp} + k_{2\perp} = -\Delta_\perp$ and the relative momentum $\frac{1}{2}(k_{2\perp} - k_{1\perp}) = P_\perp$.

In the eikonal approximation and for the transversely polarized virtual photon, the amplitude is proportional to [23,34]

$$
\propto \int d^2 x_\perp d^2 y_\perp e^{-ik_{1\perp} \cdot x_\perp - ik_{2\perp} \cdot y_\perp}
$$
$$
\times \left\langle \frac{1}{N_c} \mathrm{Tr}[U(x_\perp) U^\dagger(y_\perp)] \right\rangle \frac{\epsilon K_1(\epsilon r_\perp)}{2\pi} \frac{r_\perp^i}{r_\perp}
$$
$$
= i \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{P_\perp^i - q_\perp^i}{(P_\perp - q_\perp)^2 + \epsilon^2} F(\Delta_\perp, q_\perp), \tag{56}
$$

where $r_\perp = x_\perp - y_\perp$ and $\varepsilon^2 = z(1-z)Q^2$. The value z (or $1-z$) is the longitudinal momentum fraction of the virtual photon energy q^- carried by the quark (or antiquark).

As we already pointed out, Eq. (56) cannot depend on spin. Our key observation is that the next-to-eikonal corrections to (56) include exactly the same matrix element as (43) and is therefore sensitive to the gluon OAM function f. Going beyond the eikonal approximation, we generalize (56) as

$$
\int d^2x_\perp d^2x'_\perp d^2y_\perp d^2y'_\perp e^{-ik_{1\perp}\cdot x_\perp - ik_{2\perp}\cdot y_\perp}
$$
$$
\times \left\langle \frac{1}{N_c}\mathrm{Tr}[U(x_\perp,x'_\perp)U^\dagger(y_\perp,y'_\perp)] \right\rangle \frac{\varepsilon K_1(\varepsilon r'_\perp)}{2\pi}\frac{r'^i_\perp}{r'_\perp}, \quad (57)
$$

where we allow the quark and antiquark to change their transverse coordinates during propagation. $U(x_\perp,x'_\perp)$ is essentially the Green function and can be determined as follows.

Consider the propagation of a quark with energy $k^- = zq^-$ in the background field A^+, $A^i_\perp$. The Green function satisfies the equation[6]

$$
\left[i\frac{\partial}{\partial x^-} + \frac{1}{2k^-}D^2_{x_\perp} - gA^+(x^-,x_\perp)\right]G_{k^-}(x^-,x_\perp,x'^-,x'_\perp)
$$
$$
= i\delta(x^- - x'^-)\delta^{(2)}(x_\perp - x'_\perp). \quad (58)
$$

To zeroth order in $1/k^-$, the solution is

$$
G^0_{k^-}(x^-,x_\perp,x'^-,x'_\perp) = \theta(x^- - x'^-)\delta^{(2)}(x_\perp - x'_\perp)
$$
$$
\times \exp\left(-ig\int_{x'^-}^{x^-}dz^-A^+(z^-,x_\perp)\right). \quad (59)
$$

This is the eikonal approximation. Writing $G = G^0 + \delta G$, we find the equation for δG:

$$
\left[i\frac{\partial}{\partial x^-} - gA^+(x^-,x_\perp)\right]\delta G + \frac{1}{2k^-}D^2_{x_\perp}G^0 = 0. \quad (60)
$$

This can be easily solved as

$$
\delta G(x^-,x_\perp,x'^-,x'_\perp)
$$
$$
= \frac{i}{2k^-}\theta(x^- - x'^-)\int_{x'^-}^{x^-}dz^-U_{x^-z^-}(x_\perp)D^2_{x_\perp}\delta^{(2)}(x_\perp - x'_\perp)
$$
$$
\times U_{z^-x'^-}(x'_\perp). \quad (61)
$$

We thus obtain the desired propagator

$$
U(x_\perp,x'_\perp) \equiv G_{k^-}(\infty,x_\perp,-\infty,x'_\perp)
$$
$$
= U(x_\perp)\delta^{(2)}(x_\perp - x'_\perp)
$$
$$
+ \frac{i}{2k^-}\int_{-\infty}^{\infty}dz^-U_{\infty z^-}(x_\perp)D^2_{x_\perp}\delta^{(2)}(x_\perp - x'_\perp)
$$
$$
\times U_{z^--\infty}(x'_\perp). \quad (62)
$$

In (57), we need the Fourier transform of $U(x_\perp,x'_\perp)$,

$$
\int d^2x_\perp e^{-ik_\perp\cdot x_\perp}U(x_\perp,x'_\perp)
$$
$$
= e^{-ik_\perp\cdot x'_\perp}\left(U(x'_\perp) + \frac{i}{2k^-}\int_{-\infty}^{\infty}dz^-U_{\infty z^-}(x'_\perp)\right.
$$
$$
\left.\times(\bar{D}^2_{x'_\perp} - k^2_\perp - 2ik^i_\perp\bar{D}_{x'^i})U_{z^--\infty}(x'_\perp)\right). \quad (63)
$$

If we ignore $A_\perp$, Eq. (63) agrees with the result of Refs. [40,41] to the order of interest, although equivalence is not immediately obvious.[7] Clearly, $A_\perp$ is important for the result to be gauge invariant (covariant). The last term in Eq. (63), when substituted into (57), gives the same operator as in Eq. (43). In addition, (63) contains the operator $U_{\infty,z^-}\bar{D}^2_{x_\perp}U_{z^-,-\infty}$, which we did not encounter in the previous section. However, the matrix element of this operator does not require new functions. To see this, we write down the general parameterization to linear order in $\Delta_\perp$:

$$
\frac{4P^+}{g^2(2\pi)^3}\int d^2x_\perp d^2y_\perp e^{iq_\perp\cdot(x_\perp - y_\perp) + i\frac{\Delta_\perp}{2}(x_\perp + y_\perp)}
$$
$$
\times \int dz^-\langle\mathrm{Tr}[U_{\infty,z^-}(x_\perp)\bar{D}^2_{x_\perp}U_{z^-,-\infty}(x_\perp)U^\dagger(y_\perp)]\rangle
$$
$$
= (\kappa(x,|q_\perp|) + i\eta(x,|q_\perp|))\frac{S^+}{P^+}\epsilon^{ij}q^i_\perp\Delta^j_\perp + \cdots, \quad (64)
$$

$$
\frac{4P^+}{g^2(2\pi)^3}\int d^2x_\perp d^2y_\perp e^{iq_\perp\cdot(x_\perp - y_\perp) + i\frac{\Delta_\perp}{2}(x_\perp + y_\perp)}
$$
$$
\times \int dz^-\langle\mathrm{Tr}[U_{\infty,z^-}(x_\perp)\vec{D}^2_{x_\perp}U_{z^-,-\infty}(x_\perp)U^\dagger(y_\perp)]\rangle
$$
$$
= -(\kappa(x,|q_\perp|) + i\eta(x,|q_\perp|))\frac{S^+}{P^+}\epsilon^{ij}q^i_\perp\Delta^j_\perp + \cdots, \quad (65)
$$

where κ, η are real. Equations (64) and (65) are related by PT symmetry. By integrating by parts in Eq. (64) twice, we

---

[6]For a quark, there is an extra term in the equation at $\mathcal{O}(1/k^-)$ which depends on the gamma matrices $DD = D^2 + \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}$. We neglect this term because it gives vanishing contribution to the physical cross section to $\mathcal{O}(1/k^-)$, since $\mathrm{Tr}\sigma_{\mu\nu} = 0$.

[7]Note that the $k^2_\perp$ term comes from the expansion of the on-shell phase factor

$$
e^{-ik^+\int dz^-} = \exp\left(-i\frac{k^2_\perp}{2k^-}\int dz^-\right) \approx 1 - i\frac{k^2_\perp}{2k^-}\int dz^-.
$$

This term is proportional to the leading term and can be dropped, since it does not give any spin dependence.

can replace the operator $U_{\infty,z^-}\bar{D}^2_{x_\perp}U_{z^-,-\infty}$ with a linear combination of $U_{\infty,z^-}\vec{D}^2_{x_\perp}U_{z^-,-\infty}$ and the surface terms. The latter can depend on spin through the operator

$$i\left(q^i_\perp + \frac{\Delta^i_\perp}{2}\right)U_{\infty,z^-}\bar{D}_{x^i_\perp}U_{z^-,-\infty}, \qquad (66)$$

as in Eq. (43). We thus obtain an identity

$$\kappa + i\eta = -(\kappa + i\eta) + \frac{g}{2} - i\frac{C}{2}, \qquad (67)$$

and therefore

$$\kappa(x,|q_\perp|) = \frac{1}{4}g(x,|q_\perp|),$$

$$\eta(x,|q_\perp|) = -\frac{1}{4}C(x,|q_\perp|). \qquad (68)$$

B. Calculation of the asymmetry

We are now ready to compute the longitudinal single-spin asymmetry.

$$\frac{d\Delta\sigma}{dy_1d^2k_{1\perp}dy_2d^2k_{2\perp}} \equiv \frac{d\sigma^{\lambda=+1}}{dy_1d^2k_{1\perp}dy_2d^2k_{2\perp}}$$
$$- \frac{d\sigma^{\lambda=-1}}{dy_1d^2k_{1\perp}dy_2d^2k_{2\perp}}, \qquad (69)$$

where y_1, y_2 are the rapidities of the two jets. Our strategy is the following. We first substitute (63) into (57) and use the parameterizations (43) and (64) for the resulting matrix elements. We then square the amplitude and keep only the linear terms in S^+/k^-. The leading eikonal contribution has both the real and imaginary parts from the Pomeron and odderon exchanges, respectively:

$$\int d^2q_\perp \frac{P^i_\perp - q^i_\perp}{(P_\perp - q_\perp)^2 + \varepsilon^2}(P(\Delta_\perp, q_\perp) + i\Delta_\perp \cdot q_\perp O(q_\perp)). \qquad (70)$$

The next-to-eikonal contribution of order $1/k^-$ also contains both real and imaginary parts, as shown in Eqs. (43) and (64). When squaring the amplitude, we see that the

terms linear in S^+ arise from the interference between the leading and next-to-eikonal contributions. It turns out that the odderon O interferes with the imaginary terms in (43), which in particular include the OAM function f; while the Pomeron P interferes with the real terms in (43), which we are not interested in. The problem is that, on general grounds, one expects that the Pomeron amplitude P is numerically larger than the odderon amplitude O, and this can significantly reduce the sensitivity to the OAM function. We avoid this problem by focusing on the following two kinematic regions:

$$P_\perp \gg q_\perp, Q, \qquad Q \gg q_\perp, P_\perp \qquad (71)$$

($q_\perp$ here means the typical values of $q_\perp$ within the support of the functions P and O). In this limit, the Pomeron contribution in Eq. (70) drops out because

$$\int d^2q_\perp P(\Delta_\perp, q_\perp) = 0,$$

$$\int d^2q_\perp q^i_\perp P(\Delta_\perp, q_\perp) = 0 \qquad (72)$$

for $\Delta_\perp \neq 0$. The first integral vanishes because the $q_\perp$ integral sets the dipole size $r_\perp = x_\perp - y_\perp$ to be zero, so that $U(x_\perp)U^\dagger(x_\perp) = 1$. Thus, the integral becomes proportional to the delta function $\delta^{(2)}(\Delta_\perp)$. The second relation follows from the symmetry $P(\Delta_\perp, q_\perp) = P(\Delta_\perp, -q_\perp)$. On the other hand, the odderon contribution survives in this limit because, for example,

$$\int d^2q_\perp q^i_\perp \Delta_\perp \cdot q_\perp O(q_\perp) = \frac{\Delta^i_\perp}{2}\int d^2q_\perp q^2_\perp O(q_\perp). \qquad (73)$$

We can thus approximate, when $P_\perp \gg q_\perp, Q$,

$$\int d^2q_\perp \frac{P^i_\perp - q^i_\perp}{(P_\perp - q_\perp)^2 + \varepsilon^2}(P(\Delta_\perp, q_\perp) + i\Delta_\perp \cdot q_\perp O(q_\perp))$$
$$\approx i\left(-\frac{\Delta^i_\perp}{2P^2_\perp} + \frac{P^i_\perp P_\perp \cdot \Delta_\perp}{P^4_\perp}\right)\int d^2q_\perp q^2_\perp O(q_\perp). \qquad (74)$$

A similar result follows in the other limit $Q \gg q_\perp, P_\perp$. Equation (74) is to be multiplied by the next-to-eikonal amplitude, which reads

$$\int \frac{d^2q_\perp}{(2\pi)^2}\frac{P^i_\perp - q^i_\perp}{(P_\perp - q_\perp)^2 + \epsilon^2_f}\int d^2x'_\perp d^2y'_\perp e^{i(q_\perp+\frac{\Delta_\perp}{2})\cdot x'_\perp + i(-q_\perp+\frac{\Delta_\perp}{2})\cdot y'_\perp}\int dz^-\left\langle\frac{1}{k^-_1}\text{Tr}U_{\infty z^-}(x'_\perp)\left(k^j_{1\perp}\bar{D}'_j + \frac{i}{2}\bar{D}'^2\right)U_{z^-,-\infty}(x'_\perp)U^\dagger(y'_\perp)\right.$$

$$\left. - \frac{1}{k^-_2}\text{Tr}U(x'_\perp)U_{-\infty z^-}(y'_\perp)\left(k^j_{2\perp}D'_j + \frac{i}{2}\vec{D}'^2\right)U_{z^-\infty}(y'_\perp)\right\rangle$$

$$= \frac{i\lambda}{4}\frac{g^2(2\pi)^3}{4P^+}\int \frac{d^2q_\perp}{(2\pi)^2}\frac{P^i_\perp - q^i_\perp}{(P_\perp - q_\perp)^2 + \varepsilon^2}\left[\left(\frac{1}{k^-_1} + \frac{1}{k^-_2}\right)\epsilon_{jk}((f-g)P^j_\perp\Delta^k_\perp - (f+g)q^j_\perp\Delta^k_\perp + 2A\Delta_\perp \cdot q_\perp P^j_\perp q^k_\perp + 2\kappa q^j_\perp\Delta^k_\perp)\right.$$

$$\left. + \left(\frac{1}{k^-_1} - \frac{1}{k^-_2}\right)\epsilon_{jk}(2(f+g)P^j_\perp q^k_\perp + A\Delta_\perp \cdot q_\perp\Delta^j_\perp q^k_\perp)\right] + \cdots, \qquad (75)$$

where we keep only the imaginary part. Here, $k_1^- = zq^-$, $k_2^- = (1-z)q^-$ and $k_{1\perp} = -\frac{\Delta_\perp}{2} - P_\perp$, $k_{2\perp} = -\frac{\Delta_\perp}{2} + P_\perp$. We then expand the integrand in powers of $1/P_\perp$ or $1/Q$ and perform the angular integral over ϕ_q. Consider, for definiteness, the large-$P_\perp$ limit. At first sight, the dominant contribution comes from the $\mathcal{O}(1)$ terms proportional to $\frac{P_\perp^i P_\perp^j}{P_\perp^2}(f-g)$ and $\frac{P_\perp^i P_\perp^j}{P_\perp^2}A$. However, after the ϕ_q integral, they cancel exactly due to the sum rule (45). Thus, the leading terms are $\mathcal{O}(1/P_\perp)$ and actually come from the last line of Eq. (75), which can be evaluated as

$$
\approx \frac{i\lambda}{4} \frac{g^2(2\pi)^3}{4P^+}\left(\frac{1}{k_1^-} - \frac{1}{k_2^-}\right)\frac{\epsilon_{ij}P_\perp^j}{P_\perp^2}\int\frac{d^2q_\perp}{(2\pi)^2}q_\perp^2(f+g)
$$

$$
= -\frac{i\lambda\alpha_s(1-2z)}{32P^+q^-}\Delta G(x)\frac{\epsilon_{ij}P_\perp^j}{P_\perp^2}
$$

$$
\approx \frac{i\lambda\alpha_s(1-2z)}{64P^+q^-}L_g(x)\frac{\epsilon_{ij}P_\perp^j}{P_\perp^2}, \tag{76}
$$

where we use Eqs. (49) and (50). Multiplying Eq. (76) by Eq. (74) and restoring the prefactor, we finally arrive at

$$
\frac{d\Delta\sigma}{dy_1 d^2k_{1\perp}dy_2 d^2k_{2\perp}}
$$
$$
\approx 4\pi^4\alpha_s N_c\alpha_{em}x\sum_q e_q^2\delta(x_{\gamma^*}-1)(1-2z)(z^2+(1-z)^2)
$$
$$
\times \frac{\Delta_\perp}{P_\perp^3 Q^2}\sin\phi_{P\Delta}\left\{\begin{array}{c}-2\Delta G(x)\\L_g(x)\end{array}\right\}\int d^2q_\perp q_\perp^2 O(x,q_\perp), \tag{77}
$$

where $\phi_{P\Delta}$ is the azimuthal angle between $P_\perp$ and $\Delta_\perp$, and e_q is the electric charge of the massless quark in units of e. We also use $x = \frac{Q^2}{2P^+q^-}$. z is fixed by the dijet kinematics as

$$
z = \frac{|k_{1\perp}|e^{y_1}}{|k_{1\perp}|e^{y_1}+|k_{2\perp}|e^{y_2}}. \tag{78}
$$

In the other limit, $Q \gg q_\perp, P_\perp$, the cross section reads

$$
\frac{d\Delta\sigma}{dy_1 d^2k_{1\perp}dy_2 d^2k_{2\perp}}
$$
$$
\approx 4\pi^4\alpha_s N_c\alpha_{em}x\sum_f e_f^2\delta(x_{\gamma^*}-1)(1-2z)\frac{z^2+(1-z)^2}{z^2(1-z)^2}
$$
$$
\times \frac{P_\perp\Delta_\perp}{Q^6}\sin\phi_{P\Delta}\left\{\begin{array}{c}-2\Delta G(x)\\L_g(x)\end{array}\right\}\int d^2q_\perp q_\perp^2 O(x,q_\perp). \tag{79}
$$

The terms neglected in Eqs. (77) and (79) are suppressed by powers of $1/P_\perp$ and $1/Q$, respectively.

The above results have been obtained for the transversely polarized virtual photon. In fact, the whole contribution from the longitudinally polarized virtual photon is subleading. The only difference in the longitudinal photon case is the integral kernel

$$
\int d^2q_\perp \frac{P_\perp^i - q_\perp^i}{(P_\perp - q_\perp)^2 + \varepsilon^2} \rightarrow \int d^2q_\perp \frac{Q}{(P_\perp - q_\perp)^2 + \varepsilon^2}. \tag{80}
$$

Proceeding as before, we find that the contribution from the longitudinal photon to $\Delta\sigma$ is suppressed by factors $1/P_\perp^3$ and $1/Q^2$ compared to Eqs. (77) and (79), respectively.

We thus find that the asymmetry is directly proportional to $\Delta G(x)$. On the basis of Eq. (50), we may also say that it is proportional to $L_g(x)$. Previous direct measurements of $\Delta G(x)$ [or rather, the ratio $\langle\Delta G(x)/G(x)\rangle$ averaged over a limited interval of x] in DIS are based on longitudinal *double*-spin asymmetry [42,43]. In general, longitudinal single-spin asymmetry vanishes in QCD due to parity. Here, however, we get a nonzero result because we measure the correlation between two particles (jets) in the final state. The experimental signal of this is the $\sin\phi_{P\Delta}$ angular dependence. This is distinct from the leading angular dependence of the dijet cross section $\cos 2\phi_{P\Delta}$ [23], which has been canceled in the difference $d\Delta\sigma = d\sigma^{\lambda=1} - d\sigma^{\lambda=-1}$.

Notice that the asymmetry vanishes at the symmetric point $z = 1/2$, and the product $(1-2z)\sin\phi_{P\Delta}$ is invariant under the exchange of two jets $z \leftrightarrow 1-z$ and $k_{1\perp} \leftrightarrow k_{2\perp}$. Subleading corrections to Eq. (77) include terms proportional to $\sin 2\phi_{P\Delta}$ without a prefactor $1-2z$. These are consequences of parity. Compared to $\sin\phi_{P\Delta}$, $\sin 2\phi_{P\Delta}$ has an extra zero at $\phi_{P\Delta} = \pi/2$, or equivalently, $|k_{1\perp}| = |k_{2\perp}|$. When $z = 1/2$ and $|k_{1\perp}| = |k_{2\perp}|$, the two jets cannot be distinguished. Therefore, the $\lambda = \pm 1$ cross sections are exactly equal by parity and the asymmetry vanishes. This argument can be generalized to higher Fourier components. The most general form of longitudinal single-spin asymmetry consistent with parity is

$$
\frac{d\Delta\sigma}{dy_1 d^2k_{1\perp}dy_2 d^2k_{2\perp}}
$$
$$
= \sum_{n=0}^{\infty} c_n(z,Q,|P_\perp|,|\Delta_\perp|)\sin(2n+1)\phi_{P_\perp\Delta_\perp}
$$
$$
+ \sum_{n=1}^{\infty} d_n(z,Q,|P_\perp|,|\Delta_\perp|)\sin 2n\phi_{P_\perp\Delta_\perp}, \tag{81}
$$

where $c_n(z = \frac{1}{2}, Q, |P_\perp|, |\Delta_\perp|) = 0$.

It is very interesting that the measurement of Eq. (77) also establishes the odderon exchange in QCD, which

has long evaded detection despite many attempts in the past [44]. The connection between odderon and (transverse) single-spin aymmetries has been previously discussed in the literature [33,45–47]. However, the observable and the mechanism considered in this work are new. To estimate the cross section quantitatively, the integral $\int d^2q_\perp q_\perp^2 O(x, q_\perp)$ should be evaluated using models including the QCD evolution effects. Importantly, theory predicts [48,49] that $O(x, q_\perp)$ has no or very weak dependence on x in the linear BFKL regime. This will make the extraction of the x dependence of $\Delta G(x)$ easier.

VI. COMMENTS ON THE SMALL-x EVOLUTION EQUATION

The appearance of half-infinite Wilson line operators is quite unusual in view of the standard approaches to high-energy QCD evolution, which only deal with infinite Wilson lines $U_{\infty,-\infty}$. At the moment, little is known about the small-x evolution of these operators. Still, we can formally write down the evolution equation by assuming that the soft gluon emissions only affect Wilson lines at the end points $x^- = \pm\infty$ [50]. Defining $\mathcal{O}_{x_\perp} \equiv \int dz^- U_{\infty z^-}(x_\perp)\bar{D}U_{z^-,-\infty}(x_\perp)$ and using the technique illustrated in Ref. [50], we obtain[8]

$$
\begin{aligned}
\frac{\partial}{\partial \ln 1/x}\mathrm{Tr}[\mathcal{O}_{x_\perp}U_{y_\perp}^\dagger] &= \frac{\alpha_s N_c}{2\pi^2}\int d^2z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2}\left\{\frac{1}{N_c}\mathrm{Tr}[\mathcal{O}_{x_\perp}U_{z_\perp}^\dagger]\mathrm{Tr}[U_{z_\perp}U_{y_\perp}^\dagger] - \mathrm{Tr}[\mathcal{O}_{x_\perp}U_{y_\perp}^\dagger]\right\} \\
&+ \frac{\alpha_s N_c}{2\pi^2}\int d^2z_\perp \frac{(x_\perp - z_\perp)\cdot(y_\perp - z_\perp)}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2}\left\{\frac{1}{N_c}\mathrm{Tr}[\mathcal{O}_{x_\perp}U_{x_\perp}^\dagger]\mathrm{Tr}[U_{x_\perp}U_{y_\perp}^\dagger] - \mathrm{Tr}[\mathcal{O}_{x_\perp}U_{y_\perp}^\dagger]\right\} \\
&+ \frac{\alpha_s N_c}{2\pi^2}\int d^2z_\perp \left[\frac{(x_\perp - z_\perp)\cdot(y_\perp - z_\perp)}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2} - \frac{1}{(x_\perp - z_\perp)^2}\right] \\
&\times \left\{\frac{1}{N_c}\mathrm{Tr}[\mathcal{O}_{x_\perp}U_{z_\perp}^\dagger]\mathrm{Tr}[U_{z_\perp}U_{y_\perp}^\dagger] - \frac{1}{N_c}\mathrm{Tr}[U_{x_\perp}U_{z_\perp}^\dagger]\mathrm{Tr}[U_{z_\perp}U_{x_\perp}^\dagger\mathcal{O}_{x_\perp}U_{y_\perp}^\dagger]\right\}. \quad (82)
\end{aligned}
$$

One can show that

$$
\mathcal{O}_{x_\perp}U_{x_\perp}^\dagger = \int dz^- U_{\infty,z^-}\bar{D}U_{\infty,z^-}^\dagger \quad (83)
$$

is an element of the Lie algebra of SU(3). Therefore, its trace, which appears on the second line of the right-hand side of Eq. (82), vanishes. Note that there is no singularity at $z_\perp = y_\perp$ or $z_\perp = x_\perp$. The latter can be seen from the identity

$$
\begin{aligned}
&\frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2} + 2\frac{(x_\perp - z_\perp)\cdot(y_\perp - z_\perp)}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2} - \frac{1}{(x_\perp - z_\perp)^2} \\
&= \frac{1}{(y_\perp - z_\perp)^2}. \quad (84)
\end{aligned}
$$

The above equation is similar to the ones discussed in Refs. [51,52]. In particular, $\mathcal{O}_{x_\perp}$ and the next-to-eikonal operators in Eq. (62) are possibly related to the operator V^{pol} introduced, but unspecified, in Ref. [52]. If this is the case, the small-x behavior of $L_g(x)$ and $\Delta G(x)$ is related to that of the $g_1(x)$ structure function or the polarized quark distribution $\Delta q(x)$. This issue certainly deserves further study.

VII. CONCLUSIONS

In this paper, we first presented a general analysis of the OAM gluon distribution $L_g(x)$ by making several clarifications regarding its definition and properties. We then focused on the small-x regime and derived a novel operator representation for $L_g(x)$ in terms of half-infinite Wilson lines $U_{\pm\infty,z^-}$ and the covariant derivatives D^i. It turns out that exactly the same operators describe the polarized gluon distribution $\Delta G(x)$. Based on this, we have argued that $L_g(x)$ and $\Delta G(x)$ are proportional to each other with the relative coefficient -2. Moreover, the small-x evolution of these distributions can be related to that of the polarized quark distribution. These observations shed new light on the nucleon spin puzzle.

We have also pointed out that the same operator shows up in the next-to-eikonal approximation [40,41]. This allows us to relate the helicity and OAM distributions to observables. We have shown that single longitudinal spin asymmetry in diffractive dijet production in lepton-nucleon collisions is a sensitive probe of the gluon OAM in certain kinematic regimes.

The large-x region, on the other hand, requires a different treatment, and the first result has been recently reported in Ref. [15], to which our work is complementary. Probing the quark OAM L_q seems more difficult, but there are interesting recent developments [16,17]. Together they open up ways to access the last missing pieces in the spin decomposition formula (1), and we propose to explore this direction at the EIC.

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[8]There are terms on the right-hand side which consist of "ordinary" Wilson lines $U_{\infty,-\infty}$ and their derivatives [51]. We omit these terms because they do not have spin-dependent matrix elements.

APPENDIX A: EQUIVALENCE OF $L_g(x)$ DEFINED IN EQUATIONS (7) AND (32)

In this appendix, we show that the $L_g(x)$'s defined in Eqs. (7) and (32) are equivalent. We rewrite the operator in Eq. (9) as

$$
\begin{aligned}
&F^{+\alpha}(0)\tilde{U}_{0z}\overleftrightarrow{\bar{D}}^i(z^-)\tilde{U}_{zy}A_\alpha^{\pm\mathrm{phys}}(y^-)\\
&=\frac{1}{2}F^{+\alpha}(0)\left(\tilde{U}_{0y}D^i(y^-)+i\int_{y^-}^{z^-}d\omega^-\tilde{U}_{0\omega}gF^{+i}(\omega^-)\tilde{U}_{\omega y}\right.\\
&\quad-\bar{D}^i(0)\tilde{U}_{0y}+i\int_0^{z^-}d\omega^-\tilde{U}_{0\omega}gF^{+i}(\omega^-)\tilde{U}_{\omega y}\Bigg)A_\alpha^{\pm\mathrm{phys}}(y^-)\\
&=F^{+\alpha}(0)\left(\frac{\tilde{U}_{0y}D^i_{\mathrm{pure}}(y^-)-\bar{D}^i_{\mathrm{pure}}(0)\tilde{U}_{0y}}{2}\right.\\
&\quad\mp i\int d\omega^-\theta(\pm(\omega^--z^-))\tilde{U}_{0\omega}gF^{+i}(\omega)\tilde{U}_{\omega y}\Bigg)A_\alpha^{\pm\mathrm{phys}}(y^-).
\end{aligned}
$$

$$\tag{A1}$$

To obtain the second equality, we need to split the integral

$$
\begin{aligned}
\int_{y^-}^{z^-}d\omega^-&=\mp\int_{-\infty}^\infty d\omega^-\theta(\pm(\omega^--z^-))\\
&\quad\pm\int_{-\infty}^\infty d\omega^-\theta(\pm(\omega^--y^-)),
\end{aligned}
$$

$$\tag{A2}$$

and similarly for $\int_0^{z^-}d\omega^-$. Substituting Eq. (A1) into (9) and comparing with (7), we find

$$
\begin{aligned}
\epsilon^{ij}&\Delta_{\perp j}S^+L_g(x)\\
&=i\int\frac{dy^-}{2\pi}e^{ixP^+y^-}\langle PS|F^{+\alpha}(0)(\tilde{U}_{0y}D^i_{\mathrm{pure}}-\bar{D}^i_{\mathrm{pure}}\tilde{U}_{0y})\\
&\quad\times A_\alpha^{\pm\mathrm{phys}}(y^-)|PS\rangle.
\end{aligned}
$$

$$\tag{A3}$$

Integrating over x, we recover Eq. (4). Equation (A3) exactly agrees with the OAM defined through the WW-type Wigner distribution (32), as one can see from Eq. (34).

APPENDIX B: ARGUMENTS FOR $L_g(x)\approx-2\Delta G(x)$

In this appendix, we discuss the function $g(x,q_\perp)$ defined in Eq. (43), which accounts for the difference between $L_g(x)$ and $\Delta G(x)$ according to Eq. (49). While we cannot make rigorous statements about this nonperturbative function, we give three arguments that $g(x,q_\perp)$ is suppressed relative to the OAM function $f(x,q_\perp)$.

1. g in the parton model

First, let us evaluate f and g in the "parton model." Namely, we compute the matrix element

$$
\begin{aligned}
\int d^2x_\perp d^2y_\perp& e^{i(q_\perp+\frac{\Delta_\perp}{2})\cdot x_\perp+i(-q_\perp+\frac{\Delta_\perp}{2})\cdot y_\perp}\\
&\times\int_{-\infty}^\infty dz^-\Big\langle P+\frac{\Delta}{2}\Big|\mathrm{Tr}[U_{\infty z^-}(x_\perp)\bar{D}_iU_{z^--\infty}(x_\perp)\\
&\quad\times U^\dagger(y_\perp)]\Big|P-\frac{\Delta}{2}\Big\rangle
\end{aligned}
$$

$$\tag{B1}$$

in one-loop perturbation theory by replacing the external proton state with a superposition of single-quark states as

$$
\begin{aligned}
\langle P+\Delta/2|&....|P-\Delta/2\rangle_{\mathrm{proton}}\\
&\to\sum_f\int\frac{d\xi}{\xi}\phi_f(\xi,\Delta_\perp)\langle\xi P+\Delta/2|....|\xi P-\Delta/2\rangle_f,
\end{aligned}
$$

$$\tag{B2}$$

where $\phi_f(\xi,\Delta_\perp)$ is a weight function and f is the quark flavor. Expanding the operator to quadratic order in A^μ, we find that the S^+ dependence can arise only from the terms

$$
\sim\int dz^-\int dw^-\langle A_i(z^-,x_\perp)A^+(w^-,y_\perp)\rangle_f
$$

$$\tag{B3}$$

and

$$
\sim\int dz^-\int dw^-\langle A_i(z^-,x_\perp)A^+(w^-,x_\perp)\rangle_f.
$$

$$\tag{B4}$$

For quark matrix elements, Eq. (B3) can be evaluated as, up to a normalization factor,

$$
\begin{aligned}
&\frac{1}{(q_\perp+\frac{\Delta_\perp}{2})^2(q_\perp-\frac{\Delta_\perp}{2})^2}\bar{u}'[\xi P^+(\gamma^+\gamma^-\gamma_i+\gamma_i\gamma^-\gamma^+)\\
&\quad+q_{\perp j}(\gamma^+\gamma^j\gamma_i-\gamma_i\gamma^j\gamma^+)]u\\
&\sim\frac{1}{(q_\perp+\frac{\Delta_\perp}{2})^2(q_\perp-\frac{\Delta_\perp}{2})^2}\epsilon_{ij}\left(q_\perp^j+\frac{\Delta_\perp^j}{2}\right)\xi S^+,
\end{aligned}
$$

$$\tag{B5}$$

where we used $\bar{u}'\gamma^iu\approx i\epsilon^{ij}\Delta_j\frac{S^+}{P^+}$ and computed only the imaginary part. As for Eq. (B4), we get

$$-\delta^{(2)}\left(q_\perp - \frac{\Delta_\perp}{2}\right)\int d^2k_\perp \frac{1}{(k_\perp + \frac{\Delta_\perp}{2})^2(k_\perp - \frac{\Delta_\perp}{2})^2}\epsilon_{ij}\frac{\Delta_\perp^j}{2}\xi S^+.$$

$$(B6)$$

Because of the delta function, the factor $\Delta_\perp^j/2$ in Eq. (B6) can be replaced by $\frac{1}{2}(q_\perp^j + \frac{\Delta_\perp^j}{2})$. This shows that $g = A = 0$, and

$$f \propto \frac{1}{(q_\perp + \frac{\Delta_\perp}{2})^2(q_\perp - \frac{\Delta_\perp}{2})^2}$$
$$-\delta^{(2)}\left(q_\perp - \frac{\Delta_\perp}{2}\right)\int d^2k_\perp \frac{1}{(k_\perp + \frac{\Delta_\perp}{2})^2(k_\perp - \frac{\Delta_\perp}{2})^2}. \quad (B7)$$

It is easy to check that the sum rule (45) is satisfied to this order. This result suggests that g is a higher-order effect, suppressed at least by a factor of α_s compared to f.

2. Nonperturbative argument

Next, we give a more formal argument from another perspective. Let us simplify the notation as

$$\mathcal{O}_i(x_\perp) = \int dz^- U_{\infty,z^-}(x_\perp)\bar{D}_i U_{z^-,-\infty}(x_\perp), \quad (B8)$$

$$\mathcal{O}'(y_\perp) = U^\dagger(y_\perp), \quad (B9)$$

and consider the matrix element

$$\int d^2x_\perp d^2y_\perp e^{i(q_\perp + \frac{\Delta_\perp}{2})\cdot x_\perp + i(-q_\perp + \frac{\Delta_\perp}{2})\cdot y_\perp}$$
$$\times \left\langle P + \frac{\Delta_\perp}{2}, S \middle| \text{Tr}[\mathcal{O}_i(x_\perp)\mathcal{O}'(y_\perp)] \middle| P - \frac{\Delta_\perp}{2}, S \right\rangle$$
$$\propto -i\frac{S^+}{2P^+}\epsilon_{ij}\left[\left(q_\perp^j + \frac{\Delta_\perp^j}{2}\right)f + \left(q_\perp^i - \frac{\Delta_\perp^j}{2}\right)g\right] + \cdots. \quad (B10)$$

We observe that in covariant gauges in which the gauge field vanishes at infinity, $x^- = \pm\infty$, both $\mathcal{O}_i$ and $\mathcal{O}'$ are gauge invariant (or more properly, BRST invariant). This means that the states $\mathcal{O}_i|PS\rangle$ and $\mathcal{O}'|PS\rangle$ are "physical" in that they are annihilated by the BRST operator $Q_B(\mathcal{O}|PS\rangle) = 0$ (the Kugo-Ojima condition [53]). In much the same way as in the proof of unitarity of the S-matrix in gauge theories, we can insert the intermediate states

$$\sum_X \text{Tr}\left\langle P + \frac{\Delta_\perp}{2}, S \middle| \mathcal{O}_i(0_\perp)|X\rangle\langle X|\mathcal{O}'(0_\perp) \middle| P - \frac{\Delta_\perp}{2}, S \right\rangle\bigg|_{P_\perp^X = -q_\perp}$$

$$(B11)$$

and exclude from X the BRST exact states of the form $|X\rangle = Q_B|Y\rangle$. $|X\rangle$ are then gauge-invariant states with a positive norm and unit baryon number. A representative of

such states is the single-nucleon state, whose matrix element can be parameterized as

$$\left\langle P + \frac{\Delta_\perp}{2}, S \middle| \mathcal{O}_i(0_\perp)|P^X, S\right\rangle$$
$$= \bar{u}\left(P + \frac{\Delta_\perp}{2}, S\right)(a\gamma_i + b\Delta_{\perp i} + cq_{\perp i})u(P_\perp^X = -q_\perp, S).$$

$$(B12)$$

The structure $\sim\epsilon_{ij}S^+$ comes only from the first term

$$\bar{u}\left(P + \frac{\Delta_\perp}{2}, S\right)\gamma_i u(P_\perp^X = -q_\perp, S) \approx i\frac{S^+}{P^+}\epsilon_{ij}\left(q_\perp + \frac{\Delta_\perp}{2}\right)^j,$$

$$(B13)$$

and this means $g = 0$ for this particular contribution. We cannot extend this argument to the case where $|X\rangle$ is a multiparticle state which consists of one baryon and other hadron species whose transverse momenta add up to $-q_\perp$. Yet it seems reasonable, at least from a naive extrapolation of Eq. (B13), that the matrix element $\langle\Delta_\perp/2|\mathcal{O}_i| - q_\perp\rangle$ dominantly depends on the relative transverse momentum between the initial and final states, $q_\perp + \Delta_\perp/2$, rather than their sum, $-q_\perp + \Delta_\perp/2$. The latter contribution would come from those atypical configurations in which a baryon carries transverse momentum $+q_\perp$ and other hadrons carry $-2q_\perp$, such that their sum is $-q_\perp$. This indicates that $|f| \gg |g|$.

3. DGLAP equation

Finally, we study the double logarithmic limit of the DGLAP equation and directly show that the linear combination $L_g(x) + 2\Delta G(x)$ is parametrically suppressed compared to $\Delta G(x)$. Let us assume that $\Delta G(x)$ and $L_g(x)$ are dominant at small x. Then, from Eqs. (11) and (20), we get

$$\frac{d}{d\ln Q^2}\Delta G(x) \approx \frac{2C_A\alpha_s}{\pi}\int_x^1 \frac{dz}{z}\Delta G(z), \quad (B14)$$

$$\frac{d}{d\ln Q^2}L_g(x) \approx \frac{C_A\alpha_s}{\pi}\int_x^1 \frac{dz}{z}(L_g(z) - 2\Delta G(z)). \quad (B15)$$

We see that the linear combination $L_g(x) + 2\Delta G(x)$ evolves homogeneously:

$$\frac{d}{d\ln Q^2}(L_g(x) + 2\Delta G(x)) \approx \frac{C_A\alpha_s}{\pi}\int_x^1 \frac{dz}{z}(L_g(z) + 2\Delta G(z)).$$

$$(B16)$$

In the double logarithmic limit, Eq. (B16) can be solved by the standard technique as

$$L_g(x) + 2\Delta G(x) \sim \int \frac{dj}{2\pi i} \exp\left(jY + \frac{\xi}{j}\right) \sim e^{2\sqrt{\xi Y}}, \quad \text{(B17)}$$

where $Y = \ln 1/x$ and $\xi \equiv \frac{C_A \alpha_s}{\pi} \ln Q^2$. On the other hand, from Eq. (B14) we get

$$\Delta G(x) \sim e^{2\sqrt{2}\sqrt{\xi Y}}. \quad \text{(B18)}$$

This shows that $|L_g(x) + 2\Delta G(x)| \ll |\Delta G(x)|, |L_g(x)|$, as far as x dependence is concerned.

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