

Decays of the vector glueballFrancesco Giacosa,^{1,2,*} Julia Sammet,^{2,†} and Stanislaus Janowski^{2,‡}¹*Institute of Physics, Jan Kochanowski University, ul. Świetokrzyska 15, 25-406 Kielce, Poland,*²*Institute for Theoretical Physics, Goethe University, Max-von-Laue-Straße 1, 60438 Frankfurt am Main, Germany*

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We calculate two- and three-body decays of the (lightest) vector glueball into (pseudo)scalar, (axial-) vector, as well as pseudovector and excited vector mesons in the framework of a model of QCD. While absolute values of widths cannot be predicted because the corresponding coupling constants are unknown, some interesting branching ratios can be evaluated by setting the mass of the yet hypothetical vector glueball to 3.8 GeV as predicted by quenched lattice QCD. We find that the decay mode $\omega\pi\pi$ should be one of the largest (both through the decay chain $\mathcal{O} \rightarrow b_1\pi \rightarrow \omega\pi\pi$ and through the direct coupling $\mathcal{O} \rightarrow \omega\pi\pi$). Similarly, the (direct and indirect) decay into $\pi K K^*(892)$ is sizable. Moreover, the decays into $\rho\pi$ and $K^*(892)K$ are, although subleading, possible and could play a role in explaining the $\rho\pi$ puzzle of the charmonium state $\psi(2S)$ thanks to a (small) mixing with the vector glueball. The vector glueball can be directly formed at the ongoing BESIII experiment as well as at the future PANDA experiment at the FAIR facility. If the width is sufficiently small ($\lesssim 100$ MeV) it should not escape future detection. It should be stressed that the employed model is based on some inputs and simplifying assumptions: the value of glueball mass (at present, the quenched lattice value is used), the lack of mixing of the glueball with other quarkonium states, and the use of few interaction terms. It then represents a first step toward the identification of the main decay channels of the vector glueball, but shall be improved when corresponding experimental candidates and/or new lattice results will be available.

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The search for glueballs is an important part of experimental as well as theoretical hadronic physics [1,2]. Lattice QCD predicts a rich spectrum of glueball states below 5 GeV [3,4], but up to now no predominantly glueball state could be unambiguously assigned to one of the mesons listed in the PDG [5].

The lightest glueball predicted by lattice QCD is a scalar particle. This is definitely the glueball which has been most intensively studied both experimentally and theoretically. Various theoretical approaches were developed to understand which scalar-isoscalar resonance contains the largest gluonic amount, e.g. Refs. [2,6] and references therein. In most scenarios, either the resonance $f_0(1500)$ or the resonance $f_0(1710)$ has the largest gluonic content in its wave function.

The field corresponding to the scalar glueball is often related to the dilaton [7,8]: the corresponding nonvanishing condensate is proportional to the gluon condensate and is linked to a basic feature of QCD, the trace anomaly, which allows one to understand how a dimension enters into a classically dimensionless theory. Indeed, dilatation invariance and its anomalous breaking in the Yang-Mills sector

have been, together with chiral symmetry, the guiding principle behind the development of an effective hadronic model of QCD, the so-called extended linear sigma model (eLSM) [9,10]. The eLSM has shown to be capable to describe various hadronic masses and decays below 1.8 GeV, as the fit in Ref. [9] confirms. The eLSM can be coupled to glueballs allowing to calculate their decays, see the following discussion. The scalar glueball is automatically present in the eLSM as a dilaton and is coupled to light mesons: its assignment and its mixing with quark-antiquark scalar fields have been studied in Ref. [10], where only one assignment was found to be acceptable: $f_0(1710)$ is mostly gluonic. Besides the eLSM, evidence that $f_0(1710)$ has a large gluonic amount is recently mounting from both lattice studies [11] and from a holographic approach [12].

The pseudoscalar glueball has also received much attention (see Ref. [13] for a review), since it is linked to another important feature of QCD: the chiral anomaly [14]. Lattice QCD predicts a mass of about 2.6 GeV, but the scenario in which the pseudoscalar glueball is light (at about 1.5 GeV) was also widely investigated, e.g. Refs. [13,15] and references therein. Yet, the mismatch with the mass calculated from lattice QCD seems to be too large for this scenario to be realistic. In a recent work, the branching ratios of a putative pseudoscalar glueball with a mass of about 2.6 GeV were calculated within the eLSM mentioned above [16,17]. To this end, a chirally invariant

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interaction term coupling the pseudoscalar glueball to light mesons was considered. It was found that the channel $\pi\pi K$ is dominant and that the channels $\pi\pi\eta$ and $\pi\pi\eta'$ are sizable. Such information can be useful in the future experimental search of the pseudoscalar glueball at PANDA [18] and at BESIII [19]. (Very recently, the very same interaction Lagrangian has been used to study the branching ratios of a second excited pseudoscalar glueball with a mass of about 3.7 GeV [20].)

In this work we continue the investigation of glueballs by concentrating our attention to the vector glueball, denoted as \mathcal{O} . The vector glueball has received some attention in the past (e.g. in Refs. [21–25] and references therein, where it was studied in the context of the so-called $\rho\pi$ puzzle, see also below), but to our knowledge a systematic prediction of its decay rates has not yet been performed. Our plan is to study the vector glueball in the framework of the mentioned model of QCD, the eLSM, properly coupled to the vector glueball. As a consequence, the results are model dependent and only branching ratios can be calculated (full decay rates depend on coupling constant which cannot be calculated). Also the mass of the glueball cannot be determined within the model. We thus use the value predicted by quenched lattice QCD (3.8 GeV) [3], well above the scalar and pseudoscalar ones (the discussion concerning the use of the lattice mass as an input and the validity of our interaction terms for the evaluation of the glueball's decays are presented in Sec. IV). It is then expected that the vector glueball decays in two and three light mesons. We thus plan to evaluate, in the framework of our model, various branching ratios.

In order to obtain the interaction Lagrangians, we need to couple the vector glueball to conventional light mesons contained in the eLSM. Being glueballs chirally invariant fields, we assume that the interactions fulfill chiral invariance (and, at least for the dominant terms, also dilatation invariance). Moreover, in addition to the standard pseudoscalar, scalar, vector, and axial-vector mesons studied in Ref. [9], we will also consider pseudovector and excited vector mesons. It is important to stress from the very beginning that our approach could not yet be tested experimentally. (In particular, for what concerns heavy glueballs, there are at present no known candidates.) Moreover, we shall also neglect mixing of the glueball with conventional quarkonium states. As we discuss more in detail later on, it will be possible to model in the future such mixing when more information will be available.

With these cautionary remarks in mind, we aim to make some observations concerning the decays of the vector glueball. In particular, we will show that, in our calculational approach, one of the most important decay modes of the vector glueball is the decay into a pseudoscalar-pseudovector pair (in particular, the channel $\mathcal{O} \rightarrow b_1\pi \rightarrow \omega\pi\pi$). These decays are obtained from an interaction term (our first effective Lagrangian for \mathcal{O} , denoted as \mathcal{L}_1) which

involves the least possible number of quark-antiquark fields (only two; hence, to assure dilatation invariance, also the dilaton field is coupled). As a side effect, this analysis also offers the mathematical basis for an extension of the eLSM containing pseudovector and excited vector fields.

The second chiral Lagrangian that we build is denoted as \mathcal{L}_2 and contains the coupling of \mathcal{O} to (pseudo)scalar and (axial-)vector fields. Here, three quark-antiquark fields are at first present at each vertex, hence three-body decay of \mathcal{O} is automatically obtained. The decays of \mathcal{O} into two pseudoscalar mesons and one vector mesons are dominant. In particular, the direct decay channels $\mathcal{O} \rightarrow \omega\pi\pi$ and $\mathcal{O} \rightarrow K^*(892)K\pi$ are the largest. Both the first and the second Lagrangians predict a decay into $\omega\pi\pi$ and into $K^*(892)K\pi$ (indirect in the first case, direct in the second), which then represent the two golden channels toward a possible future detection of the vector glueball. In the end, when condensation of scalar fields is taken into account, \mathcal{L}_2 also delivers two-body decays. Here, one vector and one scalar field are the most relevant decay channels.

Finally, we shall also analyze a third interaction Lagrangian (denoted as \mathcal{L}_3), in which four quark-antiquark fields are present at each vertex. This Lagrangian breaks dilatation invariance (the coupling constant is proportional to Energy⁻²), yet we decided to study it since it delivers interesting decays into one vector and one axial-vector pair and into one vector and one pseudoscalar pair. In particular, the $\rho\pi$ and $K^*(892)K$ decay modes are relevant in connection to the so-called $\rho\pi$ -puzzle of the state $\psi(2S)$, see the discussion in Sec. IV.

The vector glueball can be constructed with (at least) three constituent gluons, hence its decays and mixing are expected to be sufficiently small to allow detection (even if in the context of our model we cannot calculate them). Even if the predicted mass of \mathcal{O} lies at about 3.8 GeV and hence above the DD threshold, we do not find a sizable decay into charmed mesons. Namely, the DD channel turns out to be small (albeit nonzero), while other channels with a potentially sizable interaction—such as the pseudoscalar-pseudovector channel mentioned above—are not kinematically allowed when charmed mesons are considered.

In the future, the PANDA experiment at FAIR [18] will be able to form glueballs and to study their decays. In particular, the vector glueball has the quantum numbers of the photon, hence it can be also directly formed at BESIII (for which an energy scan in the region 3.5 to 4 GeV with particular attention to the $\omega\pi\pi$ and $K^*(892)K\pi$ channels would be necessary). It should be mentioned that in a previous study at BES [21], a vector glueball was searched—without success—in the $\rho\pi$ channel, which is however not the favorite decay mode found in our study.

This paper is organized as follows. In Sec. II we show the quark-antiquark nonets and their properties, while in Sec. III we introduce three effective Lagrangians and calculate decay ratios. In Sec. IV we present various

discussions: the validity and the limits of the used interaction Lagrangians, the use of the mass from lattice QCD as an input, the large- N_c limit, and the so-called $\rho\pi$ puzzle of $\psi(2S)$. Finally, in Sec. V we outline our conclusions and outlooks. In the Appendixes we show relevant details of the eLSM and of the calculations.

II. CHIRAL MULTIPLETS

In this section, we concentrate on the quark-antiquark fields which represent the decay products of the vector glueball. We split our study into two parts: first, we consider (pseudo)scalar and (axial-)vector fields, which are also the basic ingredients of the eLSM when proper chiral combinations of them are taken into account. Then, in the second part we describe pseudovector mesons and excited vector mesons and the corresponding chiral combination.

A. (Pseudo)scalar and (axial-)vector quark-antiquark multiplets

The nonet of pseudoscalar fields is introduced as

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\eta_N + \pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_N - \pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \eta_S \end{pmatrix}, \quad (1)$$

which contains the renowned light pseudoscalar nonet $\{\pi, K, \eta, \eta'\}$ [5], where η and η' arise via the mixing $\eta = \eta_N \cos \theta_p + \eta_S \sin \theta_p$, $\eta' = -\eta_N \sin \theta_p + \eta_S \cos \theta_p$ with $\theta_p \simeq -44.6^\circ$ [9]. (Using other values as e.g. $\theta_p = -41.4^\circ$ [26] changes only slightly the results presented in this work.) As a next step, we introduce the matrix of scalar fields,

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sigma_N + a_0^0}{\sqrt{2}} & a_0^+ & K_S^+ \\ a_0^- & \frac{\sigma_N - a_0^0}{\sqrt{2}} & K_S^0 \\ K_S^- & \bar{K}_S^0 & \sigma_S \end{pmatrix}, \quad (2)$$

which contains the fields $\{a_0(1450), K_0^*(1430), \sigma_N, \sigma_S\}$. (Note, the scalar quark-antiquark states lie above 1 GeV [9], thus the nonet of light scalar states $\{a_0(980), K_0^*(800), f_0(500), f_0(980)\}$ is something else. A possibility is a nonet of light tetraquark states [27,28] or a nonet of dynamically generated states [29–31].) As a first approximation, the nonstrange bare field $\sigma_N \equiv |\bar{u}u + \bar{d}d\rangle/\sqrt{2}$ corresponds predominantly to the resonance $f_0(1370)$ and the bare field $\sigma_S \equiv |\bar{s}s\rangle$ predominantly to $f_0(1500)$. Finally, in the eLSM the state $f_0(1710)$ is predominantly a scalar glueball, see details in Ref. [10] in which the mixing matrix is presented.

Nowadays evidence toward a large gluonic amount in $f_0(1710)$ is increasing: besides the eLSM [10], in the recent lattice work of Ref. [11] the radiative decay $j/\psi \rightarrow \gamma G$ (where G is a pure glueball) has been analyzed and found to be in good agreement with the experimental decay rate $j/\psi \rightarrow \gamma f_0(1710)$. Moreover, the study of Ref. [12] reaches the same conclusion in the framework of holographic QCD.

The scalar and pseudoscalar matrices are combined into

$$\Phi = S + iP, \quad (3)$$

which has a simple transformation under chiral transformations $U_L(3) \times U_R(3): \Phi \rightarrow U_L \Phi U_R^\dagger$, where U_L and U_R are $U(3)$ matrices. Under parity $\Phi \rightarrow \Phi^\dagger$ and under charge conjugation $\Phi \rightarrow \Phi^t$. The matrix Φ is used as a building block in the construction of the eLSM Lagrangian, see Appendix A and Tables I and II.

We now turn to vector and axial-vector fields. The nonet of vector fields is introduced as

$$V^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N + \rho^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^{\mu-} & \frac{\omega_N - \rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \omega_S \end{pmatrix}^\mu, \quad (4)$$

and the nonet of axial-vector fields as

TABLE I. Summary of the quark-antiquark nonets and their properties.

Nonet	L	S	J^{PC}	Current	Assignment
P	0	0	0^{-+}	$P_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j i \gamma^5 q_i$	π, K, η, η'
S	1	1	0^{++}	$S_{ij} = \frac{1}{\sqrt{2}} \bar{q}_j q_i$	$a_0(1450), K_0^*(1430), f_0(1370), f_0(1510)$
V^μ	0	1	1^{--}	$V_{ij}^\mu = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^\mu q_i$	$\rho(770), K^*(892), \omega(785), \phi(1024)$
A^μ	1	1	1^{++}	$A_{ij}^\mu = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 \gamma^\mu q_i$	$a_1(1230), K_{1,A}, f_1(1285), f_1(1420)$
B^μ	1	0	1^{+-}	$B_{ij}^\mu = \frac{1}{\sqrt{2}} \bar{q}_j \gamma^5 \overleftrightarrow{\partial}^\mu q_i$	$b_1(1230), K_{1,B}, h_1(1170), h_1(1380)$
E_{ang}^μ	2	1	1^{--}	$E_{\text{ang},ij}^\mu = \frac{1}{\sqrt{2}} \bar{q}_j i \overleftrightarrow{\partial}^\mu q_i$	$\rho(1700), K^*(1680), \omega(1650), \phi(???)$

TABLE II. Transformation properties of the chiral multiplets.

Chiral multiplet	Current	$U_R(3) \times U_L(3)$	P	C
$\Phi = S + iP$	$\sqrt{2}\bar{q}_{R,j}q_{L,i}$	$U_L\Phi U_R^\dagger$	Φ^\dagger	Φ^t
$R^\mu = V^\mu - A^\mu$	$\sqrt{2}\bar{q}_{R,j}\gamma^\mu q_{R,i}$	$U_R R^\mu U_R^\dagger$	L_μ	$L^{\prime\mu}$
$L^\mu = V^\mu + A^\mu$	$\sqrt{2}\bar{q}_{L,j}\gamma^\mu q_{L,i}$	$U_L R^\mu U_L^\dagger$	R_μ	$R^{\prime\mu}$
$\tilde{\Phi}^\mu = E_{\text{ang}}^\mu - iB^\mu$	$\sqrt{2}\bar{q}_{R,j}i\hat{\partial}^\mu q_{L,i}$	$U_L\tilde{\Phi}^\mu U_R^\dagger$	$\tilde{\Phi}^{\prime\mu}$	$-\tilde{\Phi}^{\prime\prime\mu}$

$$A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N}+a_1^0}{\sqrt{2}} & a_1^+ & K_{1,A}^+ \\ a_1^- & \frac{f_{1N}-a_1^0}{\sqrt{2}} & K_{1,A}^0 \\ K_{1,A}^- & \bar{K}_{1,A}^0 & f_{1S} \end{pmatrix}^\mu. \quad (5)$$

The matrix V^μ contains the vector states $\{\rho, K^*(892), \omega, \phi\}$, while the matrix A^μ contains the axial-vector states $\{a_1(1230), K_{1,A}, f_1(1285), f_1(1420)\}$, where $K_{1,A}$ is a mixture of the two physical states $K_1(1270)$ and $K_1(1400)$, see Sec. II B. We neglect (the anyhow small) strange-nonstrange mixing, hence $\omega \equiv \omega_N$ and $f_{1N} \equiv f_1(1285)$ are purely nonstrange mesons of the type $\sqrt{1/2}(\bar{u}u + \bar{d}d)$, while $\omega_S \equiv \phi$ and $f_{1S} \equiv f_1(1285)$ are purely $\bar{s}s$ states.

Then, one defines the right-handed and left-handed combinations:

$$R^\mu = V^\mu - A^\mu \quad \text{and} \quad L^\mu = V^\mu + A^\mu. \quad (6)$$

Under chiral transformation they transform in a compact way: $R^\mu \rightarrow U_R R^\mu U_R^\dagger$, $L^\mu \rightarrow U_L L^\mu U_L^\dagger$. Details of the currents and transformations are shown in Tables I and II.

The eLSM Lagrangian includes the multiplets S , P , V , and A presented above. In addition, a dilaton/glueball field is also present in order to describe dilatation symmetry and its anomalous breaking. The details of the eLSM (together with its symmetries, most notably chiral and dilatation symmetries together with their anomalous, explicit, and spontaneous breaking terms) are briefly summarized in Appendix A and extensively presented in Refs. [9,10] for $N_f = 3$. (Previous versions of the eLSM for $N_f = 2$ are discussed in Ref. [32] while an extension to $N_f = 4$ can be found in Ref. [33]. Baryons are considered in Ref. [34], while properties at finite density are studied in Ref. [35] and at finite temperature in Ref. [36]. Recently, it was also shown that the low-energy limit of the eLSM is equivalent to chiral perturbation theory [37].)

B. Pseudovector and excited vector mesons

We aim to investigate also the decay of the vector glueball into pseudovector and excited vector mesons. To this end we introduce the matrix

$$B^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{h_{1,N}+b_1^0}{\sqrt{2}} & b_1^+ & K_{1,B}^{*+} \\ b_1^- & \frac{h_{1,N}+b_1^0}{\sqrt{2}} & K_{1,B}^{*0} \\ K_{1,B}^{*-} & \bar{K}_{1,B}^{*0} & h_{1,S} \end{pmatrix}^\mu \quad (7)$$

that describes the nonet of pseudovector resonances

$$\{b_1(1230), K_{1,B}, h_1(1170), h_1(1380)\}. \quad (8)$$

Also here, the strange-nonstrange isoscalar mixing is neglected, thus $h_{1,N} \equiv h_1(1170)$ is a purely nonstrange state, while $h_{1,S} \equiv h_1(1380)$ is a purely strange-antistrange state.

The kaonic fields $K_{1,A}$ from Eq. (5) and $K_{1,B}$ from Eq. (7) mix and generate the two physical resonances $K_1(1270)$ and $K_1(1400)$:

$$\begin{pmatrix} K_1^+(1270) \\ K_1^+(1400) \end{pmatrix}^\mu = \begin{pmatrix} \cos \varphi & -i \sin \varphi \\ -i \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} K_{1,A}^+ \\ K_{1,B}^+ \end{pmatrix}^\mu. \quad (9)$$

The mixing angle reads $\varphi = (56.3 \pm 4.2)^\circ$ [38]. The same transformations hold for $K_1^0(1270)$ and $K_1^0(1400)$, while for the other kaonic states one has to take into account that $K_1^-(1270) = K_1^+(1270)^\dagger$ and $\bar{K}_1^0(1270) = K_1^0(1270)^\dagger$ [and so for $K_1^-(1400)$].

The chiral partners of the pseudovector mesons are excited vector mesons which arise from the combination $L = 2$, $S = 1$ coupled to $J^{PC} = 1^{--}$. The corresponding fields listed in the PDG [5] are given by

$$\{\rho(1700), K^*(1680), \omega(1650), \phi(???)\}.$$

Note, $\phi(???)$ was not yet found (one expects a vector state with a mass of about 1.95 GeV from the comparison to the radially excited vector mesons). The corresponding nonet reads

$$E_{\text{ang}}^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_{\text{ang},N} + \rho_{\text{ang}}^0}{\sqrt{2}} & \rho_{\text{ang}}^+ & K_{\text{ang}}^{*+} \\ \rho_{\text{ang}}^- & \frac{\omega_{\text{ang},N} - \rho_{\text{ang}}^0}{\sqrt{2}} & K_{\text{ang}}^{*0} \\ K_{\text{ang}}^{*-} & \bar{K}_{\text{ang}}^{*0} & \omega_{\text{ang},S} \end{pmatrix}^\mu. \quad (10)$$

We then build the matrix

$$\tilde{\Phi}^\mu = E_{\text{ang}}^\mu - iB^\mu, \quad (11)$$

which under chiral transformations changes as $\tilde{\Phi}^\mu \rightarrow U_L \tilde{\Phi}^\mu U_R^\dagger$ [just as the standard (pseudo)scalar Φ , hence the name], under parity as $\tilde{\Phi}^\mu \rightarrow \tilde{\Phi}^{\prime\mu}$, and under charge conjugations as $\tilde{\Phi}^\mu \rightarrow -\tilde{\Phi}^{\prime\mu}$, see Tables I and II for details. Although not the goal of the present paper, one could use the matrix $\tilde{\Phi}^\mu$ in order to build an extension of the eLSM

which includes pseudovector and excited vector mesons. This project is left for the future.

Finally, we present in Tables I and II the summary of all relevant properties and transformations of the nonets introduced in this section.

III. DECAY OF THE VECTOR GLUEBALL INTO CONVENTIONAL MESONS

In this section we present the interaction terms describing the coupling of the vector glueball field \mathcal{O}^μ to the various quark-antiquark multiplets introduced in the previous section. Chiral symmetry as well as invariance under parity and charge conjugations are the guiding principles. In addition, dilatation invariance is assumed to hold in the two most relevant terms. A third interaction which breaks dilatation invariance and involves the Levi-Civita tensor is also considered. Branching ratios are summarized in Tables III–VI.

A. Decays into (pseudo)scalar and excited vector and pseudovector mesons

The (nontrivial) chiral Lagrangian with the minimal number of quark-antiquark fields contains the coupling of the vector glueball to (pseudo)scalar and excited (pseudo)vector mesons:

$$\mathcal{L}_1 = \lambda_{\mathcal{O},1} G \mathcal{O}_\mu \text{Tr}[\Phi^\dagger \tilde{\Phi}^\mu + \tilde{\Phi}^{\mu\dagger} \Phi]. \quad (12)$$

We introduced also the dilaton G in such a way that the interaction term has exactly dimension 4 ($\lambda_{\mathcal{O},1}$ is dimensionless) as required by dilatation invariance (we recall that only positive or vanishing powers of G are acceptable [9]). Using Table II we obtain

TABLE III. Branching ratios for the decay of the vector glueball into a pseudoscalar-pseudovector pair and into a scalar-excited-vector pair.

Quantity	Value
$\frac{\mathcal{O} \rightarrow \eta h_1(1170)}{\mathcal{O} \rightarrow b_1 \pi}$	0.17
$\frac{\mathcal{O} \rightarrow \eta h_1(1380)}{\mathcal{O} \rightarrow b_1 \pi}$	0.11
$\frac{\mathcal{O} \rightarrow \eta' h_1(1170)}{\mathcal{O} \rightarrow b_1 \pi}$	0.15
$\frac{\mathcal{O} \rightarrow \eta' h_1(1380)}{\mathcal{O} \rightarrow b_1 \pi}$	0.10
$\frac{\mathcal{O} \rightarrow K K_1(1270)}{\mathcal{O} \rightarrow b_1 \pi}$	0.75
$\frac{\mathcal{O} \rightarrow K K_1(1400)}{\mathcal{O} \rightarrow b_1 \pi}$	0.30
$\frac{\mathcal{O} \rightarrow K_0^*(1430) K^*(1680)}{\mathcal{O} \rightarrow b_1 \pi}$	0.20
$\frac{\mathcal{O} \rightarrow a_0(1450) \rho(1700)}{\mathcal{O} \rightarrow b_1 \pi}$	0.14
$\frac{\mathcal{O} \rightarrow f_0(1370) \omega(1650)}{\mathcal{O} \rightarrow b_1 \pi}$	0.034

$$\mathcal{L}_1 = \lambda_{\mathcal{O},1} G \mathcal{O}_\mu \text{Tr}[2SE_{\text{ang}}^\mu - 2PB^\mu]. \quad (13)$$

Setting the dilaton field G to its condensate $G = G_0$ [8], substituting $P \rightarrow \mathcal{P}$ as described in Appendix A, and by introducing the physical kaonic fields defined in Eq. (9), lead to the desired expressions for the two-body decays (see Appendix B for its analytic form). In particular, we have decays of the type $\mathcal{O} \rightarrow BP$ and $\mathcal{O} \rightarrow SE_{\text{ang}}$. The main decay channel is $\mathcal{O} \rightarrow b_1 \pi$. We thus expect a significant decay rate of an hypothetical vector glueball into the channel

$$\mathcal{O} \rightarrow b_1 \pi \rightarrow \omega \pi \pi. \quad (14)$$

Unfortunately, we cannot determine $\lambda_{\mathcal{O},1}$ in the present framework, but we can easily calculate various decay ratios which represent clear predictions of the present approach, see Table III. Besides the channel $b_1 \pi$, also the decays involving kaons and $K_1(1270)$ and $K_1(1400)$ are sizable.

In Table III (as well as in all other tables presented in this work) we keep for definiteness two significant digits for our results. It is difficult to estimate the actual uncertainty of our ratios, since various unknown factors enter. One source of error is related to the mass of the glueball, for which we used the quenched lattice result of 3.81 GeV (one may estimate it $\sim 10\%$ – 15% [3] see also the detailed discussion in Sec. IV B and the cautionary remarks written there). Another source of uncertainty is connected to the validity of the employed effective model; indeed, in Ref. [33] an application of the eLSM to heavy charmed mesons has found to be applicable within 10% for what concerns the calculation of decays. It seems reasonable to expect a similar accuracy in the present approach, even if it is not yet possible to test the decays of heavy glueballs since they were not yet discovered. Putting altogether, we can estimate the validity of our branching ratios to about 20%–30%. However, the interesting point is that the qualitative outcomes of our study are not strongly dependent on the precise input of the parameters. For instance, in Table I the main information ($\mathcal{O} \rightarrow b_1 \pi$ dominates) is stable.

At the present state of knowledge, the decays into scalar mesons can only be approximate. For this reason, we did not “unmix” σ_N and σ_S . In addition, we also expect a three-body decay [when G in Eq. (12) is not set to its vacuum’s expectation value]:

$$\mathcal{O} \rightarrow G b_1 \pi. \quad (15)$$

Upon identifying G with $f_0(1710)$ (the by far dominant contribution in the eLSM [10]), one obtains a very small decay ratio:

$$\frac{\Gamma_{\mathcal{O} \rightarrow f_0(1710) b_1 \pi}}{\Gamma_{\mathcal{O} \rightarrow b_1 \pi}} = 3.9 \times 10^{-6}. \quad (16)$$

There are further three-body interactions contained in \mathcal{L}_1 , but they are not kinematically allowed.

Moreover, we did not include in the table the ratio $\omega_{\text{ang},S} \equiv \phi(???)$ because the corresponding state was not yet experimentally found. Yet, assigning it to a yet hypothetical $\phi(1950)$ state, we obtain $\Gamma_{\mathcal{O} \rightarrow f_0(1510)\phi(1950)}/\Gamma_{\mathcal{O} \rightarrow b_1\pi} \approx 0.037$.

We also neglect the mixing $\mathcal{O}_\mu \text{Tr}[E_{\text{ang}}^\mu]$ arising when the field S condenses (see Appendix A). Namely, the large mass difference between \mathcal{O} and excited vector mesons assures that such mixing is negligible.

B. Coupling to (pseudo)scalar and (axial-)vector mesons

Next, we consider a chirally invariant and dilatation invariant Lagrangian which couples \mathcal{O} to three quark-antiquark states. It involves both (pseudo)scalar and (axial-)vector fields:

$$\mathcal{L}_2 = \lambda_{\mathcal{O},2} \mathcal{O}_\mu \text{Tr}[L^\mu \Phi \Phi^\dagger + R^\mu \Phi^\dagger \Phi], \quad (17)$$

where $\lambda_{\mathcal{O},2}$ is a dimensionless (unknown) coupling constant. By taking into account the expression listed in Table II, one obtains terms delivering three-body and two-body decays. The three-body decays are given by

$$\begin{aligned} \mathcal{L}_{2,\text{three-body}} = & \lambda_{\mathcal{O},2} \mathcal{O}_\mu \text{Tr}[2V^\mu(P^2 + S^2)] \\ & + 2\lambda_{\mathcal{O},2} \mathcal{O}_\mu \text{Tr}[A^\mu 2i[P, S]], \end{aligned} \quad (18)$$

hence decays of the type $\mathcal{O} \rightarrow VPP$, $\mathcal{O} \rightarrow VSS$, $\mathcal{O} \rightarrow APS$, and $\mathcal{O} \rightarrow PPS$ (the later obtained by the shift $A \rightarrow Zw\partial P$, see details in Appendix B) follow. One of the most relevant decays (the second in magnitude) is

$$\mathcal{O} \rightarrow \omega\pi\pi,$$

which we use as a reference for the ratios listed in Table IV (see Appendix C for the analytic expression). The channel $\mathcal{O} \rightarrow \pi KK^*(892)$ turns out to be the largest, followed by $\omega\pi\pi$. In the last line of Table IV we have also reported, as an example, a three-body decay into $a_0(1450)a_0(1450)\omega$, which is however very small. This is the case for all kinematically allowed VSS decays.

Quite interestingly, the most prominent decay of the Lagrangian \mathcal{L}_1 is $\mathcal{O} \rightarrow b_1\pi \rightarrow \omega\pi\pi$ (see Table III), hence it also generates a $\omega\pi\pi$ final state (the state b_1 has a dominant decay into $\omega\pi$). At a first approximation, one can write

$$\Gamma_{\mathcal{O} \rightarrow \omega\pi\pi}^{\text{tot}} \approx \Gamma_{\mathcal{O} \rightarrow \omega\pi\pi}^{\text{direct-from}\mathcal{L}_2} + \Gamma_{\mathcal{O} \rightarrow b_1\pi \rightarrow \omega\pi\pi}^{\text{indirect-from}\mathcal{L}_1}, \quad (19)$$

although strictly speaking interferences can appear (usually they are smaller than 10%, see the discussion in Ref. [17]; we will neglect such interferences in the following). Anyway, both \mathcal{L}_1 and \mathcal{L}_2 agree in predicting a strong

TABLE IV. Branching ratios for the direct three-body decay of the vector glueball into two (pseudo)scalar and one vector meson.

Quantity	Value
$\frac{\mathcal{O} \rightarrow KK\rho}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.50
$\frac{\mathcal{O} \rightarrow KK\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.17
$\frac{\mathcal{O} \rightarrow KK\phi}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.21
$\frac{\mathcal{O} \rightarrow \pi KK^*(892)}{\mathcal{O} \rightarrow \omega\pi\pi}$	1.2
$\frac{\mathcal{O} \rightarrow \eta\eta\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.064
$\frac{\mathcal{O} \rightarrow \eta\eta'\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.019
$\frac{\mathcal{O} \rightarrow \eta'\eta'\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.019
$\frac{\mathcal{O} \rightarrow \eta\eta\phi}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.039
$\frac{\mathcal{O} \rightarrow \eta\eta'\phi}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.011
$\frac{\mathcal{O} \rightarrow \eta'\eta'\phi}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.011
$\frac{\mathcal{O} \rightarrow a_0(1450)a_0(1450)\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.00029

signal into the final state $\omega\pi\pi$. Similarly, the final state into $\pi KK^*(892)$ is also relevant, since it comes directly from \mathcal{L}_2 and indirectly from \mathcal{L}_1 via the channels $KK_1(1270) \rightarrow K\pi K^*(892)$ and $KK_1(1400) \rightarrow K\pi K^*(892)$ [but the decays of $K_1(1270)$ and $K_1(1400)$ do not have a single dominating channel [5]].

Two-body decays from \mathcal{L}_2 are obtained when one of the Φ condenses, $\Phi \rightarrow \Phi_0 + \Phi$:

$$\mathcal{L}_{2,\text{two-body}} = \lambda_{\mathcal{O},2} \mathcal{O}_\mu \text{Tr}[V^\mu 2\{\Phi_0, S\}] - \lambda_{\mathcal{O},2} \mathcal{O}_\mu \text{Tr}[A^\mu 2i[\Phi_0, P]]. \quad (20)$$

Then, the decays $\mathcal{O} \rightarrow VS$, $\mathcal{O} \rightarrow AP$, and $\mathcal{O} \rightarrow PP$ follow (expressions in Appendix B). The (most relevant) decay ratios are listed in Table V. The second term in Eq. (20) is suppressed because the decay amplitudes are proportional to the chirally suppressed difference $(\phi_N - \sqrt{2}\phi_S)$ [this quantity vanishes in the $U_V(3)$ limit]. Hence both decay channels $\mathcal{O} \rightarrow AP$ and $\mathcal{O} \rightarrow PP$ are expected to be very small. In particular, $\mathcal{L}_{2,\text{two-body}}$ contains only interaction terms of the vector glueball with $K_{1,A}K$. Due to the fact that the $K_{1,A}$ is a mixture of $K_1(1270)$ and $K_1(1400)$ [Eq. (9)],

TABLE V. Branching ratios for the two-body decay of the vector glueball into one (pseudo)scalar and one (axial-)vector meson and into a kaon-kaon pair.

Quantity	Value
$\frac{\mathcal{O} \rightarrow a_0(1450)\rho}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.47
$\frac{\mathcal{O} \rightarrow f_0(1370)\omega}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.15
$\frac{\mathcal{O} \rightarrow K_1^*(1430)K^*(892)}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.30
$\frac{\mathcal{O} \rightarrow KK}{\mathcal{O} \rightarrow \omega\pi\pi}$	0.018

one would obtain decay rates into $K_1(1270)K$ and $K_1(1400)K$, which are already included in Table III. Strictly speaking, one could describe these decays only once the ratio $\lambda_{\mathcal{O},2}/\lambda_{\mathcal{O},1}$ is known. However, the contribution proportional to $\lambda_{\mathcal{O},2}$ is suppressed, hence it represents a correction to the results listed in Table III. These decay modes are then omitted from Table V. The decay into two kaons [also proportional to $(\phi_N - \sqrt{2}\phi_S)$] is small as expected.

Finally, the decay into DD (the only decay in charmed mesons which is kinematically allowed) can be obtained by using the extension of the eLSM to the four-flavor case [33]. Due to the fact that chiral symmetry can be only considered as very approximate when charmed mesons are considered, the prediction offers only a qualitative result. Anyway, the ratio

$$\frac{\Gamma_{\mathcal{O} \rightarrow DD}}{\Gamma_{\mathcal{O} \rightarrow \omega\pi\pi}} \approx 0.029$$

shows that the $\bar{D}D$ mode is also expected to be small. This result is important because it shows that the vector glueball, even if according to lattice QCD has a mass above the $\bar{D}D$ threshold, decays only rarely in charmed mesons. The (direct and indirect) $\omega\pi\pi$ and $\pi KK^*(892)$ decay modes are expected to be much larger.

C. Coupling to (axial-)vector mesons

As a last interaction term we consider an expression which breaks dilatation invariance:

$$\mathcal{L}_3 = \alpha \varepsilon_{\mu\nu\rho\sigma} \partial^\rho \mathcal{O}^\sigma \text{Tr}[L^\mu \Phi R^\nu \Phi^\dagger], \quad (21)$$

where α has dimension of energy⁻². Even if it is expected to lead to smaller decay rates than the previous two Lagrangians, the presence of an ‘‘anomalous’’ Levi-Civita tensor may point to a non-negligible interaction strength

TABLE VI. Branching ratios for the decay of the vector glueball into a vector-pseudoscalar and vector-axial-vector pair.

Quantity	Value
$\frac{\mathcal{O} \rightarrow KK^*(892)}{\mathcal{O} \rightarrow \rho\pi}$	1.3
$\frac{\mathcal{O} \rightarrow \eta\omega}{\mathcal{O} \rightarrow \rho\pi}$	0.16
$\frac{\mathcal{O} \rightarrow \eta'\omega}{\mathcal{O} \rightarrow \rho\pi}$	0.13
$\frac{\mathcal{O} \rightarrow \eta\phi}{\mathcal{O} \rightarrow \rho\pi}$	0.21
$\frac{\mathcal{O} \rightarrow \eta'\phi}{\mathcal{O} \rightarrow \rho\pi}$	0.18
$\frac{\mathcal{O} \rightarrow \rho a_1(1230)}{\mathcal{O} \rightarrow \rho\pi}$	1.8
$\frac{\mathcal{O} \rightarrow \omega f_1(1285)}{\mathcal{O} \rightarrow \rho\pi}$	0.55
$\frac{\mathcal{O} \rightarrow \omega f_1(1420)}{\mathcal{O} \rightarrow \rho\pi}$	0.82

even in the presence of an explicit breaking of dilatation symmetry.

We restrict our study to the case in which the scalar field condenses. Hence, upon setting $\Phi = \Phi_0$, we obtain

$$\mathcal{L}_{3,\text{two-body}} = 2\alpha \varepsilon_{\mu\nu\rho\sigma} \partial^\rho \mathcal{O}^\sigma \text{Tr}[A^\mu \Phi_0 V^\nu \Phi_0] \quad (22)$$

which leads to $O \rightarrow AV$ and $O \rightarrow PV$ (upon shifting A). We chose the decay channel $\mathcal{O} \rightarrow \rho\pi$ as our reference decay mode (see Appendix B for its expression). In Table VI we present the branching ratios which follow from \mathcal{L}_3 . The dominant decay modes are into $\rho\pi$, $KK^*(892)$, and $\rho a_1(1230)$ (with increasing strength). The $\rho\pi$ and $KK^*(892)$ modes are also important in the description of the $\rho\pi$ puzzle of $\psi(2S)$ described in the next section.

IV. DISCUSSIONS

In this section we discuss in detail some important issues related to our approach and our results. First, we motivate the applicability of our interaction Lagrangians, second we justify the use of glueball’s masses from lattice QCD as an input, and third we present the so-called $\rho\pi$ -puzzle and its connection to the vector glueball.

A. On the applicability of the interaction Lagrangians

The eLSM is a low-energy chiral model valid up to 1.7 GeV, while the vector glueball studied in this work has a mass of about 3.8 GeV. It should be stressed that the joint model

$$\mathcal{L}_{\text{eLSM}} + \mathcal{L}_{\mathcal{O}\text{-eLSM}} \quad (23)$$

(where $\mathcal{L}_{\mathcal{O}\text{-eLSM}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is the sum of the three interaction terms described in the previous section) should be regarded as a model suited to calculate exclusively the decays of the field \mathcal{O} . However, one should not use such a model to calculate, for instance, pion-pion scattering or analogous quantities up to the energy of about 4 GeV, since the approach is *clearly* not complete for that purpose.

Approaches which couple one heavy field to light mesons were widely used in the literature, e.g. Refs. [15,39,40] and references therein. The idea behind these approaches can be explained at best with a simple example: the decay of the scalar charmonium state χ_{c0} with a mass of 3.41 GeV into two light pseudoscalar mesons. The field χ_{c0} is flavor blind and, by requiring that the whole interaction Lagrangian is invariant under $U(3)_V$ flavor transformations, one is led to the toy-model Lagrangian

$$\mathcal{L}_{\text{toy}} = g_\chi \chi_{c0} \text{Tr}[P^t P], \quad (24)$$

where P is the 3×3 matrix of pseudoscalar mesons (see Sec. II) and g_χ is an effective coupling constant. Upon

expanding, one obtains $\mathcal{L}_{\text{toy}} = g_\chi \chi_{c0} (\vec{\pi}^2 + 2K^+ K^- + 2K^0 \bar{K}^0 + \eta^2 + \eta'^2)$. By taking into account phase space, the theoretical decay ratios read $\Gamma_{\chi_{c0} \rightarrow KK} / \Gamma_{\chi_{c0} \rightarrow \pi\pi} = 1.28$, $\Gamma_{\chi_{c0} \rightarrow \eta\eta} / \Gamma_{\chi_{c0} \rightarrow \pi\pi} = 0.32$, and $\Gamma_{\chi_{c0} \rightarrow \eta\eta'} / \Gamma_{\chi_{c0} \rightarrow \pi\pi} = 0.28$, which should be compared to the experimental results of 1.42 ± 0.14 , 0.35 ± 0.04 , and 0.23 ± 0.04 , respectively [5]. Considering that this toy approach is very simple and does not contain any violation of flavor symmetry (it involves only the dominant term in the large- N_c expansion, see below), it works remarkably well when compared to data. Quite interestingly, the toy model also predicts that $\Gamma_{\chi_{c0} \rightarrow \pi^0 \eta} = \Gamma_{\chi_{c0} \rightarrow \pi^0 \eta'} = \Gamma_{\chi_{c0} \rightarrow \eta \eta'} = 0$ [the first two results are a consequence of isospin symmetry, naturally included in $U(3)_V$ flavor symmetry]. Indeed, none of these decays has been seen in the experiment and only stringent upper values exist [5].

The discussed toy model for the resonance χ_{c0} is clearly limited, but shows an important fact: the decays of a heavy state (such as a charmonium resonance) into light mesons still respects the underlying symmetries of the light system (which is simply flavor symmetry in the present example). This is the case even if the employed toy model cannot be considered as a full hadronic model valid up to the mass of charm-anticharm states. Indeed, this principle has been utilized in a large number of papers, see for example Refs. [15,39] for what concerns the decay of the heavy charmonium state j/ψ into pseudoscalar mesons and Ref. [40] for what concerns decays of j/ψ involving scalar mesons.

In the present work, the calculation of the decays of the vector glueball follows the very same simple idea. Indeed, in decays of charmonia, one has first a conversion to gluons, which then transform to light mesons. Intuitively speaking, the glueball is dominated by gluons, a situation which is similar to the intermediate state of a charmonium decay. Moreover, glueballs have some features similar to that of ordinary mesons: they are made of heavy constituent gluons [41] and their size does not seem to be different from that of ordinary quark-antiquark states [42]. Hence, considerations based on symmetry appear to be a valid starting point to get some information on decay ratios such as the ones of the vector glueball presented in this work. Clearly, only the future experimental discovery of glueballs and/or advanced lattice calculations (see the next subsection) will be able to test this basic assumption of effective hadronic models.

In order to be more realistic, in the present work a better and more complete low-energy model with (pseudo)scalar and (axial-)vector states was used: the eLSM. As explained previously, this hadronic model is based on chiral symmetry and dilatation invariance (as well as explicit, anomalous, and spontaneous breaking of these symmetries), it contains a finite number of terms, and has been used to describe meson phenomenology up to 1.7 GeV. (It is

interesting to note that in Ref. [33], the eLSM was also applied to charmed mesons. It was found that the decays of such heavy mesons still approximately reflect chiral symmetry.) Moreover, we took into account that the heavy field that we couple to the eLSM, the vector glueball, is not only flavor blind, but also chirally blind: in this way chiral symmetry (together with its spontaneous breaking) has an influence on the determination of the decays. While a direct comparison with data is not yet possible, predictions can be obtained by the calculation of decay ratios. These predictions are model dependent and still neglect symmetry breaking terms and mixing effects. Yet, some branching ratios might be useful in the future search of the vector glueball. Moreover, the same approach can be actually applied to all glueballs listed in the lattice spectrum of Ref. [3], as the example of the vector glueball or the example of the pseudoscalar glueball show [16].

B. On the glueball masses

At present, the best theoretical method to calculate glueball's masses is lattice, since it numerically simulates the Yang-Mills (or the QCD) Lagrangian, hence it takes into account the nonperturbative nature of interactions involving gluons. The masses calculated in Ref. [3] (whose results are also reported in the “quark model” summary of the PDG [5]) were obtained in the so-called quenched approximation (no quark fluctuations). However, in Ref. [43] an unquenched calculation has been performed. In the conclusions of Ref. [43] it is written: “The most conservative interpretation of our results is that the masses in terms of lattice representations are broadly consistent with results from quenched QCD.” This is indeed a promising result for model builders. If mass shifts due to unquenching are not too large, one may be—with due care—optimistic that quark-antiquark admixtures do not spoil the presented picture. On the other hand, it must be stressed that the study of Ref. [43] has not been repeated yet by other groups (see however the very recent two-flavor study of light glueballs in Ref. [44], where the masses also do not vary much when quarks are included). A full study involving glueballs and their mixing with conventional mesons would be very useful to advance in the field. This is unfortunately not an easy task. As stated in Refs. [45,46], glueballs on the lattice become challenging in full QCD and it is very hard to determine their admixture in physical resonances. Namely, glueball signals deteriorate fast into noise, making them very hard to extract.

In connection to hadronic effective models, it should be however also stressed that the masses of glueballs which enter the Lagrangian should be the quenched ones (and not the unquenched). Namely the unquenching can be performed *within* the hadronic model. This is the case of the scalar glueball studied in Ref. [10]. We also recall that lattice is the best among many different approaches toward the calculations of the spectrum of glueballs, see

Refs. [1,47] for a list of various results. Even in the original works based on bag models, e.g. Ref. [48], the vector glueball turned out to be quite heavy (about 3–4 GeV). Quite interestingly, AdS/QCD also finds a mass of the vector state within the same range [49]. If, as we shall describe below, the width of the vector glueball turns out to be sufficiently narrow [surely, the ratio $\Gamma/(M - E_{\text{threshold}})$ is expected to be sufficiently smaller than 1], shifts due to mesonic quantum fluctuations [50] are also suppressed and shall not change drastically the mass.

Yet, in order to study mixing effects within hadronic models, first one needs to identify some promising candidates. Namely, the mixing is strongly dependent on the mass difference between nearby bare states. Its study will be possible (and highly needed) when more information will be available. As we shall discuss in the next subsection, at least in one case such mixing was estimated to be rather small.

Summarizing, it is important to say that a change in the mass of the vector glueball of about 300 MeV (above or below the value 3.8 GeV used in this paper) shall not change the overall picture. At present, the use of the well established quenched lattice value of Ref. [3] for the vector glueball seems the most reasonable choice to start with.

C. On the $\rho\pi$ puzzle

The decay of the vector glueball $\mathcal{O} \rightarrow \rho\pi$ and $\mathcal{O} \rightarrow KK^*(892)$ (which arise from \mathcal{L}_3 , see Table VI) are interesting in connection to the so-called $\rho\pi$ puzzle [22,23,25]. This puzzle has to do with the experimentally missing $\rho\pi$ and $KK^*(892)$ decays of the resonance $\psi(2S)$. This state is (predominately) a charmonium which emerges as a radial excitation of the famous j/ψ meson. Its mass is 3.6806109 GeV (quite close to the mass of the vector glueball evaluated by lattice QCD) and its decay width is very small: $\Gamma_{\psi(2S)} = 0.298$ MeV. Due to the similarity of j/ψ and $\psi(2S)$, one expects that the ratio,

$$\frac{\Gamma_{\psi(2S) \rightarrow \text{a certain light meson channel}}}{\Gamma_{j/\psi \rightarrow \text{a certain light meson channel}}} \simeq 0.14, \quad (25)$$

holds for all light channels. The value 0.14 (the 14% rule) is the ratio of the decays into e^+e^- [21]. This rule works pretty well for various decay channels, but is badly broken for $\rho\pi$ and $KK^*(892)$ channels, which are clearly seen for j/ψ but, as mentioned, are not seen for $\psi(2S)$.

A possibility to solve this puzzle invokes the presence of a nearby vector glueball. A mixing of a bare charmonium $\bar{c}c$ state and a bare glueball $\mathcal{O} \equiv ggg$ leads to two physical states:

$$\begin{pmatrix} \psi(2S) \\ \mathcal{O}' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \times \begin{pmatrix} \bar{c}c (\text{with } n=2, L=0, S=1) \\ \mathcal{O} \equiv ggg \end{pmatrix}. \quad (26)$$

Then, within this picture the state $\psi(2S)$ does not correspond to a pure charmonium, but contains (a small) glueball amount. In Ref. [23] it is estimated that $|\theta| \lesssim 2^\circ$. This is in agreement with the fact that such a glueball-quarkonium mixing is suppressed in the large- N_c limit and by the fact that the vector glueball contains (at least) three constituent gluons.

In conclusion, for what concerns the decays $\Gamma_{\mathcal{O} \rightarrow \text{light mesons}}$, the estimated small mixing with a $2S$ charmonium state has a small influence, thus justifying *a posteriori* the results presented in this work. However, a precise study of this small mixing must be left for the future (when and if a vector glueball candidate will be found in that mass region).

V. CONCLUSIONS AND OUTLOOK

In this work we have presented three chirally invariant effective interaction terms describing two-body and three-body decays of a not-yet discovered vector glueball into (pseudo)scalar, (axial-)vector and pseudo(excited-)vector mesons. While the intensity of the coupling constant cannot be determined, one can predict, in the context of our model, some decay ratios and thus determine which decay channels are expected to dominate. Hopefully, our results, even if model dependent and subject to various uncertainties (validity of the symmetries used to write the Lagrangians, the value of input's mass of the vector glueball, and the absence of mixing, see below), can be of some help in the future experimental search of the vector glueball. In particular, we have found the following outcomes. In the first two interaction terms (which are also dilatation invariant and are expected to be dominant) the main decay channels are $\mathcal{O} \rightarrow b_1\pi \rightarrow \omega\pi\pi$ (first term) as well as $\mathcal{O} \rightarrow \omega\pi\pi$ and $\mathcal{O} \rightarrow \pi KK^*(892)$ (second term), see Tables III, IV, and V for all results. Interestingly, the first and second terms predict sizable $\omega\pi\pi$ and $\pi KK^*(892)$ final states, which according to our results represent the golden channels for the identification of a vector glueball's candidate. Our third interaction term breaks dilatation invariance but was considered because it predicts decays into one vector and one pseudoscalar meson, in particular $\mathcal{O} \rightarrow \rho\pi$ and $\mathcal{O} \rightarrow KK^*(892)$ (Table VI). In turn, these channels may help to understand the $\rho\pi$ puzzle of $\psi(2S)$.

The width of the vector glueball (and of glueballs in general) cannot yet be determined theoretically. According to large- N_c arguments [51–54], a glueball decay into two mesons scales as $\Gamma_{G \rightarrow M_1 M_2} \propto N_c^{-2}$, hence it is more suppressed than Okubo-Zweig-Iizuka (OZI)-allowed decays of conventional mesons (such as $\rho \rightarrow \pi\pi$) which scale $\Gamma_{M \rightarrow M_1 M_2}^{\text{OZI-allowed}} \propto N_c^{-1}$, but less suppressed than OZI-forbidden decays (such as $j/\psi \rightarrow \text{light mesons}$ [55,56]), which scale $\Gamma_{M \rightarrow M_1 M_2}^{\text{OZI-forbidden}} \propto N_c^{-3}$. Large- N_c considerations represent only a qualitative statement (and

could be well broken for the physical value $N_c = 3$, as it is e.g. in the case of the axial anomaly), but they support the hope that (at least some) glueballs are not too wide, hence one can be cautiously optimistic that (some) glueballs can be detected in the future (in particular, at the PANDA experiment [18], designed also for that purpose). Moreover, the vector glueball studied in this work is built out of (at least) three constituent gluons, for which decay requires also the complete annihilations of three gluons, hence possibly not too large.

For what concerns the mass of the glueball, we have used 3.81 GeV as determined in Ref. [3]. Even if the uncertainty is still large, the picture concerning the main decay channels does not qualitatively change when varying the input's mass. In addition, unquenching effects do not seem to completely change the picture of glueballs [43], but future lattice studies are needed to confirm this result and to quantify deviations. In this respect, the coupling of glueballs to mesons can be a very interesting future achievement of lattice QCD. Namely, it will be possible to further constraint models such as the one described in this work.

The mixing of the glueball with other nearby quarkonium states needs also to be studied in the future. In one particular case, the mixing of the vector glueball with the predominantly charmonium state $\psi(2S)$ has been estimated to be rather small [23]. At present, the lack of candidates for the vector glueball makes a study of mixing not yet possible (in fact, mixing strongly depends on the precise value of the masses, which are not yet known). In this work, as a first, necessary step, we thus aimed to study the main interaction terms of a bare, unmixed vector glueball. As soon as candidates will be found, it will be very interesting and exciting to study mixing in more detail.

All of the remarks above show that there is a lot of room for improvement of our approach in the future. New lattice results and experimental findings will be of great help to advance in this difficult but exciting field of QCD. In this respect, this work represents a first step toward the search of the vector glueball, whose main goal

is the identification of the possible dominating decay channels.

Other glueball states can be studied by following the same procedure outlined in this work. For instance, the tensor glueball ($J^{PC} = 2^{++}$) is expected to have a mass of about 2.2 GeV (it is the second lightest glueball according to lattice QCD [3]); a good candidate could be the very narrow resonance $f_J(2220)$ [57,58] (at present the options are $J = 2$ or 4 [5]; further experimental information is needed). A future study within the eLSM should include—besides the states included in this work—also tensor mesons and their chiral partners, the pseudotensor mesons. In addition to the tensor glueball, one has a full tower of states listed by lattice QCD: pseudotensor glueball, pseudovector glueball, oddballs (glueball with exotic quantum numbers such as $J^{PC} = 0^{+-}$ and $J^{PC} = 2^{+-}$) as well as various glueballs with $J = 3$. Various branching ratios are parameter-free once the mass of the glueball is fixed and offer a useful information toward the future search of these important (and still missing) states of QCD.

As a final remark, it must be stressed that the upcoming PANDA experiment [18] is tailor made for the search of glueballs, since almost all glueballs (with the exceptions of oddballs) can be directly formed in proton-antiproton fusion processes.

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APPENDIX A: THE eLSM

The Lagrangian of the eLSM is built by requiring chiral symmetry $[U(3)_R \times U(3)_L]$, dilatation invariance, as well as invariances under charge conjugation C and parity P :

$$\begin{aligned} \mathcal{L}_{\text{mes}} = & \mathcal{L}_{\text{dil}} + \text{Tr}[(D_\mu \Phi)^\dagger (D^\mu \Phi)] - m_0^2 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 - \frac{1}{4} \text{Tr}[(L^{\mu\nu})^2 + (R^{\mu\nu})^2] \\ & + \text{Tr} \left[\left(\frac{m_1^2}{2} \left(\frac{G}{G_0}\right)^2 + \Delta \right) (L_\mu^2 + R_\mu^2) \right] + \text{Tr}[H(\Phi + \Phi^\dagger)] + c_1 (\det \Phi - \det \Phi^\dagger)^2 + i \frac{g_2}{2} \{ \text{Tr}(L_{\mu\nu}[L^\mu, L^\nu]) \\ & + \text{Tr}(R_{\mu\nu}[R^\mu, R^\nu]) \} + \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_\mu^2 + R_\mu^2) + h_2 \text{Tr}[|L_\mu \Phi|^2 + |\Phi R_\mu|^2] + 2h_3 \text{Tr}(L_\mu \Phi R^\mu \Phi^\dagger), \end{aligned} \quad (\text{A1})$$

where $D^\mu \Phi = \partial^\mu \Phi - ig_1(L^\mu \Phi - \Phi R^\mu)$ is the covariant derivative and

$$\mathcal{L}_{\text{dil}} = \frac{1}{2} (\partial_\mu G)^2 - \frac{1}{4} \frac{m_G^2}{\Lambda^2} \left(G^4 \ln \left| \frac{G}{\Lambda} \right| - \frac{G^4}{4} \right) \quad (\text{A2})$$

the dilaton (i.e. the scalar glueball) Lagrangian, see Refs. [9,10] for details.

Spontaneous breaking of chiral symmetry takes place ($m_0^2 < 0$). As a consequence, one has to perform the shift of the scalar-isoscalar fields by their vacuum expectation values ϕ_N and ϕ_S :

$$\sigma_N \rightarrow \sigma_N + \phi_N \quad \text{and} \quad \sigma_S \rightarrow \sigma_S + \phi_S. \quad (\text{A3})$$

In matrix form,

$$S \rightarrow \Phi_0 + S \quad \text{with} \quad \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\phi_N}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\phi_N}{\sqrt{2}} & 0 \\ 0 & 0 & \phi_S \end{pmatrix}. \quad (\text{A4})$$

In addition, one has also to ‘‘shift’’ the axial-vector fields,

$$\begin{aligned} \vec{a}_1^\mu &\rightarrow \vec{a}_1^\mu + Z_\pi w_\pi \partial^\mu \vec{\pi}, & K_{1,A}^{+,\mu} &\rightarrow K_{1,A}^{+,\mu} + Z_K w_K \partial^\mu K, \dots \\ f_{1,N}^\mu &\rightarrow f_{1,N}^\mu + Z_{\eta_N} w_{\eta_N} \partial^\mu \eta_N, & f_{1,S}^\mu &\rightarrow f_{1,S}^\mu + Z_{\eta_S} w_{\eta_S} \partial^\mu \eta_S, \end{aligned} \quad (\text{A5})$$

and to consider the wave-function renormalization of the pseudoscalar fields:

$$\vec{\pi} \rightarrow Z_\pi \vec{\pi}, \quad K^+ \rightarrow Z_K K^+, \dots \quad (\text{A6})$$

$$\eta_N \rightarrow Z_{\eta_N} \eta_N, \quad \eta_S \rightarrow Z_{\eta_S} \eta_S. \quad (\text{A7})$$

The constants entering into the previous expressions are

$$\begin{aligned} Z_\pi &= Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \phi_N^2}}, \\ Z_K &= \frac{2m_{K_{1,A}}}{\sqrt{4m_{K_{1,A}}^2 - g_1^2 (\phi_N + \sqrt{2}\phi_S)^2}}, \\ Z_{\eta_S} &= \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2 \phi_S^2}}, \end{aligned} \quad (\text{A8})$$

and

$$\begin{aligned} w_\pi &= w_{\eta_N} = \frac{g_1 \phi_N}{m_{a_1}^2}, \\ w_K &= \frac{g_1 (\phi_N + \sqrt{2}\phi_S)}{2m_{K_{1,A}}^2}, \\ w_{\eta_S} &= \frac{\sqrt{2}g_1 \phi_S}{m_{f_{1S}}^2}. \end{aligned} \quad (\text{A9})$$

The numerical values of the renormalization constants are $Z_\pi = 1.709$, $Z_K = 1.604$, $Z_{\eta_S} = 1.539$ [9], while those of the w -parameters are $w_\pi = 0.683$, $w_K = 0.611$, and $w_{\eta_S} = 0.554 \text{ GeV}^{-1}$. Moreover, the condensates ϕ_N and ϕ_S read

$$\begin{aligned} \phi_N &= Z_\pi f_\pi = 0.158 \text{ GeV}, \\ \phi_S &= \frac{2Z_K f_K - \phi_N}{\sqrt{2}} = 0.138 \text{ GeV}, \end{aligned} \quad (\text{A10})$$

where the standard values $f_\pi = 0.0922$ and $f_K = 0.110 \text{ GeV}$ have been used [5]. The previous expression can be summarized by the matrix replacements,

$$P \rightarrow \mathcal{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{Z_\pi}{\sqrt{2}} (\eta_N + \pi^0) & Z_\pi \pi^+ & Z_K K^+ \\ Z_\pi \pi^- & \frac{Z_\pi}{\sqrt{2}} (\eta_N - \pi^0) & Z_K K^0 \\ Z_K K^- & Z_K \bar{K}^0 & Z_{\eta_S} \eta_S \end{pmatrix}, \quad (\text{A11})$$

and

$$\begin{aligned} A^\mu &\rightarrow \mathcal{A}^\mu \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N} + a_1^0}{\sqrt{2}} & a_1^+ & K_{1,A}^+ \\ a_1^- & \frac{f_{1N} - a_1^0}{\sqrt{2}} & K_{1,A}^0 \\ K_{1,A}^- & \bar{K}_{1,A}^0 & f_{1S} \end{pmatrix}^\mu \\ &+ \frac{\partial^\mu}{\sqrt{2}} \begin{pmatrix} \frac{Z_\pi w_\pi}{\sqrt{2}} (\eta_N + \pi^0) & Z_\pi w_\pi \pi^+ & Z_K w_K K^+ \\ Z_\pi w_\pi \pi^- & \frac{Z_\pi w_\pi}{\sqrt{2}} (\eta_N - \pi^0) & Z_K w_K K^0 \\ Z_K w_K K^- & Z_K w_K \bar{K}^0 & Z_{\eta_S} w_{\eta_S} \eta_S \end{pmatrix}. \end{aligned} \quad (\text{A12})$$

In the $U_V(3)$ limit (in which all three bare quark masses are equals), one has $\Phi_N = \sqrt{2}\Phi_S$, $Z = Z_\pi = Z_K = Z_{\eta_S}$, and $w = w_\pi = w_K = w_{\eta_S}$. Hence, in this limit, $P \rightarrow \mathcal{P} = ZP$ and $A^\mu \rightarrow \mathcal{A}^\mu = A + Z w \partial^\mu P$.

The eLSM has been also enlarged to four flavors in Ref. [33]. Interestingly, charmed meson masses and large- N_c dominant decays can be described relatively well (even if one is far from the natural domain of chiral symmetry).

In the end, we also recall that the pseudoscalar glueball can be coupled to the eLSM via the chiral Lagrangian $\mathcal{L}_{\tilde{G}} = ic_{\tilde{G}\Phi} \tilde{G} (\det \Phi - \det \Phi^\dagger)$, which reflects the axial anomaly in the pseudoscalar-isoscalar sector, see details and results in Refs. [16,17]. In a recent extension, the very same Lagrangian is used to study the decay of a hypothetical excited pseudoscalar glueball [20].

APPENDIX B: EXPRESSIONS FOR TWO-BODY DECAYS

The decay $\mathcal{O} \rightarrow b_1\pi$ from \mathcal{L}_1 reads

$$\Gamma_{\mathcal{O} \rightarrow b_1\pi} = c_{\mathcal{O}b_1\pi} \frac{k_{\mathcal{O}b_1\pi}}{8\pi M_{\mathcal{O}}^2} (\lambda_1 G_0 Z_\pi)^2 \times \frac{1}{3} \left(2 + \frac{(M_{\mathcal{O}}^2 - m_\pi^2 + m_{b_1}^2)^2}{4M_{\mathcal{O}}^2 m_{b_1}^2} \right), \quad (\text{B1})$$

where $c_{\mathcal{O}b_1\pi} = 3$ is an isospin factor, $M_{\mathcal{O}} = 3.8$ GeV is the glueball mass, and m_π and m_{b_1} are the pion and b_1 masses. The quantity $k_{\mathcal{O}b_1\pi}$ is the modulus of the three-momentum of one of the two outgoing particles:

$$k_{\mathcal{O}b_1\pi} = \frac{\sqrt{M_{\mathcal{O}}^4 - 2M_{\mathcal{O}}^2(m_\pi^2 + m_{b_1}^2) + (m_\pi^2 - m_{b_1}^2)^2}}{2M_{\mathcal{O}}}. \quad (\text{B2})$$

The decays of the other channels in Table III are calculated in an analogous way, upon taking into account the change of masses, isospin factors, as well as the constants entering in the amplitudes. The two-body decays of \mathcal{L}_2 presented in Table V are calculated by using the same procedure.

We now turn to \mathcal{L}_3 . The decay $\mathcal{O} \rightarrow \rho\pi$ reads

$$\Gamma_{\mathcal{O} \rightarrow \rho\pi} = c_{\mathcal{O}\rho\pi} \frac{k_{\mathcal{O}\rho\pi}}{8\pi M_{\mathcal{O}}^2} \left(\frac{\alpha}{4} w_\pi Z_\pi \Phi_N^2 \right)^2 \left(\frac{2}{3} k_{\mathcal{O}\rho\pi}^2 M_{\mathcal{O}}^2 \right), \quad (\text{B3})$$

where $c_{\mathcal{O}\rho\pi} = 3$ and $k_{\mathcal{O}\rho\pi}$ is the modulus of the momentum in this case. The other decays $\mathcal{O} \rightarrow VP$ are calculated in the same way. The last decay that we consider is $\mathcal{O} \rightarrow \rho a_1(1230)$:

$$\Gamma_{\mathcal{O} \rightarrow \rho a_1(1230)} = c_{\mathcal{O}\rho a_1} \frac{k_{\mathcal{O}\rho a_1}}{8\pi M_{\mathcal{O}}^2} \left(\frac{\alpha}{4} \phi_N^2 \right)^2 \times \frac{1}{3} \left(6M_{\mathcal{O}}^2 + \frac{2k_{\mathcal{O}\rho a_1}^2 M_{\mathcal{O}}^2}{m_\rho^2} + \frac{2k_{\mathcal{O}\rho a_1}^2 M_{\mathcal{O}}^2}{m_{a_1}^2} \right), \quad (\text{B4})$$

where $c_{\mathcal{O}\rho a_1} = 3$ and $k_{\mathcal{O}\rho a_1}$ is the corresponding momentum. Analogous decays in Table VI are calculated in a similar way.

APPENDIX C: THREE-BODY DECAYS OF \mathcal{O} INTO TWO PSEUDOSCALAR MESONS AND A VECTOR MESON

For completeness we report the explicit expression for the three-body decay width of the process $\mathcal{O} \rightarrow P_1 P_2 V$:

$$\Gamma_{\mathcal{O} \rightarrow P_1 P_2 V} = \frac{s_{\mathcal{O} \rightarrow P_1 P_2 V}}{32(2\pi)^3 M_{\mathcal{O}}^3} \int_{(m_1+m_2)^2}^{(M_{\mathcal{O}}-m_3)^2} dm_{12}^2 \times \int_{(m_{23})_{\min}}^{(m_{23})_{\max}} |-i\mathcal{M}_{\mathcal{O} \rightarrow P_1 P_2 V}|^2 dm_{23}^2, \quad (\text{C1})$$

where (see [5])

$$(m_{23})_{\min} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2} \right)^2, \quad (\text{C2})$$

$$(m_{23})_{\max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2} \right)^2, \quad (\text{C3})$$

and

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}}, \quad E_3^* = \frac{M_{\mathcal{O}}^2 - m_{12}^2 - m_3^2}{2m_{12}}. \quad (\text{C4})$$

The quantities m_1 and m_2 refer to the masses of the two pseudoscalar states P_1 and P_2 , while m_3 is the mass of an (axial-)vector state V . We recall also that $m_{ij}^2 = (k_i + k_j)^2$ with k_1 , k_2 , and k_3 being the four-momenta of the three outgoing particles. Clearly, $p = k_1 + k_2 + k_3$, where p is the four-momentum of the vector glueball.

The amplitude $\mathcal{M}_{\mathcal{O} \rightarrow P_1 P_2 V}$ is calculated at tree level and $s_{\vec{G} \rightarrow P_1 P_2 V}$ is a symmetrization factor (it equals 1 if P_1 and P_2 are different, it equals 2 for $P_1 = P_2$). As an example, we consider the decay into $\pi^0 \pi^0 \omega$. The amplitude is

$$|-i\mathcal{M}_{\mathcal{O} \rightarrow \pi^0 \pi^0 \omega}|^2 = \frac{\lambda_2^2 Z_\pi^4}{4} \frac{1}{3} \left(2 + \frac{(M^2 + m_3^2 - m_{12}^2)^2}{4M_{\mathcal{O}}^2 m_3^2} \right), \quad (\text{C5})$$

and the symmetry factor is $s_{\mathcal{O} \rightarrow \pi^0 \pi^0 \omega} = 2$.

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