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Modified Tsallis and Weibull distributions for multiplicities in e^+e^- collisions

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Multiplicity distributions of charged particles produced in the e^+e^- collisions at energies ranging from 14 to 91 GeV are studied using Tsallis q-statistics and the recently proposed Weibull distribution functions, in both restricted rapidity windows as well as in full phase space. It is shown that Tsallis q-statistics explains the data in a statistically acceptable manner in all rapidity ranges while the Weibull distribution fails to reproduce the data in full phase space. Modifications to the distributions are proposed to establish manifold improvements in the fitting of the data.

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I. INTRODUCTION

Particle collisions at very high energies produce quark-quark, quark-gluon and gluon-gluon interactions which result in the production of a multitude of elementary particles. Several of these particles being mesons, baryons and leptons. This particle production is described in terms of several theoretical and phenomenological models derived from quantum chromodynamics. Several models use laws of fluid mechanics, statistical mechanics, thermodynamics, hydrodynamics etc. to describe the particle production. These models have been intriguingly successful. Present day high energy experiments, include several layers of detectors capable of detecting and recording the particles, both neutral and charged, produced in collisions very precisely. The distributions of these particles are then matched with predictions from various phenomenological models with an improved precision, to understand the production mechanism. Concepts from ensemble theory in statistical mechanics have been used to develop models which include statistical fluctuations as an important source of information. Distributions derived from statistics such as Poisson distribution, negative binomial distribution [1,2], Koba-Nielson-Olesen (KNO) scaling law [3] etc. have also played an important role in the understanding of multiplicity distributions. However, as more and more data at different collision energies became available, the KNO scaling violation was beginning to be observed, particularly in distributions with high multiplicity tails. Several new distributions have since been proposed. Some of these include modified negative binomial distribution [4], Krasznovszky and Wagner's [5], Tsallis [6,7], Gamma [8], the H-function extension of the negative binomial distribution [9], Log-normal [10], Weibull [11] etc. distributions. The novel approach in the Tsallis q-statistics incorporating nonextensive entropy to describe the particle production has been successfully applied to heavy-ion and p-p collisions at some energies. The nonextensive property of the entropy is quantified in terms of a parameter q which is shown to be more than unity under this assumption. The Tsallis distribution has been applied to $e+e-$ collision data to describe the multiplicity distribution. Weibull distribution is another statistical distribution which has been studied recently [11] to describe multiplicity distributions in e^+e^- collisions by S. Dash *et al.* The Weibull distribution has been also considered earlier by S. Hegyi [12,13] for e^+e^- as well as ep data from H1 experiment at Hadron-Electron Ring Accelerator.

In the present study, our focus is on investigating the multiplicity distributions, mostly in restricted rapidity windows, at different energies and to study the characteristic properties of charged particle production in e^+e^- collisions. In one of our earlier papers [14], we used Tsallis distribution to fit e^+e^- data from 34.8 to 206 GeV of energy in the full phase space and modified the Tsallis distribution to obtain the best fits as compared to several other distributions. In this paper we will limit ourselves to comparing the distributions using Tsallis q-statistics with the Weibull distribution in both restricted rapidity regions as well as in the full phase space to understand the constraints for the models used. We also propose a modification to improve the comparison between the predicted and the experimental values. In Sec. II, we give a very brief outline of probability distribution functions (PDF) of Tsallis, Weibull and their modified forms along with the references for full details. Section III presents the analyses of experimental data and the results obtained by two approaches. In approach I, we compare the Tsallis and Weibull distributions in full phase space as well as in restricted rapidity windows. Approach II describes the analysis results from the modified PDFs for Tsallis and Weibull distributions. The modification of distributions is done by convoluting PDFs from 2-jet fraction and multi-jet fractions by using appropriate weights. The 2-jet fraction for 91 GeV data has been taken from the DURHAM algorithm. Discussion and Conclusion are presented in Sec. IV.

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II. CHARGED MULTIPLICITY DISTRIBUTIONS

Charged particle multiplicity is defined as the average number of charged particles, n produced in a collision $\langle n\rangle = \sum_{n=0}^{n_{\max}} nP_n$. Angles at which these particles are produced, are measured in terms of rapidity defined as $y = -\ln\frac{E+P_L}{E-P_L}$ where E is the particle energy and P_L is the longitudinal momentum. We briefly outline the distributions used for studying the multiplicity distributions;

A. Tsallis distribution

Tsallis statistics deals with entropy in the usual Boltzman-Gibbs thermostatistics modified by introducing q-parameter and defined as;

$$S = \frac{1-\sum_a P_a^q}{q-1} \tag{1}$$

where P_a is the probability associated with microstate a and sum of the probabilities over all microstates is normalized to one; $\sum_a P_a = 1$.

Tsallis entropy is defined as;

$$S_q(A,B) = S_A + S_B + (1-q)S_A S_B \tag{2}$$

where q is entropic index with value, $q > 1$ and $1-q$ measures the departure of entropy from its extensive behavior.

In Tsallis q-statistics probability is calculated by using the partition function Z, as

$$P_N = \frac{Z_q^N}{Z}. \tag{3}$$

where Z represents the total partition function and Z_q^N represents partition function at a particular multiplicity.

For N particles, partition function can be written as,

$$Z(\beta,\mu,V) = \sum\left(\frac{1}{N!}\right)n^N(V-Nv_0)^N\Theta(V-Nv_0) \tag{4}$$

where n represents the gas density, V is the volume of the system and v_0 is the excluded volume associated to a particle. The Heaviside Θ-function limits the number of particles inside the volume V to $N < V/v_0$. $\bar{N}$, the average number of particles, is given by

$$\bar{N} = Vn[1+(q-1)\lambda(Vn\lambda-1)-2v_0 n] \tag{5}$$

where λ is related to the temperature through the parameter β as;

$$\lambda(\beta,\mu) = -\frac{\beta}{n}\frac{\partial n}{\partial\beta}. \tag{6}$$

The K-parameter is related to q and excluded volume, by

$$\frac{1}{K} = (q-1)\lambda^2 - 2\frac{v_0}{V}. \tag{7}$$

Details of the Tsallis distribution and how to find the probability distribution can be obtained from [7]. In one of our earlier papers, we have analysed the e^+e^- interactions at various energies for full phase space data and described the procedure in detail in reference [14].

B. Modified Tsallis distribution

In our earlier paper [14], we proposed to modify the multiplicity distribution in terms of two components: one due to multiplicity in 2-jet events and another due to multi-jet events. We then calculated the probability function from the weighted superposition of Tsallis distributions of these two components, as given below;

$$\begin{aligned} &P_N(\alpha:\bar{n_1},V_1,v_{01},q_1:\bar{n_2},V_2,v_{02},q_2)\\ &\quad=\alpha P_N(\bar{n_1},V_1,v_{01},q_1)\\ &\qquad+(1-\alpha)P_N(\bar{n_2},V_2,v_{02},q_2) \end{aligned} \tag{8}$$

where α is a weight factor which gives 2-jet fraction from the total events and is determined from a jet finding algorithm.

C. Weibull distribution

Weibull distribution is a continuous probability distribution which can take many shapes. It can also be fitted to non-symmetrical data.

The probability density function of a Weibull random variable is

$$P_N(N,\lambda,k) = \begin{cases} \frac{k}{\lambda}\left(\frac{N}{\lambda}\right)^{(k-1)}\exp^{-\left(\frac{N}{\lambda}\right)^k} & N\geq 0\\ 0 & N<0 \end{cases}. \tag{9}$$

The standard Weibull has characteristic value $\lambda > 0$, also known as scale factor, and shape parameter $k > 0$ for its two parameters. The two parameters for the distribution are related to the mean of function, as

$$\bar{N} = \lambda\Gamma(1+1/k). \tag{10}$$

D. Modified Weibull distribution

Modified Weibull distribution has been obtained by the weighted superposition of two Weibull distributions to produce the multiplicity distribution. We convolute the weighted distributions due to 2-jet component and multi-jet component of the events, as below;

$$P_N(\alpha : N_1, \lambda_1, k_1; N_2, \lambda_2, k_2) = \alpha P_N(N_1, \lambda_1, k_1) + (1-\alpha) P_N(N_2, \lambda_2, k_2) \tag{11}$$

where α is the weight factor for 2-jet fraction out of the total events and the remaining $1-\alpha$ is the multi-jet fraction. α is calculated from the DURHAM jet algorithm, as discussed in the next section.

III. ANALYSIS ON EXPERIMENTAL DATA & RESULTS

Experimental data on e^+e^- collisions at different collision energies from different experiments are analysed. The data used are from the experiments, TASSO [15], ALEPH [16], and DELPHI [17] at $\sqrt{s} = 14$, 22, 34.8, 44 & 91 GeV and from the restricted rapidity windows of $|y| < 0.5, 1.0, 1.5, 2.0$, where ever data are available. Charged particle multiplicity distribution in terms of probability distribution, is given by $P_n = \frac{\sigma_n}{\sigma_{\rm tot}}$, where σ_n is the cross section for multiplicity n and $\sigma_{\rm tot}$ represents the total cross section of the interaction at a center of mass energy $\sqrt{s}$. Experimentally this probability can be obtained using number of charged particles produced at specific multiplicity, n and total number of particles, $N_{\rm tot}$ produced in the process, by $P_n = \frac{n}{N_{\rm tot}}$. The experimental distribution is fitted with the predictions from Tsallis q-statistics and the Weibull distribution, as described in the following two approaches.

A. Approach I

The probability distributions using Tsallis distribution function and Weibull function are calculated using equations (3–7) & (9–10) and fitted to the experimental data. Figure 1 shows the Tsallis fits to the data and Fig. 2 shows the Weibull distributions fitted to the data in different rapidity intervals at various center of mass energies. Both Tsallis and Weibull functions are fitted to the data also in full phase space and the results are shown in Fig. 3.

We find that though Weibull shows a reasonable fitting in restricted rapidity intervals, it fails to reproduce the distributions in the high rapidity intervals and also in full phase space. While Tsallis distribution shows good fits in both full phase space and separately in each rapidity interval. A detailed comparison between the two functions

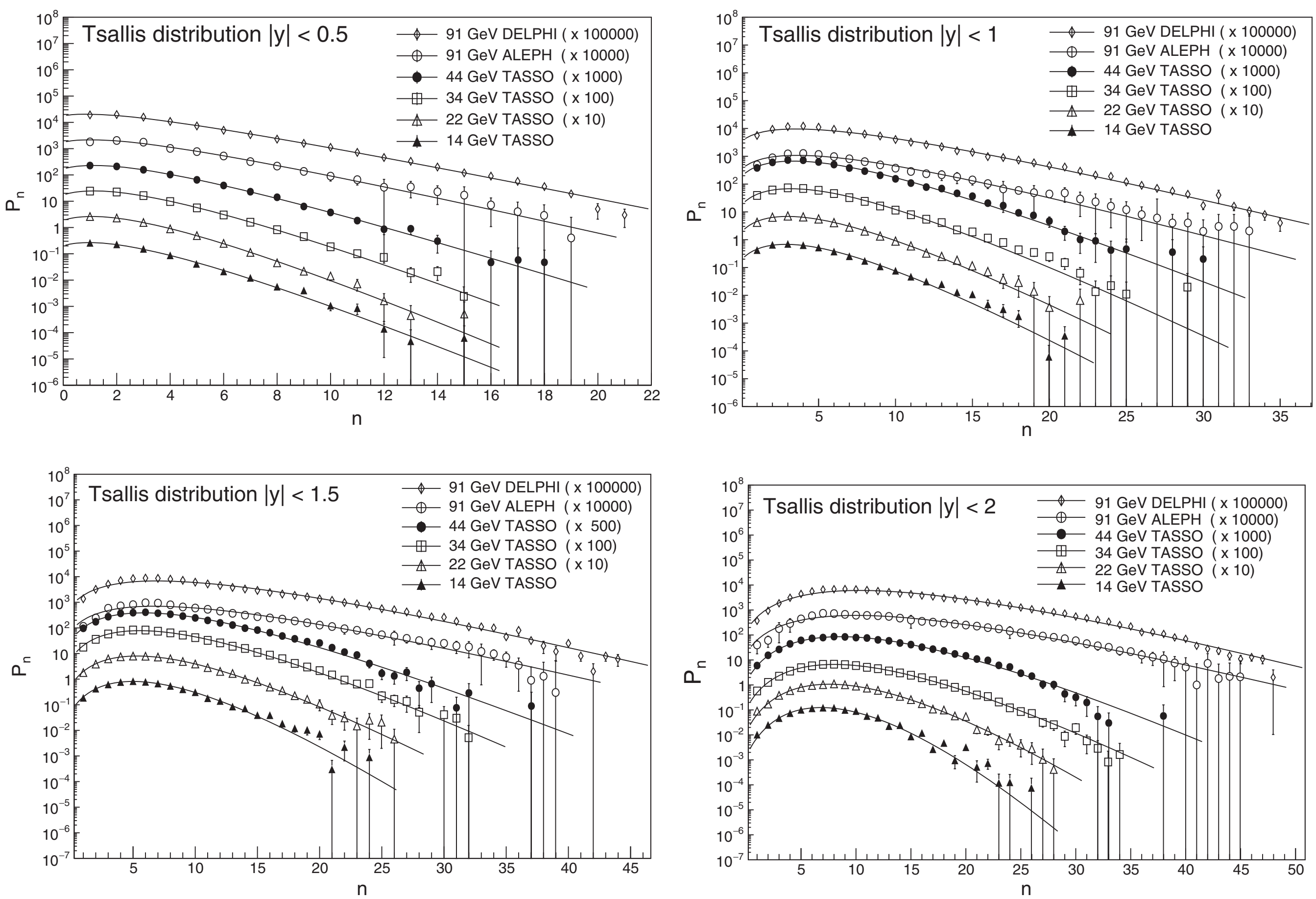

FIG. 1. Charged multiplicity distribution from top to bottom, $|y| < 0.5$, $|y| < 1$, $|y| < 1.5$ and $|y| < 2$ at $\sqrt{s} = 14$, 22, 34.8, 44 and 91 GeV. Solid lines represent the Tsallis distribution and points represent the data.

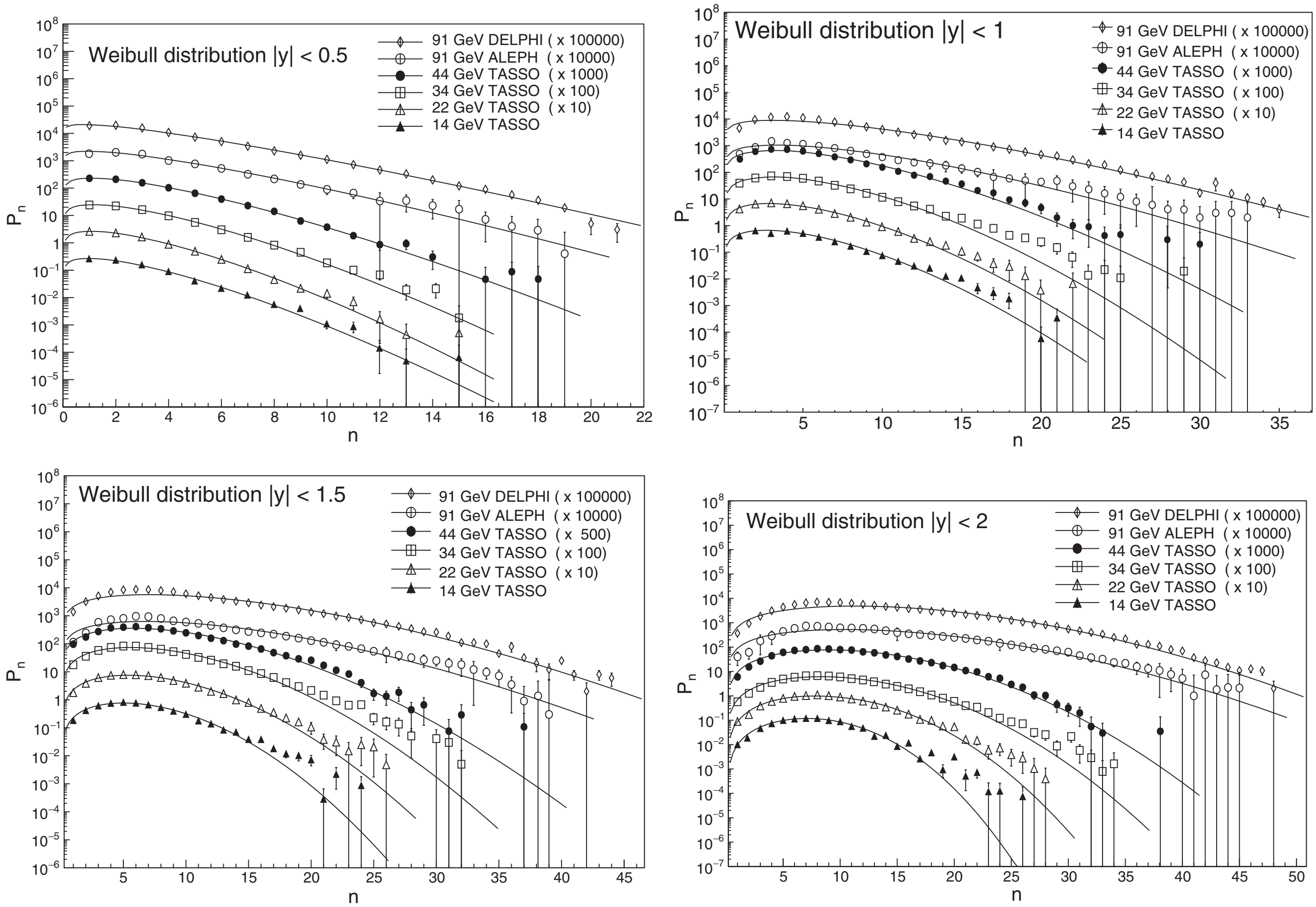

FIG. 2. Charged multiplicity distributions from top to bottom for $|y| < 0.5$, $|y| < 1$, $|y| < 1.5$ and $|y| < 2$ at $\sqrt{s} = 14$, 22, 34.8, 44 and 91 GeV. Solid lines represent the Weibull distributions fitted to the data.

is shown in Tables I and II where χ^2/ndf and p values at all energies for all rapidity intervals are given. It is observed that the χ^2/ndf values are considerably lower for the Tsallis fittings in comparison to the Weibull fittings. This is true for all rapidity intervals as well as for the full phase space. A careful examination of the p-values shows that the data at 14, 34.8 & 91 GeV (DELPHI Collaboration) are statistically excluded. For other energies, the confidence level of Weibull remains CL < 0.1% while Tsallis distribution remains good with CL > 0.1% for all cases.

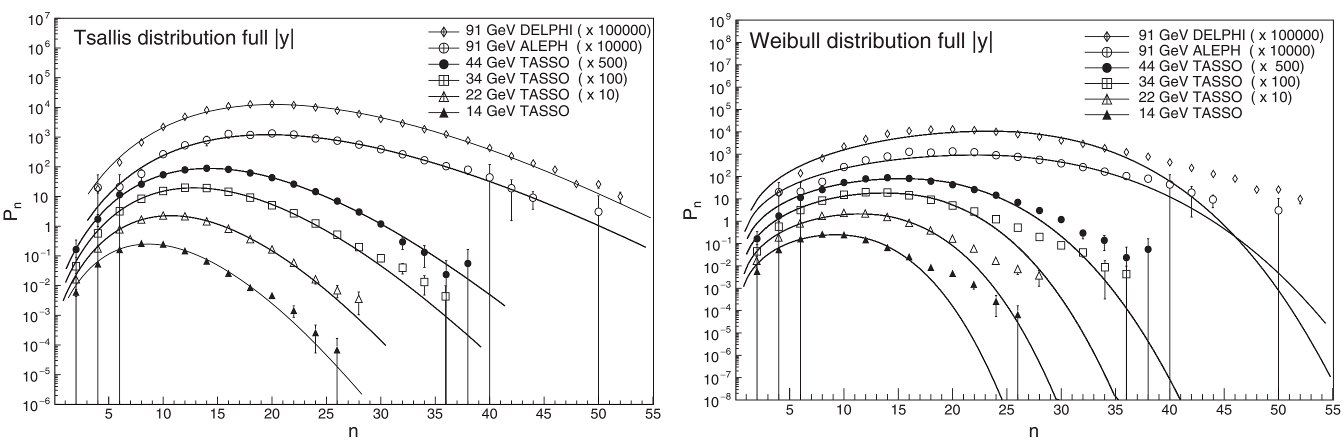

FIG. 3. Charged multiplicity distribution for full phase space at $\sqrt{s} = 14$, 22, 34.8, 44 and 91 GeV. Solid lines represent the Tsallis distribution(up) and Weibull distribution(down) fitted to the data.

TABLE I. χ^2/ndf comparison for different rapidity intervals and full phase space of Weibull distributions.

	$\|y\| < 0.5$		$\|y\| < 1$		$\|y\| < 1.5$		$\|y\| < 2$		full y	
Energy (GeV)	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value
14	17.54/11	0.0929	40.31/17	0.0012	62.55/20	0.0001	101.91/22	0.0001	105.60/10	0.0001
22	4.30/11	0.9603	26.66/18	0.0856	35.94/23	0.0418	51.06/25	0.0016	122.73/11	0.0001
34.8	25.92/12	0.0110	202.02/23	0.0001	277.37/28	0.0001	213.92/31	0.0001	611.21/15	0.0001
44	5.33/14	0.9807	77.65/24	0.0001	85.17/29	0.0001	74.78/31	0.0001	148.15/16	0.0001
91 A	6.28/16	0.9848	33.35/30	0.3076	66.63/36	0.0014	61.48/42	0.0265	34.97/19	0.0141
91 D	37.92/18	0.0040	294.51/32	0.0001	631.22/41	0.0001	756.61/45	0.0001	1524/22	0.0001

TABLE II. χ^2/ndf comparison for different rapidity intervals and full phase space of Tsallis distributions.

	$\|y\| < 0.5$		$\|y\| < 1$		$\|y\| < 1.5$		$\|y\| < 2$		full y	
Energy (GeV)	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value	χ^2/ndf	p value
14	13.96/9	0.1238	24.82/15	0.0524	33.59/18	0.0141	75.92/20	0.0001	32.62/8	0.0001
22	3.16/9	0.9576	11.46/16	0.7802	7.95/21	0.9953	12.07/23	0.9694	10.05/9	0.3465
34.8	14.65/10	0.1454	96.73/21	0.0001	73.43/26	0.0001	27.36/29	0.5523	38.35/13	0.0003
44	4.04/12	0.9827	40.38/22	0.0098	39.24/27	0.0602	37.83/29	0.1262	7.01/14	0.9347
91 A	5.23/14	0.9823	22.13/28	0.7752	39.43/34	0.2400	23.38/40	0.9833	5.71/17	0.9949
91 D	35.79/16	0.0031	184.91/30	0.0001	345.54/39	0.0001	292.72/43	0.0001	106.21/20	0.0001

Tables III and IV give fit parameters of both Weibull and Tsallis distributions for extreme rapidity intervals, $|y| < 0.5$ and $|y| < 2$. To avoid too many tables, we are not including the parameter values for other rapidity intervals. We also show the parameter values for full phase space for both the functions in Table V. A comparison of the values in Tables I and II reveals that χ^2/ndf values become worse as we go from lower rapidity to higher rapidity range in both the distributions. However the χ^2/ndf values for the Tsallis distributions are again lower by several orders, confirming that Tsallis distribution fits the data far better than Weibull.

From Tables III, IV and V, we also observe that for Weibull distribution, as expected, λ values increase with energy as well as with rapidity. Similarly for Tsallis

TABLE III. Parameters of Weibull and Tsallis functions for $|y| < 0.5$.

	Weibull	→		Tsallis		→		
Energy (GeV)	k	λ	χ^2/ndf	nV	nv_0	K	q	χ^2/ndf
14	1.37 ± 0.030	2.46 ± 0.043	17.54/11	1.053 ± 0.102	0.554 ± 0.046	3.941 ± 0.437	1.173 ± 0.162	13.96/9
22	1.41 ± 0.031	2.53 ± 0.046	4.30/11	1.455 ± 0.198	0.263 ± 0.156	4.474 ± 0.520	1.326 ± 0.324	3.16/9
34.8	1.40 ± 0.014	2.72 ± 0.021	25.92/12	1.678 ± 0.108	0.144 ± 0.081	3.919 ± 0.195	1.338 ± 0.205	14.65/10
44	1.35 ± 0.023	2.95 ± 0.043	5.33/14	1.490 ± 0.168	0.347 ± 0.251	3.076 ± 0.219	1.358 ± 0.317	4.04/12
91 A	1.25 ± 0.074	3.31 ± 0.166	6.28/16	1.365 ± 0.053	0.443 ± 0.018	2.218 ± 0.457	1.164 ± 0.103	5.23/14
91 D	1.20 ± 0.013	3.39 ± 0.052	37.92/18	1.518 ± 0.078	0.227 ± 0.078	1.924 ± 0.071	1.122 ± 0.108	35.79/16

TABLE IV. Parameters of Weibull and Tsallis functions for $|y| < 2.0$.

	Weibull	→		Tsallis		→		
Energy (GeV)	k	λ	χ^2/ndf	nV	nv_0	K	q	χ^2/ndf
14	2.58 ± 0.035	8.58 ± 0.052	101.91/22	5.514 ± 0.362	0.170 ± 0.064	22.18 ± 2.093	1.361 ± 0.358	75.92/20
22	2.44 ± 0.033	9.82 ± 0.066	51.06/25	5.097 ± 0.349	0.285 ± 0.082	12.80 ± 0.729	1.326 ± 0.316	12.07/23
34.8	2.28 ± 0.015	10.72 ± 0.037	213.92/31	5.580 ± 0.179	0.138 ± 0.041	8.931 ± 0.188	1.219 ± 0.171	27.36/29
44	2.21 ± 0.023	11.77 ± 0.073	74.78/31	5.626 ± 0.025	0.179 ± 0.011	7.946 ± 0.248	1.252 ± 0.029	37.83/29
91 A	1.96 ± 0.039	14.96 ± 0.220	61.48/42	5.537 ± 0.050	0.294 ± 0.018	4.708 ± 0.255	1.448 ± 0.131	23.38/40
91 D	2.02 ± 0.009	15.73 ± 0.053	756.61/45	5.692 ± 0.150	0.289 ± 0.051	5.015 ± 0.052	1.439 ± 0.217	292.72/43

TABLE V. Parameters of Weibull and Tsallis function for full rapidity window.

	Weibull	→		Tsallis	→			
Energy (GeV)	k	λ	χ^2/ndf	nV	nv_0	K	q	χ^2/ndf
14	3.42 ± 0.049	10.22 ± 0.055	105.60/10	6.436 ± 0.306	0.195 ± 0.035	130.15 ± 7.575	1.0001 ± 0.00004	32.62/8
22	3.61 ± 0.050	12.49 ± 0.069	122.73/11	5.863 ± 0.346	0.443 ± 0.324	100.42 ± 23.631	1.002 ± 0.001	10.05/9
34.8	3.73 ± 0.024	14.87 ± 0.036	611.21/15	8.490 ± 0.229	0.238 ± 0.024	54.21 ± 3.128	1.016 ± 0.014	38.35/13
44	3.62 ± 0.040	16.64 ± 0.072	148.15/16	9.091 ± 0.421	0.273 ± 0.048	39.15 ± 2.710	1.164 ± 0.127	7.01/14
91 A	3.67 ± 0.114	23.87 ± 0.284	34.97/19	10.633 ± 1.024	0.360 ± 0.114	23.06 ± 2.217	1.345 ± 0.273	5.71/17
91 D	4.21 ± 0.025	24.63 ± 0.076	1524/22	12.331 ± 0.266	0.205 ± 0.027	24.56 ± 0.476	1.361 ± 0.197	106.21/20

TABLE VI. Parameters of the modified Weibull function for different rapidity intervals and full rapidity window for 91 GeV.

Energy	$\|y\|$ interval	k_1	λ_1	k_2	λ_2	χ^2/ndf	p value
91 A	0.5	1.336 ± 0.142	4.067 ± 0.396	1.853 ± 0.333	2.758 ± 0.403	2.41/14	0.9998
91 A	1.0	1.947 ± 0.105	5.428 ± 0.261	1.785 ± 0.181	10.653 ± 0.529	2.78/28	0.9999
91 A	1.5	2.243 ± 0.089	8.122 ± 0.227	2.293 ± 0.116	16.544 ± 0.356	11.77/34	0.9999
91 A	2.0	2.481 ± 0.100	11.413 ± 0.300	2.781 ± 0.170	21.197 ± 0.420	11.35/40	0.9999
91 A	full $\|y\|$	3.674 ± 0.197	25.002 ± 0.350	5.643 ± 0.632	19.997 ± 0.868	13.23/17	0.8780
91 D	0.5	1.351 ± 0.022	4.253 ± 0.076	1.725 ± 0.100	2.539 ± 0.102	10.97/16	0.8113
91 D	1	1.864 ± 0.026	5.315 ± 0.053	2.001 ± 0.002	11.510 ± 0.059	70.05/30	0.0001
91 D	1.5	1.785 ± 0.014	13.05 ± 0.08	2.731 ± 0.063	7.516 ± 0.112	247.3/39	0.0001
91 D	2	2.154 ± 0.017	17.35 ± 0.084	3.064 ± 0.061	9.841 ± 0.120	144.5/43	0.0001
91 D	full $\|y\|$	4.113 ± 0.03	26.14 ± 0.05	6.275 ± 0.112	18.38 ± 0.167	582.6/20	0.0001

TABLE VII. Parameters of the modified Tsallis function for different rapidity intervals and full rapidity window for 91 GeV.

Energy	$\|y\|$ interval	nV_1	nv_{0_1}	nV_2	nv_{0_2}	K_1	K_2
91 A	0.5	1.648 ± 0.878	0.302 ± 0.170	3.899 ± 1.682	0.180 ± 0.155	8.807 ± 2.418	7.403 ± 4.288
91 A	1	2.704 ± 0.285	0.385 ± 0.192	6.041 ± 1.309	0.313 ± 0.231	8.067 ± 2.295	7.869 ± 2.758
91 A	1.5	4.504 ± 0.094	0.174 ± 0.025	4.037 ± 0.163	0.422 ± 0.037	3.914 ± 0.314	50.150 ± 30.71
91 A	2	5.354 ± 0.044	0.276 ± 0.015	28.397 ± 5.352	0.445 ± 0.187	5.056 ± 0.133	5.598 ± 0.687
91 A	full $\|y\|$	11.044 ± 1.411	0.293 ± 0.117	20.630 ± 3.613	0.117 ± 0.180	87.083 ± 70.631	53.714 ± 23.121
91 D	0.5	1.822 ± 0.224	0.116 ± 0.280	5.845 ± 0.772	3.528 ± 0.212	0.249 ± 0.074	7.814 ± 0.704
91 D	1	2.273 ± 0.175	0.445 ± 0.302	6.203 ± 0.392	6.499 ± 0.268	0.236 ± 0.065	8.307 ± 0.471
91 D	1.5	4.679 ± 0.231	0.133 ± 0.056	7.786 ± 0.288	6.801 ± 0.298	0.742 ± 0.430	14.511 ± 0.751
91 D	2	5.504 ± 0.230	0.318 ± 0.063	8.457 ± 0.251	8.709 ± 0.303	0.669 ± 0.054	22.262 ± 1.305
91 D	full $\|y\|$	9.867 ± 0.387	0.414 ± 0.044	33.642 ± 1.508	10.591 ± 1.025	0.577 ± 0.153	20.031 ± 0.390

TABLE VIII. q values of the modified Tsallis function for different rapidity intervals and full rapidity window for 91 GeV.

Energy	$\|y\|$ interval	q_1	q_2	χ^2/ndf	p value
91 A	0.5	1.025 ± 0.022	1.629 ± 0.526	2.31/10	0.9934
91 A	1	1.214 ± 0.211	1.989 ± 0.698	2.17/24	1.000
91 A	1.5	1.024 ± 0.001	1.649 ± 0.248	12.38/30	0.9981
91 A	2	1.064 ± 0.025	1.819 ± 0.899	4.88/36	1.000
91 A	full $\|y\|$	1.056 ± 0.492	1.957 ± 0.703	3.02/13	0.9979
91 D	0.5	1.279 ± 0.433	1.261 ± 0.547	10.62/12	0.5617
91 D	1	1.012 ± 0.155	1.989 ± 0.387	26.07/26	0.4593
91 D	1.5	1.292 ± 0.240	1.467 ± 0.560	43.07/35	0.1642
91 D	2	1.551 ± 0.465	1.257 ± 0.516	40.55/39	0.4019
91 D	full $\|y\|$	1.014 ± 0.108	1.947 ± 0.616	49.29/16	0.0001

distribution, the q value which measures the entropic index of the Tsallis statistics, increases with energy and is more than one in every case. This confirms the nonextensivity of the Tsallis statistics.

B. Approach II

It was observed [18] that the multiplicity distributions have a shoulderlike structure at high energies. The Tsallis and the Weibull distributions both give very high χ^2/ndf values and do not describe the data well at high energy. In our previous publication [14] we suggested to adopt the Giovannini's approach [18] whereby the multiplicity distribution is obtained by using the weighted superposition of two distributions; one accounting for the 2-jet events and another for multi-jet events. For the present work, we use this approach on both Tsallis and Weibull distributions. We call these modified Tsallis and modified Weibull distributions. The probability functions for the two cases are given in Eqs. (8) and (11). Using the modified distributions, data at 91 GeV only has been analyzed, since the shoulder structure starts showing up prominently from this energy. $\alpha = 0.657$ in the two equations is the 2-jet fraction derived from the DURHAM algorithm, as explained in references [19,20].

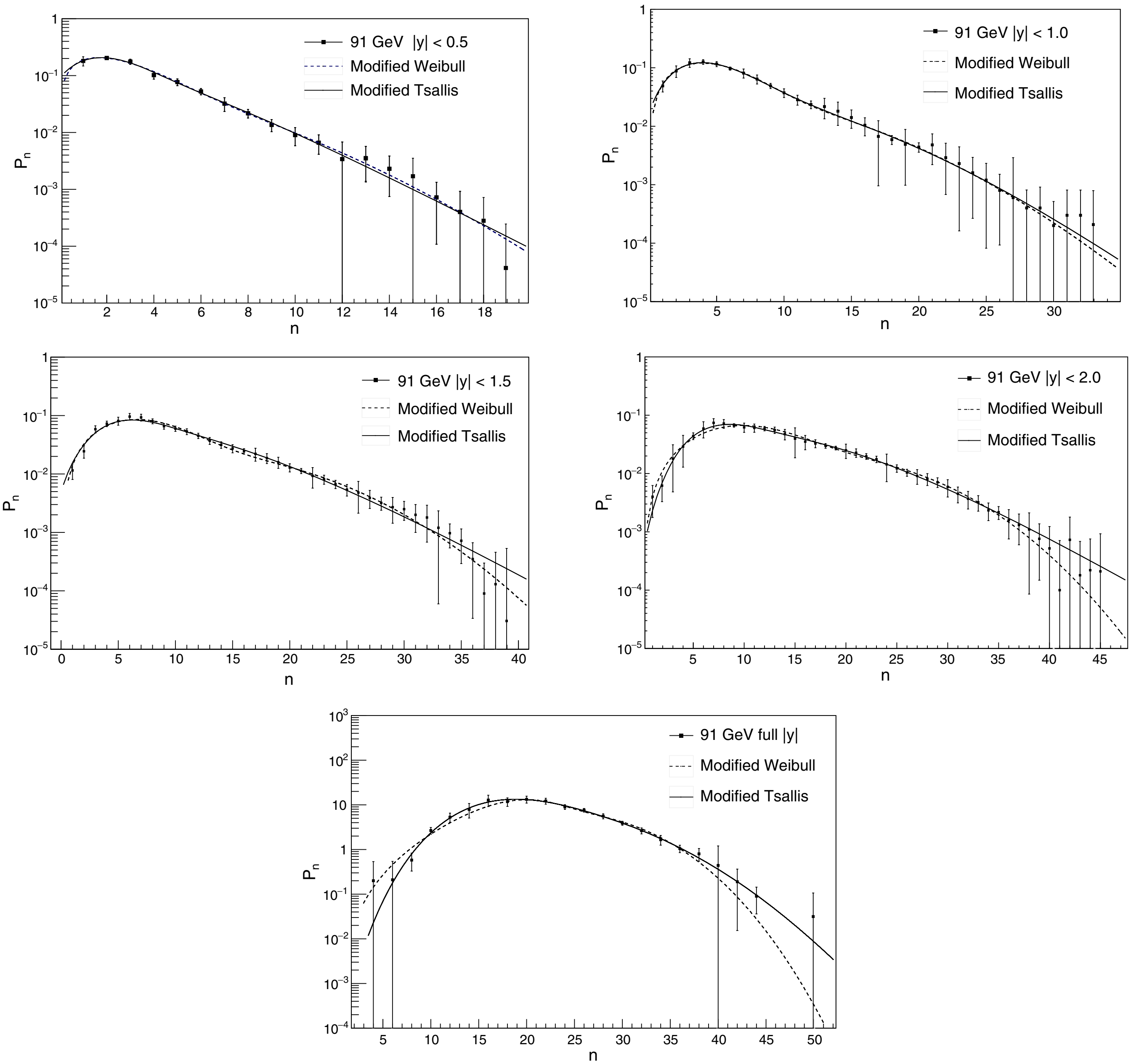

FIG. 4. Multiplicity distributions at $\sqrt{s} = 91$ GeV for the modified Tsallis distribution and the modified Weibull distribution for $|y| < 0.5$, $|y| < 1.0$, $|y| < 1.5$, $|y| < 2.0$ and full phase space from top to bottom.

Using approach II for $\sqrt{s} = 91$ GeV, data from ALEPH and DELPHI Collaborations are analyzed for various rapidity intervals as well as in the full phase space. We use these data as the shoulder structure is prominent at this energy and the data in different rapidity intervals are available. Fit parameters, χ^2/ndf and p values for both modified Weibull and modified Tsallis distributions in the four rapidity intervals and full phase space are given in Table VI, VII and VIII. Figure 4 shows the comparison of distributions for ALEPH data only. Data from DELPHI are not included in the fits to avoid cluttering of points. It may be observed from Fig. 4 and the three tables that by using this approach, the fits to the data improve enormously and the χ^2/ndf values decrease substantially. Both the Modified Weibull distribution and the modified Tsallis distribution describe the ALEPH data at 91 GeV well. However the DELPHI data at 91 GeV is statistically excluded with CL $< 0.1\%$ for the modified Weibull distribution for rapidity intervals with $|y| > 1$ and in full phase space. The modified Tsallis distribution describes well the DELPHI data in all rapidity intervals but fails in full phase space.

The fit procedure uses ROOT 5.36 from CERN to minimize the χ^2 using the library MINUIT2. The fitting procedure for Tsallis distribution involves four free parameters, namely nV, nv_0, K, q and a normalization constant. Several options of minimization (MIGRAD, MINOS, FUMILI) had to be tried to get the covariance matrix positive definite. Also n,V and v_0 are correlated, we had to choose nV and nV_0 as free parameters for meaningful fits independent of the starting values of fit parameters. In addition, nV_0 must be constrained to avoid a nonphysical situation or else the fitting may give negative values for nV_0 which is meaningless. In effect, $(V - Nv_0)$ should be constrained to be positive since Nv_0 can not be larger than V. In the Weibull distribution, the fit procedure is more straightforward without involving such constraints. However, for both modified Tsallis and modified Weibull distributions, the fit parameters are doubled. Minimization then leads to larger errors on the fit parameters, especially in $K2$ and $q2$ for Tsallis and $\lambda 2$ and $K2$ for Weibull. This may be the reason for very large p values, particularly close to unity.

IV. CONCLUSION

Detailed analysis of the data on e^+e^- collisions at energies $\sqrt{s} = 14$ to 91 GeV has been done by considering the recently proposed Weibull distribution in comparison to the Tsallis distribution. It is observed that for both Weibull and Tsallis distributions, the data at 14, 34.8 GeV and 91 GeV from DELPHI Collaboration, are statistically excluded in most of the rapidity intervals as well as in the full phase space. At other energies, the confidence level of the Weibull fit in the large rapidity intervals as well as in the full phase space remains CL $< 0.1\%$ and hence is statistically excluded, while the Tsallis distribution gives good results for all of the cases with CL $> 0.1\%$.

The shape parameter k in the Weibull distribution affects the shape of the distribution. Within a given rapidity interval, the value of k decreases slightly with increasing energy. However, it increases considerably from smaller to larger rapidity intervals, as can be seen in Tables III and IV. This behavior is related to the soft gluon emission and subsequent hadronization. The scale parameter λ of the Weibull distribution measures the width of the distribution. The larger the scale parameter, the more spread out the distribution is. This again is observed in Tables III and IV. The width of the probability distribution depends upon c.m. energy. At higher collision energies, the mean multiplicity increases and so does the number of high multiplicity events. As a result, λ is expected to increase to take into account the width of distributions. This trend is endorsed by λ values in Tables III–IV. Similar results can be observed from the modified Weibull fit distribution parameters in Table VI where λ values increase systematically from lower to higher rapidity windows. In the Tsallis distribution, the K parameter measures the deviation from Poisson distribution and is related to the variance. The definition of K is motivated by the k parameter of a negative binomial distribution, as given by Eq. (6). Tsallis statistics for $q > 1$ with excluded volume v_0 produces the multiplicity distributions that are wider than the Boltzmann-Gibbs ones. In some analyses the excluded volume is fixed between 0.3–0.4 fm^3. The corresponding value of volume V then varies from a few fm^3 to a few tens of fm^3 [7].

It is known that the multiplicity distributions at higher energies show a shoulder structure. In order to improve upon both Weibull and Tsallis fits to the data, we propose to build the multiplicity distribution by a convolution of 2-jet component and the multijet component. For the energy point at 91 GeV, the 2-jet fraction values calculated from various jet algorithms are available. We show that by appropriately weighting the multiplicity distribution with the 2-jet fraction obtained from the DURHAM algorithm for $\sqrt{s} = 91$ GeV, both Tsallis and Weibull distributions describe the data by the ALEPH experiment, very well, giving the statistically significant results. The Tsallis distribution reproduces the data well with CL $> 0.1\%$ in all rapidity windows for the data from two experiments, ALEPH and DELPHI, with an exception of failure with the DELPHI data in the full phase space. The modified Weibull distribution, also improves the fits by several orders, but fails to describe the DELPHI data in most of the rapidity windows and in the full phase space, as observed from p values in Table VI.

In the fitting of the PDFs, modified Weibull fits have four free parameters while modified Tsallis distribution has eight free parameters. Due to limited number of data points, fit parameters suffer from large errors, especially in Tsallis distribution. While the Weibull has the advantage of a smaller number of fit parameters offering a simplistic description in comparison to the Tsallis, yet the performance of Tsallis can not be undermined. It is pertinent to mention that DELPHI data has nearly five times fewer events than the ALEPH data. Thus for the subtle conclusion to be drawn, the analysis of data at higher energies from different experiments is desirable. The q value known as the entropic index in Tsallis distribution, accounts for the nonextensive thermostatistical effects in hadron production and is expected to be more than one. In the results presented in Tables III–V and VIII, the q values are found to be greater than one at all energies confirming that the Tsallis q-statistics has nonextensive behavior of entropy in both restricted rapidity intervals as well as in full phase space. We shall soon extend our analysis and comparison at higher energies using data from the large electron positron, in full phase space and by calculating the 2-jet fraction from various jet algorithms.

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