Higher-order scheme-independent series expansions of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} in conformal field theories

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We study a vectorial asymptotically free gauge theory, with gauge group G and N_f massless fermions in a representation R of this group, that exhibits an infrared (IR) zero in its beta function, β, at the coupling $\alpha = \alpha_{\mathrm{IR}}$ in the non-Abelian Coulomb phase. For general G and R, we calculate the scheme-independent series expansions of (i) the anomalous dimension of the fermion bilinear, $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$, to $O(\Delta_f^4)$ and (ii) the derivative $\beta' = d\beta/d\alpha$, to $O(\Delta_f^5)$, both evaluated at α_{IR}, where Δ_f is an N_f-dependent expansion variable. These are the highest orders to which these expansions have been calculated. We apply these general results to theories with $G = \mathrm{SU}(N_c)$ and R equal to the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations. It is shown that for all of these representations, $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$, calculated to the order Δ_f^p, with $1 \le p \le 4$, increases monotonically with decreasing N_f and, for fixed N_f, is a monotonically increasing function of p. Comparisons of our scheme-independent calculations of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} are made with our earlier higher n-loop values of these quantities, and with lattice measurements. For $R = F$, we present results for the limit $N_c \to \infty$ and $N_f \to \infty$ with N_f/N_c fixed. We also present expansions for α_{IR} calculated to $O(\Delta_f^4)$.

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I. INTRODUCTION

An important advance in the understanding of quantum field theory was the realization that the properties of a theory depend on the Euclidean energy/momentum scale μ at which they are measured. This is of particular interest in an asymptotically free non-Abelian gauge theory, in which the running gauge coupling $g(\mu)$ and the associated quantity $\alpha(\mu) = g(\mu)^2/(4\pi)$ approach zero at large μ in the deep ultraviolet (UV). We shall consider a theory of this type, with gauge group G and N_f massless fermions ψ_j, $j = 1, \ldots, N_f$, in a representation R of G. The dependence of $\alpha(\mu)$ on μ is described by the renormalization-group (RG) [1] beta function, $\beta = d\alpha(\mu)/dt$, where $dt = d\ln\mu$. The condition that the theory be asymptotically free implies that N_f must be less than a certain value, N_u, given below in Eq. (2.4). Since $\alpha(\mu)$ is small at large μ, one can self-consistently calculate β as a power series in $\alpha(\mu)$. As μ decreases from large values in the UV to small values in the infrared (IR), $\alpha(\mu)$ increases. A situation of special interest occurs if the beta function has a zero at some value away from the origin. For a given G and R, this can happen for sufficiently large N_f, while still in the asymptotically free regime. In this case, as μ decreases from large values in the UV toward $\mu = 0$ in the IR, the coupling increases but approaches the value of α at this zero in the beta function, which is thus denoted α_{IR}. Since $\beta = 0$ at $\alpha = \alpha_{\mathrm{IR}}$, the resultant theory in this IR limit is scale-invariant, and generically also conformally invariant [2,3]. A fundamental question concerns the properties of the interacting theory at such an IR fixed point (IRFP) of the renormalization group. There is convincing evidence that if α_{IR} is small enough, then the IR theory is in a (deconfined) non-Abelian Coulomb phase (NACP), also called the conformal window [4]. In terms of N_f, this phase occurs if N_f is in the interval $N_{f,cr} < N_f < N_u$, where N_u and $N_{f,cr}$ depend on G and R. Here, $N_{f,cr}$ denotes the value of N_f below which the running $\alpha(\mu)$ becomes large enough to cause spontaneous chiral symmetry breaking and dynamical fermion mass generation.

Physical quantities in the IR-limit theory at α_{IR} cannot depend on the scheme used for the regularization and subtraction procedure in renormalization. In conventional computations of these quantities, first, one expresses them as series expansions in powers of α, calculated to n-loop order; second, one computes the IR zero of the beta function at the n-loop ($n\ell$) level, denoted $\alpha_{\mathrm{IR},n\ell}$; and third, one sets $\alpha = \alpha_{\mathrm{IR},n\ell}$ in the series expansion for the given quantity to obtain its value at the IR zero of the beta function to this n-loop order. However, these conventional series expansions in powers of α, calculated to a finite order, are scheme-dependent beyond the leading one or two terms. Specifically, the terms in the beta function are scheme-dependent at loop order $\ell \ge 3$ and the terms in an anomalous dimension are scheme-dependent at loop order $\ell \ge 2$ [5]. Indeed, as is well known, the presence of scheme dependence in higher-order perturbative calculations is a general property in quantum field theory.

It is therefore of great value to use a complementary approach in which one expresses these physical quantities at $\alpha_{\rm IR}$ as an expansion in powers of a variable such that, at every order in this expansion, the result is scheme-independent. A very important property is that one can recast the expressions for physical quantities in a manner that is scheme-independent. A crucial point here is that, for a given gauge group G and fermion representation R, as N_f (formally generalized from non-negative integers to the real numbers) approaches the upper limit allowed by asymptotic freedom, denoted N_u [given by Eq. (2.4) below], the resultant value of $\alpha_{\rm IR}$ approaches zero. This means that one can equivalently express a physical quantity in a scheme-independent manner as a series in powers of the variable

$$\Delta_f = N_u - N_f = \frac{11 C_A}{4 T_f} - N_f, \qquad (1.1)$$

where C_A is the quadratic Casimir invariant for the adjoint representation, and T_f is the trace invariant for the fermion representation R [6]. Here, $\alpha_{\rm IR} \to 0 \Leftrightarrow \Delta_f \to 0$. Hence, for N_f less than, but close to N_u, this expansion variable Δ_f is reasonably small, and one can envision reliable perturbative calculations of physical quantities at this IR fixed point in powers of Δ_f. Following the original calculations of the one- and two-loop coefficients of the beta function [7–9], some early work on this was reported in [10,11].

In this paper we consider a vectorial, asymptotically free gauge theory and present scheme-independent calculations, for a general gauge group G and fermion representation R, of two physical quantities in the IR theory at $\alpha_{\rm IR}$ of considerable importance, namely (i) the anomalous dimension, denoted $\gamma_{\bar\psi\psi,{\rm IR}}$, of the (gauge-invariant) fermion bilinear $\bar\psi\psi = \sum_{j=1}^{N_f} \bar\psi_j \psi_j$ to $O(\Delta_f^4)$ and (ii) the derivative $\beta'_{\rm IR} = d\beta/d\alpha$ to $O(\Delta_f^5)$, both evaluated at $\alpha = \alpha_{\rm IR}$. These are the highest orders in powers of Δ_f to which these quantities have been calculated. We give explicit expressions for these quantities in the special cases where $G = {\rm SU}(N_c)$ and the fermion representation R is the fundamental (F), adjoint (adj), and symmetric and antisymmetric rank-2 tensors, (S_2, A_2). Our results extend our previous scheme-independent calculations of $\gamma_{\bar\psi\psi,{\rm IR}}$ to $O(\Delta_f^3)$ in [12] and of the derivative $\beta'_{\rm IR}$ to $O(\Delta_f^4)$ in [13] for general G and R, and our scheme-independent calculation of $\gamma_{\bar\psi\psi,{\rm IR}}$ to $O(\Delta_f^4)$ for $G = {\rm SU}(3)$ and $R = F$ in [14] (see also [15]). A brief report on some of our results was given in [16].

Scheme-independent series expansions of $\gamma_{\bar\psi\psi,{\rm IR}}$ and $\beta'_{\rm IR}$ can be written as

$$\gamma_{\bar\psi\psi,{\rm IR}} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j \qquad (1.2)$$

and

$$\beta'_{\rm IR} = \sum_{j=1}^{\infty} d_j \Delta_f^j, \qquad (1.3)$$

where $d_1 = 0$ for all G and R [12–14]. In general, the calculation of the coefficient κ_j in Eq. (1.2) requires, as inputs, the values of the b_ℓ for $1 \le \ell \le j+1$ and the c_ℓ for $1 \le \ell \le j$. The calculation of the coefficient d_j in Eq. (1.3) requires, as inputs, the values of the b_ℓ for $1 \le \ell \le j$. We refer the reader to [12,13] for discussions of the procedure for calculating the coefficients κ_j and d_j. We denote the truncation of these series to maximal power $j = p$ as $\gamma_{\bar\psi\psi,{\rm IR},\Delta_f^p}$ and $\beta'_{{\rm IR},\Delta_f^p}$, respectively. Where it is necessary for clarity, we will also indicate the fermion representation R in the subscript.

Our main new results here include the general expressions, for arbitrary gauge group G and fermion representation R, for the coefficient, κ_4 in Eq. (3.5) below, and for the coefficient d_5, given in Eq. (4.9) below, as well as reductions of these formulas for special cases and, for $R = F$, calculations in the LNN limit (3.21). As will be discussed further below, the derivative $\beta'_{\rm IR}$ is equivalent to the anomalous dimension of the non-Abelian field strength squared, ${\rm Tr}(F_{\mu\nu}F^{\mu\nu})$. Our present calculations make use of the newly computed five-loop coefficient in the beta function for this gauge theory for general G and R in [17], as our work in [14,15] made use of the calculation of this five-loop coefficient for the case $G = {\rm SU}(3)$ and $R = F$ in [18].

In addition to being of interest and value in their own right, our new scheme-independent calculations, performed to the highest order yet achieved, are useful in several ways. First, we will compare our results for $\gamma_{\bar\psi\psi,{\rm IR}}$ and $\beta'_{\rm IR}$ for various G and R with the values that we obtained at comparable order with the conventional n-loop approach in [19–21]. Our new results have the merit of being scheme-independent at each order in Δ_f, in contrast to scheme-dependent series expansions of $\gamma_{\bar\psi\psi,{\rm IR}}$ and $\beta'_{\rm IR}$ in powers of the IR coupling. Second, there is, at present, an intensive program to study this IR behavior on the lattice [22]. Thus, it is of considerable interest to compare our scheme-independent results for $\gamma_{\bar\psi\psi,{\rm IR}}$ for various theories with values measured in lattice simulations of these theories. We have done this in [13,14,16] (as well as in our work on conventional n-loop calculations [15,19]), and we will expand upon this comparison here. Third, we believe that our scheme-independent expansions for these physical quantities are of interest in the context of the great current resurgence of research activity on conformal field theories (CFT). Much of this current activity makes use of operator-product expansions and the associated bootstrap approach [23]. Our method of scheme-independent series expansions for physical quantities at an IR fixed point is complementary to this bootstrap approach in yielding information about a conformal field theory.

Our calculations rely on $\alpha_{\rm IR}$ being an exact zero of the beta function and thus an exact IR fixed point of the renormalization group, and this property holds in the non-Abelian Couloumb phase (conformal window). In this phase, the chiral symmetry associated with the massless fermions is preserved in the presence of the gauge interaction. However, there has also been interest in vectorial asymptotically free gauge theories that exhibit quasiconformal behavior associated with an approximate IRFP in the phase with broken chiral symmetry, which could feature a substantial value of an effective $\gamma_{\bar\psi\psi,\rm IR} \sim O(1)$ [24]. Our scheme-independent calculations are also relevant to this area of research in two ways: (i) if $N_f \lesssim N_{f,cr}$, then the effective values of quantities such as $\gamma_{\bar\psi\psi,\rm IR}$ may be close to the values calculated via the Δ_f expansion from within the NACP; (ii) combining our calculations of $\gamma_{\bar\psi\psi,\rm IR}$ with an upper bound on this anomalous dimension from conformal invariance and an assumption that this bound is saturated as $N_f \searrow N_{f,cr}$ yields an estimate of the value of $N_{f,cr}$. This is useful, since the value of $N_{f,cr}$ for a given G and R is not known exactly at present and is the subject of current investigation, including lattice studies, as discussed further below.

Although most of our paper deals with new scheme-independent results for physical quantities, one of the ouputs of our calculations is a new type of series expansion for a scheme-dependent quantity, namely $\alpha_{\rm IR}$. The conventional procedure for calculating the IR zero of a beta function at the n-loop order, which we have applied in earlier work to four-loop order for arbitrary G and R [19–21] (see also [25]) is to examine the n-loop beta function, which has the form of α^2 times a polynomial of degree $n-1$ in α, and then determine the n-loop value $\alpha_{{\rm IR},n\ell}$ as the (real, positive) root of this polynomial closest to the origin. However, in [15], we investigated the five-loop beta function for $G = {\rm SU}(3)$ and $R = F$, as calculated in the standard $\overline{\rm MS}$ scheme, and found that, over a substantial range of values of N_f in the non-Abelian Coulomb phase, it does not have any positive real root. We were able to circumvent this problem in [15] by the use of Padé approximants, but nevertheless, it is a complication for this conventional approach to calculating $\alpha_{\rm IR}$. The new calculation of $\alpha_{\rm IR}$ as an expansion in powers of Δ_f up to $O(\Delta_f^4)$ for general G and R that we present here has the advantage that it always yields a physical value, in contrast to the situation with the n-loop beta function.

The paper is organized as follows. Some relevant background and methods are discussed in Sec. II. We present our calculation of κ_4 in the scheme-independent expansion of $\gamma_{\bar\psi\psi,\rm IR}$ for general G and R in Sec. III, together with evaluations for $G = {\rm SU}(N_c)$ and $R = F, adj, S_2$, and A_2. These are compared with values from n-loop calculations and with lattice measurements. In this section we also present results for the case $R = F$ in the limit $N_c \to \infty$, $N_f \to \infty$, with N_f/N_c fixed, which we call the LNN limit. In Sec. IV we present our calculation of the coefficient d_5 in the scheme-independent expansion of $\beta'_{\rm IR}$ for general G and R, with evaluations for the above-mentioned specific representations. Section V gives an analysis of the five-loop rescaled beta function in the LNN limit and a determination of the interval over which it exhibits a physical IR zero. Section VI is devoted to the calculation of the coefficients in an expansion of $\alpha_{\rm IR}$ in powers of Δ_f up to $O(\Delta_f^4)$. Our conclusions are given in Sec. VII, and some auxiliary formulas are listed in the Appendix.

II. BACKGROUND AND METHODS

In this section we review some background and methods relevant for our calculations. The series expansion of β in powers of α is

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\alpha}{4\pi}\right)^\ell \tag{2.1}$$

where b_ℓ is the ℓ-loop coefficient. For a general operator $\mathcal{O}$, we denote the full scaling dimension as $D_\mathcal{O}$ and its free-field value as $D_{\mathcal{O},\rm free}$. The anomalous dimension of this operator, denoted $\gamma_\mathcal{O}$, is defined via the relation [26]

$$D_\mathcal{O} = D_{\mathcal{O},\rm free} - \gamma_\mathcal{O}. \tag{2.2}$$

An operator of particular interest is the (gauge-invariant) fermion bilinear, $\bar\psi\psi$. The expansion of the anomalous dimension of this operator, $\gamma_{\bar\psi\psi}$, in powers of α is

$$\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\alpha}{4\pi}\right)^\ell, \tag{2.3}$$

where c_ℓ is the ℓ-loop coefficient. As noted above, the coefficients b_1, b_2, and c_1 are scheme-independent, while the b_ℓ with $\ell \ge 3$ and the c_ℓ with $\ell \ge 2$ are scheme-dependent [5]. For a general gauge group G and fermion representation R, the coefficients b_1 and b_2 were calculated in [7,8], and b_3 and b_4 were calculated in [27,28] (and checked in [29]) in the commonly used $\overline{\rm MS}$ scheme [30]. For $G = {\rm SU}(3)$ and $R = F$, b_5 was calculated in [18] and recently, an impressive calculation of b_5 for general gauge group G and fermion representation R was presented in [17], again in the $\overline{\rm MS}$ scheme. We also make use of the c_ℓ up to loop order $\ell = 4$, calculated in [31]. Although we use these coefficients as calculated in the $\overline{\rm MS}$ scheme below, we emphasize that the main results of this paper are calculations of the quantities κ_4 and d_5 which, like all of the κ_j and d_j, are scheme-independent. We denote the n-loop β, β', and $\gamma_{\bar\psi\psi}$ as $\beta_{n\ell}$, $\beta'_{n\ell}$, and $\gamma_{\bar\psi\psi,n\ell}$. As discussed above, we denote the IR zero of $\beta_{n\ell}$ as $\alpha_{{\rm IR},n\ell}$, and the corresponding evaluations of $\beta'_{n\ell}$ and $\gamma_{\bar\psi\psi,n\ell}$ at $\alpha_{\rm IR,n\ell}$ as $\beta'_{{\rm IR},n\ell}$ and $\gamma_{\bar\psi\psi,{\rm IR},n\ell}$. The symbols $\alpha_{\rm IR}$, $\gamma_{\bar\psi\psi,{\rm IR}}$, and $\beta'_{\rm IR}$ refer to the exact values of these quantities.

For a given G and R, as N_f increases, b_1 decreases through positive values and vanishes with sign reversal at $N_f = N_u$, with

$$N_u = \frac{11C_A}{4T_f}, \tag{2.4}$$

where C_A and T_f are group invariants [6,32]. Hence, the asymptotic freedom condition yields the upper bound $N_f < N_u$.

There is a range of $N_f < N_u$ where $b_2 < 0$, so the two-loop beta function has an IR zero, at the value

$$\alpha_{\rm IR,2\ell} = -\frac{4\pi b_1}{b_2}. \tag{2.5}$$

The n-loop beta function has a double UV zero at $\alpha = 0$ and $n-1$ zeros away from the origin. Among the latter zeros of the beta function, the smallest (real, positive) zero, if there is such a zero, is the physical IR zero, $\alpha_{{\rm IR},n\ell}$, of $\beta_{n\ell}$. As N_f decreases below N_u, b_2 passes through zero to positive values as N_f decreases through

$$N_\ell = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}. \tag{2.6}$$

Hence, with N_f formally extended from nonnegative integers to nonnegative real numbers [32], $\beta_{2\ell}$ has an IR zero (IRZ) for N_f in the interval

$$I_{\rm IRZ}\colon\ N_\ell < N_f < N_u. \tag{2.7}$$

Thus, N_ℓ is the lower (ℓ) end of this interval [33]

As N_f decreases in this interval, $\alpha_{\rm IR,2\ell}$ increases. Therefore, in order to investigate the IR zero of the beta function for N_f toward the middle and lower part of $I_{\rm IRZ}$ with reasonable accuracy, one requires higher-loop calculations. These were performed in [34,35], [19–21], [15,25] for $\alpha_{{\rm IR},n\ell}$ and for the anomalous dimension of the fermion bilinear operator (see also [36,37]). Since the b_ℓ with $\ell \ge 3$ are scheme-dependent, it is necessary to determine the degree of sensitivity of the value obtained for $\alpha_{{\rm IR},n\ell}$ for $n \ge 3$ to the scheme used for the calculation. This was done in [38–41].

The nonanomalous global flavor symmetry of the theory is

$$G_{fl} = {\rm SU}(N_f)_L \otimes {\rm SU}(N_f)_R \otimes {\rm U}(1)_V. \tag{2.8}$$

This G_{fl} symmetry is preserved in the (deconfined) non-Abelian Coulomb phase. As in [12–16], we focus on this phase in the present work, since both the expansion in a small $\alpha_{\rm IR}$ and the scheme-independent expansion in powers of Δ_f start from the upper end of the interval $I_{\rm IRZ}$ in this phase. In contrast, in the phase with confinement and spontaneous chiral symmetry breaking, the gauge interaction produces a bilinear fermion condensate, $\langle\bar\psi\psi\rangle$, and this breaks G_{fl} to ${\rm SU}(N_f)_V \otimes {\rm U}(1)_V$, where ${\rm SU}(N_f)_V$ is the diagonal subgroup of ${\rm SU}(N_f)_L \otimes {\rm SU}(N_f)_R$.

We will consider the flavor-nonsinglet (fns) and flavor-singlet (fs) bilinear fermion operators $\sum_{j,k=1}^{N_f} \bar\psi_j (T_a)_{jk}\psi_k$ and $\sum_{j=1}^{N_f} \bar\psi_j\psi_j$, where here T_a with $a = 1,\ldots,N_f^2-1$ is a generator of the global flavor group ${\rm SU}(N_f)$. We will usually suppress the explicit flavor indices and thus write these operators as $\bar\psi T_a\psi$ and $\bar\psi\psi$. These have the same anomalous dimension (e.g., [42,43]), which we denote simply as the anomalous dimension for the flavor-singlet operator, $\gamma_{\bar\psi\psi}$. In vectorial gauge theories of the type considered here, these fermion bilinear operators are gauge-invariant, and hence the anomalous dimension $\gamma_{\bar\psi\psi}$ and its IR value, $\gamma_{\bar\psi\psi,{\rm IR}}$, are physical. (In contrast, in a chiral gauge theory, fermion bilinears are generically not gauge-invariant, and hence neither are their anomalous dimensions.)

Since $\alpha_{\rm IR}$ vanishes (linearly) with Δ_f as $\Delta_f \to 0$, we can express it as a series expansion in this variable, Δ_f. We thus write

$$\alpha_{\rm IR} \equiv 4\pi a_{\rm IR} = 4\pi \sum_{j=1}^{\infty} a_j \Delta_f^j. \tag{2.9}$$

The calculation of the a_j requires, as input, the b_ℓ with $1 \le \ell \le j+1$ [12,13].

A basic question concerns the part of the interval $I_{\rm IRZ}$ in which the series expansions for $\gamma_{\bar\psi\psi,{\rm IR}}$ and $\beta'_{\rm IR}$ in Eqs. (1.2) and (1.3) are reliable. We analyzed this question in [12–14,16] and concluded that these expansions for $\gamma_{\rm IR}$ and $\beta'_{\rm IR}$ should be reasonably reliable throughout much of the interval $I_{\rm IRZ}$ and non-Abelian Coulomb phase. We will use our higher-order calculations in this paper to extend this analysis here. We recall that the properties of the theory change qualitatively as N_f decreases through the value $N_{f,cr}$ and spontaneous chiral symmetry breaking occurs, with the fermions gaining dynamical masses. The (chirally symmetric) non-Abelian Coulomb phase with $N_{f,cr} < N_f < N_u$ is clearly qualitatively different from the confined phase with spontaneous chiral symmetry breaking at smaller N_f below $N_{f,cr}$. Therefore, one does not, in general, expect the small-Δ_f series expansion to hold below $N_{f,cr}$. Estimating the range of applicability of this expansion is thus connected with estimating the value of $N_{f,cr}$. For general G and R, as N_f, formally continued from the nonnegative integers to the non-negative real numbers, decreases from the upper end of the interval $I_{\rm IRZ}$ at N_u to the lower end of this interval at $N_f = N_\ell$, Δ_f increases from 0 to the maximal value

$$(\Delta_f)_{\max} = N_u - N_\ell = \frac{3C_A(7C_A + 11C_f)}{4T_f(5C_A + 3C_f)} \quad \text{for } N_f \in I_{\rm IRZ}. \tag{2.10}$$

Recall that for a function $f(z)$ that is analytic about $z = 0$ and has a Taylor series expansion

$$f(z) = \sum_{j=1}^{\infty} f_j z^j, \tag{2.11}$$

the radius of convergence of this series, z_c, can be determined by the ratio test

$$z_c = \lim_{j\to\infty} \frac{|f_{j-1}|}{|f_j|}. \tag{2.12}$$

Of course, we cannot apply the full ratio test here, since we have only calculated the κ_j and d_j to finite order. However, we can get a rough measure of the range of applicability of the series expansions in Δ_f (and also Δ_r in the LNN limit [21] discussed below) by computing the ratios κ_{j-1}/κ_j and d_{j-1}/d_j for the values of j for which we have calculated these coefficients.

The series expansion (1.2) for $\gamma_{\rm IR}$ starts at $\Delta_f = 0$, i.e., at the upper end of the non-Abelian Coulomb phase, and extends downward through this phase. Given that the theory at $\alpha_{\rm IR}$ in this phase is conformal, there is an upper bound from conformal invariance, namely [44]

$$\gamma_{\bar\psi\psi,{\rm IR}} \le 2. \tag{2.13}$$

We have used this in our earlier work [12–16,19] and we will apply it with our higher-order calculations here. As discussed in [19], in the phase with spontaneous chiral symmetry breaking (SχSB), there is a similar upper bound, $\gamma_{\bar\psi\psi,{\rm IR}} < 2$. This follows from the requirement that if $m(k)$ is the momentum-dependent running dynamical mass generated in association with the SχSB, then $\lim_{k\to\infty} m(k) = 0$ (see Eqs. (4.1)-(4.2) of [19]). Thus, if the approximate calculation of the anomalous dimension of a given quantity at a fixed value of Δ_f, computed up to order Δ_f^p, yields a value greater than 2, then we can infer that the perturbative calculation is not applicable at this value of Δ_f or equivalently, N_f.

In particular, this can give information on the extent of the non-Abelian Coulomb phase and the value of $N_{f,cr}$. The application of this bound is particularly powerful in the context of our present scheme-independent calculations because we find that the κ_j in Eq. (1.2) are positive for all of the representations considered here, and hence, for a given p, $\gamma_{{\rm IR},\Delta_f^p}$ is a monotonically increasing function of Δ_f or equivalently it increases monotonically as N_f decreases from its upper limit, N_u. If one assumes that $\gamma_{\rm IR}$ saturates its upper bound, (2.13) and if a calculation of $\gamma_{\rm IR}$ is reliable in the regime where it is approaching 2 from below, then one can, in principle, determine the value of $N_{f,cr}$, where $\gamma_{\rm IR}$ reaches this upper bound after approaching it from below. In this context, it should be mentioned that in a supersymmetric (vectorial) gauge theory (SGT) with N_f pairs of massless chiral superfields transforming according the representations R and $\bar R$ of a gauge group G, the exact expression for $\gamma_{\rm IR}$ is known [45,46], and (i) it increases monotonically with decreasing N_f in the NACP; and (ii) it saturates its upper bound (which, in the SGT case is $\gamma_{\rm IR,SGT} \le 1$) at the lower end of the non-Abelian Coulomb phase. Specifically, in this supersymmetric gauge theory, the upper and lower ends of the NACP occur at [32]

$$N_{u,{\rm SGT}} = \frac{3C_A}{2T_f}, \tag{2.14}$$

and

$$N_{\ell,{\rm SGT}} = \frac{3C_A}{4T_f} = \frac{N_u}{2}, \tag{2.15}$$

and

$$\gamma_{\bar\psi\psi,{\rm IR,SGT}} = \frac{3C_A - 2T_f N_f}{2T_f N_f} = \frac{N_u}{N_f} - 1 = \frac{\frac{2T_f}{3C_A}\Delta_f}{1 - \frac{2T_f}{3C_A}\Delta_f}. \tag{2.16}$$

Thus, $\gamma_{\bar\psi\psi,{\rm IR,SGT}}$ increases from 0 to 1 as N_f decreases from $N_{u,{\rm SGT}}$ to $N_{\ell,{\rm SGT}}$. However, it is not known if this saturation occurs in the nonsupersymmetric case. In practice, we are only able to apply this test in an approximate manner because for a given G and R, as N_f decreases toward the lower part of $I_{\rm IRZ}$, the ratio test already shows that higher-order terms in the Δ_f expansion are becoming increasingly non-negligible, so that the truncation of the infinite series (1.2) to maximal power $p = 4$ involves an increasingly great uncertainty, as does an extrapolation to $p = \infty$.

For some perspective, we note that in order to asses the accuracy of the Δ_f expansion, the coefficients $\kappa_{j,{\rm SGT}}$ were calculated for $j = 1, 2$ in [12] and were found to be in perfect agreement with the corresponding Taylor series expansion of the exact expression (2.16). This check was carried to one higher order in [16] for the case $G = {\rm SU}(N_c)$ and $R = F$ with a calculation of $\gamma_{{\rm IR,SGT},\Delta_f^3}$, and again, perfect agreement was found with the exact result. This agreement explicitly demonstrated the scheme independence of the $\kappa_{j,{\rm SGT}}$, since the calculations were carried out using inputs computed in the $\overline{\rm DR}$ scheme, while (2.16) was derived in the NSVZ scheme [45]. Furthermore, as a consequence of electric-magnetic duality [46], as $N_f \searrow N_{\ell,{\rm SGT}}$ in the non-Abelian Coulomb phase, the physics is described by a magnetic theory with coupling strength going to zero, or equivalently, by an electric theory with divergent $\alpha_{\rm IR}$. Therefore, this perfect agreement, order-by-order, between the $\kappa_{j,{\rm SGT}}$ and the expansion of the exact expression (2.16) for $\gamma_{\rm IR,SGT}$ in powers of Δ_f, showed that the Δ_f expansion in this supersymmetric gauge theory is able to treat situations with strong, as well as weak, coupling. This could not be done with conventional perturbative series expansions in powers of α [36,37].

III. CALCULATION OF $\gamma_{\bar\psi\psi,\rm IR}$ TO $O(\Delta_f^4)$

A. General G and R

The coefficients κ_j in the scheme-independent expansion of $\gamma_{\bar\psi\psi,\rm IR}$ in powers of Δ_f, Eq. (1.2), contain important information about the theory. For a general asymptotically free vectorial gauge theory with gauge group G and N_f massless fermions in a representation R, the coefficients κ_j were given in [12] up to order $j=3$, yielding the expansion of $\gamma_{\bar\psi\psi,\rm IR}$ to order Δ_f^3. It is convenient to define

$$D = 7C_A + 11C_f, \tag{3.1}$$

since this factor occurs repeatedly in denominators of various expressions. For reference, we list the κ_j for $1 \le j \le 3$ below:

$$\kappa_1 = \frac{8C_fT_f}{C_AD}, \tag{3.2}$$

$$\kappa_2 = \frac{4C_fT_f^2(5C_A+88C_f)(7C_A+4C_f)}{3C_A^2D^3}, \tag{3.3}$$

and

$$\begin{aligned}\kappa_3 = \frac{4C_fT_f}{3^4C_A^4D^5}\bigg[&3C_AT_f^2(-18473C_A^4+144004C_A^3C_f+650896C_A^2C_f^2+356928C_AC_f^3+569184C_f^4)\\ &-2560T_f^2D\frac{d_A^{abcd}d_A^{abcd}}{d_A}+45056C_AT_fD\frac{d_R^{abcd}d_A^{abcd}}{d_A}-170368C_A^2D\frac{d_R^{abcd}d_R^{abcd}}{d_A}\\ &+33\cdot 2^{10}D\left(2T_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}-13C_AT_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}+11C_A^2\frac{d_R^{abcd}d_R^{abcd}}{d_A}\right)\zeta_3\bigg].\end{aligned} \tag{3.4}$$

Here, $\zeta_s = \sum_{n=1}^\infty n^{-s}$ is the Riemann zeta function, the quantities C_A, C_f, and T_f are group invariants, the contractions $d_A^{abcd}d_A^{abcd}$, $d_R^{abcd}d_A^{abcd}$, $d_R^{abcd}d_R^{abcd}$ are additional group-theoretic quantities given in [28], and d_A is the dimension of the adjoint representation of G. In [12,13], the expression for κ_3 was given with terms written in order of descending powers of C_A. It is also useful to express this coefficient κ_3 in an equivalent form that renders certain factors of D explicit and shows the simple factorization of terms multiplying ζ_3, and we have done this in Eq. (3.4).

Our new result here for κ_4 for a general gauge group G and fermion representation R is

$$\begin{aligned}\kappa_4 = \frac{T_f^2}{3^5C_A^5D^7}\bigg[&C_AC_fT_f^2(19515671C_A^6-131455044C_A^5C_f+1289299872C_A^4C_f^2+2660221312C_A^3C_f^3\\ &+1058481072C_A^2C_f^4+6953709312C_AC_f^5+1275715584C_f^6)+2^{10}C_fT_f^2D(5789C_A^2-4168C_AC_f-6820C_f^2)\frac{d_A^{abcd}d_A^{abcd}}{d_A}\\ &-2^{10}C_AC_fT_fD(41671C_A^2-125477C_AC_f-53240C_f^2)\frac{d_R^{abcd}d_A^{abcd}}{d_A}\\ &-2^8\cdot 11^2C_A^2C_fD(2569C_A^2+18604C_AC_f-7964C_f^2)\frac{d_R^{abcd}d_R^{abcd}}{d_A}\\ &-2^{14}\cdot 3C_AT_f^2D^3\frac{d_R^{abcd}d_A^{abcd}}{d_R}+2^{13}\cdot 33C_A^2T_fD^3\frac{d_R^{abcd}d_R^{abcd}}{d_R}\\ &+2^8D\bigg[-3C_AC_fT_f^2D(4991C_A^4-17606C_A^3C_f+33240C_A^2C_f^2-30672C_AC_f^3+9504C_f^4)\\ &-2^4C_fT_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(17206C_A^2-60511C_AC_f-45012C_f^2)+40C_AC_fT_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(35168C_A^2-154253C_AC_f-88572C_f^2)\\ &-88C_A^2C_f\frac{d_R^{abcd}d_R^{abcd}}{d_A}(973C_A^2-93412C_AC_f-56628C_f^2)+1440C_AT_f^2D^2\frac{d_R^{abcd}d_A^{abcd}}{d_R}-7920C_A^2T_fD^2\frac{d_R^{abcd}d_R^{abcd}}{d_R}\bigg]\zeta_3\\ &+\frac{4505600C_AC_fD^2}{d_A}[-4T_f^2d_A^{abcd}d_A^{abcd}+2T_fd_R^{abcd}d_A^{abcd}(10C_A+3C_f)+11C_Ad_R^{abcd}d_R^{abcd}(C_A-3C_f)]\zeta_5\bigg].\end{aligned} \tag{3.5}$$

Here, d_R is the dimension of the fermion representation R. As before, we have indicated the simple factors in the prefactor and, for sufficiently simple cases, also factorizations of numbers in numerator terms. We will follow the same format for indicating numerical factorizations below. We proceed to evaluate this general expression for the gauge group $G = \mathrm{SU}(N_c)$ and several specific fermion representations R, namely the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor. As stated in the Introduction, we will use the abbreviations F, adj, S_2, and A_2 to refer to these representations. It is also worthwhile to evaluate our general formulas for other gauge groups and their representations, including orthogonal, symplectic, and exceptional groups. We will report these evaluations for other groups and their representations elsewhere. There has, indeed, been interest in conformal phases for theories with these other gauge groups [47].

The coefficients κ_1 and κ_2 are manifestly positive for all G and R. For $G = \mathrm{SU}(N_c)$ with all physical N_c, and for representations $R = F$, adj, S_2, we have found that κ_3 and κ_4 are also positive [12–16]. As one of the results in the present paper, we generalize this further to include $R = A_2$. That is, for all physical N_c and for all of these representations, we find that $\kappa_j > 0$ for $j = 3$, 4 as well as the manifestly positive cases $j = 1$, 2. Thus, extending our previous discussion in [12–16], the property that, for all of these representations R, $\kappa_j > 0$ for $1 \le j \le 4$ and for all N_c implies two important monotonicity results: (i) for these R, and with a fixed p in the interval $1 \le p \le 4$, $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ is a monotonically increasing function of Δ_f, i.e., it increases monotonically with decreasing N_f; and (ii) for these R, and with a fixed $N_f \in I_{\mathrm{IRZ}}$, $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ is a monotonically increasing function of p in the range $1 \le p \le 4$. In addition to the manifestly positive κ_1 and κ_2 and the κ_3 and κ_4 that we have shown to be positive, a plausible conjecture is that, for these R, $\kappa_j > 0$ for all $j \ge 5$. Assuming that this conjecture is valid, then three consequences are that for these representations R, (iii) for fixed N_f, $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ is a monotonically increasing function of p for all p; (iv) $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ is a monotonically increasing function of Δ_f, i.e. it increases with decreasing N_f, for all p; and hence (v) [assuming that the infinite series (1.2) converges] the quantity $\gamma_{\bar\psi\psi,\mathrm{IR}}$ defined by this infinite series, and equivalent to $\lim_{p\to\infty}\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$, is a monotonically increasing function of Δ_f, i.e., it increases monotonically with decreasing N_f.

B. $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^4}$ for $G = \mathrm{SU}(N_c)$ and $R = F$

An important special case is $G = \mathrm{SU}(N_c)$ with R being the fundamental representation. For this case, the general expression for the interval I_{IRZ}, Eq. (2.7), is [32]

$$I_{\mathrm{IRZ}}\colon\ \frac{34N_c^3}{13N_c^2-3} < N_f < \frac{11N_c}{2} \quad \text{for } R = F. \tag{3.6}$$

The factor D in Eq. (3.1) has the explicit form

$$D = \frac{25N_c^2-11}{2N_c} \quad \text{for } R = \text{fund}. \tag{3.7}$$

The general results for κ_p with $1 \le p \le 3$ in (3.2)–(3.4) from [12] take the following forms given in [13]:

$$\kappa_{1,F} = \frac{4(N_c^2-1)}{N_c(25N_c^2-11)} \tag{3.8}$$

$$\kappa_{2,F} = \frac{4(N_c^2-1)(9N_c^2-2)(49N_c^2-44)}{3N_c^2(25N_c^2-11)^3} \tag{3.9}$$

and

$$\kappa_{3,F} = \frac{8(N_c^2-1)}{3^3N_c^3(25N_c^2-11)^5}[(274243N_c^8 - 455426N_c^6 - 114080N_c^4 + 47344N_c^2 + 35574) - 4224N_c^2(4N_c^2-11)(25N_c^2-11)\zeta_3]. \tag{3.10}$$

For $\kappa_{4,F}$, we have [16]

$$\begin{aligned}\kappa_{4,F} = \frac{4(N_c^2-1)}{3^4N_c^4(25N_c^2-11)^7}[&(263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 \\ &- 290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352) \\ &+ 384(25N_c^2-11)(4400N_c^{10} - 123201N_c^8 + 480349N_c^6 - 486126N_c^4 + 84051N_c^2 + 1089)\zeta_3 \\ &+ 211200N_c^2(25N_c^2-11)^2(N_c^6 + 3N_c^4 - 16N_c^2 + 22)\zeta_5].\end{aligned} \tag{3.11}$$

We have checked that when we substitute the value $N_c = 3$ in our expression for $\kappa_{4,F}$ in Eq. (3.11), the result agrees with our previous calculation of $\kappa_{4,F}$ for this case in Eq. (9) of Ref. [14].

The explicit numerical expressions for the scheme-independent series expansions of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ to order Δ_f^4 for $R=F$ and $N_c = 2, 3, 4$ are as follows:

$$\mathrm{SU}(2):\ \gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4} = \Delta_f[0.067416 + (0.73308\times10^{-2})\Delta_f + (0.60531\times10^{-3})\Delta_f^2 + (1.62662\times10^{-4})\Delta_f^3] \tag{3.12}$$

$$\mathrm{SU}(3):\ \gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4} = \Delta_f[0.049844 + (0.37928\times10^{-2})\Delta_f + (0.23747\times10^{-3})\Delta_f^2 + (0.36789\times10^{-4})\Delta_f^3] \tag{3.13}$$

and

$$\mathrm{SU}(4):\ \gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4} = \Delta_f[0.038560 + (0.22314\times10^{-2})\Delta_f + (0.11230\times10^{-3})\Delta_f^2 + (0.126505\times10^{-4})\Delta_f^3]. \tag{3.14}$$

In these equations,

$$\Delta_f = \frac{11N_c}{2} - N_f \quad \text{for } R=F. \tag{3.15}$$

Plots of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ for $N_c=2$ and $N_c=3$ and $1\le p\le 4$ were given in [16]. These showed the two monotonicity properties mentioned above. For an extended comparison, we show the plots of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ for $2\le N_c\le 4$ and $1\le p\le 4$ in Figs. 1–3.

In Table I we list the values of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ for $1\le p\le 4$ for the SU(2), SU(3), and SU(4) theories, with N_f in the respective interval I_{IRZ} for each. For comparison, we also include the values of $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$ obtained with our earlier n-loop calculations in [19], using series expansions in powers of α evaluated at $\alpha=\alpha_{\mathrm{IR},n\ell}$ for $1\le n\le 4$ with b_3 and b_4 and c_n, $2\le n\le 4$ calculated in the $\overline{\mathrm{MS}}$ scheme. (See Table VI in [19] for a list of numerical values of values of $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$.) As discussed above, if, for a given N_c and N_f, a calculated value of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ violates the upper bound $\gamma_{\bar\psi\psi,\mathrm{IR}}\le 2$ in (2.13), this is unphysical (marked with a symbol "u" in Table I) and indicates that the perturbative calculation is not applicable for this N_f. In the case of the n-loop values $\gamma_{\mathrm{IR},n\ell}$, if this occurs at the two-loop level, it also leads to caution concerning $\gamma_{\mathrm{IR},n\ell}$ for $n=3, 4$, and this is similarly indicated with a "u". The computations of $\gamma_{\mathrm{IR},n\ell}$ in [19,25] made use of the b_n and c_n up to the $n=4$ loop level, where the scheme-dependent b_3, b_4, and c_n with $2\le n\le 4$ had been calculated in the widely used $\overline{\mathrm{MS}}$ scheme [27–29,31]. As we pointed out in [15], the five-loop beta function in the $\overline{\mathrm{MS}}$ scheme does not exhibit a physical IR zero over a substantial lower part of I_{IRZ}. We discuss this further below. For compact notation, we will often leave the subscript $\bar\psi\psi$ implicit on these and other quantities and thus write $\gamma_{\bar\psi\psi,\mathrm{IR}}\equiv\gamma_{\mathrm{IR}}$, $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}\equiv\gamma_{\mathrm{IR},n\ell}$, etc. From Eqs. (2.4) and (2.6) it follows that the respective lower and upper ends of the intervals I_{IRZ} for these theories are $(N_u, N_\ell) = (5.55, 11)$, (8.05, 16.5), and (10.61, 22) for SU(2), SU(3), and SU(4), and hence the physical intervals I_{IRZ} are $6\le N_f\le 10$ for SU(2), $9\le N_f\le 16$ for SU(3), and $11\le N_f\le 21$ for SU(4).

Since the calculation of κ_j and the resultant $\gamma_{\mathrm{IR},\Delta_f^j}$ uses information from the $(j+1)$-loop beta function from (2.1) and the j-loop expansion of $\gamma_{\bar\psi\psi}$ in (2.3), it is natural to compare the (SI) $\gamma_{\mathrm{IR},\Delta_f^p}$ with the (SD) $\gamma_{\mathrm{IR},p'\ell}$ for $p'=p$ and

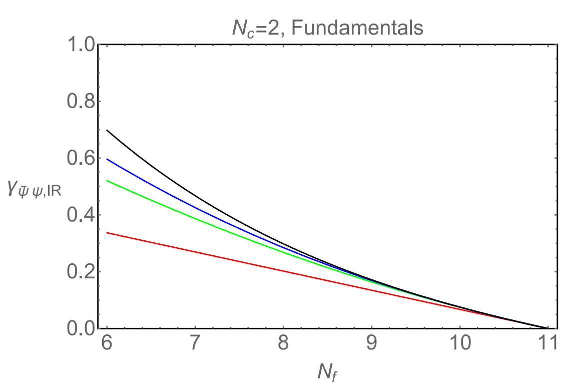

FIG. 1. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ (labeled as $\gamma_{\bar\psi\psi,\mathrm{IR}}$ on the vertical axis in this and subsequent graphs) for $N_c=2$, i.e., $G=\mathrm{SU}(2)$, and $1\le p\le 4$ as a function of $N_f\in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4}$ (black).

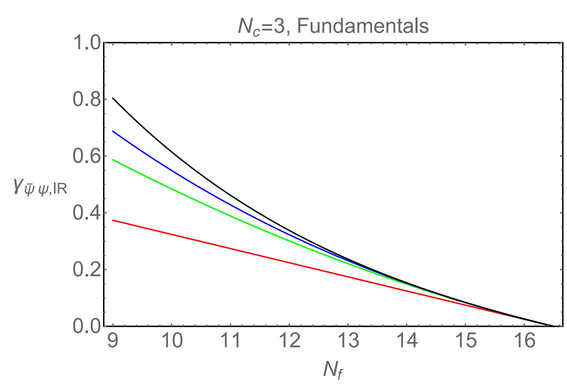

FIG. 2. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ for $N_c=4$ and $1\le p\le 4$ as a function of $N_f\in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4}$ (black).

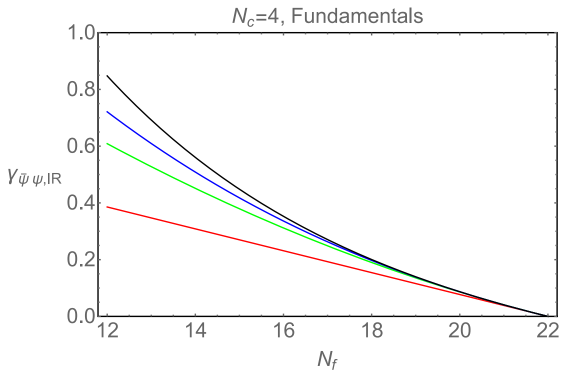

FIG. 3. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^p}$ for $N_c = 3$ and $1 \le p \le 4$ as a function of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,\mathrm{IR},F,\Delta_f^4}$ (black).

$p' = p + 1$. In the upper and middle part of the interval I_{IRZ} for a given N_c, we find that $\gamma_{\mathrm{IR},\Delta_f^4}$ is slightly larger than $\gamma_{\mathrm{IR},4\ell}$, with the difference increasing as N_f decreases below N_u, i.e., as Δ_f increases.

It is important to assess the range of applicability and reliability of these results from the Δ_f expansion. We did this in [12–14] and extend our analysis here, using our new result for κ_4. Following our discussion above on the ratio test for the determination of the radius of convergence of a Taylor series, the ratios of successive coefficients, κ_{j-1}/κ_j, give an approximate measure of the range of applicability of the Δ_f expansion for γ_{IR}. For a given G and R, this range may be compared with the maximum size of Δ_f in the interval I_{IRZ} where the scheme-independent two-loop beta function $\beta_{2\ell}$ has an IR zero. For the present case of $G = \mathrm{SU}(N_c)$ and $R = F$, the general formula (2.10) takes the form

$$R = F\colon\ (\Delta_f)_{\max} = \frac{3N_c(25N_c^2 - 11)}{2(13N_c^2 - 3)}. \tag{3.16}$$

This has the respective values

$$(\Delta_f)_{\max} = 5.45, 8.45, 11.39 \quad \text{for } N_c = 2, 3, 4. \tag{3.17}$$

We begin by reviewing the SU(3) theory, for which

$$\mathrm{SU}(3)\colon\ \frac{\kappa_{F,1}}{\kappa_{,F,2}} = 13.14, \qquad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 15.97, \qquad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 6.455. \tag{3.18}$$

As discussed in [12–14], these results suggest that for the SU(3) theory with $R = F$, the Δ_f expansion calculated to this order should be reasonably reliable over a substantial part, including the upper and middle portions, of the interval I_{IRZ} and the non-Abelian Coulomb phase.

Using our new results, we now extend this analysis to the SU(2) and SU(4) theories [and will give a further analysis in the LNN limit of Eq. (3.21)]. We find

$$\mathrm{SU}(2)\colon\ \frac{\kappa_{F,1}}{\kappa_{,F,2}} = 9.20, \qquad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 12.11, \qquad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 3.72 \tag{3.19}$$

and

$$\mathrm{SU}(4)\colon\ \frac{\kappa_{F,1}}{\kappa_{,F,2}} = 17.28, \qquad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 19.87, \qquad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 8.88. \tag{3.20}$$

Since $(\Delta_f)_{\max}$ has the respective values 5.45 and 11.39 for the SU(2) and SU(4) theories, we are led to the same conclusion for these theories that we reached for the SU(3) theory, namely that the Δ_f expansion should be reasonably reliable over a substantial portion of the respective intervals I_{IRZ}.

As discussed above, another way to assess the range of applicability of the Δ_f expansion is to check to see whether the resultant values of $\gamma_{\mathrm{IR},\Delta_f^p}$ obey the upper bound $\gamma_{\mathrm{IR}} \le 2$ in (2.13). As is evident from Table I, all of our values of $\gamma_{\mathrm{IR},\Delta_f^p}$ listed there obey this bound. This again shows the advantages of the scheme-independent Δ_f expansion as a way of calculating γ_{IR} to a given order, as compared with the conventional n-loop calculation of $\gamma_{\mathrm{IR},n\ell}$. As is also evident from Table I for each of the cases listed there, namely $N_c = 2, 3, 4$, one finds unphysically large values of $\gamma_{\mathrm{IR},n\ell}$ for values of N_f in the lower portions of the respective intervals I_{IRZ}. In [19] and later works we explained this as a consequence of the fact that, for a given G and R, as N_f decreases toward N_ℓ in the interval I_{IRZ}, the coupling α_{IR} increases from weak toward strong coupling. Thus, toward the lower end of the respective intervals I_{IRZ}, the IR coupling $\alpha_{\mathrm{IR},n\ell}$ become too large for the perturbative n-loop calculations of $\gamma_{\mathrm{IR},n\ell}$ to be applicable. In contrast, the Δ_f expansion can be applied over a considerably greater portion of the interval I_{IRZ} to yield results for $\gamma_{\mathrm{IR},\Delta_f^p}$ that obey the upper bound (2.13). We will show this further below for the LNN limit (3.21). This also demonstrates that the Δ_f expansion for γ_{IR} is able to be used in situations with substantially stronger IR coupling than is the case with the conventional expansion in powers of this coupling yielding the n-loop value $\gamma_{\mathrm{IR},n\ell}$.

We proceed to compare our values in Table I with lattice measurements. The SU(3) theory with $R = F$ and $N_f = 12$ has been the subject of many lattice measurements. In [14],

TABLE I. Values of the anomalous dimension $\gamma_{\bar{\psi}\psi,\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, i.e., $\gamma_{\bar{\psi}\psi,\mathrm{IR},F,\Delta_f^p}$, with $1 \le p \le 4$, for $G = \mathrm{SU}(N_c)$, as a function of N_c and N_f for $2 \le N_c \le 4$ and N_f in the respective intervals I_{IRZ} for each N_c. For comparison, we also include the n-loop values $\gamma_{\bar{\psi}\psi,\mathrm{IR},F,n\ell}$ with $2 \le n \le 4$ from Table VI of [19]. Values that exceed the bound $\gamma_{\bar{\psi}\psi,\mathrm{IR}} \le 2$ in Eq. (2.13) are marked as unphysical (u). For notational brevity in this and successive tables, we omit the subscript $\bar{\psi}\psi$. See text for further details.

N_c	N_f	$\gamma_{\mathrm{IR},F,2\ell}$	$\gamma_{\mathrm{IR},F,3\ell}$	$\gamma_{\mathrm{IR},F,4\ell}$	$\gamma_{\mathrm{IR},F,\Delta_f}$	$\gamma_{\mathrm{IR},F,\Delta_f^2}$	$\gamma_{\mathrm{IR},F,\Delta_f^3}$	$\gamma_{\mathrm{IR},F,\Delta_f^4}$
2	6	u	u	u	0.337	0.520	0.596	0.698
2	7	u	u	u	0.270	0.387	0.426	0.467
2	8	0.752	0.272	0.204	0.202	0.268	0.285	0.298
2	9	0.275	0.161	0.157	0.135	0.164	0.169	0.172
2	10	0.0910	0.0738	0.0748	0.0674	0.07475	0.07535	0.0755
3	9	u	u	u	0.374	0.587	0.687	0.804
3	10	u	u	u	0.324	0.484	0.549	0.615
3	11	1.61	0.439	0.250	0.274	0.389	0.428	0.462
3	12	0.773	0.312	0.253	0.224	0.301	0.323	0.338
3	13	0.404	0.220	0.210	0.174	0.221	0.231	0.237
3	14	0.212	0.146	0.147	0.125	0.148	0.152	0.153
3	15	0.0997	0.0826	0.0836	0.0748	0.0833	0.0841	0.0843
3	16	0.0272	0.0258	0.0259	0.0249	0.0259	0.0259	0.0259
4	11	u	u	u	0.424	0.694	0.844	1.029
4	12	u	u	u	0.386	0.609	0.721	0.8475
4	13	u	u	u	0.347	0.528	0.610	0.693
4	14	u	u	u	0.308	0.451	0.509	0.561
4	15	1.32	0.420	0.281	0.270	0.379	0.418	0.448
4	16	0.778	0.325	0.269	0.231	0.312	0.336	0.352
4	17	0.481	0.251	0.234	0.193	0.249	0.263	0.2705
4	18	0.301	0.189	0.187	0.154	0.190	0.197	0.200
4	19	0.183	0.134	0.136	0.116	0.136	0.139	0.140
4	20	0.102	0.0854	0.0865	0.0771	0.0860	0.0869	0.0871
4	21	0.0440	0.0407	0.0409	0.0386	0.0408	0.0409	0.0409

we compared our results for this theory with lattice measurements, so we only briefly review that discussion here. We recall that there is not, at present, a consensus among all lattice groups as to whether this theory is in an IR-conformal phase or is in a chirally broken phase [22]. There is a considerable spread of values of γ_{IR} in published papers, including the values (where uncertainties in the last digits are indicated in parentheses) $\gamma_{\mathrm{IR}} \sim 0.414(16)$ [48], $\gamma_{\mathrm{IR}} \simeq 0.35$ [49], $\gamma_{\mathrm{IR}} \simeq 0.4-0.5$ [50], $\gamma_{\mathrm{IR}} = 0.27(3)$ [51], $\gamma_{\mathrm{IR}} \simeq 0.25$ [52] (see also [53]), $\gamma_{\mathrm{IR}} = 0.235(46)$ [54], and $0.2 \lesssim \gamma_{\mathrm{IR}} \lesssim 0.4$ [55]. We refer the reader to [22,48–55] for discussions of estimates of overall uncertainties in these measurements. Our value $\gamma_{\mathrm{IR},\Delta_f^4} = 0.338$ and our extrapolated value for $\lim_{p\to\infty}\gamma_{\mathrm{IR},\Delta_f^p} = \gamma_{\mathrm{IR}}$, namely $\gamma_{\mathrm{IR}} = 0.40$, are consistent with this range of lattice measurements and are somewhat higher than our five-loop value $\gamma_{\mathrm{IR},5\ell} = 0.255$ from the conventional α series that we obtained in [15]. It is hoped that further work by lattice groups will lead to a consensus concerning whether this theory is IR conformal or not and concerning the value of γ_{IR}.

The SU(3) theory with $N_f = 10$ has been investigated on the lattice in [56], with the result $\gamma_{\mathrm{IR}} \sim 1$. While our highest-order n-loop values, namely our four-loop result, $\gamma_{\mathrm{IR},4\ell} = 0.156$ [19], and our five-loop result, $\gamma_{\mathrm{IR},5\ell} = 0.211$ obtained using Padé methods [15], are smaller than this lattice value, our extrapolated scheme-independent value, $\gamma_{\mathrm{IR}} = 0.95 \pm 0.06$ [14], is consistent with it.

There have also been a number of lattice studies of the SU(3) theory with $N_f = 8$ [57–59], which have yielded the estimate $\gamma_{\mathrm{IR}} \simeq 1$. As is evident from Fig. 3, if we were to continue the curve for $\gamma_{\mathrm{IR},\Delta_f^4}$ plotted there downward further to $N_f = 8$, the resultant value would be compatible with $\gamma_{\mathrm{IR}} \sim 1$. We note that this theory may well be in the chirally broken phase, and there is not yet a clear consensus as to whether it is in this phase or possibly near the lower end of the IR-conformal non-Abelian Coulomb phase. In this context, one may recall that if, for a given G and R, $N_f < N_{f,cr}$, so that there is spontaneous chiral symmetry breaking, then the IR zero of the beta function is only approximate, since the theory flows away from this value as the fermions gain dynamical mass and are integrated out, leaving a pure gluonic low-energy effective field theory. For such a theory, the quantity extracted from either continuum or lattice analyses as γ_{IR} is only an effective anomalous dimension that describes the renormalization-group behavior as the theory is flowing near to the approximate zero of the beta function. A general comment is that the determination of $N_{f,cr}$ relies upon effective methods to analyze the lattice data [22]; progress on this continues [48–61].

Theories with an SU(2) gauge group and $N_f = 8$ have been of interest in the context of certain ideas for physics beyond the Standard Model (SM) [62], in which the number of Dirac fermions is $N_f = N_{wk}(N_c + 1) = 8$, where $N_{wk} = 2$, corresponding to the SU(2) factor group in the SM and $N_c = 3$ colors. There have been several lattice studies of this SU(2) theory with $N_f = 8$, including [22,63,64]. These are consistent with this theory being IR-conformal, and the recent study [64] has reported the measurement $\gamma_{\rm IR} = 0.15 \pm 0.02$. For comparison, as listed in Table I, our previous higher n-loop values were $\gamma_{{\rm IR},3\ell} = 0.272$ and $\gamma_{{\rm IR},4\ell} = 0.204$ [19], and our current highest-order scheme-independent value is $\gamma_{{\rm IR},\Delta_f^4} = 0.298$. These are somewhat higher than this lattice result.

There have also been a number of lattice studies of the SU(2) theory with $N_f = 6$ [22,65–67]. From this work, it is not yet clear if this theory is IR-conformal or chirally broken. The authors of Ref. [66] obtained the range $0.26 < \gamma_{\rm IR} < 0.74$, while the authors of Ref. [67] found $\gamma_{\rm IR} \simeq 0.275$. Our higher-order scheme-independent values, as listed in Table I, in particular, $\gamma_{{\rm IR},\Delta_f^4} = 0.698$, are in agreement with the range given in [66] and are somewhat higher than the value from [67].

C. LNN limit for $G = {\rm SU}(N_c)$ and $R = F$

For $G = {\rm SU}(N_c)$ and $R = F$, it is of interest to consider the limit

$$\begin{aligned} &LNN\colon\ N_c \to \infty, \qquad N_f \to \infty \\ &\text{with} \quad r \equiv \frac{N_f}{N_c} \text{ fixed and finite} \\ &\text{and} \quad \xi(\mu) \equiv \alpha(\mu) N_c \text{ is a finite function of } \mu. \end{aligned} \tag{3.21}$$

We will use the symbol $\lim_{LNN}$ for this limit, where "LNN" stands for "large N_c and N_f" with the constraints in Eq. (3.21) imposed. This is also called the 't Hooft-Veneziano limit. Anticipating our later discussion of theories with fermions in two-index representations (adjoint and symmetric and antisymmetric rank-2 tensor), we will use the symbol $\lim_{LN}$, where "LN" stands for "large N_c", to denote the original 't Hooft limit

$$\begin{aligned} &LN\colon\ N_c \to \infty \\ &\text{with} \quad \xi(\mu) \equiv \alpha(\mu) N_c \text{ a finite function of } \mu \end{aligned} \tag{3.22}$$

and N_f fixed and finite.

Continuing our discussion of the LNN limit, as relevant to theories with fermions in the fundamental representation, we define the following quantities in this limit:

$$\xi = 4\pi x = \lim_{LNN} \alpha N_c, \tag{3.23}$$

$$r_u = \lim_{LNN} \frac{N_u}{N_c}, \tag{3.24}$$

and

$$r_\ell = \lim_{LNN} \frac{N_\ell}{N_c}, \tag{3.25}$$

with values

$$r_u = \frac{11}{2} = 5.5 \tag{3.26}$$

and

$$r_\ell = \frac{34}{13} = 2.615 \tag{3.27}$$

(to the indicated floating-point accuracy). With $I_{\rm IRZ}$: $N_\ell < N_f < N_u$, it follows that the corresponding interval in the ratio r is

$$I_{{\rm IRZ},r}\colon\ \frac{34}{13} < r < \frac{11}{2}, \quad \text{i.e.,} \quad 2.615 < r < 5.5. \tag{3.28}$$

The critical value of r such that for $r > r_{cr}$, the LNN theory is IR-conformal and for $r < r_{cr}$, it exhibits spontaneous chiral symmetry breaking, is denoted r_{cr} and is defined as

$$r_{cr} = \lim_{LNN} \frac{N_{f,cr}}{N_c}. \tag{3.29}$$

We define the scaled scheme-independent expansion parameter for the LNN limit

$$\Delta_r \equiv \frac{\Delta_f}{N_c} = r_u - r = \frac{11}{2} - r. \tag{3.30}$$

As r decreases from r_u to r_ℓ in the interval $I_{{\rm IRZ},r}$, Δ_r increases from 0 to a maximal value

$$(\Delta_r)_{\rm max} = r_u - r_\ell = \frac{75}{26} = 2.8846 \quad \text{for } r \in I_{{\rm IRZ},r}. \tag{3.31}$$

We define rescaled coefficients $\hat\kappa_{j,F}$

$$\hat\kappa_{j,F} \equiv \lim_{N_c \to \infty} N_c^j \kappa_{j,F} \tag{3.32}$$

that are finite in this LNN limit. The anomalous dimension $\gamma_{\rm IR}$ is also finite in this limit and is given by

$$R = F:\ \lim_{LNN} \gamma_{\rm IR} = \sum_{j=1}^{\infty} \kappa_{j,F}\Delta_f^j = \sum_{j=1}^{\infty} \hat{\kappa}_{j,F}\Delta_r^j. \quad (3.33)$$

From the results for κ_j, $j = 1, 2, 3$ in [12] or the special cases given above for $G = {\rm SU}(N_c)$ and $R = F$ in Eqs. (3.8)–(3.10), we have

$$\hat{\kappa}_{1,F} = \frac{2^2}{5^2} = 0.1600, \quad (3.34)$$

$$\hat{\kappa}_{2,F} = \frac{588}{5^6} = 0.037632, \quad (3.35)$$

and

$$\hat{\kappa}_{3,F} = \frac{2193944}{3^3 \cdot 5^{10}} = 0.83207 \times 10^{-2}, \quad (3.36)$$

where, as above, we indicate the factorizations of the denominators. (The numerators do not, in general, have such simple factorizations; for example, in $\kappa_{3,F}$, $2193944 = 2^3 \times 274243$.) From our new expression for κ_4, we calculate

$$\hat{\kappa}_{4,F} = \frac{210676352}{3^4 \cdot 5^{13}} + \frac{90112}{3^3 \cdot 5^{10}}\zeta_3 + \frac{11264}{3^3 \cdot 5^8}\zeta_5 = 0.36489 \times 10^{-2}. \quad (3.37)$$

Hence, numerically, to order $O(\Delta_r^4)$,

$$R = F:\ \gamma_{{\rm IR},LNN,\Delta_r^4} = \Delta_r[0.160000 + 0.037632\Delta_r + 0.0083207\Delta_r^2 + 0.003649\Delta_r^3]. \quad (3.38)$$

Using these results for $\gamma_{{\rm IR},F,\Delta_r^p}$ with $1 \le p \le 4$ for $R = F$ in the LNN limit, we can now carry out a polynomial extrapolation to $p = \infty$. To do this, we fit an expression for $\gamma_{{\rm IR},F,\Delta_r^p}$ with some subset of the p terms to a polynomial in $1/p$. We denote the resultant value generically as $\gamma_{{\rm IR},F,s}$, where here s denotes the subset of the p terms used for the extrapolation. We shall use, as a necessary condition for $\gamma_{{\rm IR},F,s}$ to be reliable, the requirement that it not differ too much from the highest-order value, $\gamma_{{\rm IR},F,\Delta_r^4}$. Quantitatively, we require that for the given subset s, $\gamma_{{\rm IR},F,s}/\gamma_{{\rm IR},F,\Delta_r^4} < 1.5$. We find that this condition is satisfied if $r \in I_{{\rm IRZ},r}$ is $r \gtrsim 3.5$, but that it is not satisfied as r decreases below this value toward the lower end of the interval $I_{{\rm IRZ},r}$ at $r_\ell = 2.615$. As an example, at $r = 4.0$, depending on the subset of terms used for the extrapolation, we obtain $\gamma_{{\rm IR},F,s}/\gamma_{{\rm IR},F,\Delta_r^4} \simeq 1.2$, while at $r = 3.6$, this ratio increases to $\simeq 1.4$. We remark that the value $r = 4.0$ corresponds to $N_f = 12$ for the SU(3) theory and $N_f = 8$ for the SU(2) theory.

Previously, in [14] we performed this analysis for the special case $G = {\rm SU}(3)$ and $R = F$ and, for that work, we studied how the extrapolated value depends on the subset of terms that one includes for the fit. We perform the corresponding analysis here for this LNN case. We study three sets of terms:

$$\text{set}_{34}: \{\gamma_{{\rm IR},F,\Delta_r^3}, \gamma_{{\rm IR},F,\Delta_r^4}\} \quad (3.39)$$

$$\text{set}_{234}: \{\gamma_{{\rm IR},F,\Delta_r^2}, \gamma_{{\rm IR},F,\Delta_r^3}, \gamma_{{\rm IR},F,\Delta_r^4}\} \quad (3.40)$$

$$\text{set}_{1234}: \{\gamma_{{\rm IR},F,\Delta_r}, \gamma_{{\rm IR},F,\Delta_r^2}, \gamma_{{\rm IR},F,\Delta_r^3}, \gamma_{{\rm IR},F,\Delta_r^4}\} \quad (3.41)$$

There are countervailing advantages of these sets of terms. The two-term set (3.39) has the advantage of using the two highest-order terms, while the three-term and four-term sets have the advantage of using progressively more terms in the fit. The fits to the sets (3.39)–(3.41) yield polynomials in the variable p^{-1} of the respective forms

$$\text{set}_{34} \Rightarrow \gamma_{{\rm IR},F,ex34,p} = s_{34,0} + s_{34,1}p^{-1} \quad (3.42)$$

$$\text{set}_{234} \Rightarrow \gamma_{{\rm IR},F,ex234,p} = s_{234,0} + s_{234,1}p^{-1} + s_{234,2}p^{-2} \quad (3.43)$$

and

$$\text{set}_{1234} \Rightarrow \gamma_{{\rm IR},F,ex1234,p} = s_{1234,0} + s_{1234,1}p^{-1} + s_{1234,2}p^{-2} + s_{1234,3}p^{-3}. \quad (3.44)$$

The extrapolated values in the limit $p \to \infty$ given by these fits are, respectively, as

$$\lim_{p\to\infty}\gamma_{{\rm IR},F,ex34,p} = s_{34,0} \equiv \gamma_{{\rm IR},F,ex34} \quad (3.45)$$

$$\lim_{p\to\infty}\gamma_{{\rm IR},F,ex234,p} = s_{234,0} \equiv \gamma_{{\rm IR},F,ex234} \quad (3.46)$$

and

$$\lim_{p\to\infty}\gamma_{{\rm IR},F,ex1234,p} = s_{1234,0}. \equiv \gamma_{{\rm IR},F,ex1234} \quad (3.47)$$

We have calculated these quantities analytically. Below, we list the corresponding expressions with coefficients given to the indicated floating-point precision:

$$\gamma_{{\rm IR},F,ex34} = 16.758754 - 11.042531r + 2.8240528r^2 - 0.32942724r^3 + 0.014595750r^4 \quad (3.48)$$

$$\gamma_{{\rm IR},F,ex234} = 27.346053 - 19.2457889r + 5.1985972r^2 - 0.63389228r^3 + 0.0291915006r^4 \quad (3.49)$$

and

TABLE II. Values of the scheme-independent γ_{IR,F,Δ_r^p} in the LNN limit (3.21) for $1 \le p \le 4$, together with $\gamma_{IR,F,n\ell}$ with $n = 2, 3, 4$ from Table V of [21] for comparison, as a function of r for $r \in I_{IRZ,r}$. Values that exceed the bound $\gamma_{IR} \le 2$ are marked as unphysical (u) or placed in parentheses. We also list the extrapolated estimate $\gamma_{IR,F,ex234}$ of $\gamma_{IR,F,\Delta_r^\infty}$ and, in the last column, the ratio $\gamma_{IR,F,ex234}/\gamma_{IR,F,\Delta_r^4}$.

r	$\gamma_{IR,F,2\ell}$	$\gamma_{IR,F,3\ell}$	$\gamma_{IR,F,4\ell}$	γ_{IR,F,Δ_r}	γ_{IR,F,Δ_r^2}	γ_{IR,F,Δ_r^3}	γ_{IR,F,Δ_r^4}	$\gamma_{IR,F,ex234}$	$\frac{\gamma_{IR,F,ex234}}{\gamma_{IR,F,\Delta_r^4}}$
2.8	u	1.708	0.190	0.432	0.706	0.870	1.064	(2.09)	1.96
3.0	u	1.165	0.225	0.400	0.635	0.765	0.908	1.645	1.82
3.2	u	0.854	0.264	0.368	0.567	0.668	0.770	1.28	1.66
3.4	u	0.656	0.293	0.336	0.502	0.579	0.650	0.993	1.53
3.6	1.853	0.520	0.308	0.304	0.440	0.497	0.5445	0.763	1.40
3.8	1.178	0.420	0.306	0.272	0.381	0.422	0.452	0.584	1.29
4.0	0.785	0.341	0.288	0.240	0.325	0.353	0.371	0.444	1.20
4.2	0.537	0.277	0.257	0.208	0.272	0.290	0.300	0.337	1.12
4.4	0.371	0.222	0.217	0.176	0.2215	0.233	0.238	0.253	1.06
4.6	0.254	0.1735	0.1745	0.144	0.1745	0.1805	0.183	0.188	1.03
4.8	0.170	0.129	0.131	0.112	0.130	0.133	0.134	0.135	1.01
5.0	0.106	0.0889	0.0900	0.0800	0.0894	0.09045	0.0907	0.0905	1.00
5.2	0.0562	0.0512	0.0516	0.0480	0.0514	0.0516	0.0516	0.0516	1.00
5.4	0.0168	0.0164	0.0164	0.0160	0.0164	0.0164	0.0164	0.0164	1.00

$$\gamma_{IR,F,ex1234} = 33.901799 - 24.4060664r + 6.71925275r^2 - 0.832708600r^3 + 0.038922001r^4. \quad (3.50)$$

Note that there are strong cancellations between individual terms for relevant values of $r \in I_{IRZ,r}$. Some examples will show the range of resultant values of extrapolations for these different choices of sets of terms used in the fits. As anticipated, for values of r in the upper part of the interval $I_{IRZ,r}$, all of the different types of extrapolation give quite similar results. For example,

$$r = 5.0 \Rightarrow \gamma_{IR,F,ex,34} = 0.0914, \qquad \gamma_{IR,F,ex234} = 0.0902, \qquad \gamma_{IR,F,ex1234} = 0.0905. \quad (3.51)$$

As r decreases in the interval $I_{IRZ,r}$, the differences between the extrapolations using the different sets of terms increase slightly, e.g., for a value roughly in the middle of this interval, namely $r = 4.0$, we find

$$r = 4.0 \Rightarrow \gamma_{IR,F,ex34} = 0.427, \qquad \gamma_{IR,F,ex234} = 0.444, \qquad \gamma_{IR,F,ex1234} = 0.456. \quad (3.52)$$

Toward the lower part of the interval $I_{IRZ,r}$, these differences increase further, but also, as discussed above, for a given r, all of the different types of extrapolations involve greater uncertainties, since each of the extrapolated values differs more from the value of highest-order explicitly calculated quantity, γ_{IR,Δ_r^4}. For example, for $r = 3.0$,

$$r = 3.0 \Rightarrow \gamma_{IR,F,ex34} = 1.335, \qquad \gamma_{IR,F,ex234} = 1.645, \qquad \gamma_{IR,F,ex1234} = 1.826. \quad (3.53)$$

The ratios of these values divided by the highest-order explicitly calculated value, γ_{IR,F,Δ_r^4}, are

$$r = 3.0 \Rightarrow \frac{\gamma_{IR,F,ex34}}{\gamma_{IR,F,\Delta_r^4}} = 1.47, \qquad \frac{\gamma_{IR,F,ex234}}{\gamma_{IR,F,\Delta_r^4}} = 1.82, \qquad \frac{\gamma_{IR,F,ex1234}}{\gamma_{IR,F,\Delta_r^4}} = 2.01. \quad (3.54)$$

Given our fiducial requirement that the ratio of the extrapolated value for $p \to \infty$ divided by the highest-order explicitly calculated value, should not be greater than 1.5 for the extrapolation to be considered reasonably reliable, it follows that we would not consider the latter two extrapolations in Eq. (3.53) to be sufficiently reliable to meet this requirement.

It is interesting to compare these scheme-independent calculations of γ_{IR,F,Δ_r^p} to order $1 \le p \le 4$ with the results from the conventional n-loop calculations as truncated expansions in $\alpha_{IR,F,n\ell}$, denoted $\gamma_{IR,F,n\ell}$ from Table V of [21] up to $n = 4$ loop order. We list our scheme-independent values together with these n-loop values in Table II. For each value of r, we also include the extrapolated value, $\gamma_{IR,F,ex234}$ for the $p \to \infty$ limit, and the ratio $\gamma_{IR,F,ex234}/\gamma_{IR,\Delta_r^4}$. We do not include the results from the $n = 5$ loop conventional calculation, because of the absence of a physical IR zero in the five-loop beta function for $2.615 < r < 4.323$ in $I_{IRZ,r}$. Although the extrapolated values $\gamma_{IR,F,ex234}$ for r values below $r = 3.5$ are included, we caution that these do not satisfy our fiducial criterion for sufficient reliability of extrapolation, since they differ by too much from our highest-order calculated values, γ_{IR,Δ_r^4}. For this reason, although we can roughly apply the method discussed in Sec. II to use

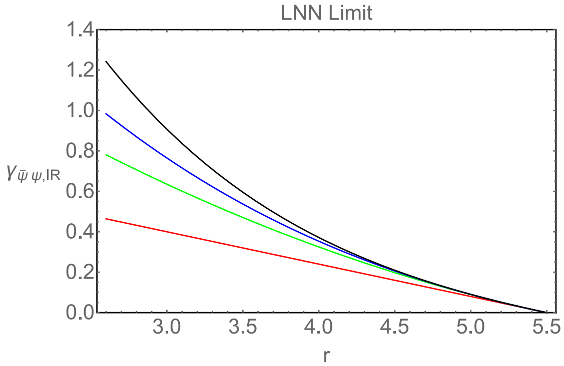

FIG. 4. Plot of $\gamma_{\mathrm{IR},F,\Delta_r^p}$ for $1 \le p \le 4$ as a function of $r \in I_{\mathrm{IRZ},r}$ in the LNN limit (3.21). From bottom to top, the curves (with colors online) refer to $\gamma_{\mathrm{IR},F,\Delta_r}$ (red), $\gamma_{\mathrm{IR},F,\Delta_r^2}$ (green), $\gamma_{\mathrm{IR},F,\Delta_r^3}$ (blue), and $\gamma_{\mathrm{IR},F,\Delta_r^4}$ (black).

the extrapolated value of γ_{IR} to estimate the lower end, r_{cr}, of the IR-conformal non-Abelian Coulomb phase [defined in Eq. (5.3)], this involves a substantial degree of uncertainty. Bearing this caveat in mind, the resulting estimate would be that $r_{cr} \sim 2.7$. If one were to pull back from the LNN limit and multiply this value of r_{cr} by a specific finite value of N_c to get an estimate of the corresponding $N_{f,cr}$, then, for example, for $N_c = 3$, i.e., $G = \mathrm{SU}(3)$, this would yield $N_{f,cr} \sim 8$. This estimate is consistent with the estimate $8 \lesssim N_{f,cr} \lesssim 9$ that we derived from our calculation of $\gamma_{\mathrm{IR},F,\Delta_f^4}$ for this theory and extrapolation to obtain $\lim_{p\to\infty} \gamma_{\mathrm{IR},F,\Delta_f^p}$ in [14]. Clearly, the lower that one goes in N_c away from the LNN limit, the greater is the error in performing this conversion from a specific r value in the LNN limit to a corresponding ratio N_f/N_c with finite N_f and N_c, so we do not perform this conversion for $N_c = 2$.

In Fig. 4 we plot $\gamma_{\mathrm{IR},F,\Delta_r^p}$, i.e., the value of γ_{IR} for $R = F$, calculated to order Δ_r^p with $1 \le p \le 4$, in the scheme-independent expansion, as a function of $r \in I_{\mathrm{IRZ},r}$. As a consequence of the positivity of the $\hat{\kappa}_{p,F}$ in Eqs. (3.34)–(3.36), for a fixed r, $\gamma_{\mathrm{IR},F,\Delta_r^p}$ is a monotonically increasing function of the order of calculation, p. As r decreases toward the lower end of the interval $I_{\mathrm{IRZ},r}$ at $r = r_\ell = 2.615$, the value of γ_{IR} calculated to the highest order in this LNN limit, namely $O(\Delta_r^4)$, is slightly greater than 1.

As we did for specific $\mathrm{SU}(N_c)$ theories above, here we proceed to investigate the range of applicability of the scheme-independent series expansion for γ_{IR} in the LNN limit (see also [68]). As is evident from Table II, all of our values of $\gamma_{\mathrm{IR},F,\Delta_r^p}$ for $1 \le p \le 4$ satisfy the bound $\gamma_{\mathrm{IR}} \le 2$. This is also true for all of our extrapolated values, $\gamma_{\mathrm{IR},F,ex234}$, except for the lowest value of r listed, namely $r = 2.8$, for which $\gamma_{\mathrm{IR},F,ex234} = 2.09$, slightly above this bound. Thus, these results in the LNN limit again demonstrate the advantage of the scheme-independent expansions, since they enable us to calculate self-consistent values of $\gamma_{\mathrm{IR},F,\Delta_r}$ over a greater range of the interval $I_{\mathrm{IRZ},r}$ than is the case with the conventional n-loop calculations. To show the latter in detail, we have explicitly listed the values of $\gamma_{\mathrm{IR},F,3\ell}$ and $\gamma_{\mathrm{IR},F,4\ell}$ for values of r where $\gamma_{\mathrm{IR},F,2\ell}$ was unphysically large.

To investigate the range of applicability of the scheme-independent expansions further, it is worthwhile to obtain an estimate of this range from ratios of successive coefficients. From the coefficients $\hat{\kappa}_{j,F}$ that we have calculated with $1 \le n \le 3$, we compute the ratios

$$\frac{\hat{\kappa}_{1,F}}{\hat{\kappa}_{2,F}} = 4.252 \tag{3.55}$$

$$\frac{\hat{\kappa}_{2,F}}{\hat{\kappa}_{3,F}} = 4.523 \tag{3.56}$$

and

$$\frac{\hat{\kappa}_{3,F}}{\hat{\kappa}_{4,F}} = 2.280. \tag{3.57}$$

Recalling that the maximal value of Δ_r in the interval $I_{\mathrm{IRZ},r}$ is 2.885 [Eq. (3.31)], these ratios are consistent with the inference that the small-Δ_r series expansion may be reasonably accurate throughout most of this interval $I_{\mathrm{IRZ},r}$.

D. $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_f^4}$ for $G = \mathrm{SU}(N_c)$ and $R = adj$

Here we present our results for the κ_j coefficients and thus $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_f^j}$ with $1 \le j \le 4$ for $G = \mathrm{SU}(N_c)$ and N_f fermions in the adjoint representation, $R = adj$. We will usually denote these as $\kappa_{j,adj}$ and $\gamma_{\bar{\psi}\psi,\mathrm{IR},adj,\Delta_f^j}$ but sometimes, when no confusion will result, we will omit this adj subscript for brevity of notation.

In this theory, Eqs. (2.4) and (2.6) yield, for the upper and lower ends of the interval I_{IRZ}, the values

$$N_{u,adj} = \frac{11}{4} = 2.75 \tag{3.58}$$

and

$$N_{\ell,adj} = \frac{17}{16} = 1.0625, \tag{3.59}$$

so this interval includes only one integral value of N_f, namely $N_f = 2$. We note that since the adjoint representation is self-conjugate, a theory with N_f Dirac fermions with $R = adj$ is equivalent to a theory with $N_{f,Maj} = 2N_f$ Majorana fermions. Hence, here, one may also allow the half-integral values $N_f = 3/2,\ 5/2$ corresponding to $N_{f,Maj} = 3,\ 5$. We have

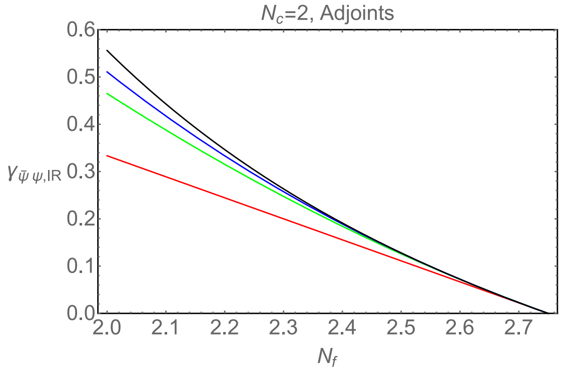

FIG. 5. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},adj,\Delta_f^p}$ for $G = \mathrm{SU}(2)$ and $1 \le p \le 4$ as a function of $N_f \in I_{\mathrm{IRZ}}$ for $R = adj$ and $N_f = 2$. From bottom to top, the curves (with colors online) refer to $\gamma_{\mathrm{IR},adj,\Delta_f}$ (red), $\gamma_{\mathrm{IR},adj,\Delta_f^2}$ (green), $\gamma_{\mathrm{IR},adj,\Delta_f^3}$ (blue), and $\gamma_{\mathrm{IR},adj,\Delta_f^4}$ (black).

$$R = adj\colon\ \Delta_f = N_u - N_f = \frac{11}{4} - N_f. \qquad (3.60)$$

For this case, the factor D in Eq. (3.1) is simply $D = 18$. In [13] we gave the coefficients $\kappa_{j,adj}$ for $1 \le n \le 3$. These are as follows:

$$\kappa_{1,adj} = \left(\frac{2}{3}\right)^2 = 0.44444, \qquad (3.61)$$

$$\kappa_{2,adj} = \frac{341}{2\cdot 3^6} = 0.23388, \qquad (3.62)$$

and

$$\kappa_{3,adj} = \frac{61873}{2^3\cdot 3^{10}} - \frac{592}{3^8 N_c^2} = 0.130978 - 0.090230 N_c^{-2}, \qquad (3.63)$$

where, as before, we indicate the simple factorizations of the denominators. The coefficient $\kappa_{4,adj}$ is

$$\kappa_{4,adj} = \frac{53389393}{2^7\cdot 3^{14}} + \frac{368}{3^{10}}\zeta_3 + \left(-\frac{2170}{3^{10}} + \frac{33952}{3^{11}}\zeta_3\right) N_c^{-2} = 0.0946976 + 0.193637 N_c^{-2}. \qquad (3.64)$$

The coefficients $\kappa_{1,adj}$ and $\kappa_{2,adj}$ are manifestly positive, and we find that for all physical N_c, the coefficients $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are also positive. Although $\kappa_{1,adj}$ and $\kappa_{2,adj}$ are independent of N_c, the coefficients $\kappa_{j,adj}$ for $j = 3, 4$ do depend on N_c. We find that $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are, respectively, monotonically increasing and monotonically decreasing functions of N_c. The $N_c \to \infty$ limits of $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are given by the respective first terms in Eqs. (3.63) and (3.64).

Thus, to order Δ_f^4, we have

$$\gamma_{\bar\psi\psi,\mathrm{IR},adj,\Delta_f^4} = \Delta_f[0.44444 + 0.23388\Delta_f + (0.13098 - 0.090230 N_c^{-2})\Delta_f^2 + (0.094698 + 0.19364 N_c^{-2})\Delta_f^3]. \qquad (3.65)$$

In Fig. 5 we show $\gamma_{\bar\psi\psi,\mathrm{IR},adj,\Delta_f^p}$ with $1 \le p \le 4$ for the SU(2) theory, as a function of N_f, formally generalized from the non-negative integers to the real numbers. In Table III we list values of $\gamma_{\bar\psi\psi,\mathrm{IR},adj,\Delta_f^p}$ with $1 \le p \le 4$ for $N_f = 2$ and $N_c = 2$ and $N_c = 3$. For comparison, we also include our n-loop values $\gamma_{\bar\psi\psi,\mathrm{IR},adj,n\ell}$ calculated in the conventional manner via power series in the coupling (in the $\overline{\mathrm{MS}}$ scheme), from Table VIII of [19].

Among SU(N_c) theories with fermions in the adjoint representation, the SU(2) theory with $N_f = 2$ (Dirac) fermions has been of particular interest [69]. In the following, for notational brevity, the subscript adj is understood implicitly. For this theory, as listed in Table III we obtain the values $\gamma_{\mathrm{IR},\Delta_f^2} = 0.465$, $\gamma_{\mathrm{IR},\Delta_f^3} = 0.511$, and $\gamma_{\mathrm{IR},\Delta_f^4} = 0.556$, which are close to our earlier higher-order n-loop calculations in [19], namely $\gamma_{\mathrm{IR},3\ell} = 0.543$ and $\gamma_{\mathrm{IR},4\ell} = 0.500$. It is of interest to compare these values with the results of lattice studies. There have been a number of such studies, and these are consistent with the conclusion that this theory is conformal in the infrared [22,70–77]. These studies have yielded a rather large range of measured values for γ_{IR}, including the following (where the published estimated uncertainties in the last digits are indicated in parentheses): $\gamma_{\mathrm{IR}} = 0.49(13)$ [70], $\gamma_{\mathrm{IR}} = 0.22(6)$ [71], $\gamma_{\mathrm{IR}} = 0.31(6)$ [72], $\gamma_{\mathrm{IR}} = 0.17(5)$ [73], $\gamma_{\mathrm{IR}} = 0.37(2)$ [74], $\gamma_{\mathrm{IR}} = 0.20(3)$ [75], and $\gamma_{\mathrm{IR}} = 0.50(26)$ [76]. (See these references and [77] for additional discussion of estimates of overall uncertainties.) Our

TABLE III. Values of the anomalous dimension $\gamma_{\mathrm{IR},adj,\Delta_f^p}$ with $1 \le p \le 4$, for $N_f = 2$ and $G = \mathrm{SU}(N_c)$ with $N_c = 2, 3$. For comparison, we also list our n-loop values, $\gamma_{\mathrm{IR},adj,n\ell}$ for this theory from Table VIII of Ref. [19].

N_c	$\gamma_{\mathrm{IR},adj,2\ell}$	$\gamma_{\mathrm{IR},adj,3\ell}$	$\gamma_{\mathrm{IR},adj,4\ell}$	$\gamma_{\mathrm{IR},adj,\Delta_f}$	$\gamma_{\mathrm{IR},adj,\Delta_f^2}$	$\gamma_{\mathrm{IR},adj,\Delta_f^3}$	$\gamma_{\mathrm{IR},adj,\Delta_f^4}$
2	0.820	0.543	0.500	0.333	0.465	0.511	0.556
3	0.820	0.543	0.523	0.333	0.465	0.516	0.553

scheme-independent calculation of $\gamma_{\rm IR}$ to $O(\Delta_f^4)$ and our earlier n-loop calculations of $\gamma_{{\rm IR},n\ell}$ up to $n=4$ loops are clearly consistent with the larger among these lattice values. Before carrying out a comparison of our results with the full set of lattice values, it will be necessary to narrow the current wide range of lattice measurements.

It is of interest to investigate the $N_c \to \infty$ limit for an $SU(N_c)$ gauge theory with fermions in the adjoint representation. Since in this case, the upper and lower ends of the interval $I_{\rm IRZ}$, given by $N_u = 11/4$ in Eq. (3.58) and $N_\ell = 17/16$ in Eq. (2.6), are independent of N_c, it follows that Δ_f is also independent of N_c. Hence, for $R = adj$,

$$\lim_{LN} \gamma_{\rm IR} = \sum_{j=1}^{\infty} \hat{\kappa}_{j,adj} \Delta_f^j \tag{3.66}$$

where

$$\hat{\kappa}_{j,adj} = \lim_{LN} \kappa_{j,adj}. \tag{3.67}$$

The values of $\hat{\kappa}_{j,adj}$ are evident from the full expressions for $\kappa_{j,adj}$ that we have given above in Eqs. (3.61)–(3.64); for example, $\hat{\kappa}_{3,adj} = 61873/(2^3 \cdot 3^{10})$.

E. $\gamma_{\bar\psi\psi,{\rm IR},\Delta_f^4}$ for $G = {\rm SU}(N_c)$ and $R = S_2$, A_2

Here we present our results for the κ_j coefficients and thus $\gamma_{\bar\psi\psi,{\rm IR},\Delta_f^j}$ with $1 \le j \le 4$ for $G = {\rm SU}(N_c)$ and N_f fermions in the symmetric and antisymmetric rank-2 tensor representations of $SU(N_c)$, S_2 and A_2. Since many formulas for these two cases are simply related to each other by sign reversals in certain terms, it is convenient to treat these cases together. As before [19], we shall use the symbol T_2 (rank-2 tensor) to refer to these cases together. (Do not confuse this use of T with our use of the symbol T in Sec. VII of Ref. [13] for the anomalous dimension of the operators $\bar\psi\sigma_{\mu\nu}\psi$ and operators $\bar\psi T_a \sigma_{\mu\nu}\psi$, where it referred to the antisymmetric Dirac tensor $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$.)

The values of N_u and N_ℓ for $R = T_2$ are [19]

$$N_{u,T_2} = \frac{11N_c}{2(N_c \pm 2)} \tag{3.68}$$

and

$$N_{\ell,T_2} = \frac{17N_c^3}{(N_c \pm 2)(8N_c^2 \pm 3N_c - 6)}, \tag{3.69}$$

so that

$$R = T_2\colon\ \Delta_f = \frac{11N_c}{2(N_c \pm 2)} - N_f. \tag{3.70}$$

The factor D in Eq. (3.1) takes the explicit form

$$R = T_2\colon\ D = \frac{18N_c^2 \pm 11N_c - 22}{N_c} \equiv \frac{F_\pm}{N_c} \tag{3.71}$$

where

$$F_\pm = 18N_c^2 \pm 11N_c - 22. \tag{3.72}$$

Both F_+ and F_- are positive-definite for the physical range $N_c \ge 2$. At the lower end of the interval $I_{\rm IRZ}$, Δ_f takes on the maximum value

$$R = T_2\colon\ (\Delta_f)_{\rm max} = \frac{3N_c F_\pm}{2(N_c \pm 2)(8N_c^2 \pm 3N_c - 6)}. \tag{3.73}$$

If $N_c = 2$, then S_2 is the same as the adjoint representation, so we focus on $N_c \ge 3$ here. For this $R = S_2$ theory, the illustrative values $N_c = 3$ and $N_c = 4$ yield the respective intervals $I_{\rm IRZ}$ $1.22 < N_f < 3.30$ and $1.35 < N_f < 3.67$. Hence, the physical integral values of N_f in these respective intervals $I_{\rm IRZ}$ are $N_f = 2, 3$ for both $N_c = 3$ and $N_c = 4$. Furthermore, the A_2 representation is the singlet if $N_c = 2$ and is the same as the conjugate fundamental, $\bar F$ if $N_c = 3$, so in the case of A_2, we restrict to $N_c \ge 3$ and focus mainly on $N_c \ge 4$. In the SU(4) theory with $R = A_2$, the interval $I_{\rm IRZ}$ is $4.945 < N_f < 11$, including the integral values $5 \le N_f \le 10$.

Here, using our general results (3.2)–(3.5), we give explicit expressions for the κ_j with $1 \le j \le 4$ for the case $G = {\rm SU}(N_c)$ and fermion representation $R = T_2$. From the general expressions for κ_j with $1 \le j \le 4$, Eqs. (3.2)–(3.5), we calculate the following. In each expression, the $+$ and $-$ signs refer to the S_2 and A_2 special cases of T_2, respectively:

$$\kappa_{1,T_2} = \frac{4(N_c \mp 1)(N_c \pm 2)^2}{N_c F_\pm} \tag{3.74}$$

$$\kappa_{2,T_2} = \frac{(N_c \mp 1)(N_c \pm 2)^3(11N_c^2 \pm 4N_c - 8)(93N_c^2 \pm 88N_c - 176)}{3N_c^2 F_\pm^3} \tag{3.75}$$

$$\kappa_{3,T_2} = \frac{(N_c \mp 1)(N_c \pm 2)^3}{2 \cdot 3^3 N_c^3 F_\pm^5}[(1670571N_c^9 \pm 7671402N_c^8 + 2181584N_c^7 \mp 25294256N_c^6 - 13413856N_c^5 \pm 17539136N_c^4 + 16707328N_c^3 \mp 3046912N_c^2 - 27320832N_c \pm 18213888) \pm 8448N_c^2(N_c \mp 2)F_\pm(3N_c^3 \pm 28N_c^2 \mp 176)\zeta_3] \quad (3.76)$$

and

$$\kappa_{4,T_2} = \frac{(N_c \mp 1)(N_c \pm 2)^4}{2^4 \cdot 3^4 N_c^4 F_\pm^7}[(4324540833N_c^{13} \pm 26924228982N_c^{12} + 30086550336N_c^{11} \mp 106026091536N_c^{10} - 224952825968N_c^9 \pm 105492861344N_c^8 + 600583055488N_c^7 \pm 45292329216N_c^6 - 1067559840512N_c^5 \pm 68261028352N_c^4 + 982655860736N_c^3 \mp 385868775424N_c^2 - 136076328960N_c \pm 54430531584) + 2^9 F_\pm(33534N_c^{11} \pm 702000N_c^{10} + 4448403N_c^9 \mp 2216812N_c^8 - 38600660N_c^7 \pm 22594304N_c^6 + 124680384N_c^5 \mp 82679040N_c^4 - 90554112N_c^3 \pm 64551168N_c^2 - 6690816N_c \pm 3345408)\zeta_3 \mp 563200N_c^2(N_c \mp 2)F_\pm^2(15N_c^5 \pm 158N_c^4 + 240N_c^3 \mp 912N_c^2 - 1056N_c \pm 2112)\zeta_5]. \quad (3.77)$$

We comment on some factors in these κ_{j,T_2} expressions. The property that the κ_{j,A_2} coefficients contain an overall factor of $(N_c - 2)$ (possibly raised to a power higher than 1), and hence vanish for $N_c = 2$, is a consequence of the fact that for $N_c = 2$, the A_2 representation is a singlet, so for SU(2), fermions in the $A_2 =$ singlet representation have no gauge interactions and hence no anomalous dimensions. Clearly, this property holds in general; i.e., the coefficients κ_{j,A_2} for all j contain an overall factor of $(N_c - 2)$ [as well as possible additional factors of $(N_c - 2)$].

As noted above, if $N_c = 2$, then the S_2 representation is the same as the adjoint representation, so the coefficients must satisfy the equality $\kappa_{j,S_2} = \kappa_{j,adj}$ for this SU(2) case, and we have checked that they do. Note that this equality requires (i) that the term proportional to ζ_3 in κ_{3,S_2} must be absent if $N_c = 2$, since $\kappa_{3,adj}$ does not contain any ζ_3 term, and, indeed, this is accomplished by the factor $(N_c - 2)$ multiplying the ζ_3 term in κ_{3,S_2}; and (ii) the term proportional to ζ_5 in κ_{4,S_2} must be absent if $N_c = 2$, since $\kappa_{4,adj}$ does not contain any ζ_5 term, and this is accomplished by the factor $(N_c - 2)$ multiplying this ζ_5 term in κ_{4,S_2}. Similarly, as we observed above, if $N_c = 3$, then the A_2 representation is the same as the conjugate fundamental representation, $\bar{F}$, so the coefficients must satisfy the equality $\kappa_{j,A_2} = \kappa_{j,F}$ for this SU(3) case, and we have checked that they do.

The resultant Δ_f expansions for $\gamma_{\bar{\psi}\psi,\mathrm{IR},S_2,\Delta_f^4}$ with $2 \le N_c \le 4$ are

$$\mathrm{SU}(2)\colon\ \gamma_{\bar{\psi}\psi,\mathrm{IR},S_2,\Delta_f^4} = \Delta_f[0.44444 + 0.23388\Delta_f + 0.10842\Delta_f^2 + 0.14311\Delta_f^3] \quad (3.78)$$

$$\mathrm{SU}(3)\colon\ \gamma_{\bar{\psi}\psi,\mathrm{IR},S_2,\Delta_f^4} = \Delta_f[0.38536 + 0.17038\Delta_f + 0.078062\Delta_f^2 + 0.060081\Delta_f^3] \quad (3.79)$$

and

$$\mathrm{SU}(4)\colon\gamma_{\bar{\psi}\psi,\mathrm{IR},S_2,\Delta_f^4} = \Delta_f[0.34839 + 0.13875\Delta_f + 0.059680\Delta_f^2 + 0.38102\Delta_f^3]. \quad (3.80)$$

For $R = A_2$, we give illustrative results for the Δ_f expansion of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ for $N_c = 4$, 5:

$$\mathrm{SU}(4)\colon\ \gamma_{\bar{\psi}\psi,\mathrm{IR},A_2,\Delta_f^4} = \Delta_f[0.090090 + (1.1114 \times 10^{-2})\Delta_f + (1.6013 \times 10^{-3})\Delta_f^2 + (2.9668 \times 10^{-4})\Delta_f^3] \quad (3.81)$$

and

$$\mathrm{SU}(5)\colon\ \gamma_{\bar{\psi}\psi,\mathrm{IR},A_2,\Delta_f^4} = \Delta_f[0.11582 + (1.7570 \times 10^{-2})\Delta_f + (2.9243 \times 10^{-3})\Delta_f^2 + (0.59791 \times 10^{-3})\Delta_f^3]. \quad (3.82)$$

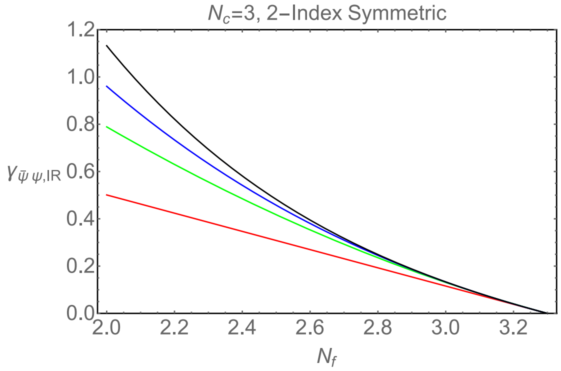

FIG. 6. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f^p}$ for $N_c = 3$ and $1 \le p \le 4$ as a function of N_f. Here, S_2 denotes the symmetric rank-2 tensor representation. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f^4}$ (black).

In Fig. 6 we present a plot of $\gamma_{\bar\psi\psi,S_2,\mathrm{IR},\Delta_f^p}$ for $G = \mathrm{SU}(3)$, $R = S_2$, and $1 \le p \le 4$, as a function of N_f. We list values of the $\gamma_{\mathrm{IR},S_2,\Delta_f^p}$ with $1 \le p \le 4$ for the SU(3) and SU(4) theories with $R = S_2$ in Table IV. In both of these theories, the interval I_{IRZ} includes the two integer values $N_f = 2, 3$. For comparison, we also include the values $\gamma_{\mathrm{IR},S_2,n\ell}$ for $2 \le n \le 4$ calculated via the conventional power series expansion to n-loop order and evaluated at $\alpha = \alpha_{\mathrm{IR},n\ell}$ from Table XI in our previous work, Ref. [19]. As is evident from this table, for a given N_c and N_f, there is reasonable agreement between the $n = 4$ loop values $\gamma_{\mathrm{IR},S_2,\Delta_f^4}$ and $\gamma_{\mathrm{IR},S_2,4\ell}$. For example, for SU(3) and $N_f = 2$, $\gamma_{\mathrm{IR},S_2,4\ell} = 1.12$ while $\gamma_{\mathrm{IR},S_2,\Delta_f^4} = 1.13$.

We next compare our calculation of $\gamma_{\bar\psi\psi,\mathrm{IR},S_2,\Delta_f^p}$ to order $p = 4$ with lattice measurements. A theory of particular interest is the SU(3) gauge theory with $N_f = 2$ flavors of fermions in the S_2 representation, and lattice studies of this theory include [78,79] (see also [22,80]). As indicated in Table IV, our higher-order scheme-independent results are $\gamma_{\mathrm{IR},\Delta_f^3} = 0.960$, and $\gamma_{\mathrm{IR},\Delta_f^4} = 1.132$, in agreement with our n-loop results from [19] for this theory, $\gamma_{\mathrm{IR},3\ell} = 1.28$ and $\gamma_{\mathrm{IR},4\ell} = 1.12$. The lattice study [78] concluded that this theory is IR-conformal and obtained $\gamma_{\mathrm{IR}} < 0.45$ [78], while Ref. [79] concluded that it is not IR-conformal and got an effective $\gamma_{\mathrm{IR}} \sim 1$ [79]. One hopes that further work by lattice groups will lead to a consensus concerning whether this theory is IR-conformal or not and concerning the value of γ_{IR}.

Regarding the range of applicability of the Δ_f expansion for these cases, we compute the following ratios of successive coefficients for the $G = \mathrm{SU}(3)$, $R = S_2$ case:

$$\frac{\kappa_{1,S_2}}{\kappa_{2,S_2}} = 2.26176 \tag{3.83}$$

$$\frac{\kappa_{2,S_2}}{\kappa_{3,S_2}} = 2.1826 \tag{3.84}$$

and

$$\frac{\kappa_{3,S_2}}{\kappa_{4,S_2}} = 1.2993. \tag{3.85}$$

The first two ratios, (3.83) and (3.84), are slightly larger than $(\Delta_f)_{\max,S_2} = 519/250 = 2.076$ in I_{IRZ} for this theory. However, the third ratio is about 40% less than this maximal value of Δ_{f,S_2}. This suggests that because of slow convergence, one must use the Δ_f expansion with caution in the lower part of the interval I_{IRZ} in this theory.

We list values of the $\gamma_{\mathrm{IR},A_2,\Delta_f^p}$ with $1 \le p \le 4$ for the SU(4) theory with $R = A_2$ and $N_f \in I_{\mathrm{IRZ}}$ for this theory in Table V. Again, for comparison, we include the values $\gamma_{\mathrm{IR},A_2,n\ell}$ for $2 \le n \le 4$ calculated via the conventional power series expansion to n-loop order and evaluated at $\alpha = \alpha_{\mathrm{IR},n\ell}$ from Table XII in our previous work [19]. As expected, the agreement between the two methods of calculation is best at the upper end of the interval I_{IRZ}, where the IRFP occurs at weak coupling. For example, for $N_f = 9$, $\gamma_{\mathrm{IR},A_2,\Delta_f^4} = 0.242$, while $\gamma_{\mathrm{IR},4\ell} = 0.232$.

It is of interest to consider the $N_c \to \infty$ (LN) limit of Eq. (3.22) for these theories with $R = S_2$ and A_2. In this LN limit, the upper ends of the interval I_{IRZ} for the S_2 and A_2 representations approach the same limit, and similarly for the lower ends:

$$\lim_{LN} N_{u,T_2} = \frac{11}{2} = 5.5 \tag{3.86}$$

TABLE IV. Values of the anomalous dimension $\gamma_{\mathrm{IR},S_2,\Delta_f^p}$ with $1 \le p \le 4$, for $G = \mathrm{SU}(N_c)$ with $N_c = 3$, 4 and $N_f = 2, 3$ (so $N_f \in I_{\mathrm{IRZ}}$). For comparison, we also include values of $\gamma_{\mathrm{IR},S_2,n\ell}$ with $2 \le n \le 4$ for this theory from Table XI in our Ref. [19]. Values that exceed the upper bound $\gamma_{\mathrm{IR}} < 2$ are marked as unphysical (u).

N_c	N_f	$\gamma_{\mathrm{IR},S_2,2\ell}$	$\gamma_{\mathrm{IR},S_2,3\ell}$	$\gamma_{\mathrm{IR},S_2,4\ell}$	$\gamma_{\mathrm{IR},S_2,\Delta_f}$	$\gamma_{\mathrm{IR},S_2,\Delta_f^2}$	$\gamma_{\mathrm{IR},S_2,\Delta_f^3}$	$\gamma_{\mathrm{IR},S_2,\Delta_f^4}$
3	2	u	1.28	1.12	0.501	0.789	0.960	1.132
3	3	0.144	0.133	0.133	0.116	0.131	0.133	0.1335
4	2	u	u	1.79	0.581	0.966	1.242	1.536
4	3	0.381	0.313	0.315	0.232	0.294	0.312	0.319

TABLE V. Values of the anomalous dimension $\gamma_{IR,A_2,\Delta_f^p}$ calculated to order $1 \le p \le 4$, for $G = SU(4)$ and $N_f \in I_{IRZ}$. For comparison, we also include values of $\gamma_{IR,A_2,n\ell}$ with $2 \le n \le 4$ for this theory from Table XII in [19]. Values that exceed the upper bound $\gamma_{IR} < 2$ are marked as unphysical (u).

N_c	N_f	$\gamma_{IR,A_2,2\ell}$	$\gamma_{IR,A_2,3\ell}$	$\gamma_{IR,A_2,4\ell}$	γ_{IR,A_2,Δ_f}	$\gamma_{IR,A_2,\Delta_f^2}$	$\gamma_{IR,A_2,\Delta_f^3}$	$\gamma_{IR,A_2,\Delta_f^4}$
4	5	u	u	u	0.5405	0.941	1.287	1.671
4	6	u	1.38	0.293	0.450	0.728	0.928	1.114
4	7	u	0.695	0.435	0.360	0.538	0.641	0.717
4	8	0.802	0.402	0.368	0.270	0.370	0.4135	0.438
4	9	0.331	0.228	0.232	0.180	0.225	0.237	0.242
4	10	0.117	0.101	0.103	0.0901	0.101	0.103	0.103

$$\lim_{LN} N_{\ell,T_2} = \frac{17}{8} = 2.125. \tag{3.87}$$

Hence, in this $N_c \to \infty$ limit, the interval I_{IRZ} is formally $2.125 < N_f < 5.5$, including the physical integer values $3 \le N_f \le 5$. Similarly, in this limit, the variable Δ_f is given by $\Delta_f = (11/2) - N_f$ and reaches a maximum value, at $N_f = N_{\ell,T_2}$, of

$$\lim_{LN} (\Delta_f)_{\max,T_2} = \frac{27}{8} = 3.375. \tag{3.88}$$

This is the $N_c \to \infty$ limit of (3.73).

As with the adjoint representation, we define

$$\hat{\kappa}_{j,T_2} = \lim_{LN} \kappa_{j,T_2}. \tag{3.89}$$

We find that

$$\hat{\kappa}_{j,S_2} = \hat{\kappa}_{j,A_2}. \tag{3.90}$$

From our general expressions for κ_{j,T_2} with $1 \le j \le 4$, we calculate

$$\hat{\kappa}_{1,T_2} = \frac{2}{3^2} = 0.2222 \tag{3.91}$$

$$\hat{\kappa}_{2,T_2} = \frac{341}{2^3 \cdot 3^6} = 0.0584705 \tag{3.92}$$

$$\hat{\kappa}_{3,T_2} = \frac{61873}{2^6 \cdot 3^{10}} = 0.016372 \tag{3.93}$$

and

$$\hat{\kappa}_{4,T_2} = \frac{53389393}{2^{11} \cdot 3^{14}} + \frac{23\zeta_3}{3^{10}} = 0.59186 \times 10^{-2}. \tag{3.94}$$

Hence,

$$\lim_{LN} \gamma_{IR,S_2,\Delta_f^p} = \lim_{LN} \gamma_{IR,A_2,\Delta_f^p} \tag{3.95}$$

and, in the limit $p \to \infty$,

$$\lim_{LN} \gamma_{IR,S_2} = \lim_{LN} \gamma_{IR,A_2}. \tag{3.96}$$

Thus, for both $R = S_2$ and $R = A_2$,

$$\lim_{LN} \gamma_{\bar{\psi}\psi,IR,T_2,\Delta_f^4} = \Delta_f[0.22222 + 0.0584705\Delta_f + 0.016372\Delta_f^2 + 0.0059186\Delta_f^3]. \tag{3.97}$$

We observe that for all of the cases we have calculated, namely $1 \le j \le 4$,

$$\hat{\kappa}_{j,T_2} = 2^{-j}\hat{\kappa}_{j,adj}. \tag{3.98}$$

One can understand this relation from the structure of the relevant group invariants, including the fact that the trace invariant $T(R)$ satisfies

$$\lim_{N_c\to\infty} \frac{T_{T_2}}{T_{adj}} = \frac{1}{2}. \tag{3.99}$$

We thus infer more generally that the relation (3.98) holds for all j. In Table VI we list the resultant common values of $\gamma_{IR,T_2,\Delta_f^p}$ for $1 \le p \le 4$ and $N_f \in I_{IRZ}$ in the LN limit. As noted above, in this LN limit, this interval consists of the integral values $N_f = 3, 4, 5$.

Concerning the range of applicability of the Δ_f expansion in this LN limit, we compute the ratios

$$\frac{\hat{\kappa}_{1,T_2}}{\hat{\kappa}_{2,T_2}} = \frac{1296}{341} = 3.8006 \tag{3.100}$$

TABLE VI. Values of the anomalous dimension $\gamma_{IR,T_2,\Delta_f^p}$ for $T_2 = S_2$ or $T_2 = A_2$, calculated to order $1 \le p \le 4$, in the limit $N_c \to \infty$ with $N_f \in I_{IRZ}$ for this limit, namely $3 \le N_f \le 5$.

N_f	γ_{IR,T_2,Δ_f}	$\gamma_{IR,T_2,\Delta_f^2}$	$\gamma_{IR,T_2,\Delta_f^3}$	$\gamma_{IR,T_2,\Delta_f^4}$
3	0.5555	0.921	1.177	1.408
4	0.333	0.465	0.520	0.550
5	0.111	0.126	0.128	0.128

$$\frac{\hat{\kappa}_{2,T_2}}{\hat{\kappa}_{3,T_2}} = \frac{220968}{61873} = 3.5713 \tag{3.101}$$

and

$$\frac{\hat{\kappa}_{3,T_2}}{\hat{\kappa}_{4,T_2}} = \frac{160374816}{53389393 + 3815424\zeta_3} = 2.76624. \tag{3.102}$$

The first two ratios, (3.100) and (3.101), are slightly greater than the maximum value $(\Delta_f)_{\max,T_2} = 3.375$, but the third ratio, (3.102), is smaller than this maximum value, suggesting that in this limit, for these tensor representations, because of slow convergence, one must use caution in applying the Δ_f expansion in the lower part of the interval I_{IRZ}. This is similar to what we found for the S_2 representation in the SU(3) theory.

IV. CALCULATION OF β'_{IR} TO $O(\Delta_f^5)$

A. General G and R

The derivative β'_{IR} is an important physical quantity characterizing the conformal field theory at α_{IR}. We denote the gauge field of the theory as A^a_μ (where a is a group index), the field-strength tensor as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g c_{abc} A^b_\mu A^c_\nu$ (where c_{abc} is the structure constant of the Lie algebra of G) and the rescaled field-strength tensor as $F^a_{\mu\nu,r} = g F^a_{\mu\nu}$, so that the gauge field kinetic term in the Lagrangian is $\mathcal{L}_g = -[1/(4g^2)] F^a_{\mu\nu,r} F^{a\mu\nu}_r$. The trace anomaly states that the trace of the energy-momentum tensor T^μ_ν satisfies the relation [81]

$$T^\mu_\mu = \frac{\beta}{16\pi\alpha^2} F^a_{\mu\nu,r} F^{a\,\mu\nu}_r. \tag{4.1}$$

Therefore, the full scaling dimension of the operator $F_{r,\mu\nu} F^{a\,\mu\nu}_r$, which we denote as D_{F^2}, satisfies [82]

$$D_{F^2} = 4 + \beta' - \frac{2\beta}{\alpha}, \tag{4.2}$$

where we use the shorthand notation $F^2 \equiv F^a_{r,\mu\nu} F^{a\,\mu\nu}_r$. We denote the anomalous dimension of F^2, γ_{F^2} via the equation [26]

$$D_{F^2} = D_{F^2,\text{free}} - \gamma_{F^2} = 4 - \gamma_{F^2} \tag{4.3}$$

and its evaluation at $\alpha = \alpha_{IR}$ as $\gamma_{F^2,IR}$. From Eq. (4.2), it follows that at a zero of the beta function away from the origin, in particular, at α_{IR}, the derivative β'_{IR} is equivalent to the anomalous dimension of the operator $F^a_{r,\mu\nu} F^{a\,\mu\nu}_r$:

$$\beta'_{IR} = -\gamma_{F^2,IR}. \tag{4.4}$$

In [13] we calculated the expansion coefficients d_j of β'_{IR} in Eq. (1.3) to order Δ_f^4 for general G and R, and to order Δ_f^5 for the special case $G = \text{SU}(3)$ and fermion representation $R = F$, the fundamental. Here we calculate the next higher-order coefficient, namely d_5, for general G and R. For this purpose, we make use of the recent computation of the five-loop beta function coefficient, b_5, in [17]. The computation in [17] was performed in the $\overline{\text{MS}}$ scheme, so that we can combine it with the scheme-independent b_1 and b_2 [7,8] and the results for b_3 and b_4 that have also been calculated in the $\overline{\text{MS}}$ scheme [27,28]. However, we again stress that since the d_n coefficients are scheme-independent, it does not matter which scheme one uses to calculate them. We first recall our previous results from Ref. [13]:

$$d_1 = 0, \tag{4.5}$$

$$d_2 = \frac{2^5 T_f^2}{3^2 C_A D}, \tag{4.6}$$

$$d_3 = \frac{2^7 T_f^3 (5C_A + 3C_f)}{3^3 C_A^2 D^2}, \tag{4.7}$$

and

$$d_4 = -\frac{2^3 T_f^2}{3^6 C_A^4 D^5} \Big[-3C_A T_f^2 (137445 C_A^4 + 103600 C_A^3 C_f + 72616 C_A^2 C_f^2 + 951808 C_A C_f^3 - 63888 C_f^4) - 5120 T_f^2 D \frac{d_A^{abcd} d_A^{abcd}}{d_A} + 90112 C_A T_f D \frac{d_R^{abcd} d_A^{abcd}}{d_A} - 340736 C_A^2 D \frac{d_R^{abcd} d_R^{abcd}}{d_A} + 8448 D \Big[C_A^2 T_f^2 (21 C_A^2 + 12 C_A C_f - 33 C_f^2) + 16 T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A} - 104 C_A T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A} + 88 C_A^2 \frac{d_R^{abcd} d_R^{abcd}}{d_A} \Big] \zeta_3 \Big]. \tag{4.8}$$

In Ref. [13] we presented the expression for d_4 with terms written in order of descending powers of C_A. It is also useful to express this coefficient d_4 in an equivalent form that renders certain factors of D explicit and shows the simple factorization of terms multiplying ζ_3, and we have done this in Eq. (4.8).

Here we present our calculation of d_5 for arbitrary G and R:

$$\begin{aligned}
d_5 = \frac{2^4 T_f^3}{3^7 C_A^5 D^7}\Bigg[& -C_A T_f^2 (39450145 C_A^6 + 235108272 C_A^5 C_f + 1043817726 C_A^4 C_f^2 + 765293216 C_A^3 C_f^3 \\
& - 737283360 C_A^2 C_f^4 + 730646400 C_A C_f^5 - 356750592 C_f^6) - 2^9 T_f^2 D \frac{d_A^{abcd} d_A^{abcd}}{d_A}(6139 C_A^2 + 2192 C_A C_f - 3300 C_f^2) \\
& + 2^9 C_A T_f D \frac{d_R^{abcd} d_A^{abcd}}{d_A}(43127 C_A^2 - 28325 C_A C_f - 2904 C_f^2) + 15488 C_A^2 D \frac{d_R^{abcd} d_R^{abcd}}{d_A}(2975 C_A^2 + 8308 C_A C_f - 12804 C_f^2) \\
& + 2^7 D [3 C_A T_f^2 D (6272 C_A^4 - 49823 C_A^3 C_f + 40656 C_A^2 C_f^2 + 13200 C_A C_f^3 + 2112 C_f^4) \\
& + 2^4 T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(19516 C_A^2 - 18535 C_A C_f - 21780 C_f^2) - 2^3 C_A T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(182938 C_A^2 - 297649 C_A C_f - 197472 C_f^2) \\
& - 88 C_A^2 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(245 C_A^2 + 62524 C_A C_f + 42108 C_f^2)]\zeta_3 \\
& + 2^{10}\cdot 55 C_A D^2 \Bigg[9 C_A T_f^2 D (C_A + 2C_f)(C_A - C_f) + 160 T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
& - 80 T_f (10 C_A + 3 C_f)\frac{d_R^{abcd} d_A^{abcd}}{d_A} - 440 C_A (C_A - 3C_f)\frac{d_R^{abcd} d_R^{abcd}}{d_A}\Bigg]\zeta_5\Bigg]. \qquad (4.9)
\end{aligned}$$

We proceed to evaluate these coefficients d_j up to $j = 5$, and hence the derivative $\beta'_{\rm IR}$ up to $O(\Delta_f^5)$ below for $G = {\rm SU}(N_c)$ and several specific representations. The coefficients d_2 and d_3 are manifestly positive for arbitrary G and R. These signs are indicated in Table VII. We discuss the signs of d_4 and d_5 for various representations below.

B. $\beta'_{{\rm IR},\Delta_f^4}$ for $G = {\rm SU}(N_c)$ and $R = F$

Here we present the evaluation of our general result (4.9) for the case $G = {\rm SU}(N_c)$ and $R = F$. For reference, we first recall our results from [13] for d_j with $2 \le j \le 4$ (and also recall that $d_1 = 0$ for all G and R):

$$d_{2,F} = \frac{2^4}{3^2(25N_c^2 - 11)}, \qquad (4.10)$$

$$d_{3,F} = \frac{2^5(13N_c^2 - 3)}{3^3 N_c (25N_c^2 - 11)^2}, \qquad (4.11)$$

and

$$\begin{aligned}
d_{4,F} = -\frac{2^4}{3^5 N_c^2 (25N_c^2 - 11)^5}[& N_c^8(-366782 + 660000\zeta_3) + N_c^6(865400 - 765600\zeta_3) \\
& + N_c^4(-1599316 + 2241888\zeta_3) + N_c^2(571516 - 894432\zeta_3) + 3993]. \qquad (4.12)
\end{aligned}$$

This coefficient can be written equivalently in a form that shows the simple factorization of the terms multiplying ζ_3:

$$\begin{aligned}
d_{4,F} = -\frac{2^4}{3^5 N_c^2 (25N_c^2 - 11)^5}[& (-366782 N_c^8 + 865400 N_c^6 - 1599316 N_c^4 + 571516 N_c^2 + 3993) \\
& + 1056 N_c^2 (25N_c^2 - 11)(25N_c^4 - 18N_c^2 + 77)\zeta_3]. \qquad (4.13)
\end{aligned}$$

In [16] we presented the expression for $d_{5,F}$ with terms ordered as descending powers of N_c. As with $d_{4,F}$, it is also useful to display this coefficient in an equivalent form that shows the simple factorizations of the terms multiplying ζ_3 and ζ_5:

$$d_{5,F} = \frac{2^5}{3^6 N_c^3 (25N_c^2-11)^7}[(-298194551N_c^{12} + 414681770N_c^{10} + 80227411N_c^8 + 210598856N_c^6 - 442678324N_c^4 + 129261880N_c^2 + 3716152) - 96(25N_c^2-11)(176375N_c^{10} - 564526N_c^8 + 1489367N_c^6 - 1470392N_c^4 + 290620N_c^2 + 968)\zeta_3 + 21120N_c^2(25N_c^2-11)^2(40N_c^6 - 27N_c^4 + 124N_c^2 - 209)\zeta_5]. \quad (4.14)$$

We have checked that when we set $N_c = 3$ in our general result for $d_{5,F}$ in Eq. (4.14), the result agrees with our earlier calculation of $d_{5,F}$ in Eq. (5.20) of Ref. [13].

As observed above, the coefficients d_2 and d_3 are manifestly positive for any G and R. We find that $d_{4,F}$ and $d_{5,F}$ are negative-definite for $G = \mathrm{SU}(N_c)$ and all physical values of $N_c \geq 2$. These results are summarized in Table VII.

We list below the explicit numerical expressions for $\beta'_{\rm IR}$ to order Δ_f^5, denoted $\beta'_{{\rm IR},{\rm SU}(N_c),F,\Delta_f^5}$, for the gauge groups $\mathrm{SU}(N_c)$ with $N_c = 2, 3, 4$, with fermions in the fundamental representation, to the indicated floating-point precision:

$$\mathrm{SU}(2):\ \beta'_{{\rm IR},F,\Delta_f^5} = \Delta_f^2[(1.99750\times10^{-2} + (3.66583\times10^{-3})\Delta_f - (3.57303\times10^{-4})\Delta_f^2 - (2.64908\times10^{-5})\Delta_f^3] \quad (4.15)$$

$$\mathrm{SU}(3):\ \beta'_{{\rm IR},F,\Delta_f^5} = \Delta_f^2[(0.83074\times10^{-2}) + (0.98343\times10^{-3}\Delta_f - (0.46342\times10^{-4})\Delta_f^2 - (0.56435\times10^{-5})\Delta_f^3] \quad (4.16)$$

and

$$\mathrm{SU}(4):\ \beta'_{{\rm IR},F,\Delta_f^5} = \Delta_f^2[(0.45701\times10^{-2}) + (0.40140\times10^{-3}\Delta_f - (0.12938\times10^{-4})\Delta_f^2 - (0.15498\times10^{-5})\Delta_f^3]. \quad (4.17)$$

In Table VIII we list the (scheme-independent) values that we calculate for $\beta'_{{\rm IR},F,\Delta_f^p}$ with $2 \leq p \leq 4$ for the illustrative gauge groups $G = \mathrm{SU}(2)$, SU(3), and SU(4), as functions of N_f in the respective intervals $I_{\rm IRZ}$ given in Eq. (2.7). For comparison, we list the n-loop values of $\beta'_{{\rm IR},F,n\ell}$ with the $2 \leq n \leq 4$ from [13,20], where $\beta'_{{\rm IR},F,3\ell}$ and $\beta'_{{\rm IR},F,4\ell}$ are computed in the $\overline{\rm MS}$ scheme. Although, for completeness, we list values of $\beta'_{{\rm IR},F,2\ell}$ with N_f extending down to the lower end of the respective intervals $I_{\rm IRZ}$ for each value of N_c, we caution that in a number of cases, including $N_f = 6$ for SU(2), $N_f = 9$ for SU(3), and $10 \leq N_f \leq 12$ for SU(4), the corresponding values of $\alpha_{{\rm IR},2\ell}$ (discussed further below) are too large for the perturbative n-loop calculations to be applicable. Moreover, since for a considerable range of values of $N_f \in I_{\rm IRZ}$ for each N_c, the five-loop beta function $\beta_{5\ell}$ calculated via the conventional power series expansion has no physical IR zero, we restrict the resultant $\beta'_{{\rm IR},F,n\ell}$ evaluations to $1 \leq n \leq 4$ loops.

In Figs. 7–9 we plot the values of $\beta'_{\rm IR}$, calculated to order Δ_f^p with $2 \leq p \leq 5$, for $R = F$ for the gauge groups SU(2), SU(3), and SU(4). In the general calculations of $\gamma_{\rm IR}$ as a series in powers of Δ_f to maximal power $p = 3$ (i.e., order Δ_f^3) in [12] and, for $G = \mathrm{SU}(3)$ and $R = F$, to maximal power $p = 4$ in [14], it was found that, for a fixed value of N_f, or equivalently, Δ_f, in the interval $I_{\rm IRZ}$, these anomalous dimensions increased monotonically as a function of p. This feature motivated our extrapolation to $p = \infty$ in [12] to obtain estimates for the exact $\gamma_{\rm IR}$. In contrast, here we find that, for a fixed value of N_f, or equivalently, Δ_f, in $I_{\rm IRZ}$, as a consequence of the fact that different coefficients d_n do not all have the same sign, $\beta'_{{\rm IR},\Delta_f^p}$ is not a monotonic function of p. Because of this nonmonotonicity, we do not attempt to extrapolate our series to $p = \infty$.

A lattice measurement of $\beta'_{\rm IR}$ has been reported in [83] for the SU(3) theory with $R = F$ and $N_f = 12$, namely $\beta'_{\rm IR} = 0.26(2)$. The earlier higher-order values calculated in [20] via n-loop expansions in the coupling are $\beta'_{{\rm IR},3\ell} = 0.2955$ and $\beta'_{{\rm IR},4\ell} = 0.282$, which agree with this lattice measurement. As indicated in Table VIII, our higher-order scheme-independent values for this theory are $\beta'_{{\rm IR},\Delta_f^3} = 0.258$, $\beta'_{{\rm IR},\Delta_f^4} = 0.239$, and $\beta'_{{\rm IR},\Delta_f^5} = 0.228$. Given the possible contributions of higher-order terms

TABLE VII. Signs of the $d_{j,R}$ coefficients for $2 \leq j \leq 5$ for gauge group $G = \mathrm{SU}(N_c)$ and fermion representations R equal to F (fundamental), adj (adjoint), S_2, and A_2 (symmetric and antisymmetric rank-2 tensor). Note that $d_1 = 0$ for all G and R. In the case $R = A_2$, we restrict to $N_c \geq 3$.

j	$d_{j,F}$	$d_{j,adj}$	d_{j,S_2}	d_{j,A_2}
2	+	+	+	+
3	+	+	+	+
4	−	+	+	− for $N_c = 3, 4, 5$ + for $N_c \geq 6$
5	−	−	−	−

TABLE VIII. Scheme-independent values of $\beta'_{\mathrm{IR},F,\Delta_f^p}$ with $2 \le p \le 4$ for $G = \mathrm{SU}(2)$, SU(3), and SU(4), as functions of N_f in the respective intervals I_{IRZ}. For comparison, we list the n-loop values of $\beta'_{\mathrm{IR},F,n\ell}$ with $2 \le n \le 5$, where $\beta'_{\mathrm{IR},F,n\ell}$ with $n = 3$, 4, 5 are computed in the $\overline{\mathrm{MS}}$ scheme. The notation ae-n means $a \times 10^{-n}$.

N_c	N_f	$\beta'_{\mathrm{IR},F,2\ell}$	$\beta'_{\mathrm{IR},F,3\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},F,4\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},F,\Delta_f^2}$	$\beta'_{\mathrm{IR},F,\Delta_f^3}$	$\beta'_{\mathrm{IR},F,\Delta_f^4}$	$\beta'_{\mathrm{IR},F,\Delta_f^5}$
2	6	6.061	1.620	0.975	0.499	0.957	0.734	0.6515
2	7	1.202	0.728	0.677	0.320	0.554	0.463	0.436
2	8	0.400	0.318	0.300	0.180	0.279	0.250	0.243
2	9	0.126	0.115	0.110	0.0799	0.109	0.1035	0.103
2	10	0.0245	0.0239	0.0235	0.0200	0.0236	0.0233	0.0233
3	9	4.167	1.475	1.464	0.467	0.882	0.7355	0.602
3	10	1.523	0.872	0.853	0.351	0.621	0.538	0.473
3	11	0.720	0.517	0.498	0.251	0.415	0.3725	0.344
3	12	0.360	0.2955	0.282	0.168	0.258	0.239	0.228
3	13	0.174	0.1556	0.149	0.102	0.144	0.137	0.134
3	14	0.0737	0.0699	0.0678	0.0519	0.0673	0.0655	0.0649
3	15	0.0227	0.0223	0.0220	0.0187	0.0220	0.0218	0.0217
3	16	2.21e-3	2.20e-3	2.20e-3	2.08e-3	2.20e-3	2.20e-3	2.20e-3
4	11	16.338	2.189	2.189	0.553	1.087	0.898	0.648
4	12	3.756	1.430	1.429	0.457	0.858	0.729	0.574
4	13	1.767	0.965	0.955	0.370	0.663	0.578	0.486
4	14	0.984	0.655	0.639	0.292	0.498	0.445	0.394
4	15	0.581	0.440	0.424	0.224	0.362	0.331	0.3045
4	16	0.348	0.288	0.276	0.1645	0.251	0.234	0.222
4	17	0.204	0.180	0.1725	0.114	0.164	0.156	0.1515
4	18	0.113	0.105	0.101	0.0731	0.0988	0.0955	0.0939
4	19	0.0558	0.0536	0.0522	0.0411	0.0520	0.0509	0.0505
4	20	0.0222	0.0218	0.0215	0.0183	0.0215	0.0213	0.0212
4	21	5.01e-3	4.99e-3	4.96e-3	4.57e-3	4.97e-3	4.96e-3	4.96e-3

in the Δ_f expansion, we consider that our scheme-independent calculation of β'_{IR} to this order is also consistent with the lattice measurement from Ref. [83].

To get a rough estimate of the range of accuracy and applicability of the series expansion for β'_{IR} for this $R = F$ case, we can compute ratios of coefficients, as discussed before. For the illustrative case of SU(3), we have

$$\frac{d_{2,F}}{d_{3,F}} = 8.447 \text{ for SU}(3), \tag{4.18}$$

$$\frac{d_{3,F}}{|d_{4,F}|} = 21.221 \text{ for SU}(3), \tag{4.19}$$

and

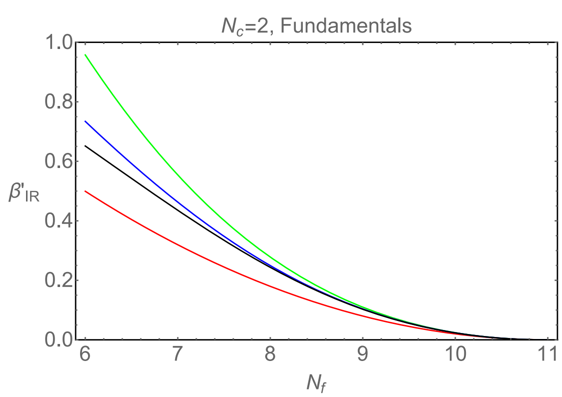

FIG. 7. Plot of $\beta'_{\mathrm{IR},F,\Delta_f^p}$ (labeled as β'_{IR} on the vertical axis) for $N_c = 2$ and $2 \le p \le 5$ as a function of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},F,\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},F,\Delta_f^4}$ (blue), and $\beta'_{\mathrm{IR},F,\Delta_f^5}$ (black).

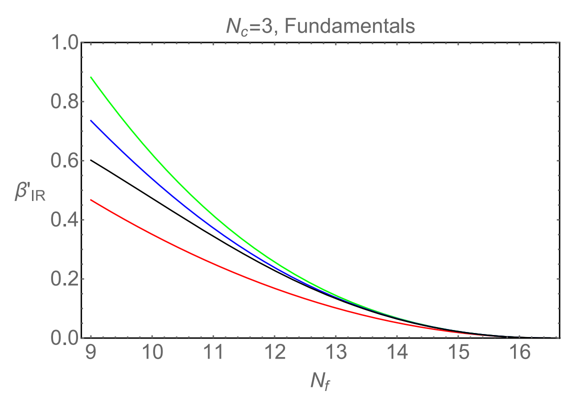

FIG. 8. Plot of $\beta'_{\mathrm{IR},F,\Delta_f^p}$ for $N_c = 3$ and $2 \le p \le 5$ as a function of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},F,\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},F,\Delta_f^4}$ (blue), and $\beta'_{\mathrm{IR},F,\Delta_f^5}$ (black).

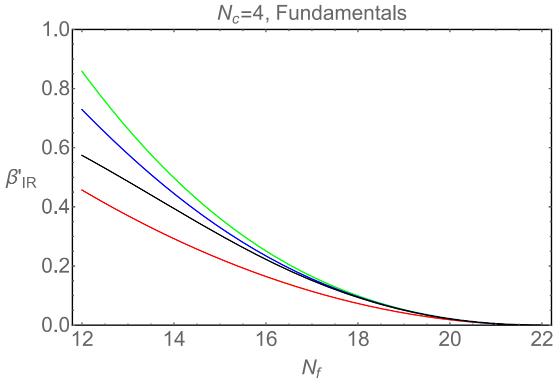

FIG. 9. Plot of $\beta'_{\mathrm{IR},F,\Delta_f^p}$ for $N_c = 4$ and $2 \le p \le 5$ as a function of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},F,\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},F,\Delta_f^4}$ (blue), and $\beta'_{\mathrm{IR},F,\Delta_f^5}$ (black).

$$\frac{|d_{4,F}|}{|d_{5,F}|} = 8.2115 \text{ for SU}(3). \tag{4.20}$$

Since $N_u = 16.5$ and $N_\ell = 153/19 = 8.053$ in this SU(3) theory, the maximal value of Δ_f in the interval I_{IRZ}, as given by (3.16), is

$$(\Delta_f)_{\max} = \frac{321}{38} = 8.447 \quad \text{for SU}(3), \qquad N_f \in I_{\mathrm{IRZ}}. \tag{4.21}$$

Therefore, these ratios suggest that the small-Δ_f expansion may be reasonably reliable in most of this interval, I_{IRZ} and the associated non-Abelian Coulomb phase.

C. $\beta'_{\mathrm{IR},\Delta_f^5}$ in LNN limit

The appropriately rescaled beta function that is finite in the LNN limit is

$$\beta_\xi = \frac{d\xi}{dt} = \lim_{LNN} N_c \beta, \tag{4.22}$$

where $\xi = 4\pi x = \lim_{LNN} \alpha N_c$ was defined in Eq. (3.21). This has the series expansion

$$\beta_\xi \equiv \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell = -2\xi \sum_{\ell=1}^{\infty} \tilde{b}_\ell \xi^\ell \tag{4.23}$$

where

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell}. \tag{4.24}$$

and $\tilde{b}_\ell = \hat{b}_\ell/(4\pi)^\ell$. The $\hat{b}_\ell$ with $1 \le \ell \le 4$ were analyzed in [20,21] and are listed for the reader's convenience in the Appendix.

From the recent calculation of b_5 in [17], for general G and R, in the $\overline{\mathrm{MS}}$ scheme [17], we calculate

$$\begin{aligned}
\hat{b}_5 &= \frac{8268479}{3888} + \frac{38851}{162}\zeta_3 - \frac{121}{6}\zeta_4 - 330\zeta_5 \\
&\quad + \left(-\frac{11204369}{5184} - \frac{231619}{648}\zeta_3 + \frac{77}{6}\zeta_4 + \frac{4090}{9}\zeta_5\right) r \\
&\quad + \left(\frac{3952801}{7776} + \frac{33125}{108}\zeta_3 - \frac{241}{6}\zeta_4 - \frac{1630}{9}\zeta_5\right) r^2 \\
&\quad + \left(-\frac{5173}{432} - \frac{1937}{81}\zeta_3 + 7\zeta_4 + \frac{20}{3}\zeta_5\right) r^3 \\
&\quad + \left(\frac{61}{486} - \frac{52}{81}\zeta_3\right) r^4 \\
&= 2050.932 - 2105.880r + 645.7474r^2 \\
&\quad - 26.2309r^3 - 0.64618r^4.
\end{aligned} \tag{4.25}$$

(In this expression although ζ_4 could be expressed explicitly as $\zeta_4 = \pi^4/90$, we leave it in abstract form to be parallel with the ζ_3 and ζ_5 terms.) We find that this coefficient $\hat{b}_5$ is positive throughout the entire asymptotically free interval $0 \le r < 5.5$. (Considered formally as a function of $r \in \mathbb{R}$, $\hat{b}_5$ is negative for $r < -58.609$, positive for $-58.609 < r < 14.336$, and negative for $r > 14.336$, where the numbers are quoted to the given floating-point accuracy.)

Since the derivative $d\beta_\xi/d\xi$ satisfies the relation

$$\frac{d\beta_\xi}{d\xi} = \frac{d\beta}{d\alpha} \equiv \beta', \tag{4.26}$$

it follows that β' is finite in the LNN limit (3.21). In terms of the variable x defined in Eq. (3.23), we have

$$\beta' = -2 \sum_{\ell=1}^{\infty} (\ell+1) \hat{b}_\ell x^\ell. \tag{4.27}$$

Because β'_{IR} is scheme-independent and is finite in the LNN limit, one is motivated to calculate the LNN limit of the scheme-independent expansion (1.3). For this purpose, in addition to the rescaled quantities Δ_r defined in Eq. (3.30), we define the rescaled coefficient

$$\hat{d}_{j,F} = \lim_{LNN} N_c^j d_{j,F}, \tag{4.28}$$

which is finite. Then each term

$$\lim_{LNN} d_{j,F} \Delta_f^j = (N_c^j d_{j,F}) \left(\frac{\Delta_f}{N_c}\right)^j = \hat{d}_{j,F} \Delta_r^j \tag{4.29}$$

is finite in this limit. Thus, writing $\lim_{LNN} \beta'_{\mathrm{IR}}$ as $\beta'_{\mathrm{IR},LNN}$ for this $R = F$ case, we have

TABLE IX. Scheme-independent values of $\beta'_{\mathrm{IR},\Delta_r^p}$ for $2 \le p \le 5$ in the LNN limit (3.21) as functions of $r = 5.5 - \Delta_r$. For comparison, we also list the n-loop values $\beta'_{\mathrm{IR},n\ell}$ with $2 \le n \le 5$, where $\beta'_{\mathrm{IR},n\ell}$ with $n = 3, 4, 5$ are computed in the $\overline{\mathrm{MS}}$ scheme. The notation ae-n means $a \times 10^{-n}$.

r	$\beta'_{\mathrm{IR},2\ell}$	$\beta'_{\mathrm{IR},3\ell}$	$\beta'_{\mathrm{IR},4\ell}$	$\beta'_{\mathrm{IR},\Delta_r^2}$	$\beta'_{\mathrm{IR},\Delta_r^3}$	$\beta'_{\mathrm{IR},\Delta_r^4}$	$\beta'_{\mathrm{IR},\Delta_r^5}$
2.8	8.100	1.918	1.913	0.518	1.004	0.851	0.583
3.0	3.333	1.376	1.379	0.444	0.830	0.717	0.535
3.2	1.856	1.006	1.003	0.376	0.676	0.596	0.4755
3.4	1.153	0.7395	0.729	0.314	0.542	0.486	0.410
3.6	0.752	0.542	0.527	0.257	0.426	0.388	0.342
3.8	0.500	0.393	0.378	0.2055	0.327	0.303	0.276
4.0	0.333	0.279	0.267	0.160	0.243	0.229	0.214
4.2	0.219	0.193	0.184	0.120	0.174	0.166	0.159
4.4	0.139	0.128	0.122	0.0860	0.119	0.115	0.112
4.6	0.0837	0.0792	0.0766	0.0576	0.0756	0.0737	0.0726
4.8	0.0460	0.0445	0.0435	0.0348	0.0433	0.0426	0.0423
5.0	0.0215	0.0212	0.0208	0.0178	0.0209	0.0207	0.0206
5.2	0.714e-2	0.710e-2	0.706e-2	0.640e-2	0.707e-2	0.704e-2	0.704e-3
5.4	0.737e-3	0.736e-3	0.7356e-3	0.7111e-3	0.7358e-3	0.7355e-3	0.7355e-3

$$\beta'_{\mathrm{IR},LNN} = \sum_{j=1}^{\infty} d_{j,F}\Delta_f^j = \sum_{j=1}^{\infty} \hat{d}_{j,F}\Delta_r^j. \qquad (4.30)$$

We denote the value of $\beta'_{\mathrm{IR},LNN}$ obtained from this series calculated to order $O(\Delta_f^p)$ as $\beta'_{\mathrm{IR},LNN,\Delta_f^p}$.

From Eqs. (4.5)–(4.8), we find that the approach to the LNN limits for $\hat{d}_{j,F}$ involves correction terms that vanish like $1/N_c^2$. This is the same property that was found in [20,21] and, in the same way, it means that the approach to the LNN limit for finite N_c and N_f with fixed $r = N_f/N_c$ is rather rapid, as discussed in [21]. In [13] we gave the $\hat{d}_{j,F}$ for $1 \le n \le 4$; in addition to $\hat{d}_1 = 0$ (which holds for any G and R), these are

$$\hat{d}_{2,F} = \frac{2^4}{3^2 \cdot 5^2} = 0.0711111, \qquad (4.31)$$

$$\hat{d}_{3,F} = \frac{416}{3^3 \cdot 5^4} = 2.465185 \times 10^{-2}, \qquad (4.32)$$

and

$$\hat{d}_{4,F} = \frac{5868512}{3^5 \cdot 5^{10}} - \frac{5632}{3^4 \cdot 5^6}\zeta_3 = -(2.876137 \times 10^{-3}). \qquad (4.33)$$

Here we give the next higher coefficient:

$$\hat{d}_{5,F} = -\frac{9542225632}{3^6 \cdot 5^{14}} - \frac{1444864}{3^5 \cdot 5^9}\zeta_3 + \frac{360448}{3^5 \cdot 5^8}\zeta_5 = -(1.866490 \times 10^{-3}). \qquad (4.34)$$

In these equations we have indicated the simple factorizations of the denominators that were already evident in the general analytic expressions (4.5)–(4.8). Although the numerical coefficients in the numerators of terms in Eq. (4.34) do not, in general, have simple factorizations, they do contain various powers of 2; for example, in $\hat{d}_{5,F}$, $1444864 = 2^{10} \cdot 17 \cdot 83$, etc. Thus, numerically, to order Δ_r^5, for the LNN limit of this theory with $R = F$, we have

$$\beta'_{\mathrm{IR},LNN} = \Delta_r^2[7.1111 \times 10^{-2} + (2.4652 \times 10^{-2})\Delta_r - (2.8761 \times 10^{-3})\Delta_r^2 - (1.8665 \times 10^{-3})\Delta_r^3 + O(\Delta_r^4)], \qquad (4.35)$$

where the coefficients are given to the indicated floating-point precision. We may again calculate ratios of successive magnitudes of these coefficients to get a rough estimate of the range over which the small-Δ_r expansion is reliable in this LNN limit. We find

$$\frac{\hat{d}_{2,F}}{\hat{d}_{3,F}} = 2.885, \qquad (4.36)$$

$$\frac{\hat{d}_{3,F}}{|\hat{d}_{4,F}|} = 8.571, \qquad (4.37)$$

and

$$\frac{|\hat{d}_{4,F}|}{|\hat{d}_{5,F}|} = 1.541. \qquad (4.38)$$

For $r \in I_{\mathrm{IRZ},r}$, the maximal value of Δ_r is $(\Delta_r)_{\max} = 75/26 = 2.885$. The first two ratios, (4.36) and (4.37), suggest that the Δ_r expansion for β'_{IR} may be reasonably reliable over a reasonable fraction of the interval $I_{\mathrm{IRZ},r}$. From the third ratio, (4.38), we infer that the expansion is expected to be more accurate in the upper portion of the interval $I_{\mathrm{IRZ},r}$ than the lower portion.

In Ref. [13] we presented a comparison of these scheme-independent calculations of $\beta'_{\mathrm{IR},LNN}$ calculated up to the Δ_r^4 order with the results of conventional n-loop calculations, denoted $\beta'_{\mathrm{IR},n\ell,LNN}$, computed up to the $n=4$ loop order for which the b_n were known at that time. We refer the reader to [13] for details of this discussion. Here we shall extend this comparison to the Δ_r^5 order. In Table IX we list the numerical values of these conventional n-loop calculations up to $n=4$, in comparison with our scheme-independent results calculated to $O(\Delta_r^p)$ with p up to 5. (The conventional 4-loop values $\beta'_{\mathrm{IR},4\ell}$ for some values of r toward the lower part of $I_{\mathrm{IRZ},r}$ supersede the corresponding entries in Table II of [13].) Both $\beta'_{\mathrm{IR},n\ell}$ and $\beta'_{\mathrm{IR},\Delta_r^n}$ use, as inputs, the coefficients of the beta function up to loop order n, although $\beta'_{\mathrm{IR},\Delta_r^n}$ does this in a scheme-independent manner. We see that, especially for r values in the upper part of the interval $I_{\mathrm{IRZIRZ},r}$, the results are rather close, and, furthermore, as expected, for a given r, the higher the loop level n and the truncation order p in the respective calculations of $\beta'_{\mathrm{IR},n\ell}$ in the $\overline{\mathrm{MS}}$ scheme and the scheme-independent $\beta'_{\mathrm{IR},\Delta_r^p}$, the better the agreement between these two results. Toward the lower end of the interval $I_{\mathrm{IRZ},r}$, both the conventional expansion of β'_{IR} and the scheme-independent expansion of β'_{IR} in powers of Δ_r become less reliable, and hence it is understandable that the results differ from each other in this lower part of $I_{\mathrm{IRZ},r}$.

D. $\beta'_{\mathrm{IR},\Delta_f^5}$ for $G=\mathrm{SU}(N_c)$ and $R=adj$

Here we calculate the d_j and hence $\beta'_{\mathrm{IR},\Delta_f^j}$ for j up to $j=5$ in the $\mathrm{SU}(N_c)$ gauge theory with fermion representation $R=adj$. As was discussed above, in this case, the interval I_{IRZ} contains the single Dirac value, $N_f=2$. For this value of N_f, Eq. (3.60) yields $\Delta_f=3/4$. We recall that the d_j for $2\le j\le 4$ are [13]

$$d_{2,adj}=\left(\frac{2}{3}\right)^4=0.19753, \tag{4.39}$$

$$d_{3,adj}=\frac{2^8}{3^7}=0.11706, \tag{4.40}$$

and

$$d_{4,adj}=\frac{46871}{2^2\cdot 3^{12}}+\frac{2368}{3^{10}N_c^2}=0.022049+0.040102N_c^{-2}. \tag{4.41}$$

Here, from our new general result (4.9) for d_5, we obtain the next coefficient for this case of the adjoint representation:

$$d_{5,adj}=-\frac{7141205}{2^3\cdot 3^{16}}+\frac{5504}{3^{12}}\zeta_3-\left(\frac{30928}{3^{14}}+\frac{465152}{3^{13}}\zeta_3\right)N_c^{-2}=-(0.828739\times 10^{-2})-0.357173N_c^{-2}. \tag{4.42}$$

While the $d_{j,adj}$ with $2\le j\le 4$ are positive-definite, we thus find that $d_{5,adj}$ is negative-definite. These results on signs are listed in Table VII. In the $N_c\to\infty$ (LN) limit of Eq. (3.22), the values of $\hat{d}_{j,adj}$ can be read off directly from our general results in Eqs. (4.39)–(4.42); for example, $\hat{d}_{4,adj}=46871/(2^2\cdot 3^{12})$, etc.

With these coefficients, one can again compute ratios to obtain a crude idea of the region over which the small-Δ_f series expansion is reliable. We have

$$\frac{d_{2,adj}}{d_{3,adj}}=\frac{3^3}{2^4}=1.687 \tag{4.43}$$

and, taking the large-N_c limit for simplicity,

$$\lim_{N_c\to\infty}\frac{d_{3,adj}}{d_{4,adj}}=\frac{3^5\cdot 2^{10}}{46871}=5.309 \tag{4.44}$$

$$\lim_{N_c\to\infty}\frac{d_{4,adj}}{|d_{5,adj}|}=\frac{7593102}{7141205-3566592\zeta_3}=2.6606. \tag{4.45}$$

Since $\Delta_f=0.75$ for $N_f=2$, these ratios indicate that the small-Δ_f expansion should be reasonably accurate here.

E. $\beta'_{\mathrm{IR},\Delta_f^5}$ for $G=\mathrm{SU}(N_c)$ and $R=S_2$, A_2

Here we present our results for the d_j coefficients and hence $\beta'_{\mathrm{IR},\Delta_f^j}$ with j up to 5 for $G=\mathrm{SU}(N_c)$ and N_f fermions in the symmetric and antisymmetric rank-2 tensor representations, S_2 and A_2. As before with $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$, since many formulas for these two cases are simply related to each other by sign reversals in certain terms, it is convenient to treat these two cases together, denoting them collectively as T_2. We recall that for $R=A_2$, we restrict to $N_c\ge 3$.

From our general formulas (4.5)–(4.9), we obtain the following, where the upper and lower signs refer to the S_2 and A_2 special cases of T_2, respectively, and $F_\pm$ was defined in Eq. (3.72):

$$d_{2,T_2}=\frac{2^3(N_c\pm 2)^2}{3^2F_\pm} \tag{4.46}$$

$$d_{3,T_2}=\frac{2^4(N_c\pm 2)^3(8N_c^2\pm 3N_c-6)}{3^3N_cF_\pm^2} \tag{4.47}$$

$$d_{4,T_2} = \frac{(N_c \pm 2)^3}{2\cdot 3^5 N_c^2 F_\pm^5}[(1265517N_c^9 \pm 6305850N_c^8 + 8455112N_c^7 \mp 18825808N_c^6 - 47225264N_c^5 \pm 61021088N_c^4 + 70598528N_c^3 \mp 72131840N_c^2 + 3066624N_c \mp 2044416) \pm 8448N_c^2(N_c \mp 2)(18N_c^2 \pm 11N_c - 22)(12N_c^3 \mp 9N_c^2 \pm 308)\zeta_3] \tag{4.48}$$

and

$$d_{5,T_2} = \frac{(N_c \pm 2)^4}{2\cdot 3^6 N_c^3 F_\pm^7}[(-578437605N_c^{13} \mp 2353001022N_c^{12} - 1643220810N_c^{11} \pm 1685855300N_c^{10} + 12567177608N_c^9 \pm 29240054768N_c^8 - 75390007296N_c^7 \mp 70417381376N_c^6 + 243309040128N_c^5 \mp 27199484928N_c^4 - 228577603584N_c^3 \pm 143780184064N_c^2 - 38053396480N_c \pm 15221358592) + 2^7F_\pm(125388N_c^{11} \pm 372762N_c^{10} - 7324047N_c^9 \mp 9682414N_c^8 + 52934332N_c^7 \mp 12735976N_c^6 - 192234240N_c^5 \pm 112670976N_c^4 + 164609280N_c^3 \mp 111598080N_c^2 + 2973696N_c \mp 1486848)\zeta_3 + 2^{10}\cdot 55N_c^2(N_c \mp 2)F_\pm^2(\mp 87N_c^5 + 259N_c^4 \pm 1134N_c^3 - 3600N_c^2 \mp 5016N_c + 10032)\zeta_5]. \tag{4.49}$$

We find that, in addition to the manifestly positive d_{2,T_2}, the coefficient d_{3,T_2} is also positive for all relevant N_c. Here, by "relevant N_c", we mean $N_c \geq 2$ for S_2 and $N_c \geq 3$ for A_2. In contrast, while d_{4,S_2} is positive for all relevant N_c, we find that d_{4,A_2} is negative for $N_c = 3$, 4, 5, passes through zero at $N_c = 5.515$, and is positive for $N_c \geq 6$. Further, we find that d_{5,S_2} and d_{5,A_2} are both negative for their respective physical ranges, $N_c \geq 2$ and $N_c \geq 3$. These sign properties are listed in Table VII.

Some general comments are in order concerning these d_{j,T_2} expressions. These are analogous to the comments that we made for the κ_{j,T_2} coefficients. The property that all of the d_{j,A_2} coefficients contain an overall factor of $(N_c - 2)$ (possibly raised to a power higher than 1), and hence vanish for $N_c = 2$, is a consequence of the fact that for $N_c = 2$, the A_2 representation is a singlet, so for SU(2), fermions in the A_2 = singlet representation have no gauge interactions and do not contribute to the beta function or $\beta'_{\rm IR}$.

Furthermore, if $N_c = 2$, then the S_2 representation is the same as the adjoint representation, so the coefficients must satisfy the equality $d_{j,S_2} = d_{j,adj}$ for this SU(2) case, and we have checked that they do. This equality requires (i) that the term proportional to ζ_3 in d_{4,S_2} must be absent if $N_c = 2$, since $d_{4,adj}$ does not contain any ζ_3 term, and this is accomplished by the factor of $(N_c - 2)$ multiplying the ζ_3 term in d_{4,S_2}; and (ii) the term proportional to ζ_5 in d_{5,S_2} must be absent if $N_c = 2$, since $d_{5,adj}$ does not contain any ζ_5 term, and this is accomplished by the factor $(N_c - 2)$ multiplying this ζ_5 term in d_{5,S_2}. Similarly, as observed before, if $N_c = 3$, then the A_2 representation is the same as the conjugate fundamental representation, $\bar{F}$, so the coefficients must satisfy the equality $d_{j,A_2} = d_{j,F}$ for this SU(3) case, and we have checked that they do.

In the LN limit (3.22), as discussed above in the case of the anomalous dimension $\gamma_{{\rm IR},T_2}$, the upper ends of the interval $I_{\rm IRZ}$ for the S_2 and A_2 theories approach the same value, N_{u,T_2}, given in Eq. (3.86), and similarly the lower ends of this interval for these S_2 and A_2 theories approach the same value, N_{ℓ,T_2}, given in Eq. (3.87). We denote

$$\hat{d}_{j,T_2} = \lim_{LN} d_{j,T_2}, \tag{4.50}$$

and we find

$$\hat{d}_{j,S_2} = \hat{d}_{j,A_2}, \tag{4.51}$$

which we denote simply as $\hat{d}_{j,T_2}$. Hence,

$$\lim_{LN} \beta'_{{\rm IR},S_2} = \lim_{LN} \beta'_{{\rm IR},A_2}. \tag{4.52}$$

Further, again in analogy with Eq. (3.98) and for the same reasons concerning group invariants in the LN limit, we have

$$\hat{d}_{j,T_2} = 2^{-j}\hat{d}_{j,adj}. \tag{4.53}$$

From our general expressions, we calculate

$$\hat{d}_{2,T_2} = \frac{2^2}{3^4} = 0.049383 \tag{4.54}$$

$$\hat{d}_{3,T_2} = \frac{2^5}{3^7} = 1.46319 \times 10^{-2} \tag{4.55}$$

$$\hat{d}_{4,T_2} = \frac{46871}{2^6 \cdot 3^{12}} = 1.37806 \times 10^{-3} \tag{4.56}$$

and

$$\hat{d}_{5,T_2} = -\frac{7141205}{2^8 \times 3^{16}} + \frac{172}{3^{12}}\zeta_3 = -(2.58981 \times 10^{-4}). \tag{4.57}$$

To estimate the region over which the Δ_f expansion converges, we calculate the ratios of adjacent coefficients. We have

$$\frac{d_{2,T_2}}{d_{3,T_2}} = \frac{3N_c(18N_c^2 \pm 11N_c - 22)}{(N_c \pm 2)(8N_c^2 \pm 3N_c - 6)} \tag{4.58}$$

and similarly for the ratios $d_{j-1,T_2}/d_{j,T_2}$ for $j = 4, 5$. For the LN limit,

$$\frac{\hat{d}_{2,T_2}}{\hat{d}_{3,T_2}} = \left(\frac{3}{2}\right)^3 = 3.375 \tag{4.59}$$

$$\frac{\hat{d}_{3,T_2}}{\hat{d}_{4,T_2}} = \frac{497664}{46871} = 10.618 \tag{4.60}$$

and

$$\frac{\hat{d}_{4,T_2}}{|\hat{d}_{5,T_2}|} = 5.321. \tag{4.61}$$

Since formally, $(\Delta_f)_{\max} = 3.375$ from Eq. (3.88) and $\Delta_f = 5.5$ for $N_f = 2$, these ratios indicate that the Δ_f expansion for the LN limit of this $R = T_2$ case should be reasonably accurate in the interval $I_{\rm IRZ}$ for this case.

V. IR ZERO OF β_ξ IN THE LNN LIMIT

In this section we analyze the zeros of the rescaled five-loop beta function in the LNN limit. This elucidates further the result that we first found for a finite value of N_c, namely $N_c = 3$, in [15], that for SU(3), the five-loop beta function only has a physical IR zero in the upper range of the interval $I_{\rm IRZ}$. We denote the n-loop rescaled beta function (4.22) in this LNN limit as $\beta_{\xi,n\ell}$, and its IR zero (if such a zero exists) as $\xi_{{\rm IR},n\ell} = 4\pi x_{{\rm IR},n\ell}$. The analytic expressions of $\xi_{{\rm IR},2\ell}$ and $\xi_{{\rm IR},3\ell}$ were given in [21], together with numerical values of $\xi_{{\rm IR},n\ell}$ for $1 \le n \le 4$. Here we extend these results to the five-loop level, using the coefficient $\hat{b}_5$ in Eq. (4.25). As noted before, we use the $\hat{b}_n$ with $3 \le n \le 5$ calculated in the $\overline{\rm MS}$ scheme. The reader is referred to [21] for analysis of these zeros up to the four-loop level.

In general, the IR zero of the n-loop beta function, $\beta_{\xi,n\ell}$, is the positive real root closest to the origin (if such a root exists) of the equation

$$\sum_{\ell=1}^{n} \hat{b}_\ell x^{\ell-1} = 0, \tag{5.1}$$

of degree $n-1$ in the variable x. The roots of Eq. (5.1) depend on the $n-1$ ratios $\hat{b}_\ell/\hat{b}_1$ for $2 \le \ell \le n$. In particular, at the five-loop level, Eq. (5.1) is the quartic equation

$$\hat{b}_1 + \hat{b}_2 x + \hat{b}_3 x^2 + \hat{b}_4 x^3 + \hat{b}_5 x^4 = 0. \tag{5.2}$$

To analyze the roots of this equation, it is natural to start with r in the vicinity of $r_u = 11/2$, where $\hat{b}_1 \to 0$ and hence one solution of Eq. (5.2) approaches zero, matching the behavior of $x_{{\rm IR},n\ell}$ for $2 \le n \le 4$ in this limit. As we reduce r from the value r_u in the interval $I_{{\rm IRZ},r}$, we can thus calculate how the physical IR root, $x_{{\rm IR},5\ell} = \xi_{{\rm IR},5\ell}/(4\pi)$, changes. We find that, in contrast to the behavior of the IR zero of the lower-loop beta functions $\beta_{\xi,n\ell}$ with $2 \le n \le 4$, here at the five-loop level, as r decreases past a certain value r_{cx}, Eq. (5.2) (with $\hat{b}_n$, $n = 3, 4, 5$ calculated in the $\overline{\rm MS}$ scheme) ceases to have a physical IR zero. We find that the value of r_{cx} is

$$r_{cx} = 4.32264, \tag{5.3}$$

TABLE X. Values of the IR zero $\xi_{{\rm IR},n\ell}$ of the $\beta_{\xi,n\ell}$ function in the LNN limit for $2 \le n \le 5$ and $r \in I_r$. Notation u (unphysical) means that there is no physical IR zero $\xi_{{\rm IR},5\ell}$ of the 5-loop beta function.

r	$\xi_{{\rm IR},2\ell}$	$\xi_{{\rm IR},3\ell}$	$\xi_{{\rm IR},4\ell}$	$\xi_{{\rm IR},5\ell}$
2.8	28.274	3.573	3.323	u
3.0	12.566	2.938	2.868	u
3.2	7.606	2.458	2.494	u
3.4	5.174	2.076	2.168	u
3.6	3.731	1.759	1.873	u
3.8	2.774	1.489	1.601	u
4.0	2.095	1.252	1.349	u
4.2	1.586	1.041	1.115	u
4.4	1.192	0.8490	0.9003	1.0353
4.6	0.8767	0.6725	0.7038	0.7439
4.8	0.6195	0.5083	0.5244	0.5364
5.0	0.4054	0.3538	0.3603	0.3630
5.2	0.2244	0.2074	0.2089	0.2093
5.4	0.06943	0.06769	0.06775	0.06776

to the indicated floating-point accuracy. This is determined as the relevant root of the discriminant of Eq. (5.2), which is a polynomial of degree 15 in the variable r. (The discriminants of the corresponding equations at loop levels 3 and 4 are polynomials of degree 3 and 8 in r.) For example, for the illustrative value $r = 5$, near to the upper end of the interval $I_{\mathrm{IRZ},r}$, Eq. (5.2) has the solutions in x, expressed in terms of $\xi = 4\pi x$: $\xi = 0.36300$, 1.69540, and $-1.48884 \pm 1.08446i$. Of these, we identify the first as the IR zero, $\xi_{\mathrm{IR},5\ell}$. As r decreases and approaches r_{cx} from above, the two real roots approach a common value, $\xi \simeq 1.312$ and as r decreases below r_{cx}, Eq. (5.2) has only two complex-conjugate pairs of solutions, roots, but no real positive solution. In Table X we list our new results for $\xi_{\mathrm{IR},5\ell}$, in comparison with the previously calculated values of $\xi_{\mathrm{IR},n\ell}$ in the LNN limit with $2 \le n \le 4$ from Table III of [21]. Although we list $\xi_{\mathrm{IR},n\ell}$ values extending to the lower part of the interval $I_{\mathrm{IRZ},r}$ for completeness, it is clear that a number of these values are too large for the perturbative calculations to be reliable. For values of r where the five-loop beta function (calculated in the $\overline{\mathrm{MS}}$ scheme) has no physical IR zero, we denote this as unphysical (u).

We note that the absence of a physical IR zero in the five-loop beta function (calculated in the $\overline{\mathrm{MS}}$ scheme) for N_f values in the lower portion of the interval I_{IRZ} does not necessarily imply that higher-loop calculations would yield similarly unphysical results. We gave an example of this in Sec. VIII of the second paper in [38], using an illustrative exact beta function. In this example, it was shown that a certain order of truncation of the Taylor series expansion in powers of α for this beta function did not yield any physical IR zero, but higher orders did converge toward this zero.

VI. Δ_f EXPANSION FOR α_{IR} TO $O(\Delta_f^4)$

A. General G and R

Since the exact α_{IR} (and also the n-loop approximation to this exact α_{IR}) vanishes as functions of Δ_f, it follows that one can expand it as a power series in this variable. This expansion was given above as Eq. (2.9), and it was noted that the calculation of the coefficient a_j requires, as input, the ℓ-loop beta function coefficients b_ℓ with $1 \le \ell \le j+1$. We denote the truncation of this infinite series (2.9) to maximal power $j = p$ as $\alpha_{\mathrm{IR},\Delta_f^p}$. Here we present a calculation of this series to $O(\Delta_f^4)$, which is the highest order to which it has been calculated. Since α_{IR} is scheme-dependent, it follows that the a_j coefficients in Eq. (2.9) are also scheme-dependent, in contrast to the scheme-independent coefficients κ_j and d_j in Eqs. (1.2) and (1.3). Nevertheless, it is still worthwhile to calculate these coefficients a_j and the resultant finite-order approximations $\alpha_{\mathrm{IR},\Delta_f^p}$, for several reasons. First, this method has the advantage that $\alpha_{\mathrm{IR},\Delta_f^p}$ is always physical and thus avoids the problem that we found in [15] and have further studied above, that the five-loop beta function calculated in the $\overline{\mathrm{MS}}$ scheme does not have a physical IR zero in the lower part of the interval I_{IRZ}. In [14], for the special case $G = \mathrm{SU}(3)$ and $R = F$, we presented the a_j (denoted $\tilde{a}_j$ there) for $1 \le j \le 4$.

Here, as a new result, we present the expressions for the a_j for arbitrary G and R, for $1 \le j \le 4$. For this purpose, we use the n-loop beta function coefficients b_n with $3 \le n \le 5$ calculated in the $\overline{\mathrm{MS}}$ scheme. In particular, our result for a_4 makes use of the recently calculated five-loop beta function for general G and R [17].

For general G and R, recalling the definition of the denominator factor $D = 7C_A + 11C_f$ in Eq. (3.1), we find

$$a_1 = \frac{4T_f}{3C_A D} \tag{6.1}$$

$$a_2 = \frac{2T_f^2(-287C_A^2 + 1208C_A C_f + 924C_f^2)}{3^3 C_A^2 D^3} \tag{6.2}$$

$$\begin{aligned} a_3 = \frac{2T_f}{3^5 C_A^4 D^5}\Bigg[& C_A T_f^2(-71491C_A^4 + 372680C_A^3 C_f + 2102252C_A^2 C_f^2 + 835560C_A C_f^3 + 836352C_f^4) \\ & - 2560T_f^2 D\frac{d_A^{abcd}d_A^{abcd}}{d_A} + 45056C_A T_f D\frac{d_R^{abcd}d_A^{abcd}}{d_A} - 170368C_A^2 T_f D\frac{d_R^{abcd}d_R^{abcd}}{d_A} \\ & + 4224D\left[3C_A^2 T_f^2 D(C_A - C_f) + 16T_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A} - 104C_A T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A} + 88C_A^2\frac{d_R^{abcd}d_R^{abcd}}{d_A}\right]\zeta_3\Bigg] \end{aligned} \tag{6.3}$$

and

$$a_4 = \frac{T_f^2}{2\cdot 3^7 C_A^5 D^7}\Bigg[C_A T_f^2(194849725C_A^6 - 684457480C_A^5 C_f + 4175949036C_A^4 C_f^2 + 13292017040C_A^3 C_f^3$$
$$+ 2617931536C_A^2 C_f^4 + 8758858944C_A C_f^5 + 85865472C_f^6)$$
$$+ 2^{10} T_f^2 D \frac{d_A^{abcd} d_A^{abcd}}{d_A}(21287C_A^2 - 5504C_A C_f - 19140C_f^2)$$
$$+ 2^{10} C_A T_f D \frac{d_R^{abcd} d_A^{abcd}}{d_A}(-194005C_A^2 + 253231C_A C_f + 136488C_f^2)$$
$$+ 2^8\cdot 11^2 C_A^2 D \frac{d_R^{abcd} d_R^{abcd}}{d_A}(917C_A^2 - 40412C_A C_f + 26796C_f^2)$$
$$- 2304D[C_A T_f^2 D(15456C_A^4 - 75039C_A^3 C_f + 45716C_A^2 C_f^2 + 23848C_A C_f^3 + 2112C_f^4)$$
$$+ 16T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(8610C_A^2 - 15037C_A C_f - 14036C_f^2) - 8C_A T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(95984C_A^2 - 190355C_A C_f - 135036C_f^2)$$
$$+ 88C_A^2 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(3199C_A^2 - 26004C_A C_f - 17908C_f^2)]\zeta_3$$
$$+ 337920C_A D^2\Bigg[-9C_A T_f^2 D(C_A - C_f)(C_A + 2C_f) - 160T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}$$
$$+ 80T_f(10C_A + 3C_f)\frac{d_R^{abcd} d_A^{abcd}}{d_A} + 440C_A(C_A - 3C_f)\frac{d_R^{abcd} d_R^{abcd}}{d_A}\Bigg]\zeta_5\Bigg]. \tag{6.4}$$

We next specialize to the case $G = \mathrm{SU}(N_c)$ and give explicit reductions of these general formulas for the representations of interest here.

B. $R = F$

For $R = F$, our general results (6.1)–(6.4) reduce to the following expressions:

$$a_{1,F} = \frac{4}{3(25N_c^2 - 11)} \tag{6.5}$$

$$a_{2,F} = \frac{4(548N_c^4 - 1066N_c^2 + 231)}{3^3 N_c (25N_c^2 - 11)^3} \tag{6.6}$$

$$a_{3,F} = \frac{2^3}{3^5 N_c^2 (25N_c^2 - 11)^5}[(730529N_c^8 - 1105385N_c^6 - 719758N_c^4 + 389235N_c^2 + 52272)$$
$$+ 1584N_c^2(25N_c^2 - 11)(25N_c^4 - 18N_c^2 + 77)\zeta_3] \tag{6.7}$$

and

$$a_{4,F} = \frac{2^2}{3^7 N_c^3 (25N_c^2 - 11)^7}[(2783259085N_c^{12} - 7278665930N_c^{10} + 4578046419N_c^8 - 1719569282N_c^6$$
$$+ 2905511455N_c^4 - 1137735654N_c^2 + 1341648)$$
$$+ 288(25N_c^2 - 11)(548025N_c^{10} - 1857036N_c^8 + 4694107N_c^6 - 5482510N_c^4 + 1098130N_c^2 + 2904)\zeta_3$$
$$- 190080N_c^2(25N_c^2 - 11)^2(40N_c^6 - 27N_c^4 + 124N_c^2 - 209)\zeta_5]. \tag{6.8}$$

We have checked that setting $N_c = 3$ in our new a_4 coefficient in Eq. (6.8) yields agreement with the value that we obtained previously for this special case in [Eq. (14) of] Ref. [14].

We comment next on the signs of these coefficients. The coefficient a_1 is manifestly positive for arbitrary group G and fermion representation R. We find that $a_{2,F}$ and $a_{3,F}$ are also positive for all physical $N_c \geq 2$. In contrast, we find that $a_{4,F}$ is negative for $N_c = 2$ and positive for $N_c \geq 3$. With N_c generalized from positive integers to positive real numbers in the range $N_c \geq 2$, we calculate that as N_c increases through the value $N_c = 2.1184$ (given to the indicated accuracy), $a_{4,F}$ passes through zero with positive slope.

We list below the explicit numerical expressions for $\alpha_{\rm IR}$ to order Δ_f^4, for $N_c = 2$, 3, 4 and $R = F$, given to the indicated floating-point precision:

$$\mathrm{SU}(2)\colon\ \alpha_{\mathrm{IR},F,\Delta_f^4} = \Delta_f[(0.18826 + (0.62521\times 10^{-2})\Delta_f + (0.70548\times 10^{-2})\Delta_f^2 - (0.45387\times 10^{-4})\Delta_f^3] \tag{6.9}$$

$$\mathrm{SU}(3)\colon\ \alpha_{\mathrm{IR},F,\Delta_f^4} = \Delta_f[(0.078295 + (2.2178\times 10^{-3})\Delta_f + (1.1314\times 10^{-3})\Delta_f^2 + (2.1932\times 10^{-5})\Delta_f^3] \tag{6.10}$$

and

$$\mathrm{SU}(4)\colon\ \alpha_{\mathrm{IR},F,\Delta_f^4} = \Delta_f[(0.043072 + (0.97619\times 10^{-3})\Delta_f + (0.33823\times 10^{-3})\Delta_f^2 + (0.71999\times 10^{-5})\Delta_f^3]. \tag{6.11}$$

In Figs. 10–12 we show $\alpha_{\mathrm{IR},F,\Delta_f^p}$ for $N_c = 2$, 3, 4 and $1 \leq p \leq 4$ as a function of N_f. Note that in Fig. 10 the curves for $p = 3$ and $p = 4$ are so close as to be indistinguishable for this range of N_f.

In Table XI we compare the values of the IR zero of the n-loop beta function for $1 \leq n \leq 4$ from [19] with our values of $\alpha_{\mathrm{IR},F,\Delta_f^p}$ for $1 \leq p \leq 4$ and $N_c = 2, 3, 4$. Since the calculation of $\alpha_{\mathrm{IR},n\ell}$ uses the ℓ-loop beta function coefficients b_ℓ with $1 \leq \ell \leq n$, while the calculation of $\alpha_{\mathrm{IR},\Delta_f^p}$ uses the b_ℓ for $1 \leq \ell \leq p+1$, the closest comparison is of $\alpha_{\mathrm{IR},n\ell}$ with $\alpha_{\mathrm{IR},\Delta_f^{n-1}}$, which both use n-loop information from the beta function. Although, for completeness, we include values of $\alpha_{\mathrm{IR},2\ell}$ for N_f extending down to the lower end of the respective intervals I_{IRZ} for each value of N_c, we caution that in a number of cases, including $N_f = 6$ for SU(2), $N_f = 9$ for SU(3), and $10 \leq N_f \leq 12$ for SU(4), these values of $\alpha_{\mathrm{IR},2\ell}$ are too large for the perturbative

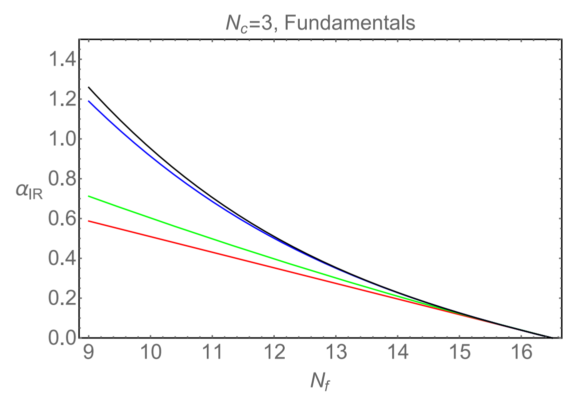

FIG. 11. Plot of $\alpha_{\mathrm{IR},F,\Delta_f^p}$ with $1 \leq p \leq 4$ for $G = \mathrm{SU}(3)$, as functions of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\alpha_{\mathrm{IR},F,\Delta_f}$ (red), $\alpha_{\mathrm{IR},F,\Delta_f^2}$ (green), $\alpha_{\mathrm{IR},F,\Delta_f^3}$ (blue), and $\alpha_{\mathrm{IR},F,\Delta_f^4}$ (black).

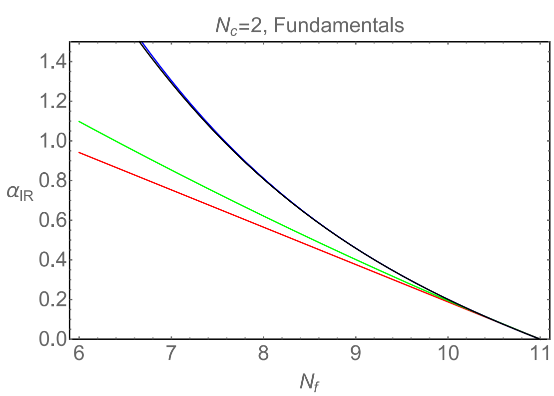

FIG. 10. Plot of $\alpha_{\mathrm{IR},F,\Delta_f^p}$ (denoted as α_{IR} on the vertical axis) with $1 \leq p \leq 4$ for $G = \mathrm{SU}(2)$, as functions of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\alpha_{\mathrm{IR},F,\Delta_f}$ (red), $\alpha_{\mathrm{IR},F,\Delta_f^2}$ (green), $\alpha_{\mathrm{IR},F,\Delta_f^3}$ (blue), and $\alpha_{\mathrm{IR},F,\Delta_f^4}$ (black). Note that the curves for $\alpha_{\mathrm{IR},F,\Delta_f^3}$ and $\alpha_{\mathrm{IR},\Delta_f^4}$ are so close as to be indistinguishable in this figure.

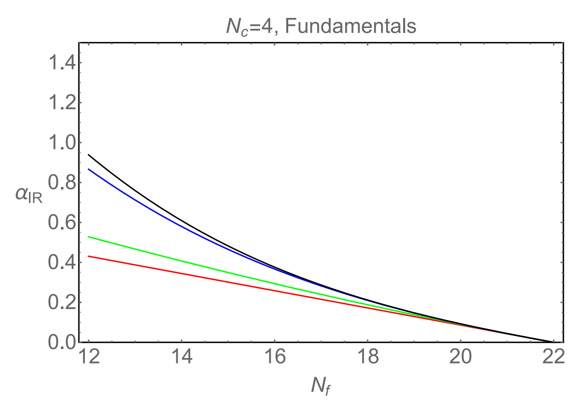

FIG. 12. Plot of $\alpha_{\mathrm{IR},F,\Delta_f^p}$ with $1 \leq p \leq 4$ for $G = \mathrm{SU}(4)$, as functions of $N_f \in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\alpha_{\mathrm{IR},F,\Delta_f}$ (red), $\alpha_{\mathrm{IR},F,\Delta_f^2}$ (green), $\alpha_{\mathrm{IR},F,\Delta_f^3}$ (blue), and $\alpha_{\mathrm{IR},F,\Delta_f^4}$ (black).

TABLE XI. Values of $\alpha_{\mathrm{IR},\Delta_f^p}$ with $1 \le p \le 4$ for $N_c = 2, 3, 4$ and $R = F$, as functions of $N_f \in I_{\mathrm{IRZ}}$, together with $\alpha_{\mathrm{IR},2\ell}$ and $\overline{\mathrm{MS}}$ values of n-loop $\alpha_{\mathrm{IR},n\ell}$ with $3 \le n \le 4$ from [19], for comparison.

N_c	N_f	$\alpha_{\mathrm{IR},2\ell}$	$\alpha_{\mathrm{IR},3\ell}$	$\alpha_{\mathrm{IR},4\ell}$	$\alpha_{\mathrm{IR},\Delta_f}$	$\alpha_{\mathrm{IR},\Delta_f^2}$	$\alpha_{\mathrm{IR},\Delta_f^3}$	$\alpha_{\mathrm{IR},\Delta_f^4}$
2	6	11.42	1.645	2.395	0.941	1.098	1.979	1.951
2	7	2.83	1.05	1.21	0.753	0.853	1.305	1.293
2	8	1.26	0.688	0.760	0.565	0.621	0.8115	0.808
2	9	0.595	0.418	0.444	0.377	0.402	0.458	0.457
2	10	0.231	0.196	0.200	0.188	0.1945	0.202	0.2015
3	9	5.24	1.028	1.072	0.587	0.712	1.19	1.26
3	10	2.21	0.764	0.815	0.509	0.603	0.913	0.952
3	11	1.23	0.578	0.626	0.431	0.498	0.686	0.706
3	12	0.754	0.435	0.470	0.352	0.397	0.500	0.509
3	13	0.468	0.317	0.337	0.274	0.301	0.350	0.353
3	14	0.278	0.215	0.224	0.196	0.210	0.227	0.228
3	15	0.143	0.123	0.126	0.117	0.122	0.126	0.126
3	16	0.0416	0.0397	0.0398	0.0391	0.0397	0.0398	0.0398
4	11	14.00	0.972	0.943	0.474	0.592	1.042	1.1475
4	12	3.54	0.754	0.759	0.431	0.528	0.867	0.939
4	13	1.85	0.6035	0.628	0.388	0.467	0.713	0.7605
4	14	1.16	0.489	0.521	0.345	0.407	0.580	0.610
4	15	0.783	0.397	0.428	0.3015	0.349	0.465	0.483
4	16	0.546	0.320	0.345	0.258	0.294	0.367	0.376
4	17	0.384	0.254	0.271	0.215	0.240	0.282	0.2865
4	18	0.266	0.194	0.205	0.172	0.188	0.210	0.211
4	19	0.175	0.140	0.145	0.129	0.138	0.147	0.148
4	20	0.105	0.091	0.092	0.0861	0.09005	0.0928	0.0929
4	21	0.0472	0.044	0.044	0.0431	0.04405	0.0444	0.0444

n-loop calculations to be reliable. Concerning the comparison of the higher-order n-loop values of $\alpha_{\mathrm{IR},n\ell}$ with our values of $\alpha_{\mathrm{IR},F,\Delta_f^p}$, we see that for a given N_c and N_f, at the upper end of the non-Abelian Coulomb phase, the values of $\alpha_{\mathrm{IR},\Delta_f^{n-1}}$ and $\alpha_{\mathrm{IR},n\ell}$ are quite close to each other, but as N_f decreases in this NACP in the interval I_{IRZ}, $\alpha_{\mathrm{IR},\Delta_f^{n-1}}$ becomes slightly larger than $\alpha_{\mathrm{IR},n\ell}$.

In the LNN limit, for the IR zero of the rescaled beta function, we write

$$\xi_{\mathrm{IR}} = 4\pi \sum_{j=1}^{\infty} \hat{a}_{j,F} \Delta_r^j \qquad (\text{LNN limit}), \tag{6.12}$$

where

$$\hat{a}_{j,F} = \lim_{LNN} N_c^{j+1} a_{j,F}. \tag{6.13}$$

From our results for $a_{j,F}$, we calculate

$$\hat{a}_{1,F} = \frac{4}{3 \cdot 5^2} = 0.053333 \tag{6.14}$$

$$\hat{a}_{2,F} = \frac{2192}{3^3 \cdot 5^6} = 0.519585 \times 10^{-2} \tag{6.15}$$

$$\hat{a}_{3,F} = \frac{5844232}{3^5 \cdot 5^{10}} + \frac{1408}{3^3 \cdot 5^6}\zeta_3 = 0.647460 \times 10^{-2} \tag{6.16}$$

and

$$\hat{a}_{4,F} = \frac{2226607268}{3^7 \cdot 5^{13}} + \frac{935296}{3^4 \cdot 5^{10}}\zeta_3 - \frac{45056}{3^4 \cdot 5^8}\zeta_5 = 0.778770 \times 10^{-3}. \tag{6.17}$$

Thus, in the LNN limit, the expansion of ξ_{IR}, to $O(\Delta_r^4)$ is

$$\xi_{\mathrm{IR},\Delta_r^4} = 4\pi\Delta_r[0.053333 + (0.519585 \times 10^{-2})\Delta_r + (0.647460 \times 10^{-2})\Delta_r^2 + (0.778770 \times 10^{-3})\Delta_r^3]. \tag{6.18}$$

C. $R = adj$

For $R = adj$, our general results (6.1)–(6.4) reduce to the following expressions:

$$a_{1,adj} = \frac{2}{3^3 N_c} = \frac{0.074747}{N_c} \tag{6.19}$$

$$a_{2,adj} = \frac{205}{2^2 \cdot 3^7 N_c} = \frac{0.023434}{N_c} \tag{6.20}$$

$$a_{3,adj} = \frac{49129}{2^4 \cdot 3^{11} N_c} - \frac{296}{3^9 N_c^3} = \frac{0.017333}{N_c} - \frac{0.015038}{N_c^3} \tag{6.21}$$

and

$$a_{4,adj} = \left(\frac{38811689}{2^8 \cdot 3^{15}} - \frac{40}{3^9}\zeta_3\right)\frac{1}{N_c} + \left(-\frac{3157}{3^{13}} + \frac{25616}{3^{12}}\zeta_3\right)\frac{1}{N_c^3} = \frac{0.0081230}{N_c} + \frac{0.055960}{N_c^3}. \tag{6.22}$$

The coefficients $a_{j,adj}$ with $j = 1,\ 2,\ 4$ are manifestly positive, and we find that $a_{3,adj}$ is also positive for all $N_c \geq 2$.

Since for the adjoint representation, $R = adj$, the upper and lower boundaries of the interval $I_{\rm IRZ}$, $N_{u,T_2} = 11/2$ in Eq. (3.58) and $N_{\ell,adj} = 17/16$ in (3.59), are independent of N_f, it follows that $\Delta_f = N_u - N_f$ is also independent of N_c. From the general formula (2.9), in the LN limit of a theory with fermions in a two-index representation R_2, including the adjoint and symmetric and antisymmetric tensors, we can write

$$\xi_{\rm IR} = 4\pi \sum_{j=1}^{\infty} \hat{a}_{j,R_2} \Delta_f^j \ \ ({\rm LN\, limit}), \tag{6.23}$$

where

$$\hat{a}_{j,R_2} = \lim_{LN} N_c a_{j,R_2}. \tag{6.24}$$

From our calculations above, setting $R_2 = adj$, we have

$$\hat{a}_{1,adj} = \frac{2}{3^3} = 0.074747 \tag{6.25}$$

$$\hat{a}_{2,adj} = \frac{205}{2^2 \cdot 3^7} = 0.023434 \tag{6.26}$$

$$\hat{a}_{3,adj} = \frac{49129}{2^4 \cdot 3^{11}} = 0.017333 \tag{6.27}$$

and

$$\hat{a}_{4,adj} = \frac{38811689}{2^8 \cdot 3^{15}} - \frac{40}{3^9}\zeta_3 = 0.0081230. \tag{6.28}$$

D. $R = S_2,\ A_2$

For R equal to the symmetric or antisymmetric rank-2 tensor representations, S_2 and A_2, we give the reductions of our general results (6.1)–(6.4) next. As before, it is convenient to consider these together, since many terms differ only by sign reversal. As above, the upper and lower signs refer to the S_2 and A_2 representations, respectively. Also, as before, for A_2, we require that $N_c \geq 3$. Recalling the definition of the denominator factor $F_\pm$ in Eq. (3.72), we have

$$a_{1,T_2} = \frac{2(N_c \pm 2)}{3F_\pm} \tag{6.29}$$

$$a_{2,T_2} = \frac{(N_c \pm 2)^2 (1845 N_c^4 \pm 3056 N_c^3 - 5188 N_c^2 \mp 3696 N_c + 3696)}{2 \cdot 3^3 N_c F_\pm^3} \tag{6.30}$$

$$a_{3,T_2} = \frac{(N_c \pm 2)^2}{2^2 \cdot 3^5 N_c^2 F_\pm^5}[(3979449 N_c^9 \pm 16999002 N_c^8 + 761444 N_c^7 \mp 52233472 N_c^6 - 3099440 N_c^5 \pm 11578144 N_c^4 - 16368000 N_c^3 \pm 36440448 N_c^2 - 40144896 N_c \pm 26763264) \mp 12672 N_c^2 (N_c \mp 2) F_\pm (12 N_c^3 \mp 9 N_c^2 \pm 308)\zeta_3] \tag{6.31}$$

and

$$a_{4,T_2} = \frac{(N_c \pm 2)^3}{2^5 \cdot 3^7 N_c^3 F_\pm^7}[(28293721281 N_c^{13} \pm 156860406306 N_c^{12} + 13832572748 N_c^{11} \mp 547968555432 N_c^{10} - 929147053664 N_c^9 \pm 428226859968 N_c^8 + 2279581786496 N_c^7 \pm 586028410624 N_c^6 - 4633121830656 N_c^5 \pm 143588589056 N_c^4 + 4686268342272 N_c^3 \mp 2321839534080 N_c^2 - 27476951040 N_c \pm 10990780416) - 2304 F_\pm (131220 N_c^{11} \pm 695898 N_c^{10} - 6916683 N_c^9 \mp 10687114 N_c^8 + 60333108 N_c^7 \mp 12100440 N_c^6 - 239418432 N_c^5 \pm 140804928 N_c^4 + 208053120 N_c^3 \mp 140560640 N_c^2 + 2973696 N_c \mp 1486848)\zeta_3 + 1013760 N_c^2 (N_c \mp 2) F_\pm^2 (\pm 87 N_c^5 - 259 N_c^4 \mp 1134 N_c^3 + 3600 N_c^2 \pm 5016 N_c - 10032)\zeta_5]. \tag{6.32}$$

The same general comments that we made before concerning factors in the κ_{j,T_2} and d_{j,T_2} coefficients also apply here. Thus, for arbitrary j, the a_{j,A_2} coefficients contain at least one overall factor of (N_c-2) and hence vanish for $N_c=2$, as a result of the fact that for $N_c=2$, the A_2 representation is a singlet, so for SU(2), fermions in the $A_2=$ singlet representation are free fields and hence make no contribution to the beta function. Moreover, if $N_c=2$, then the S_2 representation is the same as the adjoint representation, so the a_j coefficients must satisfy the equality $a_{j,S_2}=a_{j,adj}$ for this SU(2) case, and we have checked that they do. Similarly, if $N_c=3$, then the A_2 representation is the same as the conjugate fundamental representation, $\bar{F}$, so these coefficients must satisfy the equality $a_{j,A_2}=a_{j,F}$ for this SU(3) case, and we have checked that they do.

We next consider the LN limit of the theory with fermions in the S_2 or A_2 representations. Using the definition (6.24) with $R_2=S_2$ and $R_2=A_2$, we find that

$$\hat{a}_{j,S_2}=\hat{a}_{j,A_2} \tag{6.33}$$

so we denote these simply as $\hat{a}_{j,T_2}$. In general, for the same group-theoretical reasons as led to the LN relation $\hat{\kappa}_{j,T_2}=2^{-j}\hat{\kappa}_{j,adj}$ in Eq. (3.98) and the LN relation $\hat{d}_{j,T_2}=2^{-j}\hat{d}_{j,adj}$ in Eq. (4.53), we have, in the LN limit,

$$\hat{a}_{j,T_2}=2^{-j}\hat{a}_{j,adj}. \tag{6.34}$$

Explicitly, we calculate

$$\hat{a}_{1,T_2}=\frac{1}{3^3}=0.05333 \tag{6.35}$$

$$\hat{a}_{2,T_2}=\frac{205}{2^4\cdot 3^7}=0.58585\times 10^{-2} \tag{6.36}$$

$$\hat{a}_{3,T_2}=\frac{49129}{2^7\cdot 3^{11}}=2.16668\times 10^{-3} \tag{6.37}$$

and

$$\hat{a}_{4,T_2}=\frac{38811689}{2^{12}\cdot 3^{15}}-\frac{5}{2\cdot 3^9}\zeta_3=0.50769\times 10^{-3}. \tag{6.38}$$

VII. CONCLUSIONS

In conclusion, in this paper we have presented a number of new results on scheme-independent calculations of various quantities in an asymptotically free vectorial gauge theory having an IR zero of the beta function. We have presented scheme-independent series expansions of the anomalous dimension $\gamma_{\bar\psi\psi,\mathrm{IR}}$ to $O(\Delta_f^4)$ and the derivative of the beta function, β'_{IR}, to $O(\Delta_f^5)$ for a theory with a general gauge group G and N_f fermions in a representation R of G. We have given reductions of our general formulas for theories with $G=\mathrm{SU}(N_c)$ and R equal to the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations. We have compared our scheme-independent calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and β'_{IR} with previous n-loop values of these quantities calculated via series expansions in powers of the coupling. For a number of specific theories we have also compared our new scheme-independent calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and β'_{IR} with lattice measurements. We have shown that for all of the representations we have studied, and for the full range $1\le p\le 4$ for which we have performed calculations, $\gamma_{\bar\psi\psi,\mathrm{IR}}$ calculated to $O(\Delta_f^p)$, denoted $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$, increases monotonically with decreasing N_f (i.e., increasing Δ_f) and, for a fixed N_f, $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$, increases monotonically with the order p. For the representation $R=F$, we have presented results for the limit $N_c\to\infty$ and $N_f\to\infty$ with N_f/N_c fixed. These higher-order results have been applied to obtain estimates of the lower end of the (IR-conformal) non-Abelian Coulomb phase. We have confirmed and extended our earlier finding that our expansions in powers of Δ_f should be reasonably accurate throughout a substantial portion of the non-Abelian Coulomb phase. We have also given expansions for α_{IR} calculated to $O(\Delta_f^4)$ which provide a useful complementary approach to calculating α_{IR}. Our scheme-independent calculations of physical quantities at a conformal IR fixed point yield new information about the properties of a conformal field theory.

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APPENDIX: SERIES COEFFICIENTS FOR β_ξ AND $\gamma_{\bar\psi\psi}$ IN THE LNN LIMIT

For reference, we list here the rescaled series coefficients for β_ξ and $\gamma_{\bar\psi\psi}$ in the LNN limit (3.21). From the (scheme-independent) one-loop and two-loop coefficients in the beta function [7,8], it follows that in the LNN limit the $\hat{b}_\ell$ with $\ell=1,\,2$ are

$$\hat{b}_1=\frac{1}{3}(11-2r) \\ =3.667-0.667r \tag{A1}$$

and

$$\hat{b}_2=\frac{1}{3}(34-13r) \\ =11.333-4.333r. \tag{A2}$$

The coefficients b_3 and b_4 have been calculated in the $\overline{\rm MS}$ scheme [27,28]. With these inputs, one has [21]

$$\hat{b}_3 = \frac{1}{54}(2857 - 1709r + 112r^2) = 52.907 - 31.648r + 2.074r^2 \quad (A3)$$

and

$$\hat{b}_4 = \left(\frac{150473}{486} + \frac{44}{9}\zeta_3\right) - \left(\frac{485513}{1944} + \frac{20}{9}\zeta_3\right)r + \left(\frac{8654}{243} + \frac{28}{3}\zeta_3\right)r^2 + \left(\frac{130}{243}\right)r^3 = 315.492 - 252.421r + 46.832r^2 + 0.5350r^3. \quad (A4)$$

The behavior of these coefficients $\hat{b}_\ell$ as functions of r was discussed in [21] for $1 \le \ell \le 4$. The positivity of $\hat{b}_1$ is equivalent to the asymptotic freedom of the theory, and requires r to lie in the interval $0 \le r < 11/2$. The existence of an IR zero in the two-loop beta function is equivalent to the condition that $\hat{b}_2 < 0$, which, in turn, is equivalent to the condition that $r \in I_{\mathrm{IRZ},r}$ as given in Eq. (3.28). In this interval, $\hat{b}_3$ is negative-definite, while $\hat{b}_4$ is negative for $2.615 < r < 3.119$ and positive for $3.119 < r < 5.5$ [21].

For the coefficients $\hat{c}_\ell$ in Eq. (3.33), from [31] and references therein, one has [21]

$$\hat{c}_1 = 3, \quad (A5)$$

$$\hat{c}_2 = \frac{203}{12} - \frac{5}{3}r, \quad (A6)$$

$$\hat{c}_3 = \frac{11413}{108} - \left(\frac{1177}{54} + 12\zeta_3\right)r - \frac{35}{27}r^2, \quad (A7)$$

and

$$\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81}r + \frac{899}{162}r^2 - \frac{83}{81}r^3 + \left(\frac{1157}{9} - \frac{889}{3}r + 20r^2 + \frac{16}{9}r^3\right)\zeta_3 + r(66 - 12r)\zeta_4 + (-220 + 160r)\zeta_5. \quad (A8)$$

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[4] Fully nonperturbative evidence for the non-Abelian Coulomb phase comes from lattice simulations, as discussed below. Furthermore, in the case where $G = \mathrm{SU}(N_c)$ and the fermions are in the fundamental representation, one can take the limit in Eq. (3.21), namely $N_c \to \infty$ and $N_f \to \infty$ with $r = N_f/N_c$ fixed and finite. In this case, α_{IR} can be made arbitrarily small so that strength of the gauge coupling at the IR fixed point approaches arbitrarily close to zero.

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