Exact resurgent trans-series and multibion contributions to all orders

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The full resurgent trans-series is found exactly in near-supersymmetric $\mathbb{C}P^1$ quantum mechanics. By expanding in powers of the supersymmetry-breaking deformation parameter, we obtain the first and second expansion coefficients of the ground-state energy. They are an absolutely convergent series of nonperturbative exponentials corresponding to multibions with perturbation series on those backgrounds. We obtain all multibion exact solutions for a finite time interval in the complexified theory. We sum the semiclassical multibion contributions that reproduce the exact result supporting the resurgence to all orders. We also discuss the similar resurgence structure in $\mathbb{C}P^{N-1}$ $(N > 2)$ models. This is the first result in the quantum-mechanical model where the resurgent trans-series structure is verified to all orders in nonperturbative multibion contributions.

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I. INTRODUCTION

The path integral has been extremely useful in many areas of quantum physics through perturbative and nonperturbative analysis. It is crucial to understand contributions from all of the complex saddle points based on the thimble analysis in the path integral in order to give a proper foundation of quantum theories. The resurgence theory gives a stringent relation between a divergent perturbation series and a nonperturbative exponential term, which often allows for the reconstruction of one from the other [1–5]. Resurgence was originally developed by studying ordinary differential equations and provides a trans-series, which contains infinitely many nonperturbative exponentials and divergent perturbation series [6]. The intimate relation between these infinitely many nonperturbative contributions and perturbative ones is expected to provide an unambiguous definition of quantum theories. A mathematically rigorous foundation of the path integral can now be envisaged [7–9]. Resurgence has been most precisely studied recently in quantum mechanics (QM) to systematically yield relations between nonperturbative and perturbative contributions [10–29], two-dimensional quantum field theories (QFTs) [30–41], four-dimensional QFTs [42–48], supersymmetric (SUSY) gauge theories [49–53], the matrix models, and topological string theory [54–62].

In the resurgent trans-series for theories with degenerate vacua, one needs to take account of configurations called "bions" consisting of an instanton and an anti-instanton [2,10], which give imaginary ambiguities that cancel those of non-Borel-summable perturbation series. Recently, single-bion configurations were identified as saddle points in the complexified path integral [21]. Exact solutions of the holomorphic equations of motion (complex and real bion solutions) were found in the complexified path integral of double-well, sine-Gordon, and $\mathbb{C}P^1$ quantum-mechanical models with fermionic degrees of freedom (incorporated as the parameter ϵ) [21,25]. $\mathbb{C}P^1$ quantum mechanics is a dimensional reduction of the two-dimensional $\mathbb{C}P^1$ sigma model, which shows asymptotic freedom, dimensional transmutation, and the existence of instantons akin to four-dimensional QCD. Contributions from these solutions were evaluated based on Lefschetz-thimble integrals and it was shown that the combined contributions vanish for the SUSY case $\epsilon = 1$, in conformity with the exact results of SUSY [25]. On the other hand, for the non-SUSY case $\epsilon \neq 1$, the result contains the imaginary ambiguity, which is expected to be canceled by that arising from the Borel resummation of perturbation series.

Trans-series generically contain high powers of nonperturbative exponentials, which may correspond to multiple bions. Non-SUSY models including $\mathbb{C}P^{N-1}$ quantum mechanics have been worked out explicitly to several low orders, but it has been difficult to explicitly reveal the full trans-series to all powers of nonperturbative exponentials and to ascertain their resurgence structure [2,19,20]. Localization in SUSY models helped to uncover the full trans-series, but so far their resurgence structures have been found to be trivial without imaginary ambiguities [49,53].

The purpose of this work is to present and verify the complete resurgence structure of the trans-series in

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$\mathbb{C}P^1$ QM (and partly $\mathbb{C}P^{N-1}$ QM), focusing on the near-SUSY regime $\epsilon \approx 1$ where we can obtain exact results which exhibit resurgence structure to infinitely high powers of nonperturbative exponentials. We will show that the contributions from an infinite tower of multibion solutions yield all of these nonperturbative exponentials. This is the first result revealing the thimble structure of all of the complex saddle points, which is useful not only to understand the resurgence structure in quantum theories but also to study complex path integrals, including the real-time formalism and finite-density systems in condensed and nuclear matter [63–68].

II. EXACT GROUND-STATE ENERGY

We first consider the (Lorentzian) $\mathbb{C}P^1$ quantum mechanics described by the Lagrangian

$$g^2 L = G[|\partial_t \varphi|^2 - |m\varphi|^2 + i\bar{\psi}\mathcal{D}_t\psi] - \epsilon \frac{\partial^2 \mu}{\partial\varphi\partial\bar{\varphi}}\psi\bar{\psi}, \quad (1)$$

where φ is the inhomogeneous coordinate, $G = \partial_\varphi \partial_{\bar{\varphi}} \log(1+|\varphi|^2)$ is the Fubini-Study metric, $\mathcal{D}_t = \partial_t + \partial_t\varphi\partial_\varphi \log G$ is the pull-back of the covariant derivative, and $\mu = m|\varphi|^2/(1+|\varphi|^2)$ is the moment map associated with the $U(1)$ symmetry $\varphi \to e^{i\theta}\varphi$. The parameter ϵ is the boson-fermion coupling and the Lagrangian becomes supersymmetric at $\epsilon = 1$. Since the fermion number $F = G\psi\bar{\psi}$ commutes with the Hamiltonian, the Hilbert space can be decomposed into two subspaces with $F = 1$ and $F = 0$. By projecting quantum states onto the subspace which contains the ground state ($F = 1$), we obtain the bosonic Lagrangian

$$L = \frac{|\partial_t\varphi|^2}{(g^2(1+|\varphi|^2)^2)} - V, \quad (2)$$

with the potential

$$V = \frac{1}{g^2}\frac{m^2|\varphi|^2}{(1+|\varphi|^2)^2} - \epsilon m \frac{1-|\varphi|^2}{1+|\varphi|^2}. \quad (3)$$

We note that $\theta(\equiv -2\arctan|\varphi|) = 0, \pi$ are global and metastable vacua, respectively.

For $\epsilon = 1$, the ground-state wave function Ψ_0 preserving SUSY is given as a zero-energy solution of the Schrödinger equation

$$H_{\epsilon=1}\Psi_0 = \left[-g^2(1+|\varphi|^2)^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V_{\epsilon=1}\right]\Psi_0 = 0. \quad (4)$$

It is exactly solved as

$$\Psi_0 = \langle\varphi|0\rangle = \exp(-\mu/g^2). \quad (5)$$

For $\epsilon \approx 1$, the leading-order correction to the ground-state wave function can be obtained by expanding the Schrödinger equation with respect to small $\delta\epsilon \equiv \epsilon - 1$ as $\langle\varphi|\delta\Psi\rangle$. Correspondingly, the ground-state energy E can also be expanded:

$$E = \delta\epsilon E^{(1)} + \delta\epsilon^2 E^{(2)} + \cdots. \quad (6)$$

These expansion coefficients can be determined by the standard Rayleigh-Schrödinger perturbation theory as

$$E^{(1)} = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle}, \quad (7)$$

$$E^{(2)} = -\frac{\langle \delta\Psi|H_{\epsilon=1}|\delta\Psi\rangle}{\langle 0|0\rangle}, \ldots, \quad (8)$$

with $\delta H = H - H_{\epsilon=1}$. We find that these coefficients $E^{(i)}$ are real without imaginary ambiguities and can be expanded in absolutely convergent power series with respect to the nonperturbative exponential $\exp(-2m/g^2)$,

$$E^{(i)} = \sum_{p=0}^{\infty} E_p^{(i)} \exp(-2pm/g^2), \quad (9)$$

where the zeroth term $E_0^{(i)}$ corresponds to the perturbative contributions on the trivial vacuum (perturbative vacuum). The coefficients of $E^{(1)}$ [25] are

$$E_0^{(1)} = -m + g^2, \qquad E_p^{(1)} = -2m, \qquad (p \geq 1). \quad (10)$$

If the coefficients of $E^{(2)}$ are expanded in powers of g^2, they give factorially divergent asymptotic series, which can be Borel-resummed. Hence, we rewrite the coefficient in the form of the Borel transform (see Appendix A for the details of the calculations) as

$$E_0^{(2)} = g^2 + 2m \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}}, \quad (11)$$

$$E_p^{(2)} = 2m \int_0^\infty dt e^{-t} \left\{ \frac{(p+1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right\} + 4mp^2 \left(\gamma + \log\frac{2m}{g^2} \pm \frac{\pi i}{2} \right), \qquad (p \geq 1). \quad (12)$$

Note that the imaginary ambiguities associated to the Borel resummation are manifest in the first term of $E_p^{(2)}$ with $g^2 \pm i0$, which is compensated by the imaginary part $\pm i\pi/2$ in the last term of $E_{p+1}^{(2)}$, reproducing the original real $E^{(2)}$ precisely. In the present case, we only have poles in the Borel plane, while cuts are expected for general cases. We also note that in Ref. [27] the perturbation series on a zero-bion background including the level number information has been shown to give all p-bion contributions.

We can now recognize the full resurgence structure to all orders of the nonperturbative exponential: the imaginary ambiguity of the non-Borel summable divergent perturbation series on the p-bion background in the first term of $E_p^{(2)}$ is canceled by the imaginary ambiguity of the classical contribution of the $(p+1)$-bion contribution in the last term of $E_{p+1}^{(2)}$. We note the absence of powers of g^2 in the imaginary ambiguity, which will allow us to recover non-Borel summable perturbation series on the p-bion background completely from the $(p+1)$-bion contribution through the dispersion relation, without computing perturbative corrections around the multibion background explicitly. Moreover, if we observe that $E^{(2)}/m$ is an even function of m/g^2, we can also understand the presence of the Borel-summable part [second term of the first line in Eq. (12)]. Thus, all of the terms can now be reproduced through the resurgence relation and the sign change of m/g^2, if we can compute all of the semiclassical p-bion contributions.

III. MULTIBION SOLUTIONS

Nonperturbative contributions to the ground-state energy come from the saddle points of the path integral $Z = \int \mathcal{D}\varphi\mathcal{D}\tilde{\varphi}e^{-S_E} \sim e^{-\beta E}$ (for large β), where we have complexified the degrees of freedom by regarding $\varphi \equiv \varphi_R^{\mathbb{C}} + i\varphi_I^{\mathbb{C}}$ and $\tilde{\varphi} \equiv \varphi_R^{\mathbb{C}} - i\varphi_I^{\mathbb{C}}$ as independent holomorphic variables, and imposed the periodic boundary condition $\varphi(\tau+\beta) = \varphi(\tau)$ and for $\tilde{\varphi}$. The Euclidean action

$$S_E = \int_0^\beta d\tau[\partial_\tau\varphi\partial_\tau\tilde{\varphi}/(g^2(1+\varphi\tilde{\varphi})^2) + V(\varphi\tilde{\varphi})] \quad (13)$$

has two conserved Noether charges associated with the complexification of the Euclidean time translation $\tau \to \tau + a$ and the phase rotation $(\varphi, \tilde{\varphi}) \to (e^{ib}\varphi, e^{-ib}\tilde{\varphi})$ $(a, b \in \mathbb{C})$. Using the corresponding conservation laws, we can obtain the following solution of the equation of motion with a nontrivial contribution in a $\beta \to \infty$ limit:

$$\varphi = e^{i\phi_c}\frac{f(\tau-\tau_c)}{\sin^2\alpha}, \qquad \tilde{\varphi} = e^{-i\phi_c}\frac{f(\tau-\tau_c)}{\sin^2\alpha}, \quad (14)$$

where (τ_c, ϕ_c) are complex moduli parameters associated with the symmetry and $f(\tau)$ is the elliptic function

$$f(\tau) = \mathrm{cs}(\Omega\tau, k) \equiv \mathrm{cn}(\Omega\tau, k)/\mathrm{sn}(\Omega\tau, k), \quad (15)$$

which satisfies the differential equation

$$(\partial_\tau f)^2 = \Omega^2(f^2+1)(f^2+1-k^2). \quad (16)$$

Solutions are characterized by two integers (p, q) for the period

$$\beta = \frac{(2pK + 4iqK')}{\Omega}, \quad (17)$$

where $2K(k)$ and $4iK'$ $(K' \equiv K(\sqrt{1-k^2}))$ are the periods of the doubly periodic functions. The parameters (α, Ω, k) are given in terms of the period β, and their asymptotic forms for large β (see Appendix B for the details of the calculations) are given by

$$k \approx 1 - 8e^{-\frac{\omega\beta - 2\pi iq}{p}},$$
$$\Omega \approx \omega\left(1 + 8\frac{\omega^2+m^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi iq}{p}}\right),$$
$$\cos\alpha \approx \frac{m}{\omega}\left(1 - \frac{8m^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi iq}{p}}\right), \quad (18)$$

where $\omega = m\sqrt{1+2\epsilon g^2/m}$ and (p, q) are arbitrary integers such that $0 \le q < p$. The asymptotic value of the action for the (p, q) solution is given by

$$S \approx pS_{\text{bion}} + 2\pi i\epsilon l, \qquad S_{\text{bion}} = \frac{2m}{g^2} + 2\epsilon\log\frac{\omega+m}{\omega-m}, \quad (19)$$

where we have ignored the vacuum value of the action. The imaginary part $2\pi i\epsilon l$ is related to the so-called hidden topological angle [39] and the integer l is zero or the greatest common divisor of p and $2q$ depending on the value of $\mathrm{Im}\tau_c$. We see that the integer p is the number of bions, and that the nth kink and antikink are located at τ_n^+ and τ_n^-, with

$$\tau_n^\pm = \tau_c + \frac{n-1}{\omega p}(\omega\beta - 2\pi iq) \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2-m^2}. \quad (20)$$

There are p bions (pairs of kink-antikink) equally spaced on S^1. In Fig. 1, we depict the profile of the complexified height function

$$\Sigma = \frac{(1-\varphi\tilde{\varphi})}{(1+\varphi\tilde{\varphi})} \quad (21)$$

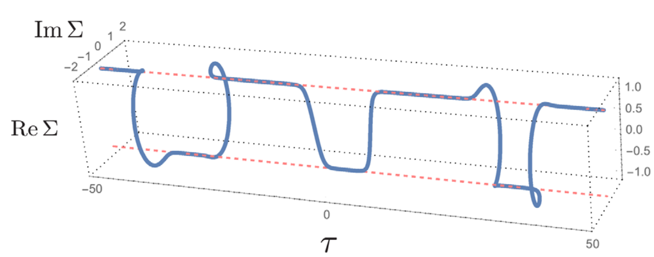

FIG. 1. Multibion solution: Kink profile of $\Sigma(\tau) = (1-\varphi\tilde{\varphi})/(1+\varphi\tilde{\varphi})$ for $(p, q) = (3, 1)$, $\epsilon = 1$, $m = 1$, $g = 1/200$, $\beta = 100$ and $\tau_c = 0$. $\Sigma = \pm 1$ (dashed lines) correspond to north and south poles (global and local minima) of $\mathbb{C}P^1$.

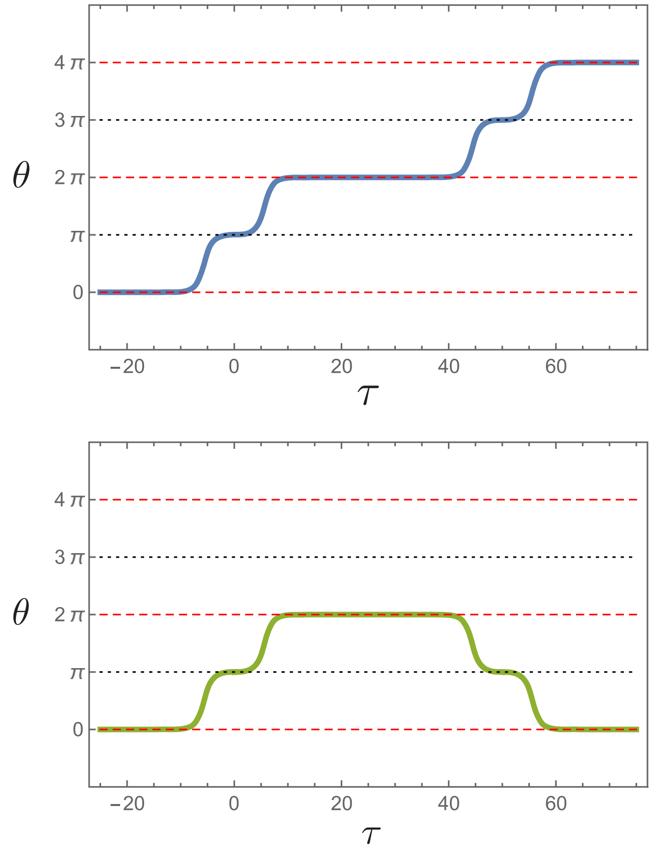

FIG. 2. Multibion solution: $\theta = -2\arctan|\varphi|$ for $(p, q) = (2, 0)$ (top) and for $(p, q) = (2, 1)$ (bottom). The other parameters are the same as those in Fig. 1. $\theta = 0, 2\pi, \ldots$ and $\theta = \pi, 3\pi, \ldots$ correspond to north and south poles of $\mathbb{C}P^1$.

of the $(p, q) = (3, 1)$ solution. It illustrates that general (p, q) solutions are intrinsically complex, and are not a mere repetition of single (real or complex) bions. In Fig. 2 we depict other solutions $[(p, q) = (2, 0)$ and $(p, q) = (2, 1)]$ in terms of $\theta = -2\arctan|\varphi|$, which visualizes patterns of transition between the (metastable) vacua. Although our solutions are not solutions of the SUSY theory with fermions, they are composite configurations of instantons and anti-instantons which are typically non-BPS. This fact implies that the non-BPS configurations play a vital role in the semiclassics in the path integral formalism of quantum theories.

IV. MULTIBION CONTRIBUTIONS

The contributions from the p-bion solutions can be calculated by performing the Lefschetz thimble integral associated with the saddle points. In the weak-coupling limit $g \to 0$, we can use the Gaussian approximation for the fluctuation modes from the saddle points, except for the nearly massless modes parametrized by the quasimoduli parameters (τ_i, ϕ_i). Thus, we can simplify the Lefschetz thimble analysis by reducing the degrees of freedom onto the quasimoduli space.

The leading-order contributions come from the region around the saddle points, where all the kinks are well separated in the weak-coupling limit. Therefore, the effective potential can be approximated by that for well-separated kinks,

$$S_E \to V_{\rm eff} = -m\epsilon\beta + \sum_{i=1}^{2p}\left(\frac{m}{g^2} + V_i\right), \tag{22}$$

where V_i is the asymptotic interaction potential between neighboring kink-antikink pairs [34],

$$\frac{V_i}{m} = \epsilon_i(\tau_i - \tau_{i-1}) - \frac{4}{g^2}e^{-m(\tau_i-\tau_{i-1})}\cos(\phi_i - \phi_{i-1}), \tag{23}$$

with $\tau_{2n-1} = \tau_i^-$, $\tau_{2n} = \tau_i^+$, $\tau_0 = \tau_{2p} - \beta$, $\phi_0 = \phi_{2p}$ (mod 2π), $\epsilon_{2n-1} = 0$, and $\epsilon_{2n} = 2\epsilon$. We find that the saddle points of $V_{\rm eff}$ are consistent with $\tau_n^\pm$ in Eq. (20) for large β and small g. We introduce a Lagrange multiplier σ to impose the periodicity as

$$2\pi\delta\left(\sum_i \tau_i - \beta\right) = m\int d\sigma \exp\left[im\sigma\left(\sum_i \tau_i - \beta\right)\right]. \tag{24}$$

By generalizing the Lefschetz thimble analysis in Ref. [25] to the multibion contribution

$$Z_p \propto \int \prod_{i=1}^{2p} d\tau_i d\phi_i, \exp(-V_{\rm eff}), \tag{25}$$

we obtain the following p-bion contribution to the partition function (see Appendix C for the details of the calculations):

$$\frac{Z_p}{Z_0} \approx -\frac{2im\beta}{p}e^{-\frac{2pm}{g^2}}\operatorname*{Res}_{\sigma=0}\left[e^{-im\beta\sigma}\prod_{i=1}^{2p} I_i\right], \tag{26}$$

with

$$I_i = \frac{2m}{g^2}\left(\frac{2m}{g^2}e^{\pm\frac{\pi i}{2}}\right)^{i\sigma-\epsilon_i}\frac{\Gamma((\epsilon_i - i\sigma)/2)}{\Gamma(1-(\epsilon_i - i\sigma)/2)}. \tag{27}$$

The sign $\pm$ is associated with $\arg[g^2] = \pm 0$. This gives a polynomial of β, whose leading term is of order β^p,

$$\frac{Z_p}{Z_0} \approx \frac{1}{p!}\left[\frac{2m\beta\Gamma(\epsilon)}{\Gamma(1-\epsilon)}e^{-\frac{2m}{g^2}\mp\pi i\epsilon}\left(\frac{2m}{g^2}\right)^{2(1-\epsilon)}\right]^p, \tag{28}$$

consistent with the dilute gas approximation: $Z_p/Z_0 = (Z_1/Z_0)^p/p! + \mathcal{O}(\beta^{p-1})$. From the p-bion contribution (26) and the perturbative contribution ($p = 0$), the ground-state energy $E = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z$ can be obtained as

$$E = E_0 - \lim_{\beta\to\infty} \frac{1}{\beta} \log\left(1 + \sum_{p=1}^{\infty} \frac{Z_p}{Z_0}\right). \tag{29}$$

By taking the logarithm, contributions of high powers of β such as β^p for $p > 1$ should be canceled, and the ground-state energy is obtained from the remaining contributions of order β. Fortunately, most of these contributions with high powers of β disappear near the SUSY case thanks to the zero in $1/\Gamma(1 - (\epsilon_i - i\sigma)/2)$. As a result, we find that the first derivative is proportional to β and gives the near-SUSY ground-state energy $E^{(1)}$,

$$E_p^{(1)} = -e^{\frac{2pm}{g^2}} \lim_{\epsilon\to 1} \lim_{\beta\to\infty} \frac{1}{\beta} \frac{\partial}{\partial\epsilon} \frac{Z_p}{Z_0} = -2m, \tag{30}$$

verifying the exact result (10). The second derivative in ϵ turns out to be quadratic in β, and

$$E_p^{(2)} = -\frac{e^{\frac{2pm}{g^2}}}{2} \lim_{\epsilon\to 1} \lim_{\beta\to\infty} \frac{1}{\beta} \left[\partial_\epsilon^2 \frac{Z_p}{Z_0} - \sum_{i=1}^{p-1} \partial_\epsilon \frac{Z_{p-i}}{Z_0} \partial_\epsilon \frac{Z_i}{Z_0}\right] \tag{31}$$

is calculated as

$$E_p^{(2)} = 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{\pi i}{2}\right), \tag{32}$$

in complete agreement with the exact result (12). We have obtained the classical contributions to all orders of multi-bions, which provide all of the terms needed for the full resurgence structure of our model, although it is difficult to check the divergent perturbation series on the p-bion background directly, except for the trivial vacuum ($p = 0$).

V. PERTURBATION SERIES ON THE TRIVIAL VACUUM

We obtain the perturbation series on the trivial background ($p = 0$) by using the Bender-Wu method [26,69]. We first expand the energy and the wave function as

$$E = m\sum_l A_l \eta^{2l}, \qquad \Psi = \exp(-x^2)\sum_{l,k} B_{l,k}\eta^{2l}x^{2k}, \tag{33}$$

with $|\varphi| = \eta x$ and $\eta^2 = g^2/m$. Then, the Schrödinger equation reduces to a (Bender-Wu) recursive equation for A_l and $B_{l,k}$, which gives the leading asymptotic behavior (see Appendix D for the details of the calculations) as

$$A_l \sim -\frac{\Gamma(l + 2(1-\epsilon))}{2^{l-1}\Gamma(1-\epsilon)^2} \qquad (\text{for large } l). \tag{34}$$

Since the coefficients A_l grow factorially for large l, we obtain the perturbative part of the ground-state energy by using the Borel resummation

$$E_0 \sim \frac{2m}{\Gamma(1-\epsilon)^2} \int_0^\infty dt e^{-t} t^{2(1-\epsilon)} \left(t - \frac{2m}{g^2}\right)^{-1}. \tag{35}$$

The Borel resummation gives a finite result with the imaginary ambiguity

$$\mathrm{Im}E_0 = \mp \frac{2\pi m}{\Gamma(1-\epsilon)^2} \left(\frac{g^2}{2m}\right)^{2(\epsilon-1)} e^{-\frac{2m}{g^2}}, \tag{36}$$

with $-$ ($+$) on the right-hand side for $\mathrm{Im}g^2 = +0$ (-0). This imaginary ambiguity of the perturbation series in the trivial vacuum ($p = 0$) cancels that of the single-bion sector [Eq. (28) with $p = 1$]. Therefore, combining these two contributions gives an unambiguous real result. This result explicitly verifies the resurgence for arbitrary values of ϵ including the non-SUSY case, although only to the leading order of the nonperturbative exponential.

For the near-SUSY case, we can obtain the perturbation series on the trivial vacuum exactly to all orders in g^2, by exactly solving the Bender-Wu recursion relation to the second order of $\delta\epsilon$ as

$$E_0 = (g^2 - m)\delta\epsilon - 2m \sum_{l=2}^{\infty} \Gamma(l) \left(\frac{g^2}{2m}\right)^l \delta\epsilon^2 + \cdots. \tag{37}$$

This agrees completely with the exact results $E_0^{(1)}$ in Eq. (10) and $E_0^{(2)}$ in Eq. (11) after Borel resummation.

VI. SUMMARY AND DISCUSSION

In conclusion:

(i) We have derived the exact expansion coefficients of the ground-state energy to the second order of the SUSY-breaking deformation parameter $\delta\epsilon$. The result shows a resurgent trans-series structure to all orders of the nonperturbative exponential.

(ii) We have derived nonperturbative multibion contributions with imaginary ambiguities in the weak-coupling limit and found that they agree with the corresponding parts in the exact result.

(iii) At least for near-SUSY $\mathbb{C}P^1$ QM, by assuming the cancellation of imaginary ambiguities (resurgence structure) and an even function of m/g^2, we have recovered the entire trans-series which agrees with the exact result of the near-SUSY case.

(iv) With the Bender-Wu recursion relation, we have obtained the perturbation series on the zero-bion vacuum to all orders, which gives an imaginary ambiguity when Borel-resummed, and have verified the cancellation with that of the single-bion sector for a general deformation parameter ϵ including the non-SUSY case.

The exact result in Eq. (12) shows that the imaginary ambiguities have no g^2 corrections in $\mathbb{C}P^1$ QM. This fact enabled us to recover the entire trans-series from the semiclassical multibion contributions only. In other models such as sine-Gordon QM, imaginary ambiguities from the multibion contribution have perturbative corrections in powers of g^2 [20]. Then these perturbative corrections are needed in order to recover the full resurgent trans-series.

The same resurgence structure exists in $\mathbb{C}P^{N-1}$ models with $N > 2$. Similarly to $N = 2$, we obtain a $O(\delta\epsilon^2)$ perturbative contribution with the imaginary ambiguity

$$\mathrm{Im}E_0^{(2)} = \mp \frac{N^2\pi}{2}\sum_{i=1}^{N-1} m_i A_i e^{-\frac{2m_i}{g^2}}, \tag{38}$$

where $A_i = \prod_{j=1,j\neq i}^{N-1} \frac{m_j}{m_j - m_i}$ and the mass parameters m_i are reduced from the two-dimensional $\mathbb{C}P^{N-1}$ model with twisted boundary conditions. We also calculate the $O(\delta\epsilon^2)$ single-bion contribution

$$E_1^{(2)} = \sum_{i=1}^{N-1} N^2 m_i A_i e^{-\frac{2m_i}{g^2}}\left(\gamma + \log\frac{2m_i}{g^2} \pm \frac{\pi i}{2}\right). \tag{39}$$

The imaginary ambiguities cancel each other. As for the convergence of the $\delta\epsilon$ expansion, we observe that each of the p-bion semiclassical contributions has a convergent expansion for any p.

Focusing on the near-SUSY regime can be extended to the solvable models (including localizable SUSY theories [49,53] and quasi-solvable models [28]) by softly breaking the solvable condition and expanding the physical quantities with respect to the deformation parameter. This is because these models have a similar resurgence property as the present $\mathbb{C}P^1$ model, where the resurgence structure becomes trivial without cancellation of the imaginary ambiguity at localization-applicable or quasi-exactly-solvable regimes. We also notice that the localization technique is applicable in $\mathbb{C}P^{N-1}$ QM to compute the first-order ground-state energy $E^{(1)}$ but not the second order. Recent results on volume independence [41] should be useful in extending our study to QFT, which may also require a more refined thimble analysis, as has been studied intensively [64–68].

Regarding non-SUSY gauge theories, complex instanton solutions were discussed in gauge theories with complexified gauge groups decades ago [70,71]. It would be important to discuss contributions from these complex solutions in terms of resurgence theory.

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APPENDIX A: EXACT GROUND-STATE ENERGY

In this appendix we show details of the calculations in Sec. II. The leading-order correction to the ground-state wave function and energy for $\mathbb{C}P^1$ quantum mechanics in Eqs. (1)and (3) can be obtained by solving the $\mathcal{O}(\delta\epsilon)$ part of the Schrödinger equation,

$$H_{\epsilon=1}|\delta\Psi\rangle = \left(E^{(1)} + m\frac{1-|\varphi|^2}{1+|\varphi|^2}\right)|0\rangle, \tag{A1}$$

$$E^{(1)} = g^2 - m\coth\frac{m}{g^2}$$
$$= -m + g^2 - \sum_{p=1}^{\infty} 2me^{-\frac{2pm}{g^2}}. \tag{A2}$$

From this expanded form, we can read off the expansion coefficients in Eq. (10). The above differential equation can be exactly solved as

$$\langle\varphi|\delta\Psi\rangle = e^{-\frac{\mu}{g^2}}\int_0^{\mu}\frac{d\mu'}{\mu'(\mu'-m)}\left(\mu' - m\frac{1-e^{\frac{2\mu'}{g^2}}}{1-e^{\frac{2m}{g^2}}}\right). \tag{A3}$$

Then we find the second-order correction to the ground-state energy as

$$E^{(2)} = -\frac{\langle\delta\Psi|H_{\epsilon=1}|\delta\Psi\rangle}{\langle 0|0\rangle}$$
$$= g^2 - 2m\frac{\cosh\frac{m}{g^2}}{\sinh^3\frac{m}{g^2}}\int_0^m d\mu\frac{\sinh^2\frac{\mu}{g^2}}{\mu}. \tag{A4}$$

Using the hyperbolic cosine integral $\mathrm{Chi}(z)$ defined by

$$\mathrm{Chi}(z) = \gamma + \log z - \int_0^z\frac{dt}{t}(1-\cosh t), \tag{A5}$$

we can rewrite $E^{(2)}$ as

$$E^{(2)} = g^2 - m\frac{\cosh\frac{m}{g^2}}{\sinh^3\frac{m}{g^2}}\left[\mathrm{Chi}\left(\frac{2m}{g^2}\right) - \gamma - \log\frac{2m}{g^2}\right]. \tag{A6}$$

By using the relation

$$\mathrm{Chi}\left(\frac{2m}{g^2}\right) = -\frac{1}{2}\int_0^\infty dt e^{-t}\left(\frac{e^{\frac{2m}{g^2}}}{t-\frac{2m}{g^2\pm i0}} + \frac{e^{-\frac{2m}{g^2}}}{t+\frac{2m}{g^2}}\right) \mp \frac{\pi i}{2} \tag{A7}$$

$E^{(2)}$ can be expanded as

$$\begin{aligned} E^{(2)} &= g^2 + 2m\int_0^\infty dt e^{-t}\frac{1}{t-\frac{2m}{g^2\pm i0}} \\ &+ 4m\sum_{p=1}^\infty e^{-\frac{2pm}{g^2}}\left[p^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{\pi i}{2}\right)\right. \\ &\left.+\frac{1}{2}\int_0^\infty dt e^{-t}\left(\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right)\right]. \end{aligned} \tag{A8}$$

From this expanded form, we can read off the expansion coefficients Eqs. (11) and (12).

APPENDIX B: MULTIBION SOLUTIONS

In this appendix we summarize the basic properties of the multibion solution (14),

$$\begin{aligned} \varphi &= e^{i\phi_c}\frac{f(\tau-\tau_c)}{\sin\alpha}, \\ \tilde{\varphi} &= e^{-i\phi_c}\frac{f(\tau-\tau_c)}{\sin\alpha}, \\ f(\tau) &= \mathrm{cs}(\Omega\tau, k), \end{aligned} \tag{B1}$$

where the parameters are related as

$$\begin{aligned} k^2 &= 1-\tan^2\alpha\left(\cos^2\alpha-\frac{m^2}{\Omega^2}\right), \\ \Omega &= \omega\sqrt{1-(1+\sec^2\alpha)\left(1-\frac{m^2}{\omega^2}\sec^2\alpha\right)}, \end{aligned} \tag{B2}$$

with

$$\omega = m\sqrt{1+\frac{2\epsilon g^2}{m}}. \tag{B3}$$

This is a periodic solution, whose period is given by

$$\beta = \oint\frac{df}{\partial_\tau f} = \frac{1}{\Omega}\oint\frac{df}{\sqrt{(f^2+1)(f^2+1-k^2)}}, \tag{B4}$$

where we have used the relation $(\partial_\tau f)^2 = \Omega^2(f^2+1)(f^2+1-k^2)$. There are four branch points corresponding to the turning points $(\partial_\tau\varphi = \partial_\tau\tilde{\varphi} = 0)$,

$$f = \pm i, \qquad \pm i\sqrt{1-k^2}. \tag{B5}$$

Let us introduce two branch cuts on the lines from $\pm i$ to $\pm i\sqrt{1-k^2}$ in the complex f plane. Let C_A be the cycle from $\mathrm{Re}f = -\infty$ to $\mathrm{Re}f = \infty$ which does not pass through the branch cuts and let C_B be the cycle surrounding the two branch points $\pm i\sqrt{1-k^2}$. Their periods are

$$\beta_A = \frac{2K(k)}{\Omega}, \qquad \beta_B = \frac{4iK(\sqrt{1-k^2})}{\Omega}, \tag{B6}$$

where $K(k) = F(\pi/2, k)$ is the complete elliptic integral of the first kind,

$$F(x,k) = \int_0^x\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{B7}$$

The period of the solution winding the cycle $pC_A + qC_B (p,q\in\mathbb{Z})$ is given by

$$\beta = \frac{2pK(k)+4qiK(\sqrt{1-k^2})}{\Omega}, \qquad p,q\in\mathbb{Z}. \tag{B8}$$

Solving this equation and Eq. (B3), we can determine the parameters (α,Ω,k) for each pair of integers (p,q). The $\beta\to\infty$ limit of the $(p,q)=(1,0)$ solution is given by the known one-bion solution for an infinite time interval with

$$E = m\epsilon, \qquad k=1, \qquad \cos\alpha = \frac{m}{\omega}, \qquad \Omega = \omega. \tag{B9}$$

We need the $\beta\to\infty$ limit, keeping (p,q) fixed. Expanding the period with respect to $\delta k = k-1$, we find that

$$\beta = \frac{1}{\omega}\left[-p\log\left(-\frac{\delta k}{8}\right)+2\pi iq\right]+\mathcal{O}(\delta k\log\delta k). \tag{B10}$$

Therefore, the asymptotic form of δk for large β is

$$\delta k \approx -8e^{-\frac{\omega\beta-2\pi iq}{p}}. \tag{B11}$$

We can also show that the asymptotic forms of the other parameters are

$$\begin{aligned} \delta\alpha &\approx \left(\frac{4m^2}{\omega^2-m^2}\right)^{\frac{3}{2}}e^{-\frac{\omega\beta-2\pi iq}{p}}, \\ \delta\Omega &= 8\omega\frac{\omega^2+m^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi iq}{p}}, \\ \delta E &= \frac{8\omega^2}{g^2}\frac{\omega^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi iq}{p}}. \end{aligned} \tag{B12}$$

We read Eq. (18) from these equations. Note that Eq. (B10) implies that the solution exists only for $0\le q<p$ in the large-β limit.

The action for this solution is given by

$$S_{\rm sol} = \int_0^\beta d\tau L_{\rm sol} = -m\epsilon\beta + \frac{\Omega}{g^2}\oint dfX(f), \qquad (B13)$$

where the function $X(f)$ can be written as

$$\begin{aligned} X &= -\frac{\partial}{\partial f}\left[\frac{\omega^2-m^2}{\Omega^2}\left\{\frac{F(x,k)}{\cos^2\alpha} - \tan^2\alpha\Pi(\cos^2\alpha,x,k)\right\}\right. \\ &\quad \left. + E(x,k) - \sqrt{\frac{f^2+1-k^2}{f^2+1}}\frac{f\cos^2\alpha}{f^2+\sin^2\alpha}\right] \\ &= -i\frac{\partial}{\partial f}\left[\frac{\omega^2-m^2}{\Omega^2}\left\{F(y,k') - \Pi\left(\frac{1-k^2}{\sin^2\alpha},y,k'\right)\right\}\right. \\ &\quad \left. + F(y,k') - E(y,k') + i\frac{f\sqrt{(f^2+1)(f^2+1-k^2)}}{f^2+\sin^2\alpha}\right], \end{aligned} \qquad (B14)$$

with $x = \arcsin\sqrt{\frac{1}{f^2+1}}$, $y = \arcsin\sqrt{\frac{-f^2}{1-k^2}}$, and $k' = \sqrt{1-k^2}$. Then we obtain

$$\begin{aligned} X(f) &= \frac{1}{\sqrt{(f^2+1)(f^2+1-k^2)}}\left[-\frac{1-k^2}{\sin^2\alpha}\right. \\ &\quad \left. + (f^2+1)(f^2+1-k^2)\frac{2\sin^2\alpha}{(f^2+\sin^2\alpha)^2}\right]. \end{aligned} \qquad (B15)$$

There are contributions from C_A, C_B, and the poles at $f = \pm i\sin\alpha$ (more precisely, integration cycles should be defined on the torus with two punctures):

$$S = -m\epsilon\beta + pS_A + qS_B + 2\pi i l S_{\rm res}, \qquad l \in \mathbb{Z}. \qquad (B16)$$

Explicitly, $S_{\rm res} = \epsilon$ and S_A and S_B are given by

$$\begin{aligned} S_A &= \frac{2\Omega}{g^2}\left[\frac{\omega^2-m^2}{\Omega^2}\left\{\frac{K(k)}{\cos^2\alpha} - \tan^2\alpha\Pi(\cos^2\alpha,k)\right\} + E(k)\right], \\ S_B &= \frac{4i\Omega}{g^2}\left[\frac{\omega^2-m^2}{\Omega^2}\left\{K(k') - \Pi\left(\frac{1-k^2}{\sin^2\alpha},k'\right)\right\}\right. \\ &\quad \left. + K(k') - E(k')\right], \end{aligned} \qquad (B17)$$

where $E(k) = E(\frac{\pi}{2},k)$ and $\Pi(a,k) = \Pi(a,\frac{\pi}{2},k)$ are the complete elliptic integrals of the second and third kind,

$$\begin{aligned} E(x,k) &= \int_0^x d\theta\sqrt{1-k^2\sin^2\theta}, \\ \Pi(a,x,k) &= \int_0^x \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\frac{1}{1-a\sin^2\theta}, \end{aligned} \qquad (B18)$$

and $k' = \sqrt{1-k^2}$. For large β,

$$S \approx -m\epsilon\beta + p\left[\frac{2\omega}{g^2} + 2\epsilon\log\frac{\omega+m}{\omega-m}\right] + 2\pi i\epsilon l, \qquad (B19)$$

from which we read Eq. (19). This implies that the integer p corresponds to the number of bions.

Focusing on the region around

$$\tau \approx \frac{n}{p}\beta \qquad (n = 0,1,\ldots,p-1), \qquad (B20)$$

we can approximate the solution for large β as

$$h(\tau) \approx \sqrt{\frac{\omega^2}{\omega^2-m^2}}\left[\sinh\left\{\omega\left(\tau - \frac{n\beta}{p}\right) + \frac{2\pi inq}{p}\right\}\right]^{-1}, \qquad (B21)$$

where we have used $\mathrm{cs}(x,1) = 1/\sinh x$. Therefore, the solution in this region looks like the single-bion configuration,

$$\begin{aligned} \varphi &\approx (e^{\omega(\tau-y_n^+)} + e^{-\omega(\tau+\tilde{y}_n^-)})^{-1}, \\ \tilde{\varphi} &\approx (e^{\omega(\tau-\tilde{y}_n^+)} + e^{-\omega(\tau+y_n^-)})^{-1}, \end{aligned} \qquad (B22)$$

with

$$\omega y_n^\pm = \omega\tau_c + i\phi_c + \frac{n}{p}\omega\beta - \frac{2\pi inq}{p} \pm \log\sqrt{\frac{4\omega^2}{\omega^2-m^2}} \quad (\mathrm{mod}\ 2\pi i), \qquad (B23)$$

$$\omega\tilde{y}_n^\pm = \omega\tau_c - i\phi_c + \frac{n}{p}\omega\beta - \frac{2\pi inq}{p} \pm \log\sqrt{\frac{4\omega^2}{\omega^2-m^2}} \quad (\mathrm{mod}\ 2\pi i). \qquad (B24)$$

From this asymptotic form, we can read off the positions $\tau_n^\pm = (y_{n-1}^\pm + \tilde{y}_{n-1}^\pm)/2$ and phases $\phi_n^\pm = (y_{n-1}^\pm - \tilde{y}_{n-1}^\pm)/2i$ of the component kinks. The nth kink $(+)$ and antikink $(-)$ locations [Eq. (20)] are given by

$$\tau_n^\pm = \tau_c + \frac{n-1}{\omega p}(\omega\beta - 2\pi iq) \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2-m^2}. \qquad (B25)$$

The poles of the Lagrangian are located at

$$\begin{aligned} \omega\tau_{\mathrm{pole},n\pm} &\approx \omega\tau_c + \frac{n}{p}\omega\beta - \frac{2\pi inq}{p} \\ &\quad \pm \mathrm{arccosh}\sqrt{\frac{\omega^2}{\omega^2-m^2}} + \frac{\pi i}{2} \qquad (\mathrm{mod}\ \pi i). \end{aligned} \qquad (B26)$$

These poles pass through the real τ axis for certain values of $\mathrm{Im}\tau_0$, at which the value of the action jumps discontinuously. When one of the poles—for example, $\tau_{\mathrm{pole},n+}$—is on the real τ axis, then $\tau_{\mathrm{pole},n'+}$ with $n' = n + kp/\gcd(p,2q)$ $[k = 0, \ldots, \gcd(p,2q) - 1]$ are also on the real τ axis, where $\gcd(p,2q)$ is the greatest common divisor of p and $2q$. Therefore, the discontinuity of the action when the poles pass through the real τ axis is

$$\Delta S = \pm 2\pi i \epsilon \gcd(p,2q). \tag{B27}$$

APPENDIX C: MULTIBION CONTRIBUTIONS

In this appendix we explicitly evaluate the quasimoduli integral for the chain of p kinks and p antikinks alternately aligned on S^1 with period β. The effective potential consists of the nearest-neighbor interactions

$$V_{\mathrm{eff}} = -m\epsilon\beta + \sum_{i=1}^{2p}\left(\frac{m}{g^2} + V_i\right), \tag{C1}$$

where V_i is the interaction potential

$$V_i = m\epsilon_i(\tau_i - \tau_{i-1}) - \frac{4m}{g^2}e^{-m(\tau_i-\tau_{i-1})}\cos(\phi_i - \phi_{i-1}), \tag{C2}$$

where (τ_i, ϕ_i) are quasimoduli parameters corresponding to the position and phase of the ith (anti)kink $(\tau_0 = \tau_{2p} - \beta, \phi_0 = \phi_{2p} \bmod 2\pi)$ and

$$\epsilon_i = \begin{cases} 2\epsilon & \text{for } i \in 2\mathbb{Z}, \\ 0 & \text{for } i \in 2\mathbb{Z}+1. \end{cases} \tag{C3}$$

It is convenient to redefine the relative quasimoduli parameters as

$$\begin{aligned} z_i &= m(\tau_i - \tau_{i-1}) + i(\phi_i - \phi_{i-1}), \\ \tilde{z}_i &= m(\tau_i - \tau_{i-1}) - i(\phi_i - \phi_{i-1}). \end{aligned} \tag{C4}$$

Note that the imaginary parts of z_i and $\tilde{z}_i$ are phases defined modulo 2π. The complex variables z_i and $\tilde{z}_i$ are subject to the constraints

$$\sum_{i=1}^{2p}\frac{z_i + \tilde{z}_i}{2} = m\beta, \qquad \sum_{i=1}^{2p}\frac{z_i - \tilde{z}_i}{2i} = 0 \qquad (\mathrm{mod}\ 2\pi), \tag{C5}$$

which are expressed by the integral forms of delta functions as functions of σ and s,

$$\begin{aligned} &\delta\left(\sum_{i=1}^{2p}\frac{z_i+\tilde{z}_i}{2} - m\beta\right) \\ &= \int\frac{d\sigma}{2\pi}\exp\left[i\sigma\left(\sum_{i=1}^{2p}\frac{z_i+\tilde{z}_i}{2} - m\beta\right)\right], \\ &\times\sum_{n=-\infty}^{\infty}\delta\left(\sum_{i=1}^{2p}\frac{z_i-\tilde{z}_i}{2i} - 2\pi n\right) \\ &= \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\exp\left(s\frac{z_i-\tilde{z}_i}{2}\right). \end{aligned} \tag{C6}$$

The saddle points $(q = 0, 1, \ldots, p-1)$ which give nontrivial contributions to the ground-state energy are located at

$$z_i = \begin{cases} -\log\left(\sqrt{\left(\frac{\epsilon g^2}{4m}\right)^2 + e^{-\frac{m\beta - 2\pi i q}{p}}} + \frac{\epsilon g^2}{4m}\right) \approx \log\frac{2m}{\epsilon g^2} & (\text{for } i \in 2\mathbb{Z}), \\ -\log\left(\sqrt{\left(\frac{\epsilon g^2}{4m}\right)^2 + e^{-\frac{m\beta - 2\pi i q}{p}}} - \frac{\epsilon g^2}{4m}\right) \approx -\log\frac{2m}{\epsilon g^2} + \frac{m\beta - 2\pi i q}{p} & (\text{for } i \in 2\mathbb{Z}+1). \end{cases} \tag{C7}$$

We note that the Lagrange multiplier σ is expressed in terms of the other parameters on the saddle points. This is consistent with the weak-coupling limit $g^2 \to 0$ of Eq. (20) with $z_{2n}/\omega = \tau_n^+ - \tau_n^-$ and $z_{2n+1}/\omega = \tau_{n+1}^- - \tau_n^+$.

The p-bion contribution to the partition function is given by

$$\frac{Z_p}{Z_0} = \frac{1}{p}\int\prod_{i=1}^{2p}\left[d\tau_i \wedge d\phi_i\frac{2m^2}{\pi g^2}\exp\left(-\frac{m}{g^2} - V_i\right)\right], \tag{C8}$$

where the factor $\frac{2m^2}{\pi g^2}$ is the one-loop determinant from the massive modes around each kink and the factor $1/p$ is inserted since the bions are indistinguishable. The integration measure can be rewritten as

$$\begin{aligned} \prod_{i=1}^{2p} d\tau_i \wedge d\phi_i &= m d\tau_c \wedge d\phi_c \wedge \left(\prod_{i=1}^{2p}\frac{i}{2m}dz_i \wedge d\tilde{z}_i\right) \\ &\times\delta\left(\sum_{i=1}^{2p}\frac{z_i+\tilde{z}_i}{2} - m\beta\right) \\ &\times\delta\left(\sum_{i=1}^{2p}\frac{z_i-\tilde{z}_i}{2i} - 2\pi n\right), \end{aligned} \tag{C9}$$

where τ_c and ϕ_c are the overall moduli parameters. We can rewrite the p-bion contribution as

$$\frac{Z_p}{Z_0} = \frac{2\pi m\beta}{p}e^{-\frac{2pm}{g^2}}\sum_{s=-\infty}^{\infty}\frac{1}{4\pi^2}\int d\sigma e^{-im\beta\sigma}\prod_{i=1}^{2p}I_i, \tag{C10}$$

where

$$I_i = \frac{im}{\pi g^2}\int dz_i \wedge d\tilde{z}_i \exp(-\mathcal{V}_i)\exp(-\tilde{\mathcal{V}}_i), \tag{C11}$$

with

$$\mathcal{V}_i = -\frac{2m}{g^2}e^{-z_i} + \frac{1}{2}(\epsilon_i - s - i\sigma)z_i,$$
$$\tilde{\mathcal{V}}_i = -\frac{2m}{g^2}e^{-\tilde{z}_i} + \frac{1}{2}(\epsilon_i + s - i\sigma)\tilde{z}_i. \tag{C12}$$

We can show that the p-bion contribution satisfies the following differential equation:

$$\left[\prod_{i=1}^{2p}\frac{s^2-(\epsilon_i - i\hat{\sigma})^2}{4} - \left(\frac{2m}{g^2}\right)^{4p}e^{-2m\beta}\right]\left(\frac{1}{\beta}\frac{Z_p}{Z_0}\right) = 0,$$
$$\hat{\sigma} = \frac{i}{m}\frac{\partial}{\partial\beta}. \tag{C13}$$

There are $4p$ linearly independent solutions, whose asymptotic forms for large β are given by

$$\frac{1}{\beta}\frac{Z_p}{Z_0} \approx \beta^q e^{-(2\epsilon\pm s)m\beta} \quad \text{or} \quad \beta^q e^{\pm sm\beta}$$
$$(q = 0, \ldots, p-1). \tag{C14}$$

Since the leading behavior of the p-bion contribution for large β should be $Z_p/Z_0 \approx \beta^p$, the above asymptotic solutions imply that the term with $s=0$ gives the leading contribution for large β. In the following, we only consider the term with $s=0$. For fixed values of σ, the saddle points of $\mathcal{V}_i$ and $\tilde{\mathcal{V}}_i$ are

$$z_i = \log\left(\frac{4m}{g^2}\frac{1}{\epsilon_i - i\sigma}\right) + \pi i(2l_i - 1),$$
$$\tilde{z}_i = \log\left(\frac{4m}{g^2}\frac{1}{\epsilon_i - i\sigma}\right) + \pi i(2\tilde{l}_i - 1), \tag{C15}$$

where $l_i, \tilde{l}_i$ are integers labeling the saddle points. It is convenient to shift the integration contour for σ so that $\mathrm{Re}(\epsilon_i - i\sigma) > 0$ for all i. Then, the integration over the thimble $\mathcal{J}_{l_i,\tilde{l}_i}$ associated with the saddle point labeled by $(l_i, \tilde{l}_i)$ gives

$$\int_{\mathcal{J}_{l_i,\tilde{l}_i}} dz_i \wedge d\tilde{z}_i e^{-\mathcal{V}_i - \tilde{\mathcal{V}}_i}$$
$$= \left(\frac{2m}{g^2}e^{\pi i(l_i+\tilde{l}_i-1)}\right)^{i\sigma-\epsilon_i}\Gamma\left(\frac{\epsilon_i - i\sigma}{2}\right)^2. \tag{C16}$$

The saddle points which contribute to the partition function can be determined by the Lefschetz thimble method. In Ref. [25], we have shown that when $\epsilon_i - i\sigma$ is a positive real number, the thimbles which contribute to the partition function are

$$\mathcal{C}_{\mathbb{R}} = \begin{cases} \mathcal{J}_{1,1} - \mathcal{J}_{1,0} & \mathrm{Im}g^2 = +0, \\ \mathcal{J}_{0,1} - \mathcal{J}_{0,0} & \mathrm{Im}g^2 = -0. \end{cases} \tag{C17}$$

As long as $\mathrm{Re}(\epsilon_i - i\sigma) > 0$, we can show that the same thimbles provide contributions to the partition function. Thus, we obtain

$$I_i = \frac{2m}{g^2}\left(\frac{2m}{g^2}e^{\pm\frac{\pi i}{2}}\right)^{i\sigma-\epsilon_i}\frac{\Gamma(\frac{\epsilon_i - i\sigma}{2})}{\Gamma(1-\frac{\epsilon_i - i\sigma}{2})}, \tag{C18}$$

where we have used the reflection formula for the gamma function

$$\sin(\pi x)\Gamma(x) = \frac{\pi}{\Gamma(1-x)}. \tag{C19}$$

Then, the contour integral for the p-bion contribution

$$\frac{Z_p}{Z_0} \approx \frac{m\beta}{p}e^{-\frac{2pm}{g^2}}\int\frac{d\sigma}{2\pi}e^{-i\sigma m\beta}\prod_{i=1}^{2p}I_i|_{s=0} \tag{C20}$$

can be evaluated by picking up the poles at $\sigma = -2ik$ and $\sigma = -2i(\epsilon + k)$ $(k \in \mathbb{Z}_{\geq 0})$. In the $\beta \to \infty$ limit, the pth-order pole at $\sigma = 0$ gives the leading-order term (26),

$$\frac{Z_p}{Z_0} \approx -\frac{im\beta}{p}e^{-\frac{2pm}{g^2}}\operatorname*{Res}_{\sigma=0}\left[e^{-im\sigma\beta}\prod_{i=1}^{2p}I_i|_{s=0}\right]$$
$$= -\frac{im\beta}{p!}e^{-\frac{2pm}{g^2}}\lim_{\sigma\to 0}\left(\frac{\partial}{\partial\sigma}\right)^{p-1}$$
$$\times\left[\frac{8im^2}{g^4}e^{-\frac{i\sigma m\beta}{p}}\left(\frac{2m}{g^2}e^{\pm\frac{\pi i}{2}}\right)^{2(i\sigma-\epsilon)}\right.$$
$$\left.\times\frac{\Gamma(\epsilon - \frac{i\sigma}{2})}{\Gamma(1-\epsilon+\frac{i\sigma}{2})}\frac{\Gamma(1-\frac{i\sigma}{2})}{\Gamma(1+\frac{i\sigma}{2})}\right]^p. \tag{C21}$$

The leading-order term (28) is

$$\frac{Z_p}{Z_0} \approx \frac{1}{p!}\left[\frac{2m\beta\Gamma(\epsilon)}{\Gamma(1-\epsilon)}e^{-\frac{2m}{g^2}\mp\pi i\epsilon}\left(\frac{2m}{g^2}\right)^{2(1-\epsilon)}\right]^p. \tag{C22}$$

This is consistent with the dilute gas approximation. In the supersymmetric case $\epsilon = 1$, Z_p/Z_0 vanishes due to the factor $1/\Gamma(1-\epsilon)$. In the near-SUSY case, we obtain

$$\lim_{\epsilon\to 1}\frac{\partial}{\partial\epsilon}\frac{Z_p}{Z_0} \approx 2m\beta e^{-\frac{2pm}{g^2}}, \tag{C23}$$

where we have used

$$\lim_{x\to 0}\partial_x \frac{1}{\Gamma(x)} = 1. \tag{C24}$$

Then we obtain

$$\lim_{\epsilon\to 1}\partial_\epsilon E = \lim_{\epsilon\to 1}\partial_\epsilon\left[E_{\rm pert} - \lim_{\beta\to\infty}\frac{1}{\beta}\log\left(1+\sum_{p=1}^{\infty}\frac{Z_p}{Z_0}\right)\right]$$
$$= \lim_{\epsilon\to 1}\partial_\epsilon E_{\rm pert} - \sum_{p=1}^{\infty} e^{-\frac{2pm}{g^2}}(2m+\mathcal{O}(g^2)). \tag{C25}$$

This is consistent with the exact result. Using the relation

$$\frac{1}{p!}\lim_{\epsilon\to 1}\partial_\epsilon^2 \lim_{\sigma\to 0}\partial_\sigma^{p-1}\left[\frac{X}{\Gamma(1-\epsilon+\frac{i\sigma}{2})}\right]^p$$
$$= \lim_{\epsilon\to 1}\lim_{\sigma\to 0} p\left(\frac{i}{2}X\right)^{p-1}[(p+1)\gamma - 2(p-1)i\partial_\sigma - 2\partial_\epsilon]X, \tag{C26}$$

we can show that

$$\frac{1}{2}\lim_{\epsilon\to 1}\partial_\epsilon^2\frac{Z_p}{Z_0} \approx -2m\beta e^{-\frac{2pm}{g^2}}\left[2p^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{\pi i}{2}\right) - (p-1)m\beta\right]. \tag{C27}$$

Therefore, the second-order coefficient of the ground-state energy in Eq. (20) is given by

$$\frac{1}{2}\lim_{\epsilon\to 1}\partial_\epsilon^2 E = \frac{1}{2}\lim_{\epsilon\to 1}\lim_{\beta\to\infty}\left(-\frac{1}{\beta}\frac{Z\partial_\epsilon^2 Z - (\partial_\epsilon Z)^2}{Z^2}\right)$$
$$= \sum_p 4me^{-\frac{2pm}{g^2}}p^2\left[\gamma+\log\frac{2m}{g^2}\pm\frac{\pi i}{2}+\mathcal{O}(g^2)\right]. \tag{C28}$$

APPENDIX D: PERTURBATION SERIES ON THE TRIVIAL VACUUM

In this appendix we derive the perturbative part of the ground-state energy by using the Bender-Wu method. Since the ground state is invariant under the phase rotation $\varphi \to e^{ib}\varphi$, the corresponding wave function Ψ is a function of $|\varphi|$. By redefining the wave function and the coordinate as

$$\Psi = e^{-x^2}\psi(x), \qquad |\varphi| = \eta x, \qquad \eta \equiv \frac{g}{\sqrt{m}}, \tag{D1}$$

the Schrödinger equation can be rewritten as

$$m\left[-\frac{1}{4}(1+\eta^2x^2)^2\left\{\frac{\partial^2}{\partial x^2}+(1-4x^2)\frac{1}{x}\frac{\partial}{\partial x}\right\} + V(x)\right]\psi = E\psi, \tag{D2}$$

where the potential is

$$V(x) = (1-x^2)(1+\eta^2x^2)^2 + \frac{x^2}{(1+\eta^2x^2)^2} - \epsilon\frac{1-\eta^2x^2}{1+\eta^2x^2}. \tag{D3}$$

Let us expand the energy and the wave function with respect to η:

$$\frac{E}{m} = \sum_{l=0}^{\infty} A_l \eta^{2l}, \qquad \psi = \sum_{l=0}^{\infty}\psi_l(x)\eta^{2l}. \tag{D4}$$

Then, the Schrödinger equation $(\tilde{H}-E)\psi = 0$ can be expanded as

$$0 = \frac{1}{4}\sum_{i=0}^{4}\binom{4}{i}x^{2i}\left[\psi''_{l-i}+(1-4x^2)\frac{1}{x}\psi'_{l-i} - 4(1-x^2)\psi_{l-i}\right]$$
$$+\sum_{i=0}^{l}A_l(\psi_{l-i}+2x^2\psi_{l-i-1}+x^4\psi_{l-i-2})$$
$$+(\epsilon - x^2)\psi_l - x^4\epsilon\psi_{l-2}, \tag{D5}$$

where $\psi_l = 0$ for $l < 0$. Setting $\psi_0 = 1$, we can solve these equations order by order. It is not difficult to show that ψ_l are polynomials of the form

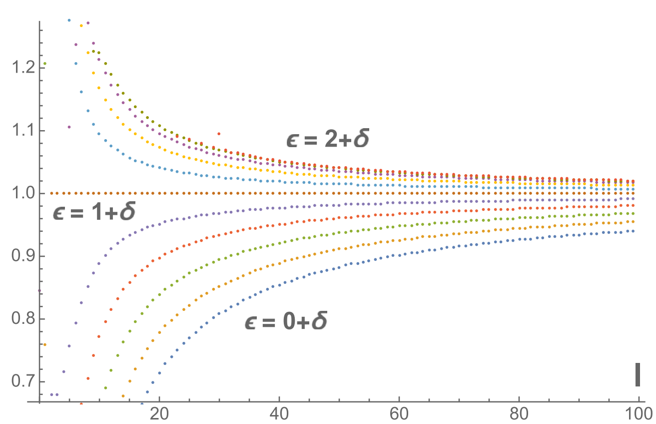

FIG. 3. The asymptotic behavior of the ratio $A_l/[\frac{1}{2^{l-1}}\frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon)^2}]$ $(l \le 100)$ for $0 \le \epsilon \le 2$ $(\epsilon = n/5, n = 0, \ldots, 10)$. δ is a regularization parameter $(\delta = 10^{-10})$.

$$\psi_l = \sum_{k=0}^{2l} B_{l,k} x^{2k}. \tag{D6}$$

We can always fix the normalization of the wave function as $\Psi(x=0) = 1$, i.e., $B_{0,0} = 1$, $B_{l,0} = 0 (l \neq 0)$. The Schrödinger equation reduces to

$$\begin{aligned} 0 = \sum_{i=0}^{4} \binom{4}{i} [&(k-i+1)^2 B_{l-i,k-i+1} \\ &- (2k-2i+1) B_{l-i,k-i} + B_{l-i,k-i-1}] \\ &+ \sum_{i=1}^{l} A_i (B_{l-i,k} + 2B_{l-i-1,k-1} + B_{l-i-2,k-2}) \\ &- B_{l,k-1} + \epsilon (B_{l,k} - B_{l-2,k-2}), \end{aligned} \tag{D7}$$

where $B_{l,k} = 0$ if $l < 0$, $k < 0$, $k > 2l$. As shown in Fig. 3, the asymptotic behavior (34) for $0 < \epsilon < 2$ is consistent with

$$A_l \sim -\frac{1}{2^{l-1}} \frac{\Gamma(l + 2(1-\epsilon))}{\Gamma(1-\epsilon)^2}. \tag{D8}$$

The Borel resummation of the right-hand side gives

$$\begin{aligned} &-m \sum_{l=1}^{\infty} \frac{1}{2^{l-1}} \frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon)^2} \eta^{2l} \\ &\overset{\text{Borel}}{=} -\frac{g^2}{\Gamma(1-\epsilon)^2} \int_0^\infty dt e^{-t} \sum_{l=0}^{\infty} t^{2(1-\epsilon)} \left(\frac{\eta^2}{2} t\right)^l \\ &= \frac{2m}{\Gamma(1-\epsilon)^2} \int_0^\infty dt e^{-t} \frac{t^{2(1-\epsilon)}}{t - \frac{2m}{g^2}}. \end{aligned} \tag{D9}$$

Therefore, the imaginary ambiguity [Eq. (22)] from the perturbative part is

$$\mathrm{Im} E_0 = \mp \frac{2\pi m}{\Gamma(1-\epsilon)^2} \left(\frac{g^2}{2m}\right)^{2(\epsilon-1)} e^{-\frac{2m}{g^2}}. \tag{D10}$$

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