

Analog rotating black holes in a magnetohydrodynamic inflowSousuke Noda^{*} and Yasusada Nambu[†]*Department of Physics, Graduate School of Science, Nagoya University, Chikusa, Nagoya 464-8602, Japan*Masaaki Takahashi[‡]*Department of Physics and Astronomy, Aichi University of Education, Kariya, Aichi 448-8542, Japan*

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We present a model of the analog geometry in a magnetohydrodynamic (MHD) flow. For the MHD flow with magnetic pressure-dominated and gas pressure-dominated conditions, we obtain the magnetoacoustic metric for the fast MHD mode. For the slow MHD mode, on the other hand, the wave is governed by the advective-type equation without an isotropic dispersion term. Thus, the “distance” perpendicular to the wave propagation is not defined, and the magnetoacoustic metric cannot be introduced. To investigate the properties of the magnetoacoustic geometry for the fast mode, we prepare a two-dimensional axisymmetric inflow and examine the behavior of magnetoacoustic rays which is a counterpart of the MHD waves in the eikonal limit. We find that the magnetoacoustic geometry is classified into three types depending on two parameters characterizing the background flow: analog spacetimes of rotating black holes, ultra spinning stars with ergoregions, and spinning stars without ergoregions. We address the effects of the magnetic pressure on the effective geometries.

DOI: [10.1103/PhysRevD.95.104055](https://doi.org/10.1103/PhysRevD.95.104055)**I. INTRODUCTION**

Black holes predicted by Einstein’s general relativity are characterized by the event horizon from which even light rays cannot escape. A rotating black hole has the ergo-region where light rays cannot propagate to the counter-rotating direction. When we consider the wave scattering, rotating black holes can amplify waves, and this phenomenon is called superradiance [1–3]. From the analysis of the wave scattering, it turns out that superradiance with some boundary conditions can evoke instabilities of waves. For example, considering the scattering of massive fields in the Kerr spacetime or massless fields in the Kerr-AdS spacetime, the wave is confined around the black hole due to a wall of effective potential for waves, and the superradiant scattering occurs repeatedly. Then the amplitude of the wave finally grows exponentially in time. This kind of instability is called the superradiant instability [4]. Even if there is no potential wall for confinement, the wave also suffers from the instability if there exists a reflective inner boundary such as the surface of stars (ergoregion instability) [5–11].

To examine the properties of superradiance associated with rotating black holes, we can use analog models of black holes. Analogs based on fluid models are called acoustic black holes. If a flow of the fluid has a sonic surface, the acoustic wave cannot pass through the sonic

surface from the downstream to the upstream. Namely, the sonic surface behaves as the black hole horizon for the acoustic wave (acoustic horizon). In such a situation, the perturbation of the velocity potential of the fluid obeys the Klein-Gordon equation in a “curved” spacetime of which geometry is defined by the acoustic metric [12]. That is why this kind of analog model is called an acoustic black hole (see also Ref. [13]). Although the motivation of the original work by Unruh [12] was to investigate the Hawking radiation from an acoustic black hole in laboratories, some papers later examined the feature of superradiance for the acoustic wave (acoustic superradiance) as well. To realize the rotating acoustic black hole in laboratories, several experimental setups or models have been proposed [14–25]. The draining bathtub model [14] is one example of such flows, and its acoustic metric has a structure similar to a rotating black hole. In this model, the flow is two dimensional, and the acoustic horizon is a closed circle where the radial velocity of the background flow exceeds the sound velocity in the fluid. The analog structure of the ergoregion also exists.

There are several works that consider the acoustic black holes in astrophysical situations. The original idea of the acoustic geometry in relativistic fluids was proposed by Moncrief [26] to examine the stability of the accretion flow onto a Schwarzschild black hole. In Moncrief’s analysis, the acoustic wave in the accretion flow was shown to satisfy the Klein-Gordon-type wave equation. Some papers also apply the idea of the acoustic black holes to astrophysical phenomena [27–34]. In particular, Abraham *et al.* [28] study the axially symmetric accretion flow onto a Kerr black hole and

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discuss a situation similar to the draining bathtub model with acoustic superradiance. When we apply the concept of acoustic black holes to some astrophysical problems, we should consider the effects of the magnetic field. In fact, due to the angular-momentum transportation caused by the magnetic viscosity, matter in the accretion disk can fall efficiently onto the central compact object [35–38]. Hence, when we discuss acoustic black holes in magnetized fluids such as in accretion disks, we need to take into account the magnetic field and generalize the acoustic black hole to the magnetohydrodynamic (MHD) case.

In this paper, we discuss the acoustic geometry in a MHD flow (magnetoacoustic geometry). As the eikonal equation of the MHD waves is the fourth-order differential equation for the phase of the velocity perturbation, it is not possible to define the quadratic magnetoacoustic metrics, and the MHD wave equation cannot be written in the form of the Klein-Gordon equation in curved spacetimes. To circumvent this problem, we focus on the magnetic pressure-dominated and the gas pressure-dominated cases of magnetized fluids. In such restricted cases, it is possible to expand the eikonal equation to define the quadratic magnetoacoustic metric. As a model of background flows, we prepare a stationary and axisymmetric two-dimensional ideal MHD inflow with a sink at the center. In our model, the magnetoacoustic property is characterized by two parameters relating to the sound velocity in the fluid and the angular velocity of the magnetic field line at the radial Alfvén point where the radial velocity of the fluid coincides with the radial component of the Alfvén velocity. Using these parameters, the magnetoacoustic geometry is classified into three types according to the values of these two parameters.

This paper is organized as follows. In Sec. II, we review one of the draining bathtub-type models and the superradiant scattering of the acoustic waves. In Sec. III, we first define the magnetoacoustic metric by considering the magnetic pressure-dominated case and the gas pressure-dominated case. Then we introduce the stationary and axisymmetric background MHD flow. In Sec. IV, we present possible magnetoacoustic geometries for our MHD flow. Section V is devoted to a summary of our paper. We use units $c = G = \hbar = 1$ throughout this paper.

II. ACOUSTIC BLACK HOLE FOR PERFECT FLUID

A. Draining bathtub model

We review an analog rotating black hole with two-dimensional flows, which is essentially the same as the draining bathtub model [14]. We consider an irrotational perfect fluid in a flat spacetime. The basic equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla p}{\rho} + \mathbf{F}_{\text{ex}} = 0, \quad (2)$$

$$\mathbf{v} = \nabla \Phi, \quad (3)$$

where the fluid velocity $\mathbf{v}$ is represented by the velocity potential Φ and $\mathbf{F}_{\text{ex}}$ is an external force to control the flow. We assume the pressure p and the density ρ of the fluid obey the polytropic equation of state $p \propto \rho^\Gamma$. We consider an axisymmetric stationary inflow and introduce the polar coordinates (R, ϕ) . Then the velocity and the external force are

$$\mathbf{v} = v^R(R) \mathbf{e}_{\hat{R}} + v^\phi(R) \mathbf{e}_{\hat{\phi}}, \quad \mathbf{F}_{\text{ex}} = F_{\text{ex}}(R) \mathbf{e}_{\hat{R}}, \quad (4)$$

where $\mathbf{e}_{\hat{R}}$ and $\mathbf{e}_{\hat{\phi}}$ are the orthonormal basis vectors. As the flow is two dimensional, it does not have a z component ($v^z = 0, F_{\text{ex}}^z = 0$). From the continuity Eq. (1), we have

$$R \rho v^R = \text{const}, \quad (5)$$

and from the azimuthal component of Eq. (2),

$$R v^\phi = \text{const} \equiv L, \quad (6)$$

where L is the angular momentum of the flow. The radial component of Eq. (2) is

$$v^R \frac{dv^R}{dR} - \frac{v^\phi}{R} + \frac{1}{\rho} \frac{dp}{d\rho} + F_{\text{ex}} = 0. \quad (7)$$

This equation yields the relation between v^R and F_{ex} because the pressure is related to the density as $p \propto \rho^\Gamma$ and the density is described by v^R and R through the relation (5). For a given radial velocity profile $v^R(R)$, this relation defines an appropriate form of $F_{\text{ex}}(R)$. We choose the radial velocity as¹

$$v^R \propto R^{-1/2}. \quad (8)$$

The sound velocity is

$$c_s \equiv \sqrt{\frac{\partial p}{\partial \rho}} \propto \Gamma^{1/2} R^{-(\Gamma-1)/4}. \quad (9)$$

To define an acoustic black hole, the background flow has to be a transonic inflow, and the downstream of the sonic point needs to be supersonic. Since the sonic point $R = R_s$ is given by

¹In some papers on the draining bathtub model, for example [14], the radial velocity is chosen as $v^R \propto R^{-1}$ to make $\rho = \text{const}$. The reason why we choose the nonstandard radial velocity is that we will compare it with our MHD model discussed in Sec. III, which has the radial velocity $\propto R^{-1/2}$.

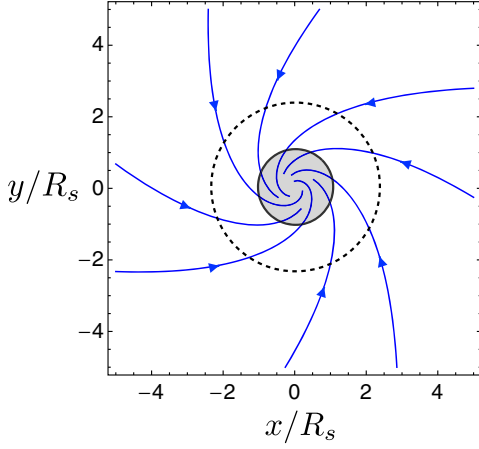


FIG. 1. Draining bathtub flow in the xy plane with $\Gamma = 4/3$, $c_s(R_s) = 1$, $L = 1.2R_s$, where R_s is the radius of the acoustic horizon defined by Eq. (10). The solid curves are streamlines of the flow. The grey region represents the inside of the acoustic horizon, and the dotted circle is the ergosurface for the acoustic waves.

$$c_s = |v^R|, \quad (10)$$

we consider the polytropic index within the interval $0 < \Gamma < 3$ for the radial velocity (8). By normalizing all quantities at the sonic point R_s , we obtain v and c_s as functions of R as

$$\begin{aligned} v^R &= -c_s(R_s) \left(\frac{R}{R_s} \right)^{-1/2}, \\ v^\phi &= \frac{L}{R}, \\ c_s &= c_s(R_s) \left(\frac{R}{R_s} \right)^{-(\Gamma-1)/4}. \end{aligned} \quad (11)$$

Figure 1 shows the structure of this flow.

Now, we add small perturbations δv , $\delta\Phi$, $\delta\rho$ to the background flow. We assume that the perturbations are independent of the z coordinate. Combining the first-order perturbed equations of (1)–(3), the perturbation of the velocity potential $\delta\Phi$ satisfies the following wave equation:

$$\begin{aligned} \frac{\partial}{\partial t} \left[c_s^{-2} \rho \left(\frac{\partial \delta\Phi}{\partial t} + \mathbf{v} \cdot \nabla \delta\Phi \right) \right] + \nabla \cdot \left\{ \left[c_s^{-2} \rho \left(\frac{\partial \delta\Phi}{\partial t} + \mathbf{v} \cdot \nabla \delta\Phi \right) \right] \right. \\ \left. \times \mathbf{v} - \rho \nabla \delta\Phi \right\} = 0, \quad \frac{\partial \delta\Phi}{\partial z} = 0. \end{aligned} \quad (12)$$

Equation (12) can be written as

$$\frac{\partial}{\partial x^\mu} \left(f^{\mu\nu} \frac{\partial \delta\Phi}{\partial x^\nu} \right) = 0, \quad \mu, \nu = 0, 1, 2, \quad (13)$$

with the matrix $f^{\mu\nu}$ defined by

$$f^{\mu\nu}(t, \mathbf{x}) = \frac{\rho}{c_s^2} \begin{pmatrix} -1 & -v^i \\ -v^i & c_s^2 \delta^{ij} - v^i v^j \end{pmatrix}, \quad i, j = 1, 2. \quad (14)$$

We introduce a matrix $s^{\mu\nu}$ as

$$\sqrt{|s|} s^{\mu\nu} = f^{\mu\nu}, \quad s = \det s_{\mu\nu}. \quad (15)$$

The matrix $s^{\mu\nu}$ and its inverse are given by

$$\begin{aligned} s^{\mu\nu}(t, \mathbf{x}) &= \frac{1}{\rho c_s} \begin{pmatrix} -1 & -v^j \\ -v^i & c_s^2 \delta^{ij} - v^i v^j \end{pmatrix}, \\ s_{\mu\nu}(t, \mathbf{x}) &= \frac{\rho}{c_s} \begin{pmatrix} -(c_s^2 - v^2) & -v^j \\ -v^i & \delta^{ij} \end{pmatrix}. \end{aligned} \quad (16)$$

Then, Eq. (12) becomes

$$\frac{1}{\sqrt{|s|}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|s|} s^{\mu\nu} \frac{\partial \delta\Phi}{\partial x^\nu} \right) = \square_s \delta\Phi = 0, \quad (17)$$

where the d'Alembertian $\square_s$ is defined with the metric $s_{\mu\nu}$. Equation (17) is the Klein-Gordon equation for a scalar field $\delta\Phi$ in the curved spacetime with the “acoustic metric” $s_{\mu\nu}$. This metric represents the effective geometry for the acoustic waves. For transonic background flows, it can be shown that the geometry represented by $s_{\mu\nu}$ has a similar structure to black hole spacetimes. The acoustic interval (acoustic line element) is defined as

$$\begin{aligned} ds^2 &\equiv s_{\mu\nu} dx^\mu dx^\nu = \frac{\rho}{c_s} [-(c_s^2 - v^2) dt^2 - 2\mathbf{v} \cdot d\mathbf{x} dt + d\mathbf{x}^2], \\ d\mathbf{x} &= (dx, dy). \end{aligned} \quad (18)$$

In the polar coordinates (R, ϕ) , the acoustic interval (18) becomes

$$\begin{aligned} ds^2 &= \frac{\rho}{c_s} \{ -[c_s^2 - ((v^R)^2 + (v^\phi)^2)] dt^2 - 2v^R dt dR \\ &\quad - 2v^\phi R dt d\phi + dR^2 + R^2 d\phi^2 \}. \end{aligned} \quad (19)$$

To examine the characteristics of this geometry, we rewrite the metric (19) by applying the following coordinate transformation:

$$\begin{aligned} dt &\rightarrow dt + \frac{v^R}{c_s^2 - (v^R)^2} dR, \\ d\phi &\rightarrow d\phi + \frac{v^R v^\phi}{c_s^2 - (v^R)^2} \frac{dR}{R}. \end{aligned} \quad (20)$$

Then we can write the acoustic interval in the following form:

$$ds^2 = \frac{\rho}{c_s} \left\{ -[c_s^2 - ((v^R)^2 + (v^\phi)^2)]dt^2 - 2v^\phi R d\phi dt + \frac{dR^2}{1 - (v^R/c_s)^2} + R^2 d\phi^2 \right\}, \quad (21)$$

where we used (t, R, ϕ) as the new coordinates after the coordinate transformation (20). From the similarity between the acoustic metric (21) and the metric of rotating black holes, we can introduce the acoustic horizon and ergosurface. The RR component and the tt component of the acoustic metric provide the following conditions:

$$|v^R| = c_s \quad (\text{acoustic horizon}), \quad (22)$$

$$\sqrt{(v^R)^2 + (v^\phi)^2} = c_s \quad (\text{acoustic ergosurface}). \quad (23)$$

The acoustic horizon is a circle where the radial velocity v^R coincides with the sound velocity and the acoustic wave cannot propagate outward from the inside of the acoustic horizon. The acoustic ergoregion is defined by $\sqrt{(v^R)^2 + (v^\phi)^2} \geq c_s$, and the wave propagation against the flow is not possible. For the background flow (8), the radius of the acoustic horizon R_H is R_s (the sonic point, see Fig. 1). The radius of the ergosurface R_E is determined as the solution of the following equation:

$$\left(\frac{R}{R_s}\right)^{(5-\Gamma)/2} - \left(\frac{R}{R_s}\right) - \left(\frac{L}{c_s(R_s)R_s}\right)^2 = 0. \quad (24)$$

B. Acoustic superradiance in the draining bathtub flow

We consider the solution of the Klein-Gordon Eq. (17) with the draining bathtub flow (11). The wave function $\delta\Phi$ can be separated as

$$\delta\Phi = \frac{\psi(R)}{R^{(7-\Gamma)/16}} e^{i(-\omega t + m\phi)}, \quad m = \pm 0, \pm 1, \pm 2, \dots, \quad (25)$$

where $\omega > 0$ is the frequency of the acoustic wave and m is the azimuthal quantum number. The radial part of this function satisfies the following differential equation:

$$-\frac{d^2\psi}{dR_{\text{tort}}^2} + V_{\text{eff}}(R; \omega, m, \Gamma)\psi = 0, \quad (26)$$

where the tortoise coordinate R_{tort} is introduced by $\partial/\partial R_{\text{tort}} = g(R) \cdot \partial/\partial R$, $g(R) = 1 - (v^R/c_s)^2$ and the effective potential $V_{\text{eff}}(R; \omega, m, \Gamma)$ is given by

$$V_{\text{eff}}(R; \omega, m, \Gamma) = -\frac{1}{c_s^2} \left(\omega - \frac{mv^\phi}{R} \right)^2 - g(R) \left[\frac{(7-\Gamma)(9+\Gamma)}{4096} \frac{g(R)}{R^2} - \frac{7-\Gamma}{16R} \frac{dg(R)}{dR} - \frac{m^2}{R^2} \right]. \quad (27)$$

Since $g = 0$ at the acoustic horizon $R = R_H$ and v^ϕ approaches zero at a point R_f far from the acoustic horizon, the asymptotic forms of the effective potential V_{eff} are

$$V_{\text{eff}} \approx \begin{cases} -\frac{1}{c_s^2} \left(\omega - \frac{mv^\phi}{R} \right)^2 & \text{for } R \rightarrow R_H \\ -\frac{\omega^2}{c_s^2} & \text{for } R \rightarrow R_f. \end{cases} \quad (28)$$

Therefore, the WKB solutions of Eq. (26) are

$$\psi = \begin{cases} \exp\left(-i \int^{R_{\text{tort}}} \frac{1}{c_s} \left(\omega - \frac{mv^\phi}{R} \right) dR_{\text{tort}}\right) & \text{for } R \sim R_H \\ C_{\text{in}} \exp\left(-i \int^{R_{\text{tort}}} \frac{\omega}{c_s} dR_{\text{tort}}\right) + C_{\text{out}} \exp\left(i \int^{R_{\text{tort}}} \frac{\omega}{c_s} dR_{\text{tort}}\right) & \text{for } R \sim R_f. \end{cases} \quad (29)$$

We take a purely ingoing solution near the acoustic horizon. From the conservation of the Wronskian $W = \psi^*(d\psi/dR_{\text{tort}}) - \psi(d\psi^*/dR_{\text{tort}})$ at $R = R_H$ and $R = R_f$, we obtain the relation between the reflection and the transmission rates of the acoustic wave:

$$\left| \frac{C_{\text{out}}}{C_{\text{in}}} \right|^2 + \frac{c_s(R_H)}{c_s(R_f)} \cdot \frac{\omega - m\Omega_H}{\omega} \left| \frac{1}{C_{\text{in}}} \right|^2 = 1, \quad (30)$$

where $\Omega_H \equiv v^\phi(R_H)/R_H$ is the angular velocity of the background flow at the acoustic horizon. From the relation (30), we see that the reflection rate $|C_{\text{out}}/C_{\text{in}}|^2$ can exceed

unity when the frequency ω satisfies the superradiant condition

$$0 < \omega < m\Omega_H = m \frac{v^\phi(R_H)}{R_H} = \frac{mL}{(R_H)^2} \quad \text{with } mL > 0. \quad (31)$$

Now we consider the short wavelength limit of the wave (the eikonal limit). Although the amplification factor of scattered waves cannot be determined by the analysis with the eikonal limit, we can discuss the condition of superradiance. In the leading order of the eikonal approximation,

the phase function S of the wave function $\delta\Phi \propto e^{iS}$ satisfies the eikonal equation

$$s^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0. \quad (32)$$

We introduce the acoustic rays as integral curves of the vector field defined by

$$k^\mu \equiv \frac{dx^\mu}{d\lambda} = s^{\mu\nu} \frac{\partial S}{\partial x^\nu}, \quad (33)$$

where λ is an affine parameter. The acoustic rays propagate in the direction perpendicular to the acoustic wave front. For the axisymmetric stationary flow, the phase function S can be separated as $S = -\omega t + m\phi + S_R(R)$, and Eq. (32) with the metric (21) yields

$$s^{tt}\omega^2 - 2s^{t\phi}\omega m + s^{\phi\phi}m^2 + s^{RR}\left(\frac{dS_R}{dR}\right)^2 = 0. \quad (34)$$

Then using the relation (33), we obtain the equation for the radial component of the acoustic rays,

$$\begin{aligned} \left(\frac{dR}{d\lambda}\right)^2 &= -s^{RR}(s^{tt}\omega^2 - 2s^{t\phi}\omega m + s^{\phi\phi}m^2) \\ &= \frac{1}{c_s^2}(\omega - V^+)(\omega - V^-) \geq 0, \end{aligned} \quad (35)$$

where $V^\pm$ are effective potentials for the acoustic rays defined by

$$V^\pm \equiv m \frac{-s_{t\phi} \pm \sqrt{(s_{t\phi})^2 - s_{tt}s_{\phi\phi}}}{s_{\phi\phi}} = m \frac{v^\phi \pm \sqrt{c_s^2 - (v^R)^2}}{R}. \quad (36)$$

From Eq. (35), the interval of R determined by $V^-(R) \leq \omega \leq V^+(R)$ is forbidden for the propagation of the acoustic rays (Fig. 2). At $R = R_H$, $V^+ = V^- = m\Omega_H$.

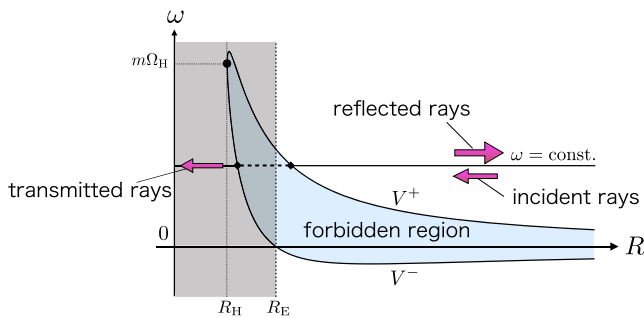


FIG. 2. The effective potentials $V^\pm$ with the parameters $\Gamma = 4/3$, $c_s(R_s) = 1$, $L = 1.2R_s$ and $m = 20$. The location of the acoustic horizon is R_H . The region between the acoustic horizon R_H and the acoustic ergosurface R_E corresponds to the acoustic ergoregion.

The interpretation of superradiance in terms of acoustic rays is as follows: Owing to the wave effect (i.e., tunneling), the incident rays with frequency (energy) satisfying the superradiant condition (31) can appear in the negative energy state region by penetrating the potential barrier. The transmitted rays finally fall into the acoustic horizon. In the course of this process, the rays reflected by the potential wall are amplified.

III. MAGNETOACOUSTIC GEOMETRY

Now, we investigate the effective geometry defined for the MHD waves in the eikonal limit. To define the magnetoacoustic metric, we consider the magnetic pressure-dominated and the gas pressure-dominated cases. Then, to investigate the structure of the magnetoacoustic geometry, we prepare an axisymmetric stationary solution of MHD inflow.

A. Eikonal equation and magnetoacoustic metric

The basic equations for the ideal MHD are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (37)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (38)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (39)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla p}{\rho} + \frac{1}{4\pi\rho} \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{F}_{\text{ex}}^{\text{MHD}} = 0, \quad (40)$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{F}_{\text{ex}}^{\text{MHD}}$ is an external force to control the MHD flow. The barotropic equation of state $p = p(\rho)$ is also assumed. We decompose all quantities to the stationary background part and the perturbed part. Assuming that the wavelength of perturbations is small compared to the scale of spatial variation of the background quantities, the equations of the first-order perturbations are written as follows:

$$\frac{D\delta\rho}{Dt} + \rho \nabla \cdot \delta \mathbf{v} = 0, \quad (41)$$

$$\nabla \cdot \delta \mathbf{B} = 0, \quad (42)$$

$$\frac{D\delta \mathbf{B}}{Dt} = \nabla \times (\delta \mathbf{v} \times \mathbf{B}), \quad (43)$$

$$\frac{D\delta \mathbf{v}}{Dt} + \frac{c_s^2}{\rho} \nabla \delta \rho + \frac{1}{4\pi\rho} \mathbf{B} \times (\nabla \times \delta \mathbf{B}) = 0, \quad (44)$$

where $\mathbf{v}$, $\mathbf{B}$, ρ are background quantities and the Lagrange derivative is introduced by $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$. Then, applying D/Dt to Eq. (44) and using Eqs. (41) and (43), we obtain the wave equation for $\delta\mathbf{v}$,

$$\frac{D^2 \delta\mathbf{v}}{Dt^2} - c_s^2 \nabla(\nabla \cdot \delta\mathbf{v}) + \mathbf{V}_A \times \nabla \times (\nabla \times (\delta\mathbf{v} \times \mathbf{V}_A)) = 0, \quad (45)$$

where $\mathbf{V}_A = \mathbf{B}/\sqrt{4\pi\rho}$ is the Alfvén velocity, which represents the propagating velocity of the transverse MHD wave mode along the magnetic field lines.

Hereafter, we consider two-dimensional flows in the xy plane and assume that all the vectors have no z components. By applying the Helmholtz theorem, the velocity perturbation can be described by using two scalar functions Φ , Ψ as

$$\delta\mathbf{v} = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} + \begin{pmatrix} \partial_y \Psi \\ -\partial_x \Psi \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (46)$$

Substituting this into Eq. (45), we obtain

$$\frac{D^2}{Dt^2} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \left[\begin{pmatrix} c_s^2 \nabla^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \hat{L}_1^2 & \hat{L}_1 \hat{L}_2 \\ \hat{L}_1 \hat{L}_2 & \hat{L}_2^2 \end{pmatrix} \right] \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad (47)$$

where $\hat{L}_1$ and $\hat{L}_2$ are the following derivative operators:

$$\begin{aligned} \hat{L}_1 &= (V_A)_y \partial_x - (V_A)_x \partial_y, \\ \hat{L}_2 &= (V_A)_x \partial_x + (V_A)_y \partial_y, \end{aligned} \quad (48)$$

and they are approximately commutable with each other since the spatial derivatives of the background quantities are assumed to be small. We define the magnetoacoustic metric as the components of the eikonal equation for the MHD waves. We substitute the following form of the wave function into Eq. (47):

$$\Phi = |\Phi| e^{iS}, \quad \Psi = |\Psi| e^{iS}, \quad (49)$$

where S is the phase of the wave functions. Up to the leading order of the eikonal approximation, we obtain

$$\begin{aligned} & \left(\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S \right)^2 \begin{pmatrix} |\Phi| \\ |\Psi| \end{pmatrix} \\ &= \left[\begin{pmatrix} c_s^2 |\nabla S|^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (\hat{L}_1 S)^2 & (\hat{L}_1 S)(\hat{L}_2 S) \\ (\hat{L}_1 S)(\hat{L}_2 S) & (\hat{L}_2 S)^2 \end{pmatrix} \right] \\ & \times \begin{pmatrix} |\Phi| \\ |\Psi| \end{pmatrix}. \end{aligned} \quad (50)$$

For Eq. (50) to have nontrivial solutions, we obtain the following eikonal equation²:

$$\begin{aligned} & \left(\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S \right)^4 - (c_s^2 + V_A^2) |\nabla S|^2 \left(\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S \right)^2 \\ & + (c_s^2 |\nabla S|^2) (\mathbf{V}_A \cdot \nabla S)^2 = 0, \end{aligned} \quad (51)$$

where we used relations $(\hat{L}_1 S)^2 + (\hat{L}_2 S)^2 = V_A^2 |\nabla S|^2$ and $\hat{L}_2 S = \mathbf{V}_A \cdot \nabla S$. We rewrite the eikonal equation as

$$\begin{aligned} & \left(\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S \right)^2 \\ &= \frac{V_M^2 |\nabla S|^2}{2} \left[1 \pm \sqrt{1 - 4 \left(\frac{c_s V_A}{V_M^2} \right)^2 \left(\frac{\mathbf{b} \cdot \nabla S}{|\nabla S|} \right)^2} \right], \end{aligned} \quad (52)$$

where $V_M \equiv \sqrt{c_s^2 + V_A^2}$, and $\mathbf{b} \equiv \mathbf{V}_A/V_A$ represents the direction of background magnetic field lines. The plus and minus signs correspond to the fast and the slow magnetoacoustic wave modes, respectively. Note that there is no transverse Alfvén mode in our analysis, as we are considering two-dimensional flows. Since the eikonal Eq. (52) does not have the quadratic form with respect to the derivative term of the phase S , it is not possible to introduce the quadratic (Riemannian) acoustic metric. In order to circumvent this problem, we introduce a parameter

$$\eta \equiv \left(\frac{c_s V_A}{V_M^2} \right)^2. \quad (53)$$

If we regard η as a small parameter, it is possible to expand the square root term in Eq. (52) as

$$\sqrt{1 - 4\eta \left(\frac{\mathbf{b} \cdot \nabla S}{|\nabla S|} \right)^2} = 1 - 2\eta \left(\frac{\mathbf{b} \cdot \nabla S}{|\nabla S|} \right)^2 + O(\eta^2). \quad (54)$$

The condition $\eta \ll 1$ can be satisfied for $(c_s/V_A)^2 \ll 1$ or $(V_A/c_s)^2 \ll 1$ because the value of the parameter is

²As Eq. (51) is the quartic form for $\partial S/\partial x^\mu$, it can be written as

$$M^{\mu\nu\lambda\sigma} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \frac{\partial S}{\partial x^\lambda} \frac{\partial S}{\partial x^\sigma} = 0,$$

where $M^{\mu\nu\lambda\sigma}$ is the coefficient of the eikonal equation. If we define a function $F(x, y)$ as $F(x, y) = (M^{\mu\nu\lambda\sigma} y_\mu y_\nu y_\lambda y_\sigma)^{1/4}$ with $y_\mu = \partial_\mu S$, we see that this function satisfies the condition $F(x, \alpha y) = \alpha F(x, y)$. This kind of function defines a distance $ds = F(x, dx)$ of the Finsler geometry [39], which is a generalization of the Riemannian geometry. Since the null condition $ds = 0$ corresponds to the eikonal Eq. (51), the motion of MHD waves may provide an effective Finsler geometry.

$$\eta \approx \begin{cases} (c_s/V_A)^2 \ll 1 & (\text{magnetic pressure-dominated case}) \\ (V_A/c_s)^2 \ll 1 & (\text{gas pressure-dominated case}). \end{cases} \quad (55)$$

Then Eq. (52) becomes

$$\left(\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S\right)^2 \approx \begin{cases} V_M^2 (|\nabla S|^2 - \eta(\mathbf{b} \cdot \nabla S)^2) & (\text{fast mode}) \\ \eta V_M^2 (\mathbf{b} \cdot \nabla S)^2 & (\text{slow mode}). \end{cases} \quad (56)$$

Note that in both the magnetic pressure-dominated and gas pressure-dominated cases, the fast wave mode propagates almost isotropically like sound waves. This is because the right-hand side of the eikonal equation for the fast mode consists of the isotropic term $V_M^2 |\nabla S|^2$ and the small correction term $\eta V_M^2 (\mathbf{b} \cdot \nabla S)^2$, which represents anisotropic effects due to the magnetic field. By assigning the coefficients of these equations to the components of matrices $M_{\text{fast}}^{\mu\nu}$ and $M_{\text{slow}}^{\mu\nu}$, Eqs. (56) can be written as

$$M_{\text{fast}}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0 \quad (\text{fast mode}), \quad (57)$$

$$M_{\text{slow}}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0 \quad (\text{slow mode}), \quad (58)$$

where $M_{\text{fast}}^{\mu\nu}$ and $M_{\text{slow}}^{\mu\nu}$ are

$$M_{\text{fast}}^{\mu\nu} = \begin{pmatrix} -1 & -v^i \\ -v^i & V_M^2 \delta^{ij} - (v^i v^j + \eta V_M^2 b^i b^j) \end{pmatrix},$$

$$M_{\text{slow}}^{\mu\nu} = \begin{pmatrix} -1 & -v^i \\ -v^i & -(\eta V_M^2 b^i b^j) \end{pmatrix},$$

$$i, j = 1, 2. \quad (59)$$

The matrix $M_{\text{fast}}^{\mu\nu}$ has an inverse, and we obtain the magnetoacoustic line element $ds_{\text{fast}}^2 = (M_{\text{fast}})_{\mu\nu} dx^\mu dx^\nu$ as

$$ds_{\text{fast}}^2 \propto -[(V_M^2 - \mathbf{v}^2) - \eta(\mathbf{b} \cdot \mathbf{v})^2] dt^2 - 2[v_i + \eta b_i(\mathbf{b} \cdot \mathbf{v})] dt dx^i + (\delta_{ij} + \eta b_i b_j) dx^i dx^j. \quad (60)$$

We write this metric in the polar coordinates (R, ϕ) and apply the following coordinate transformation:

$$dt \rightarrow dt + \frac{v^R}{V_M^2 - \eta V_M^2 (b^R)^2 - (v^R)^2} dR,$$

$$d\phi \rightarrow d\phi + \frac{v^R v^\phi + \eta b^R b^\phi V_M^2}{V_M^2 - \eta V_M^2 (b^R)^2 - (v^R)^2} \frac{dR}{R}. \quad (61)$$

Then the magnetoacoustic metric (60) can be written as

$$ds_{\text{fast}}^2 \propto -[(V_M^2 - \mathbf{v}^2) - \eta(\mathbf{b} \cdot \mathbf{v})^2] dt^2 - 2[v^\phi + \eta b^\phi(\mathbf{b} \cdot \mathbf{v})] R dt d\phi + \frac{dR^2}{1 - \eta(b^R)^2 - (v^R/V_M)^2} + [1 + \eta(b^\phi)^2] R^2 d\phi^2, \quad (62)$$

where t and ϕ are new coordinates after the coordinate transformation (61). As in the case of the acoustic black holes for perfect fluids, the effective horizon and ergosurface for MHD waves (magnetoacoustic horizon and ergosurface) are defined as points where the following conditions hold:

$$(v^R)^2 = V_M^2 - \eta V_M^2 (b^R)^2 \quad (\text{magnetoacoustic horizon}), \quad (63)$$

$$\mathbf{v}^2 = V_M^2 - \eta(\mathbf{b} \cdot \mathbf{v})^2 \quad (\text{magnetoacoustic ergosurface}). \quad (64)$$

Note that these conditions can be obtained without the coordinate transformation (61), as well by following the definition of the black hole horizon and the ergosurface in general relativity. Namely, we can derive the horizon condition (63) by searching the surface where a killing vector $\xi \equiv \xi_{(t)} + (v^\phi/R)\xi_{(\phi)}$ becomes null. The condition for ergosurface (64) is given by $\xi_{(t)}^2 = 0$. According to the eikonal Eqs. (56), the fast mode propagating in the radial direction ($\nabla S \propto \mathbf{e}_R$) has the propagating velocity $V_M^2 - \eta V_M^2 (b^R)^2$. Hence, the magnetoacoustic horizon defined by relation (63) is the one-way boundary for the fast mode propagation. While inside of the magnetoacoustic ergoregion defined by $\mathbf{v}^2 \geq V_M^2 - \eta(\mathbf{b} \cdot \mathbf{v})^2$, the fast mode cannot propagate in the counter direction of the background flow velocity $\mathbf{v}$.

For the slow mode, on the other hand, it is not possible to introduce the magnetoacoustic metric because the matrix $M_{\text{slow}}^{\mu\nu}$ does not have an inverse due to the lack of the isotropic term $\propto |\nabla S|^2 = (\partial_x S)^2 + (\partial_y S)^2$, and the inner product between vectors cannot be defined. We can rewrite the eikonal Eq. (58) for the slow mode as

$$\frac{\partial S}{\partial t} + \mathbf{v}_\pm \cdot \nabla S = 0, \quad (65)$$

where $\mathbf{v}_\pm \equiv \mathbf{v} \pm (c_s V_A/V_M) \mathbf{b}$. Since the eikonal Eq. (65) is the advective type, we see that the propagation is restricted along $\mathbf{v}_\pm$, and plus and minus signs correspond to outgoing and ingoing waves, respectively. Although the magnetoacoustic line element is not defined, it can be possible to discuss the motion of rays of the slow mode.

B. Background MHD flow and magnetoacoustic black holes

As the background MHD flow, we consider a stationary and axisymmetric inflow. We assume that the equation of state is polytropic $p \propto \rho^\Gamma$ with $0 \leq \Gamma \leq 4$.³ To obtain the background MHD flow, we basically follow Weber and Davis [40] who derived the stationary and axisymmetric solution of the ideal MHD flow. The difference between our treatment and theirs is that we assume a profile of the radial fluid velocity $v^R(R)$ by introducing an appropriate external force. The basic equations for the background flow are

$$\nabla \cdot (\rho \mathbf{v}) = 0, \quad (66)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (67)$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad (68)$$

$$(\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla p}{\rho} + \frac{1}{4\pi\rho} \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{F}_{\text{ex}}^{\text{MHD}} = 0. \quad (69)$$

Since the flow is axisymmetric, the components of $\mathbf{v}$ and $\mathbf{B}$ depend only on R as

$$\begin{aligned} \mathbf{v} &= v^R(R) \mathbf{e}_{\hat{R}} + v^\phi(R) \mathbf{e}_{\hat{\phi}}, \\ \mathbf{B} &= B^R(R) \mathbf{e}_{\hat{R}} + B^\phi(R) \mathbf{e}_{\hat{\phi}}. \end{aligned} \quad (70)$$

Equations (66) and (67) are solved as

$$\rho v^R R = \text{const} < 0, \quad R B^R = \text{const} > 0. \quad (71)$$

The azimuthal component of the induction Eq. (68) yields

$$\frac{d}{dR} [v^\phi B^R - v^R B^\phi] = 0, \quad (72)$$

and we obtain the following relation:

$$v^R B^\phi - v^\phi B^R = \text{const} \equiv -\Omega_F R B^R, \quad (73)$$

where we used Eq. (71) to define the conserved quantity Ω_F , which is the angular velocity of the magnetic field line. In addition to this quantity, the conservation of the angular momentum of the fluid L is also derived from the azimuthal component of Eq. (69) as

$$R v^\phi - \frac{B^R}{4\pi\rho v^R} R B^\phi = \text{const} \equiv L. \quad (74)$$

From Eqs. (73) and (74), the azimuthal component of the fluid velocity v^ϕ is obtained as a function of the radial components v^R and B^R :

³In the MHD case, we can consider $\Gamma = 0$ ($c_s = 0$), and the magnetoacoustic geometries belong to the class on the $\alpha = 0$ axis in Fig. 5. The upper bound $\Gamma \leq 4$ is determined by the condition $\eta \ll 1$. According to Eq. (89), for $\Gamma > 4$, the condition $\eta \ll 1$ is not satisfied for small R in the background flow.

$$v^\phi = \Omega_F R \frac{M_A^2 L R^{-2} \Omega_F^{-1} - 1}{M_A^2 - 1}, \quad (75)$$

where we introduced the radial Alfvénic Mach number $M_A^2(R) \equiv 4\pi\rho(v^R)^2/(B^R)^2 = (v^R/V_A^R)^2$. To construct the solution, we require that the background inflow is the trans-magnetosonic flow. Such a flow should pass through the radial Alfvén point $R = R_*$ where $M_A^2 = 1$ and the denominator in Eq. (75) becomes zero. Hence, the numerator should go to zero simultaneously to keep the azimuthal component v^ϕ finite. Therefore, the relation between the conserved quantities L and Ω_F is obtained as

$$L = \Omega_F R_*^2. \quad (76)$$

Taking this condition into account, v^ϕ can be reduced to

$$v^\phi = \frac{\Omega_F R_*}{v_*^R} \frac{v^R - (R/R_*) v_*^R}{M_A^2 - 1}, \quad (77)$$

where $v_*^R = v^R(R_*)$. In the same way, the azimuthal component of the magnetic field is obtained from Eqs. (73) and (75) with the relation (76),

$$B^\phi = -\frac{B^R \Omega_F}{v_*^R R_*} \frac{R^2 - R_*^2}{M_A^2 - 1}. \quad (78)$$

To obtain the background quantities as functions of R , we first consider the radial components v^R and B^R . From Eq. (67), the radial component of the magnetic field is

$$B^R = B_*^R \frac{R_*}{R}, \quad (79)$$

where $B_*^R \equiv B^R(R_*) > 0$. The radial component of Eq. (69) is

$$v^R \frac{dv^R}{dR} - \frac{v^\phi}{R} + \frac{1}{\rho} \frac{dp}{dR} + \frac{1}{4\pi\rho} \frac{B^\phi}{R} \frac{d}{dR} (R B^\phi) + F_{\text{ex}}^{\text{MHD}} = 0. \quad (80)$$

This equation determines $v^R(R)$ because v^ϕ and B^ϕ are already expressed by v^R from (77) and (78), and the pressure $p = p(\rho)$ can be written with v^R through the polytropic equation of state and the conservation law (71). We can assume an arbitrary profile of v^R by choosing $F_{\text{ex}}^{\text{MHD}}(R)$ to satisfy Eq. (80). We choose the draining bathtub-type inflow, which behaves as⁴

⁴Writing the radial velocity as $v^R \propto (R/R_*)^{-\sigma}$, we see that the same classification of the magnetoacoustic geometry is obtained for $0 < \sigma < 1$. In this paper, we choose $\sigma = 1/2$ to make calculations simpler.

$$v^R = v_*^R \left(\frac{R}{R_*} \right)^{-1/2}, \quad (81)$$

where $v_*^R \equiv v^R(R_*) < 0$ is the radial fluid velocity at the radial Alfvén point. This profile is different from the standard bathtub model $v^R \propto 1/R$. The reason why we do not take the standard one is that for $v^R \propto 1/R$, the radial Alfvénic Mach number M_A becomes constant and the radial Alfvén point does not exist. Substituting v^R [Eq. (81)] into Eqs. (77) and (78), we obtain the azimuthal components v^ϕ and B^ϕ ,

$$v^\phi = -\beta v_*^R \left[\left(\frac{R}{R_*} \right)^{1/2} + \left(\frac{R}{R_*} \right)^{-1/2} + 1 \right], \quad (82)$$

$$B^\phi = -\beta B_*^R \left[1 + \left(\frac{R}{R_*} \right)^{1/2} \right] \left[1 + \left(\frac{R}{R_*} \right)^{-1} \right], \quad (83)$$

where we introduced a parameter

$$\beta \equiv \Omega_F R_* / v_*^R. \quad (84)$$

This constant represents the ratio of the angular velocity of the magnetic field lines to the radial component of the fluid velocity v^R at the radial Alfvén point R_* . The density profile $\rho(R)$ is also obtained from Eq. (71). Although the obtained flow has a singularity at $R = 0$ and the magnetic field is not divergence free there, we evaluate the properties of the MHD waves in the region, except for this singularity. Using the quantities of the background flow, we obtain the sound velocity c_s and Alfvén velocity V_A as functions of R/R_* :

$$c_s^2 = \alpha^2 (v_*^R)^2 \left(\frac{R}{R_*} \right)^{-(\Gamma-1)/2}, \quad (85)$$

$$V_A^2 = (v_*^R)^2 \left\{ \left(\frac{R}{R_*} \right)^{-3/2} + \beta^2 \left(\frac{R}{R_*} \right)^{1/2} \left[1 + \left(\frac{R}{R_*} \right)^{1/2} \right]^2 \times \left[1 + \left(\frac{R}{R_*} \right)^{-1} \right]^2 \right\}, \quad (86)$$

where we introduced another parameter,

$$\alpha \equiv -c_{s*} / v_*^R > 0, \quad (87)$$

which is the ratio of the sound velocity to the radial component of the flow velocity v^R at the radial Alfvén point R_* . The structure of the MHD flow obtained here is shown in Fig. 3. Note that our MHD flow is not reduced to the draining bathtub model Eq. (11) for any values of (α, β) . This is because the condition (76) is imposed via the relation between L and Ω_F in this magnetized fluid system. Hence, our two-dimensional MHD flow is not just the generalization of the draining bathtub model for the perfect fluids.

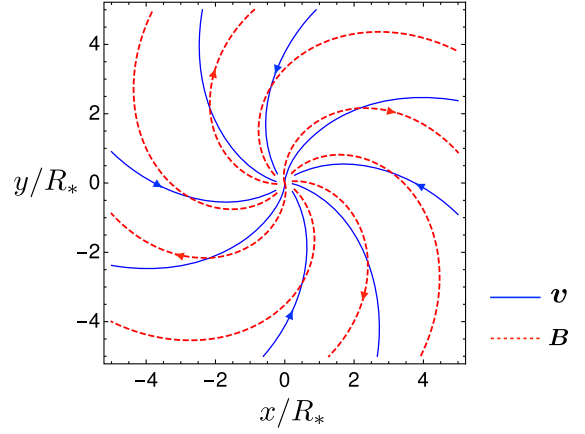


FIG. 3. Background MHD inflow with $\Gamma = 4/3$ and $\beta = 0.08$. The solid arrows (blue) are the stream lines of velocity $\mathbf{v}$, and the dotted arrows (red) represent the magnetic field lines of $\mathbf{B}$. The shapes of stream lines and magnetic field lines depend only on values of the parameter β .

In order to apply this background flow to the magnetoacoustic metric (62), the following condition must be satisfied:

$$\eta(R) = \left(\frac{c_s V_A}{V_M^2} \right)^2 \ll 1 \quad \text{for all } R. \quad (88)$$

Since $\eta = (c_s V_A / V_M^2)^2 \ll 1$ holds for both the $(V_A / c_s)^2 \ll 1$ and $(c_s / V_A)^2 \ll 1$ cases, we need to check this condition for our background MHD flow. From (85) and (86), the ratios $(c_s / V_A)^2$ at $R \rightarrow 0$ and $R \rightarrow \infty$ are

$$\begin{aligned} \left(\frac{c_s}{V_A} \right)^2 &\xrightarrow{R \rightarrow 0} \frac{\alpha^2}{1 + \beta^2} \left(\frac{R}{R_*} \right)^{(4-\Gamma)/2}, \\ \left(\frac{c_s}{V_A} \right)^2 &\xrightarrow{R \rightarrow \infty} \frac{\alpha^2}{\beta^2} \left(\frac{R}{R_*} \right)^{-(\Gamma+2)/2}. \end{aligned} \quad (89)$$

For the polytropic index $0 \leq \Gamma \leq 4$, these values approach zero, and only the magnetic pressure-dominated case $(c_s / V_A)^2 \ll 1$ can be realized in our model (see Fig. 4). As the criterion for the condition (88), we use the maximum

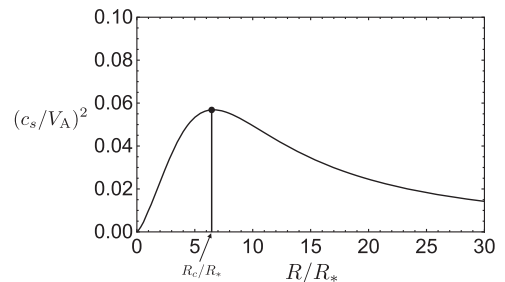


FIG. 4. R dependence of $(c_s / V_A)^2$ for $\Gamma = 4/3$ and $(\alpha, \beta) = (0.1, 0.04)$. Note that $(c_s / V_A)^2$ becomes maximum at $R = R_c$.

value $\eta_c \equiv (c_s/V_A)^2|_{R=R_c}$ where R_c is a maximum point of $(c_s/V_A)^2$.

IV. CLASSIFICATION OF THE MAGNETOACOUSTIC GEOMETRY

Now, we discuss the magnetoacoustic geometry introduced in the previous section. As a probe of the geometry, we investigate the motion of the magnetoacoustic rays.

A. Fast mode

With the eikonal equation of the MHD waves, Eq. (57), we introduce the magnetoacoustic rays as the integral curve of the vector k^μ defined by

$$k^\mu \equiv \frac{dx^\mu}{d\lambda} = M_{\text{fast}}^{\mu\nu} \frac{\partial S}{\partial x^\nu}, \quad (M_{\text{fast}})_{\mu\nu} k^\mu k^\nu = 0. \quad (90)$$

In the magnetic pressure-dominated case, the magnetoacoustic metric (57) becomes

$$\begin{aligned} ds_{\text{fast}}^2 &\propto -[(V_M^2 - \mathbf{v}^2) - \eta(\mathbf{b} \cdot \mathbf{v})^2] dt^2 \\ &\quad - 2[v^\phi + \eta b^\phi(\mathbf{b} \cdot \mathbf{v})] R dt d\phi \\ &\quad + \frac{dR^2}{1 - \eta(b^R)^2 - (v^R/V_M)^2} \\ &\quad + [1 + \eta(b^\phi)^2] R^2 d\phi^2 \end{aligned} \quad (91)$$

with $\eta \approx (c_s/V_A)^2 \ll 1$. As the geometry defined by the metric (91) is stationary and axisymmetric, there exist two Killing vectors $\xi_{(t)}$ and $\xi_{(\phi)}$, and the conserved quantities associated with them are

$$\omega \equiv -k_\mu \xi_{(t)}^\mu, \quad m \equiv k_\mu \xi_{(\phi)}^\mu. \quad (92)$$

The phase of the eikonal equation is separated as $S = -\omega t + m\phi + S_R(R)$. We can obtain the equation for the radial motion of the magnetoacoustic rays in the same way discussed in Sec. II for the acoustic rays. Then the eikonal Eq. (90) provides the following radial equation for magnetoacoustic rays:

$$\left(\frac{dR}{d\lambda}\right)^2 = \frac{1}{V_A^2} [1 - \eta(b^R)^2] (\omega - V^+) (\omega - V^-), \quad (93)$$

$$\begin{aligned} V^\pm &= m \frac{-(M_{\text{fast}})_{t\phi} \pm \sqrt{(M_{\text{fast}})_{t\phi}^2 - (M_{\text{fast}})_{\phi\phi} (M_{\text{fast}})_{tt}}}{(M_{\text{fast}})_{\phi\phi}} \\ &= \frac{m v^\phi + \eta b^\phi(\mathbf{b} \cdot \mathbf{v}) \pm \sqrt{V_M^2 - (v^R)^2 - \eta[(v^R)^2 - (b^\phi)^2 V_M^2]}}{R \frac{1 + \eta(b^\phi)^2}{1 + \eta(b^\phi)^2}}. \end{aligned} \quad (94)$$

The magnetoacoustic rays are allowed to move in the R region determined by $\omega \leq V^-(R)$ or $V^+(R) \leq \omega$. The zero point of the square-root term in the effective potentials $V^\pm$ provides the condition for the magnetoacoustic horizon (63) as

$$\begin{aligned} 0 &= V_M^2 - (v^R)^2 - \eta[(v^R)^2 - (b^\phi)^2 V_M^2] \\ &\approx V_M^2 - (v^R)^2 - \eta(b^R)^2 V_M^2, \end{aligned} \quad (95)$$

where we used the relation $V_M^2 = (v^R)^2$ in the first-order term of η . Likewise, the condition for the magnetoacoustic ergosurface (64) is reproduced from the condition $V^- = 0$ by using the relation $V_M^2 = \mathbf{v}^2$ in the first-order term of η . As the components of the magnetoacoustic metric $(M_{\text{fast}})_{\mu\nu}$ depend on α and β through the background flow $\mathbf{v}$ and $\mathbf{B}$, the properties of the effective potentials $V^\pm$ are determined by the parameter sets (α, β) . We classify the magnetoacoustic geometry by examining the existence of the magnetoacoustic horizon and ergoregion using conditions (63) and (64). Let us define the following functions of R :

$$\begin{aligned} H(R; \alpha, \beta) &\equiv V_M^2 - \eta(b^R)^2 V_M^2 - (v^R)^2, \\ E(R; \alpha, \beta) &\equiv V_M^2 - \eta(\mathbf{b} \cdot \mathbf{v})^2 - \mathbf{v}^2. \end{aligned} \quad (96)$$

If these functions have intersections with the R axis, the magnetoacoustic geometry has the magnetoacoustic horizon and ergoregion. The functions (96) have one minimum at $R_h = R_h(\alpha, \beta)$ and $R_e = R_e(\alpha, \beta)$, which satisfy $dH/dR|_{R_h} = 0$ and $dE/dR|_{R_e} = 0$. Conditions for functions (96) to touch the R axis are given by

$$H(R_h(\alpha, \beta); \alpha, \beta) = 0, \quad E(R_e(\alpha, \beta); \alpha, \beta) = 0. \quad (97)$$

These relations (97) determine boundaries between different types of geometries in the $\alpha\beta$ plane. We show the results for $\Gamma = 4/3$ in Fig. 5 because we obtain the same classification of the magnetoacoustic geometry for all Γ within $0 \leq \Gamma \leq 4$. Figure 5 shows the classification of possible shapes of the effective potentials. For the validity of the approximation adopted in our analysis, we focus only on the region $\eta_c \leq 0.1$ in the left panel, and we find that the magnetoacoustic geometries are classified into three types—(a), (b), and (c)—depending on the existence of the magnetoacoustic horizon and the magnetoacoustic ergoregion. From the shape of the effective potentials $V^\pm$, we see $R = 0$ is surrounded by the potential wall with infinite height, and the incident rays cannot reach the origin. This behavior of the effective potential is caused by the magnetic pressure; for small R , the V_M^2 term in the effective potential (94) becomes dominant and

$$V^\pm \approx \pm \frac{m}{R} V_M \approx \pm m \sqrt{1 + \beta^2} \frac{|v_*^R|}{R_*} \left(\frac{R}{R_*}\right)^{-7/4}. \quad (98)$$

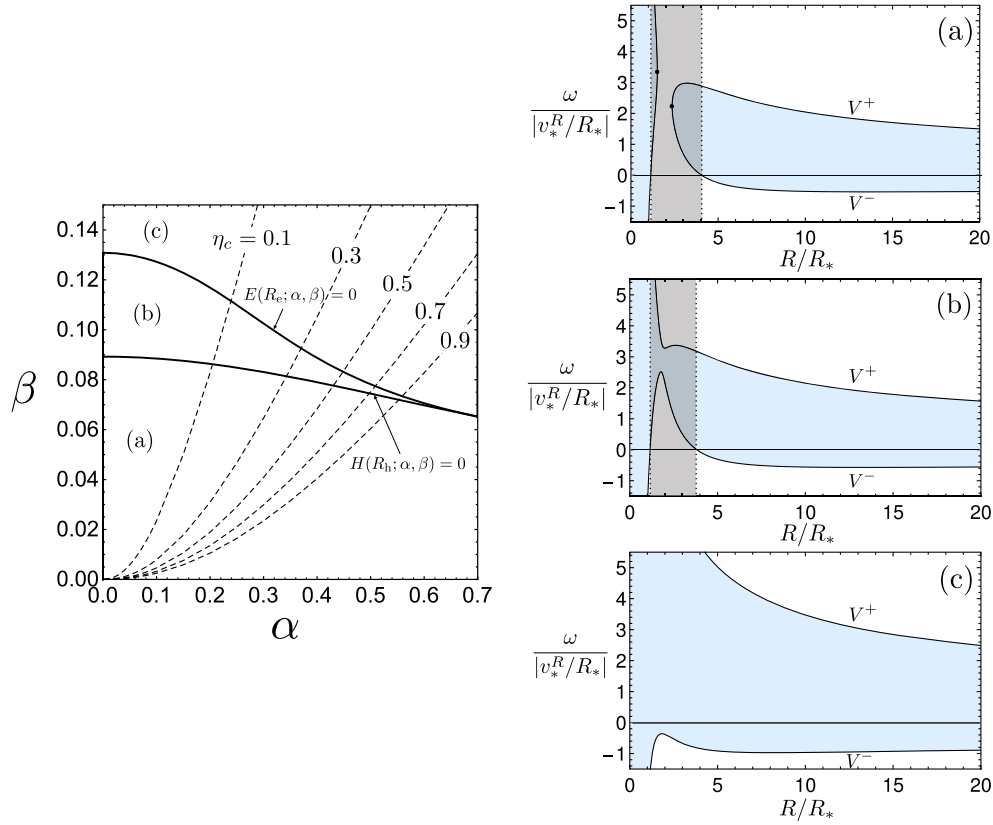


FIG. 5. The classification of the magnetoacoustic geometry for the fast mode with $\Gamma = 4/3$ and $m = 20$. In the left panel, the solid lines are $H(R_h(\alpha, \beta); \alpha, \beta) = 0$ (lower line) and $E(R_e(\alpha, \beta); \alpha, \beta) = 0$ (upper line). The dashed lines are contours $\eta_c(\alpha, \beta) = 0.1, 0.3, 0.5, 0.7, 0.9$. There are three types—(a), (b), and (c)—of the effective potentials $V^\pm$ classified by values of the parameters (α, β) . The right panels are typical effective potentials for (a): $(\alpha, \beta) = (0.1, 0.085)$; (b): $(\alpha, \beta) = (0.1, 0.089)$; and (c): $(\alpha, \beta) = (0.1, 0.14)$. The grey regions in the right panels represent the magnetoacoustic ergoregions for the fast mode.

This explains the wall of the effective potential in the vicinity of $R = 0$. Although we only consider $\beta \geq 0$ in Fig. 5, it is possible to treat the $\beta < 0$ region by changing the sign of the azimuthal quantum number m . Now, we discuss geometrical properties of each type.

For type (a), both the magnetoacoustic horizon and the ergoregion exist. Thus, the magnetoacoustic geometries are analogs of rotating black holes. In this case, by checking the value of the potentials at the magnetoacoustic horizon, we obtain the following superradiant condition:

$$\begin{aligned} \omega < -m \frac{M_{t\phi}^{\text{fast}}}{M_{\phi\phi}^{\text{fast}}} \Big|_{R_H} &\approx m \frac{v^\phi}{R} \left(1 + \eta \left(\frac{v^R}{v^\phi} \right) b^R b^\phi \right) \Big|_{R=R_H} \\ &= -\frac{m\Omega_F}{R_H} \left[\left(\frac{R_H}{R_*} \right)^{1/2} + \left(\frac{R_H}{R_*} \right)^{-1/2} + 1 \right] \\ &\quad \times \left(1 + \eta \left(\frac{v^R}{v^\phi} \right) b^R b^\phi \right) \Big|_{R=R_H} \end{aligned} \quad (99)$$

with $m\Omega_F < 0$.

From the $t\phi$ component of the magnetoacoustic metric (91), the signs of v^ϕ and b^ϕ determine the direction of the black hole's spin. As the signs of v^ϕ and b^ϕ are the same

and the sign of Ω_F is the opposite, for negative Ω_F ($\beta > 0$), the magnetoacoustic black hole rotates counterclockwise as shown in Fig. 3. Therefore, the additional condition $m\Omega_F < 0$ means that the incident magnetoacoustic rays should fall along prograde orbits $m > 0$ for superradiance.

For the flow belonging to type (b), the magnetoacoustic horizon disappears, and the potential wall becomes the inner boundary of the magnetoacoustic ergoregion. Hence, the flows are analogs of the ultra-spinning compact stars which evoke the ergoregion instability [5–9]. The analog ergoregion instability for the magnetoacoustic rays is possible, as rays are confined in the magnetoacoustic ergoregion where rays have negative energies (the grey region between two vertical dotted lines in the right panels of Fig. 5). Finally, the magnetoacoustic rays can propagate to outside of the magnetoacoustic ergoregion with positive energies via wave tunneling. For type (c), superradiance does not occur because there is no magnetoacoustic ergoregion.

B. Slow mode

As discussed in Sec. III, it is not possible to introduce the magnetoacoustic metric for the slow mode. However, we

can discuss the motion of the magnetoacoustic rays from the eikonal Eq. (58). The directions of the propagation are given by the characteristics $\mathbf{v}_{\pm} = \mathbf{v} \pm (c_s V_A/V_M) \mathbf{b} \approx \mathbf{v} \pm c_s \mathbf{b}$. For the ingoing wave, $v_-^R = v^R - c_s b^R < 0$ because $v^R < 0$ and $b^R > 0$ and the ingoing wave propagates towards $R = 0$. On the other hand, for the outgoing wave, v_+^R is given by

$$\begin{aligned} v_+^R &= v^R + c_s b^R \\ &= |v_*^R| \left(\frac{R}{R_*} \right)^{-1/2} \\ &\quad \times \left[-1 + \frac{\alpha}{\sqrt{1 + \beta^2 (1 + \sqrt{R/R_*})^2 (1 + R/R_*)^2}} \right] \\ &\quad \times \left(\frac{R}{R_*} \right)^{(3-\Gamma)/4}. \end{aligned} \quad (100)$$

For the polytropic index within $0 \leq \Gamma \leq 3$, the limiting behavior of v_+^R is

$$\lim_{R \rightarrow 0} v_+^R = -\infty, \quad \lim_{R \rightarrow \infty} v_+^R = -0, \quad (101)$$

and we have checked that $v_+^R < 0$ for all R under the condition $\eta_c < 0.1$. However, for $3 < \Gamma \leq 4$, the sign of v_+^R becomes positive in the vicinity of the origin of the flow $R = 0$, and hence there exists a point where $v_+^R = 0$. Thus, the “outgoing” waves propagate towards this point from the outer region and wind around a circle because the azimuthal component v_+^ϕ is not zero even at this point. Anyway, we do not have horizonlike structure associated with the slow mode.

V. CONCLUDING REMARKS

We have discussed the magnetoacoustic geometry defined for the MHD waves in an axisymmetric stationary inflow. We found that for the magnetic pressure-dominated

and the gas pressure-dominated cases, the magnetoacoustic metric can be introduced only for the fast mode. For the slow mode, it is not possible to introduce effective geometries because its propagation is restricted along lines determined by the background fluid flow and the magnetic field. As the background MHD flow, we considered the axisymmetric stationary inflow characterized by two parameters α and β specified at the radial Alfvén point. Then, the property of the magnetoacoustic geometry for this background flow can be classified into three types corresponding to rotating black holes, ultraspinning compact objects, and rotating stars.

In order to have a complete understanding of the magnetoacoustic geometry without the assumption adopted in this paper, we need to introduce the magnetoacoustic metric for the general MHD wave equation. Moreover, in our MHD inflow model, there is no Alfvén wave mode since we have restricted our analysis to two-dimensional flows. However, the Alfvén wave mode should be important when we consider three-dimensional MHD flows. In particular, in the high-energy astrophysical phenomena (e.g., active galactic nuclei, gamma-ray bursts), we expect that the study of the wave propagation in three-dimensional MHD flows is essential to understand the energy transportation, the formation, and evolution of some astrophysical systems. Therefore, analysis of the magnetoacoustic geometries including the Alfvén mode in astrophysical situations would provide us with important and interesting topics. We leave these problems to our future works.

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