

Hamiltonian analysis of the cuscutonHenrique Gomes^{*} and Daniel C. Guariento[†]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*

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The cuscuton was introduced in the context of cosmology as a field with infinite speed of propagation. It has been claimed to resemble Hořava gravity in a certain limit, and it is a good candidate for an ether theory in which a time-dependent cosmological constant appears naturally. The analysis of its properties is usually performed in the Lagrangian framework, which makes issues like the counting of its dynamical degrees of freedom less clear-cut. Here we perform a Hamiltonian analysis of the theory. We show that the homogeneous limit with local degrees of freedom has singular behavior in the Hamiltonian framework. In other frames, it has an extra scalar degree of freedom. The homogeneous field has regular behavior only if defined *a priori* as a spatially constant field in a constant mean curvature foliation and contributing with a single global degree of freedom. Lastly, we find conditions on the cuscuton potential for the resulting lapse function to be nonzero throughout evolution.

DOI: [10.1103/PhysRevD.95.104049](https://doi.org/10.1103/PhysRevD.95.104049)**I. INTRODUCTION**

One of the main observational issues facing cosmology and gravitational theories are the phases of accelerated expansion of the Universe: the many forms of the cosmological constant problem, dark energy, inflation. Moreover, given the failure of particle physics to account for the observed value of the vacuum energy, a quantum gravity theory is expected to at least give some insight on a solution to this problem. There have been several attempts at tackling one form or another of dark energy properties, mostly concentrating on two approaches: modifying general relativity or considering an additional field that would be responsible for these large-scale differences while retaining the well observed short-distance predictions of general relativity.

There is a considerable overlap between these approaches, as many modified gravity theories admit a description in the so-called “Einstein frame,” in which they are formulated as general relativity coupled to a scalar field. In fact, an economic and elegant way for scalar-tensor theories of gravity to take advantage of this fact when interpreting modifications of gravity is to adopt the effective description of a time-dependent cosmological constant and to provide constraints on this variability [1]. When formulating this system, the most general scalar-tensor theory of gravity that generates only second-order equations of motion is the Horndeski theory [2], also referred to in the literature as the generalized Galileons [3]. There are candidates to generalizing this theory to include higher-order spatial derivatives, known as beyond-Horndeski theories [4].

A simple yet noteworthy subset of Horndeski theories is the extreme form of *k*-essence known as the *cuscuton* [5]. It

is defined as the limit of *k*-essence in which the sound speed, as defined by derivatives of the Lagrangian with respect to the kinetic term of the scalar field, becomes infinite [6]. Interestingly, in flat or cosmological backgrounds, as well as in some other special cases [7], the equations of motion of the cuscuton are degenerate, effectively becoming only a source term in the Hamiltonian constraint of general relativity. Its behavior is akin to a time-dependent cosmological constant, without any propagating excitations. This degenerate behavior in the homogeneous limit has prompted the characterization of the cuscuton as a field that adds no additional degrees of freedom to a gravitational system, and, because the homogeneity condition selects a preferred foliation, the cuscuton has since been identified with similarly degenerate limits of modified gravity theories with broken Lorentz invariance, such as Hořava-Lifshitz gravity [5]. Correspondence with Einstein-ether theory is still a subject of debate [8].

The cuscuton is in fact not the unique subclass of Horndeski that possesses such a degenerate behavior. It was recently shown that a more general subclass of fields, known as *kinetic gravity braiding* [9], also produces degenerate equations of motion under certain limits, and a class of exact solutions of general relativity has been found with this field as source [10].

All these developments motivate one to ask: does the cuscuton in fact possess no propagating degrees of freedom? If this is generically true, a theory of cosmological perturbations need not worry about infinite propagation speeds, since no excitations would travel in general. Conversely, if there are in fact cases in which the cuscuton does propagate degrees of freedom, one needs to be aware of whether these cases pose a problem for causality and an incompatibility with Lorentz invariance in general.

^{*}hgomes@perimeterinstitute.ca[†]dguariento@perimeterinstitute.ca

The purpose of this work is to address this question. To do that in a rigorous setting, we formulate the cuscuton system coupled to general relativity formally using the Hamiltonian formalism and infer its dynamics by using the algebra of constraints to count the degrees of freedom.

The paper is organized as follows: in Sec. II we state the generic inhomogeneous cuscuton system coupled to general relativity in the Hamiltonian language. Section III discusses the symplectic geometry of this system and the constraint algebra which leads to the counting of degrees of freedom. The homogeneous limit and its pathologies are explored in Sec. IV. We present our conclusions and a brief discussion of the results in Sec. V. Throughout this work, latin indices run from 1 to 3, an overhead dot represent time derivatives, and we use a mostly plus Lorentzian signature.

II. HAMILTONIAN CUSCUTON DYNAMICS

The cuscuton Lagrangian [5] is a particular case of k -essence [11], in its turn defined as the subset of the Horndeski [2] action with only $L_2 \neq 0$. Its scalar sector is given in the language of generalized Galileons [4] by

$$L_2 = \int d^3x \sqrt{-g} [\mu^2 \sqrt{|2X|} - V(\phi)], \quad (1)$$

with g the determinant of the spacetime metric, and with μ a coupling constant. The kinetic term X is defined in terms of the first derivatives of ϕ as

$$X = \frac{1}{2} \left[\frac{1}{N^2} (\dot{\phi} - \xi^i \nabla_i \phi)^2 - \gamma^{ij} \nabla_i \phi \nabla_j \phi \right], \quad (2)$$

where we have performed the Arnowitt-Deser-Misner (ADM) decomposition of the spacetime metric in terms of the lapse N , shift ξ^i and spatial metric γ_{ij} , so that $\sqrt{-g} = N\sqrt{\gamma}$. From here on, we use the metric γ to raise and lower indices as per the ADM notation [12]. Also note that the modulus in Eq. (1), while ensuring that the Lagrangian is always real, also defines two branches depending on the sign of the kinetic term X . Since the main interest of the cuscuton is its cosmological applications, the interesting region should be the one in which the homogeneous field constitutes a spacelike surface. Therefore, for the remainder of this work, we consider only the branch $X > 0$.¹

The canonical momentum associated with ϕ is

$$\begin{aligned} \pi_\phi &\equiv \frac{\delta L_2}{\delta \dot{\phi}} \\ &= \frac{\sqrt{\gamma}}{N} \frac{\mu^2}{\sqrt{2X}} (\dot{\phi} - \xi^i \nabla_i \phi), \end{aligned} \quad (3)$$

and, in general, this definition does not introduce any primary constraints. The expression (3) may be solved for $\dot{\phi}$, giving

$$\dot{\phi} = \xi^i \nabla_i \phi \pm N \pi_\phi \sqrt{\frac{\nabla^i \phi \nabla_i \phi}{\pi_\phi^2 - \gamma \mu^4}}. \quad (4)$$

One should notice, however, that the solution with the minus sign above is spurious. If one inserts the negative branch of Eq. (4) into Eq. (3), one obtains $\pi_\phi = -\pi_\phi$. Therefore, the only reasonable solution is the one with the plus sign. Expressing the kinetic term X in terms of π_ϕ as

$$X = \frac{1}{2} \nabla^i \phi \nabla_i \phi \left(\frac{\gamma \mu^4}{\pi_\phi^2 - \gamma \mu^4} \right), \quad (5)$$

the Legendre transform will yield the following cuscuton Hamiltonian:

$$\begin{aligned} H_\phi &= \int d^3x \pi_\phi \dot{\phi} - L_2 \\ &= \int d^3x \left[N \sqrt{\nabla^i \phi \nabla_i \phi (\pi_\phi^2 - \gamma \mu^4)} + N \sqrt{\gamma} V + \xi_i \pi_\phi \nabla^i \phi \right]. \end{aligned} \quad (6)$$

For ease of comparison with the corresponding terms that come from the gravitational sector when computing the Hamiltonian and momentum constraints, the Hamiltonian (6) may be cast as a sum of a super-Hamiltonian \mathcal{H}_ϕ and a supermomentum \mathcal{H}_ϕ^i , that is,

$$H_\phi = \int d^3x (N \mathcal{H}_\phi + \xi_i \mathcal{H}_\phi^i), \quad (7)$$

where

$$\mathcal{H}_\phi = \sqrt{\nabla^i \phi \nabla_i \phi (\pi_\phi^2 - \gamma \mu^4)} + \sqrt{\gamma} V, \quad (8)$$

$$\mathcal{H}_\phi^i = \pi_\phi \nabla^i \phi. \quad (9)$$

Ignoring surface terms (by assuming that the spatial manifold is compact without boundary), the equations of motion of this field then read²

$$\begin{aligned} \dot{\pi}_\phi &\equiv \{\pi_\phi, H\} = -\frac{\delta H_\phi}{\delta \phi} \\ &= \nabla_i \left(N \nabla^i \phi \sqrt{\frac{\pi_\phi^2 - \gamma \mu^4}{\nabla^a \phi \nabla_a \phi}} + \xi^i \pi_\phi \right) - N \sqrt{\gamma} \frac{dV}{d\phi}, \end{aligned} \quad (10)$$

¹See Appendix B for comments on the negative branch.

²See Appendix C for the derivation of the full variation.

$$\begin{aligned}\dot{\phi} &\equiv \{\phi, H\} = \frac{\delta H_\phi}{\delta \pi_\phi} \\ &= \xi^i \nabla_i \phi + N \pi_\phi \sqrt{\frac{\nabla^i \phi \nabla_i \phi}{\pi_\phi^2 - \gamma \mu^4}}.\end{aligned}\quad (11)$$

The gravitational sector of the Hamiltonian in the ADM decomposition [12] reads

$$\begin{aligned}H_g &= \int d^3x \left\{ N \left[\frac{1}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} \mathcal{R} \right] + 2\xi_a \nabla_b \pi^{ab} \right\} \\ &= \int d^3x (N \mathcal{H}_g + \xi_a \mathcal{H}_g^a),\end{aligned}\quad (12)$$

where \mathcal{R} is the 3-Ricci scalar and we have defined

$$\mathcal{H}_g \equiv \frac{1}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} \mathcal{R}, \quad (13)$$

$$\mathcal{H}_g^a \equiv -2\nabla_b \pi^{ab}, \quad (14)$$

so the full Hamiltonian of the cuscuton field minimally coupled to the gravitational field reads

$$\begin{aligned}H &= H_g + H_\phi \\ &= \int d^3x [N(\mathcal{H}_g + \mathcal{H}_\phi) + \xi_a(\mathcal{H}_g^a + \mathcal{H}_\phi^a)].\end{aligned}\quad (15)$$

Clearly, the Hamiltonian has cyclic variables, imposing $\pi_N = 0 = \pi_\xi$. With the full action in mind, we may now consider the Hamiltonian and momentum constraints. Computing the Poisson brackets of the Hamiltonian (15) with the lapse and shift momenta, we find

$$\begin{aligned}\{H, \pi_N\} &= \frac{1}{\sqrt{\gamma}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} \mathcal{R} \\ &\quad + \sqrt{\nabla^i \phi \nabla_i \phi (\pi_\phi^2 - \gamma \mu^4)} + \sqrt{\gamma} V \\ &= 0,\end{aligned}\quad (16)$$

$$\{H, \pi_\xi^a\} = 2\nabla_b \pi^{ab} + \pi_\phi \nabla^a \phi = 0, \quad (17)$$

and the Poisson brackets of the full Hamiltonian (15) with the fields yield the equations of motion [12]. Together with Eqs. (10) and (11), the remaining equations of motion are

$$\begin{aligned}\dot{\gamma}_{ij} &\equiv \{\gamma_{ij}, H\} \\ &= 2\frac{N}{\sqrt{\gamma}} \left(\pi_{ij} - \frac{1}{2} \pi \gamma_{ij} \right) + 2\nabla_{(i} \xi_{j)},\end{aligned}\quad (18)$$

$$\begin{aligned}\dot{\pi}^{ij} &\equiv \{\pi^{ij}, H\} \\ &= -N\sqrt{\gamma} \left(\mathcal{R}^{ij} - \frac{1}{2} \gamma^{ij} \mathcal{R} \right) + \frac{N}{\sqrt{\gamma}} \left[\frac{\gamma^{ij}}{2} \left(\pi^{cd} \pi_{cd} - \frac{1}{2} \pi^2 \right) \right. \\ &\quad \left. - 2 \left(\pi^{ic} \pi_c^j - \frac{1}{2} \pi \pi^{ij} \right) \right] + \sqrt{\gamma} (\nabla^i \nabla^j N - \gamma^{ij} \nabla^c \nabla_c N) \\ &\quad + \nabla_c (\pi^{ij} \xi^c) - 2\pi^c (i \nabla_c \xi^i) + \frac{N}{2} \left[\nabla^i \phi \nabla^j \phi \sqrt{\frac{\pi_\phi^2 - \gamma \mu^4}{\nabla^a \phi \nabla_a \phi}} \right. \\ &\quad \left. + \gamma^{ij} \left(\gamma \mu^4 \sqrt{\frac{\nabla^a \phi \nabla_a \phi}{\pi_\phi^2 - \gamma \mu^4}} - V \sqrt{\gamma} \right) \right].\end{aligned}\quad (19)$$

Now, the constraints (16) and (17) might have first and second class components. A nontrivial second class part might imply that refoliation invariance be partly gauge fixed. In that case, the theory would come equipped with a ‘‘preferred frame.’’ In order to ascertain whether this is the case, we now analyze the symplectic geometry of the full theory.

III. SYMPLECTIC GEOMETRY OF THE CUSCUTON

Before we start, it is useful at this point to set the notation for a smearing of a phase-space density S over space:

$$S(N) \equiv \int d^3x N(x) S[g, \pi; x]. \quad (20)$$

The Poisson bracket between arbitrary smearings of the Hamiltonian constraint of general relativity (13) with itself defines the following algebra:

$$\begin{aligned}\mathcal{A}_g(N_1, N_2) &\equiv \{\mathcal{H}_g(N_1), \mathcal{H}_g(N_2)\} \\ &= \mathcal{H}_g^a(N_1 \nabla_a N_2 - N_2 \nabla_a N_1),\end{aligned}\quad (21)$$

which corresponds to the lapse component of the algebra of deformations of general relativity in the ADM formulation. In order to determine whether the Hamiltonian constraint of the cuscuton field coupled to gravity in fact spoils refoliation invariance by selecting a preferred foliation or restricting the choice of foliations with respect to general relativity, we must determine the algebra defined by the Hamiltonian constraint of the full theory and compare it with Eq. (21).

For that purpose, we calculate the Poisson bracket between arbitrary smearings of the Hamiltonian constraint of the full theory (16), that is,

$$\mathcal{A}(N_1, N_2) \equiv \{(\mathcal{H}_\phi + \mathcal{H}_g)(N_1), (\mathcal{H}_\phi + \mathcal{H}_g)(N_2)\}. \quad (22)$$

Using definition (20), and dropping the dependence of \mathcal{A} on the smearings, we get the following property,

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_g + \{\mathcal{H}_\phi(N_1), \mathcal{H}_\phi(N_2)\} \\ &\quad + \{\mathcal{H}_g(N_1), \mathcal{H}_\phi(N_2)\} - \{\mathcal{H}_g(N_2), \mathcal{H}_\phi(N_1)\}.\end{aligned}\quad (23)$$

The off-diagonal terms vanish, that is,

$$\{\mathcal{H}_g(N_1), \mathcal{H}_\phi(N_2)\} - \{\mathcal{H}_g(N_2), \mathcal{H}_\phi(N_1)\} = 0. \quad (24)$$

This happens because there are only derivatives of the configuration variables, and H_g has no π_ϕ dependence, and H_ϕ has no metric momenta; this implies no integration by parts needs to be done to remove derivatives from delta functions in these brackets, and thus the lapses appear without derivatives. Antisymmetry then implies the vanishing of Eq. (24).

We now compute the final term. Remembering that $V(\phi)$ contains no derivatives of the metric, already excluding terms that are linear in N , and using the results from Eq. (C2), we obtain

$$\begin{aligned} \{\mathcal{H}_\phi(N_1), \mathcal{H}_\phi(N_2)\} &= \pi_\phi \nabla^a \phi (N_1 \nabla_a N_2 - N_2 \nabla_a N_1) \\ &= \mathcal{H}_\phi^a (N_1 \nabla_a N_2 - N_2 \nabla_a N_1), \end{aligned} \quad (25)$$

which is of course, the same algebra that the gravitational scalar constraints obey in Eq. (21). The deformation algebra of the full Hamiltonian then is

$$\mathcal{A} = (\mathcal{H}_\phi^a + \mathcal{H}_g^a)(N_1 \nabla_a N_2 - N_2 \nabla_a N_1). \quad (26)$$

Thus, we obtain a consistent first class system of constraints. The number of physical degrees of freedom can be obtained from the usual counting: the dimension of the phase space Ω in a field theory in N -dimensional space subject to M first class constraints and S second class constraints is given by [13]

$$\dim(\Omega) = \frac{1}{2}(N - 2M - S). \quad (27)$$

In this case, the constraints are the same as in ADM, but we have an extra scalar field, thus we obtain a 3-dimensional space of physical degrees of freedom per space point.

IV. THE HOMOGENEOUS LIMIT

The modulus in the square root in the cuscuton Lagrangian (1) ensures that the Hamiltonian (6) remains real even if $X < 0$. However, the positive branch presents a problem on the surface $\pi_\phi = \mu^2 \sqrt{\gamma}$, as can be seen from the fact that Eq. (11) becomes divergent there. This can be solved if the numerator vanishes at the same rate; however, it can be seen from the momentum equation of motion (10) that a homogeneous solution with $\nabla_i \phi = 0$ may also pose a problem, in the form of the ill-defined limit inside the divergence. Since most applications of the cuscuton involve its cosmological evolution and perturbations around a homogeneous background, in the general theory one needs to check what happens if and when the Hamiltonian flow takes the system across these surfaces. As a first step, let us analyze the behavior of the dynamical system when we impose homogeneity as a constraint.

A. Weakly imposing homogeneity

Consider the following constraint, which represents a homogeneous cuscuton field on shell:

$$\Phi_h \equiv \nabla_i \phi \approx 0. \quad (28)$$

Since this constraint is a pure divergence, any smearing ζ^i of Eq. (28) must obey $\nabla_i \zeta^i \neq 0$ as other components of the smeared constraint automatically vanish.

If one considers imposing the constraint (28) onto the Hamiltonian (6), the equation of motion (10) becomes undetermined. In order to contain this potential loss of predictability, one may define the unit vector

$$f^i \equiv \frac{\nabla^i \phi}{\sqrt{\nabla^a \phi \nabla_a \phi}}, \quad (29)$$

in which case Eq. (10) may be cast in the form

$$\dot{\pi}_\phi = \nabla_i \left(N f^i \sqrt{\pi_\phi^2 - \gamma \mu^4} + \xi^i \pi_\phi \right) - N \sqrt{\gamma} \frac{dV}{d\phi}. \quad (30)$$

By setting $\nabla_i \phi = 0$, and assuming that f^i remains regular in this limit, the equations of motion (30) and (11) reduce to

$$\dot{\pi}_\phi = \nabla_i \left(N f^i \sqrt{\pi_\phi^2 - \gamma \mu^4} + \xi^i \pi_\phi \right) - N \sqrt{\gamma} \frac{dV}{d\phi}, \quad (31)$$

$$\dot{\phi} = 0. \quad (32)$$

Notice that the right-hand side of the equation of motion for ϕ vanishes, effectively decoupling the field from its momentum.

Assuming there are no boundaries, we can easily check that the constraint (28) is propagated:

$$\begin{aligned} &\{\Phi_h(\zeta^i), H(N)\} \\ &= \int d^3x \sqrt{\gamma} \left[(\zeta^i \nabla_i \phi) \frac{1}{2} \gamma^{ab} \dot{\gamma}_{ab} - (\nabla_i \zeta^i) \dot{\phi} \right] \approx 0, \end{aligned} \quad (33)$$

so the system remains on the constraint surface over the course of evolution. However, there is no equation of motion for ϕ , and Eq. (30) has no predictive power due to the presence of the undetermined vector f^i .

A possible way around this difficulty is to consider the effect of applying the homogeneity condition in the definition of the canonical momentum in order to define an alternative constraint that might be dynamically equivalent to homogeneity, but without the singularities introduced by the constraint (28). If we consider Eq. (3), introducing Eq. (28) into it results in the following constraint:

$$\Phi_{h'} = \pi_\phi - \mu^2 \sqrt{\gamma} \approx 0. \quad (34)$$

Clearly, it commutes with itself, but it is second class with respect to (28):

$$\{\Phi_h(x), \Phi_h(\zeta^i)\} = -\nabla_i \zeta^i(x) \neq 0. \quad (35)$$

Thus the two constraints are second class. We conclude therefore that weakly imposing homogeneity either leads to divergences or to a second class set of constraints.

B. Strongly imposing homogeneity

Imposing second class constraints requires a projection onto the constraint surface. This is equivalent to using variables where ϕ is a spatial constant, and π_ϕ as well.³ Alternatively, we will impose the constraint $\nabla_i \phi = 0$ strongly, explicitly on the Lagrangian (1) and show that this also results in the same system. Denoting in this particular case the constrained field by $\varphi(t)$ and its associated momentum by π_φ , we may cast the Lagrangian (1) as

$$L_2 = \int d^3x N \sqrt{\gamma} \left[\mu^2 \frac{\dot{\varphi}}{N} - V(\varphi) \right]. \quad (36)$$

Due to the spatial constancy of φ , this can be rewritten as

$$L_2 = \mu^2 \dot{\varphi} \int d^3x \sqrt{\gamma} - V(\varphi) \int d^3x \sqrt{\gamma} N, \quad (37)$$

so the associated momentum reads

$$\pi_\varphi = \mu^2 \int d^3x \sqrt{\gamma}. \quad (38)$$

This definition introduces the following primary constraint:

$$\Phi_\varphi \equiv \pi_\varphi - \mu^2 \int d^3x \sqrt{\gamma} \approx 0. \quad (39)$$

Note that now both φ and its associated momentum π_φ are spatial constants, and from the linearity of L_2 in $\dot{\varphi}$, its Hamiltonian is just the potential term. Absorbing the primary constraint into the ones we had previously found, the total Hamiltonian now reads

$$\begin{aligned} H &= H_\varphi + H_g + k \Phi_\varphi \\ &= k \left(\pi_\varphi - \mu^2 \int d^3x \sqrt{\gamma} \right) \\ &\quad + \int d^3x [N(\mathcal{H}_g + V\sqrt{\gamma}) + \xi_a \mathcal{H}_g^a]. \end{aligned} \quad (40)$$

However, the constraints composing the Hamiltonian are not first class. Indeed, we find the following expression for the algebra between the k and N generators:

³Note, however, that in the previous system π_ϕ is a density, which is point dependent. We would have to redefine the variables so that $\pi_\phi \rightarrow \pi_\phi / \sqrt{g}$ is the new momentum, and is a spatial constant.

$$\{\Phi_\varphi, (\mathcal{H}_\varphi + \mathcal{H}_g)(x)\} \approx \frac{dV}{d\varphi} \sqrt{\gamma}(x) - \frac{\mu^2}{2} \pi(x). \quad (41)$$

For this to propagate, we need the right-hand side to vanish, which provides us with a condition that the trace of the momentum be independent of the spatial coordinates. This is known in the literature as the constant mean curvature (CMC) condition, which we denote as

$$\Phi_{\text{CMC}} = \pi - \frac{2\sqrt{\gamma}}{\mu^2} \frac{dV}{d\varphi} \approx 0. \quad (42)$$

Now we must insert this further constraint into the system, to see how it behaves. Propagating Eq. (42), we cast its evolution as

$$\begin{aligned} \{\Phi_{\text{CMC}}, H\} &\equiv \dot{\Phi}_{\text{CMC}} \\ &= \left(\pi^{ij} - \frac{\sqrt{\gamma}}{\mu^2} \frac{dV}{d\varphi} \gamma^{ij} \right) \dot{\gamma}_{ij} + \gamma_{ij} \dot{\pi}^{ij} - \frac{2\sqrt{\gamma}}{\mu^2} \frac{d^2V}{d\varphi^2} \dot{\varphi}. \end{aligned} \quad (43)$$

Denoting the trace-free part of the momentum as σ^{ij} , that is,

$$\sigma^{ij} \equiv \pi^{ij} - \frac{1}{3} \pi \gamma^{ij}, \quad (44)$$

and substituting the constraints into Eq. (43), we find

$$\begin{aligned} &\left[\frac{12}{\gamma} \sigma^{ab} \sigma_{ab} + 6V + \left(\frac{2}{\mu^2} \frac{dV}{d\varphi} \right)^2 - 12\nabla^2 \right] N \\ &+ k \left(\frac{2}{\mu^2} \frac{d^2V}{d\varphi^2} - \frac{3}{2} \mu^2 \right) \approx 0. \end{aligned} \quad (45)$$

This is a lapse-fixing equation, which together with the CMC constraint (42) determines a gauge fixing for the system.

Since with our definition the spectrum of the Laplacian is strictly negative, if

$$12\sigma^{ab} \sigma_{ab} + 6V + \left(\frac{2}{\mu^2} \frac{dV}{d\varphi} \right)^2 > 0, \quad (46)$$

the operator

$$\Delta \equiv \left[12\sigma^{ab} \sigma_{ab} + 6V + \left(\frac{2}{\mu^2} \frac{dV}{d\varphi} \right)^2 - 12\nabla^2 \right] \quad (47)$$

is invertible. In that case, as long as $\left(\frac{2}{\mu^2} \frac{d^2V}{d\varphi^2} - \frac{3}{2} \mu^2 \right) \neq 0$, we have a unique lapse for each choice of k , which we will call N_0 . Nonetheless, if the left-hand side of Eq. (46) is negative, Δ can have a phase-space-dependent finite-dimensional kernel (since Δ is elliptic and we are in a compact manifold). In that case, there would be a finite-dimensional degeneracy of N_0 .

The constrained homogeneous cuscuton second class Hamiltonian system can be written as

$$H = k_1 \Phi_\varphi + k_2 \Phi_{\text{CMC}} + V(\varphi) \int d^3x N \sqrt{\gamma} + \int d^3x [N \mathcal{H}_g + \xi_a \mathcal{H}_g^a]. \quad (48)$$

This yields the foliation-fixed dynamics, which we will describe in the next section.

C. Second class system dynamics

Since this is a second class system, to find the equations of motion in the constraint surface we have two options: either (i) we use a Dirac bracket with the inverse of the (highly nonlocal) operator Δ from Eq. (47), or (ii) we find conjugate variables that can solve for the two second class constraints, $\mathcal{H}_g + \mathcal{H}_\varphi$ and Φ_{CMC} . Setting these variables strongly to the value of their solution, the Dirac bracket coincides with the standard Poisson one in the remaining variables. In general, these solutions will also be highly nonlocal.

A convenient choice is to use the spatial conformal factor of the metric and the trace of the momentum. We take the trace of the momentum to be conformally invariant, so the transformations read

$$\gamma_{ij} = e^{4\omega} \bar{\gamma}_{ij}, \quad (49)$$

$$\begin{aligned} \pi^{ij} &= e^{-4\omega} \bar{\pi}^{ij} \\ &= e^{-4\omega} \bar{\sigma}^{ij} + \frac{1}{3} e^{-4\omega} \bar{\pi} \bar{\gamma}^{ij}. \end{aligned} \quad (50)$$

The best way to understand this decomposition without introducing densities into the configuration variables is to use a reference metric and a reference density, η_{ij} , $\sqrt{\eta}$, such that $e^{12\omega} = \gamma/\eta$, so that $\bar{\gamma} = \eta$.

Inserting this decomposition and substituting the CMC constraint (42) into the Hamiltonian constraint, we find a Lichnerowicz-York-type equation for this system [14,15]:

$$\begin{aligned} 8e^{2\omega} \bar{\nabla}^2 \omega - e^\omega \bar{\mathcal{R}} + e^{-7\omega} \bar{\sigma}^{ab} \bar{\sigma}_{ab} \\ - e^{5\omega} \left[\frac{1}{3\mu^2} \left(\frac{dV}{d\varphi} \right)^2 + V \right] = 0. \end{aligned} \quad (51)$$

According to the Lichnerowicz-York method, this will have unique solutions as long as $6V + \left(\frac{2}{\mu^2} \frac{dV}{d\varphi}\right)^2 > 0$, which is also a sufficient condition for the positivity of Eq. (46). This is an interesting substitution of the standard condition for the unique solutions of the LY equation, which deserves to be further explored.

Under (44), the smeared momentum constraint reads

$$\begin{aligned} \int d^3x \pi^{ij} \mathcal{L}_\xi \gamma_{ij} &\rightarrow \int d^3x \left(\sigma^{ij} \mathcal{L}_\xi \gamma_{ij} + \frac{1}{3} \pi \gamma^{ij} \mathcal{L}_\xi \gamma_{ij} \right) \\ &= \int d^3x (\sigma^{ij} \mathcal{L}_\xi \gamma_{ij} + \pi \nabla^k \xi_k) \\ &\doteq \int d^3x (\sigma^{ij} \mathcal{L}_\xi \gamma_{ij}), \end{aligned} \quad (52)$$

where the last term in the middle equation vanishes for when $\frac{\pi}{\sqrt{g}}$ is a spatial constant, as $\nabla^k \xi_k = \gamma^{ij} \mathcal{L}_\xi \gamma_{ij}$ is a pure divergence. Under the conformal transformations induced by Eqs. (49) and (50), we now get

$$\int d^3x (\sigma^{ij} \mathcal{L}_\xi \gamma_{ij}) \rightarrow \int d^3x (\bar{\sigma}^{ij} \mathcal{L}_\xi \bar{\gamma}_{ij}), \quad (53)$$

since the extra term containing the derivative of ω is multiplied by $\bar{\gamma}^{ij} \sigma_{ij}$ and thus vanishes. This decouples the momentum and scalar constraints.

Finally, the complete system of equations of motion then reads

$$\dot{\varphi} = k, \quad (54)$$

$$\dot{\pi}_\varphi = -\frac{dV}{d\varphi} \int d^3x N_0 e^{6\omega}, \quad (55)$$

$$\dot{\omega} - \mathcal{L}_\xi \omega = \frac{1}{6} \left(\bar{\nabla}_a \xi^a - \frac{N_0}{\mu^2} \frac{dV}{d\varphi} \right), \quad (56)$$

$$\dot{\bar{\gamma}}_{ij} - \mathcal{L}_\xi \bar{\gamma}_{ij} = 2N_0 e^{-6\omega} \bar{\sigma}_{ij} - \frac{2}{3} \bar{\nabla}_a \xi^a \bar{\gamma}_{ij}, \quad (57)$$

$$\begin{aligned} \dot{\bar{\sigma}}_{ij} - \mathcal{L}_\xi \bar{\sigma}_{ij} &= -\frac{1}{3} \bar{\nabla}_a \xi^a \bar{\sigma}_{ij} + 2N_0 e^{-2\omega} \bar{\gamma}^{ab} \bar{\sigma}_{ia} \bar{\sigma}_{jb} \\ &\quad + e^{-2\omega} \left\{ \bar{\nabla}_i \bar{\nabla}_j N_0 - 4\bar{\nabla}_{(i} \omega \bar{\nabla}_{j)} N_0 \right. \\ &\quad - \frac{1}{3} (\bar{\nabla}^2 N_0 - 4\bar{\nabla}_a \omega \bar{\nabla}^a N_0) \bar{\gamma}_{ij} \\ &\quad - N_0 \left[\bar{\mathcal{R}}_{ij} - \frac{1}{3} \bar{\mathcal{R}} \bar{\gamma}_{ij} - 2\bar{\nabla}_i \bar{\nabla}_j \omega + 4\bar{\nabla}_i \omega \bar{\nabla}_j \omega \right. \\ &\quad \left. \left. + \frac{2}{3} (\bar{\nabla}^2 \omega - 2\bar{\nabla}_a \omega \bar{\nabla}^a \omega) \bar{\gamma}_{ij} \right] \right\}. \end{aligned} \quad (58)$$

Note that no source term appears in the evolution equation for $\bar{\sigma}_{ij}$. This can also be interpreted as a consequence of the energy-momentum tensor associated with the cuscuton being that of a perfect fluid (like any k -essence), and therefore possessing only terms that are proportional to the metric.

V. CONCLUSIONS

In this work we have formulated the cuscuton field, an extreme form of k -essence, in the Hamiltonian formalism, in order to count the degrees of freedom propagated by the field. By computing the algebra of constraints in the full system, we have found no evidence that a general field requires a preferred foliation. However, the field equations become singular in the homogeneous limit, so a transition to or from a homogeneous regime implies the loss of predictivity of the equations of motion. We have also shown that even if

homogeneity is imposed weakly as a constraint on the system, the system becomes second class. A way to resolve this difficulty is to impose strong homogeneity of the cuscuton field. Requiring that the dynamics be described with no dependence of the field with respect to spatial variables, this theory selects a foliation. Indeed, we then show that the resulting theory is constrained to a CMC foliation whose propagation is ensured by a lapse-fixing equation, which effectively fixes the gauge for this theory and ensures that it is well defined. In this gauge, there are in fact no dynamical evolution equations for the cuscuton, and the time derivative of the field appears as a Lagrange multiplier which parametrizes the patch of lapse functions that solve the CMC equation. No local excitations of the field can be activated, and therefore no worries about superluminal propagation need ensue.

An important consequence that our analysis shows is the fact that, because the homogeneous limit is fundamentally discontinuous from the generic cuscuton action, the two limits must be treated as different dynamical systems.

This means that the gauge freedom one usually has when treating a regular scalar field to choose a homogeneous foliation is not at all guaranteed here, since a gauge freedom is an equivalence on field space. In the $3+1$ setting, it was shown by Kuchař [16] that, for standard source terms that are regular functions in phase space, the algebra of constraints should follow the ADM algebra. Indeed, this means that for matter fields with regular, everywhere timelike gradients, one should be able to explore this gauge freedom to go to the frame in which the foliation is defined by the regular values of the field, and in which one can therefore assume $\nabla_i\varphi = 0$. However, these gauge symmetries need not be present everywhere in field space, if the source is not regular. In particular, the problem is that the cuscuton source has a square root and is in fact not regular, so we need to explicitly look at the gauge properties in field space separately at the different regularity domains. Indeed, we find that, away from the surfaces $\nabla_i\varphi = 0$ and $\pi_\varphi = \mu$, the standard refoliation symmetry is maintained, which means that the field has an extra local scalar degree of freedom in this part of phase space (with respect to pure gravity).

In other words, one should not be *a priori* able to exploit gauge freedom to go to the pre-image of the branching point in field space, since the theory does not necessarily have the same properties there. The standard assumption that gauge symmetry may be exploited to select an arbitrary foliation (e.g., one in which $\nabla_i\varphi = 0$) is couched on regularity of the sources in phase space—in which case, as mentioned above, one can indeed show that refoliation symmetry extends seamlessly to general relativity (GR) with sources [16]. In fact, we find that the theory is only consistent in a subset of the space in the pre-image of the branching point. Namely, only where it has no dynamical local degrees of freedom and defines the CMC surface.

Moreover, if one performs perturbation analysis around a homogeneous cuscuton background, one must carry out the analysis within the limits of a CMC foliation, and without allowing for local perturbations on the cuscuton field itself. By allowing the cuscuton to depart from homogeneity, it effectively becomes an entirely different system with nothing of the constraint structure that appears in the homogeneous case. In this latter case, it acquires an extra locally propagating scalar degree of freedom.

Still regarding the inhomogeneous case, the presence of two singular surfaces in phase space which are not limit surfaces—and which may in principle be reached in finite time by the Hamiltonian flow of a generic initial condition—may in fact render the inhomogeneous cuscuton a nonphysical model, in the sense that any generic initial condition surface may lead to an ill-defined system. However, a full analysis of the dynamical evolution of the inhomogeneous cuscuton is beyond the scope of this work, and will be the subject of future study.

The homogeneous cuscuton and the structure it generates has an interesting consequence. The CMC condition (42) establishes a connection between the trace of the gravitational momentum and the cuscuton potential. This means that each choice of solution for the potential corresponds to a different function of the momentum to appear in the Hamiltonian [17]. Gravitational theories which propose a dependency on the trace of the gravitational momentum include time-asymmetric modifications of general relativity [18] and the linking theory connecting shape dynamics and general relativity [14,19]. If these theories can be shown to display the same constraint structure as the cuscuton for a suitable choice of potentials, this in turn means that each cuscuton potential corresponds to an Einstein-frame description of a modified gravity theory with some dependence on the trace of the gravitational momentum.⁴

The ubiquity of CMC foliations in many different attempts to implement in GR a “time function”—i.e., a clock synchronization which is integrable and characterized by the constant value of some scalar field—raises questions: could it be related to the role that such foliations play in allowing a duality with spatially conformally invariant theories [18–20]? Or it could be related to the fact that, up to second order in gravitational momenta, there are no first class extensions of the ADM scalar constraint [21], and $\pi = \text{constant}$ is the only spatially covariant, first class constraint, which is purely second class with respect to \mathcal{H}_g [given in Eq. (13)] over the entire ADM phase space [18]. Of course, these two reasons are not mutually exclusive, and have in fact been argued to be related [18].

Lastly, we comment on the substitution of York time by the appearance of the term $6V + (\frac{2}{\mu^2} \frac{dV}{d\varphi})^2$. In the context of

⁴In the case of shape dynamics, this will only be a correspondence over a given patch of the conformal phase space which can be covered by the ADM gauge section.

cosmology, York time is the Hubble parameter H up to a negative constant,⁵

$$\tau = -(4\pi G)^{-1}H = -2M_{\text{pl}}^2 H, \quad (59)$$

where G is the gravitational constant and M_{pl} the reduced Planck mass. The monotonicity of τ , a requirement for it to function as a time parameter, is guaranteed if the equation of state parameter w obeys $w \geq -1$, with equality only in the limiting case when the kinetic contribution vanishes exactly, a scenario corresponding to a first approximation to slow-roll inflation. Here, we could potentially adjust the potential $V(\varphi)$ to probe different domains, making it more flexible than York time. These questions remain for further study.

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APPENDIX A: A QUICK PRIMER ON SYMPLECTIC GEOMETRY

In the Hamiltonian formalism, the symplectic flow of constraints generates motion in phase space. Because the Poisson bracket acts as a derivative,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h, \quad (A1)$$

we can see the linear operator $\{f, \cdot\}$ as a kind of vector field in phase space. Thus, the *symplectic flow* of a given phase-space function f is defined as $v_f \equiv \{f, \cdot\}$. It will act on other functions as a directional derivative $v_f[h]$ and measure how much h changes in the direction of v_f . That is, it measures how other phase-space functions change under “evolution” through the action of the corresponding phase-space function f . The phase-space function f implicitly defines a surface through the regular value theorem, provided certain regularity assumptions are satisfied. In the presence of a metric, one would usually say that the differential one-form df is “perpendicular” to the surface $f^{-1}(0)$, because its dual vector field df^\sharp is defined as $X[f] = df(X) =: g(df^\sharp, X)$, i.e., $g^{-1}df = (df)^\sharp$, which obviously vanishes for any vector field tangent to $f^{-1}(0)$. In the case of symplectic geometry, one does not define the analogous operation through the use of a

⁵For a more thorough study of cosmology in York time, see Ref. [22].

metric but a symplectic two-form, usually denoted by ω . Explicitly,

$$\omega(v_f, \cdot) := df, \quad (A2)$$

and furthermore

$$\omega(v_f, v_h) = \{f, h\}. \quad (A3)$$

Now, just as a vector field can be tangential to a given manifold, so can symplectic flows. Suppose then that a surface \mathcal{N} in phase space is given by the intersection of regular manifolds defined by the inverse values of the functions χ^I , i.e., $\mathcal{N} := \{(q, p) | \chi^I(q, p) = 0 \forall I\}$. Then \mathcal{N} will be said to be *first class* if for all phase-space functions f such that f vanishes on \mathcal{N} , i.e., $df(X) = 0$ for all $X \in T\mathcal{N}$, then $v_f[\chi^I](p, q) = 0$ for all I and $(p, q) \in \mathcal{N}$. The statement is equivalent to the much simpler statement that $\{f, \chi^I\} = a_I \chi^I$, since this will indeed be zero whenever we are on the surface. The geometric translation is indeed very simple: \mathcal{N} is first class if all symplectic flows v_f of functions f that vanish on the surface \mathcal{N} are tangent to the surface.

By contrast, we can define a second class manifold (or set of regular functions χ^I) if all symplectic flows (of functions that vanish on the surface) take us out of the surface (i.e., are not tangent to it).

Gauge fixings are further constraints imposed to be second class with the symmetry generator one would like to fix. Its flow is transverse to the gauge orbits, and it does not change the physical degree of freedom count (a theory with one set of n first class constraints has the same number of degrees of freedom than a theory with $2n$ second class constraints). On the other hand, imposition of a further first class constraint will change the degree of freedom counting, consisting in an actual dynamical reduction.

APPENDIX B: EQUATIONS OF MOTION IN THE $X < 0$ REGION

It can immediately be seen from the definition of X in Eq. (2) that the region $X < 0$ does not intersect with the homogeneous region. However, it is a simple exercise to demonstrate that this region of the cusp is well behaved, at least in the sense of the constraint algebra.

Let us consider $X < 0$ in the Lagrangian (1). The conjugate momentum in this case reads

$$\pi_\phi = -\frac{\sqrt{\gamma}}{N} \frac{\mu^2}{\sqrt{-2X}} (\dot{\phi} - \xi^i \nabla_i \phi), \quad (B1)$$

so when we cast $\dot{\phi}$ and X in terms of the momentum we find

$$\dot{\phi} = \xi^i \nabla_i \phi - N \pi_\phi \sqrt{\frac{\nabla^i \phi \nabla_i \phi}{\pi_\phi^2 + \gamma \mu^4}}, \quad (B2)$$

$$X = -\frac{1}{2}\nabla^i\phi\nabla_i\phi\left(\frac{\gamma\mu^4}{\pi_\phi^2 + \gamma\mu^4}\right). \quad (\text{B3})$$

The Hamiltonian is then given by

$$H_\phi = \int d^3x \left[N\sqrt{\gamma}V - N\sqrt{\nabla^i\phi\nabla_i\phi(\pi_\phi^2 + \gamma\mu^4)} + \xi_i\pi_\phi\nabla^i\phi \right], \quad (\text{B4})$$

and the equations of motion read

$$\dot{\pi}_\phi = -\nabla_i \left(N\nabla^i\phi\sqrt{\frac{\pi_\phi^2 + \gamma\mu^4}{\nabla^a\phi\nabla_a\phi}} - \xi^i\pi_\phi \right) - N\sqrt{\gamma}\frac{dV}{d\phi}, \quad (\text{B5})$$

$$\dot{\phi} = \xi^i\nabla_i\phi - N\pi_\phi\sqrt{\frac{\nabla^i\phi\nabla_i\phi}{\pi_\phi^2 + \gamma\mu^4}}. \quad (\text{B6})$$

Including the gravitational field, the full Hamiltonian constraint reads

$$\begin{aligned} \{H, \pi_N\} &= \frac{1}{\sqrt{\gamma}} \left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) - \sqrt{\gamma}\mathcal{R} \\ &\quad - \sqrt{\nabla^i\phi\nabla_i\phi(\pi_\phi^2 + \gamma\mu^4)} + \sqrt{\gamma}V \\ &= 0, \end{aligned} \quad (\text{B7})$$

while the momentum constraint remains unaltered. From these results, it becomes clear that the negative region obeys the same deformation algebra (25) as the positive region.

APPENDIX C: FULL VARIATION OF THE CUSCUTON HAMILTONIAN

In this Appendix we present all components of the variation of the cuscuton Hamiltonian. The components of the variation of the supermomentum term \mathcal{H}_ϕ^i with respect to the field variables are

$$\begin{aligned} \int d^3x\delta(\xi^i\mathcal{H}_i^\phi) &= \int d^3x[\pi_\phi\nabla_i\phi\delta\xi^i + \xi^i\delta(\pi_\phi\nabla_i\phi)] \\ &= \int d^3x[\pi_\phi\nabla_i\phi\delta\xi^i + \xi^i\nabla_i\phi\delta\pi_\phi + \pi_\phi\xi^i\nabla_i(\delta\phi)] \\ &= \int d^3x[\pi_\phi\nabla_i\phi\delta\xi^i + \xi^i\nabla_i\phi\delta\pi_\phi - \nabla_i(\xi^i\pi_\phi)\delta\phi + \nabla_i(\pi_\phi\xi^i\delta\phi)] \\ &= \int d^3x[\pi_\phi\nabla_i\phi\delta\xi^i + \xi^i\nabla_i\phi\delta\pi_\phi - \nabla_i(\xi^i\pi_\phi)\delta\phi] + \oint d^2s_i\pi_\phi\xi^i\delta\phi. \end{aligned} \quad (\text{C1})$$

Likewise, the variation of the super-Hamiltonian term \mathcal{H}_ϕ reads

$$\begin{aligned} \int d^3x\delta(N\mathcal{H}_\phi) &= \int d^3x \left\{ \mathcal{H}_\phi\delta N + N\delta \left[\sqrt{\nabla^j\phi\nabla_j\phi(\pi_\phi^2 - \gamma\mu^4)} + \sqrt{\gamma}V \right] \right\} \\ &= \int d^3x \left\{ \mathcal{H}_\phi\delta N + N \left[\frac{\delta\mathcal{H}_\phi}{\delta\phi}\delta\phi + \frac{\delta\mathcal{H}_\phi}{\delta(\nabla\phi)}\delta(\nabla\phi) + \frac{\delta\mathcal{H}_\phi}{\delta\pi_\phi}\delta\pi_\phi + \left(\frac{\delta\mathcal{H}_\phi}{\delta\gamma^{ij}} - \frac{\delta\mathcal{H}_\phi}{\delta\gamma}\gamma\gamma_{ij} \right) \delta\gamma^{ij} \right] \right\} \\ &= \int d^3x \left\{ \mathcal{H}_\phi\delta N + N\sqrt{\gamma}\frac{dV}{d\phi}\delta\phi + N\sqrt{\frac{\pi_\phi^2 - \gamma\mu^4}{\nabla^j\phi\nabla_j\phi}}\nabla^i\phi\nabla_i(\delta\phi) + N\pi_\phi\sqrt{\frac{\nabla^j\phi\nabla_j\phi}{\pi_\phi^2 - \gamma\mu^4}}\delta\pi + N \left(\frac{\delta\mathcal{H}_\phi}{\delta\gamma^{ij}} - \frac{\delta\mathcal{H}_\phi}{\delta\gamma}\gamma\gamma_{ij} \right) \delta\gamma^{ij} \right\} \\ &= \int d^3x \left\{ \left[\sqrt{\nabla^a\phi\nabla_a\phi(\pi_\phi^2 - \gamma\mu^4)} + \sqrt{\gamma}V \right] \delta N + \left[N\sqrt{\gamma}\frac{dV}{d\phi} - \nabla_i \left(N\nabla^i\phi\sqrt{\frac{\pi_\phi^2 - \gamma\mu^4}{\nabla^a\phi\nabla_a\phi}} \right) \right] \delta\phi + N\pi_\phi\sqrt{\frac{\nabla^a\phi\nabla_a\phi}{\pi_\phi^2 - \gamma\mu^4}}\delta\pi \right. \\ &\quad \left. + \frac{N}{2} \left[\nabla_i\phi\nabla_j\phi\sqrt{\frac{\pi_\phi^2 - \gamma\mu^4}{\nabla^a\phi\nabla_a\phi}} + \gamma_{ij} \left(\gamma\mu^4\sqrt{\frac{\nabla^a\phi\nabla_a\phi}{\pi_\phi^2 - \gamma\mu^4}} - V\sqrt{\gamma} \right) \right] \delta\gamma^{ij} \right\} + \oint d^2s_i N\nabla^i\phi\sqrt{\frac{\pi_\phi^2 - \gamma\mu^4}{\nabla^a\phi\nabla_a\phi}}\delta\phi. \end{aligned} \quad (\text{C2})$$

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