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Loop quantum gravity simplicity constraint as surface defect in complex Chern-Simons theory

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The simplicity constraint is studied in the context of four-dimensional spinfoam models with a cosmological constant. We find that the quantum simplicity constraint is realized as the two-dimensional surface defect in $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory in the construction of spinfoam amplitudes. By this realization of the simplicity constraint in Chern-Simons theory, we are able to construct the new spinfoam amplitude with a cosmological constant for an arbitrary simplicial complex (with many 4-simplices). The semiclassical asymptotics of the amplitude is shown to correctly reproduce the four-dimensional Einstein-Regge action with a cosmological constant term.

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I. INTRODUCTION

There has been significant development recently in including a cosmological constant in loop quantum gravity (LQG) [1–6].[1] A new covariant formulation of LQG has been developed, which gives a nice relation between the covariant LQG in four dimensions and Chern-Simons theory on a 3-manifold. In this new formalism, the spinfoam vertex amplitude is constructed by using the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory on a 3-sphere with a Wilson graph. This new formulation using Chern-Simons theory evolves from the earlier formulation using quantum groups [12–14].

This work focuses on the spinfoam amplitude constructed from the new formalism. In particular, this work studies the quantum application of the *simplicity constraint* to the spinfoam amplitude in the presence of a cosmological constant. It turns out that the simplicity constraint is realized as the two-dimensional (2d) surface defect in $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory used in constructing spinfoam amplitudes. By this realization of the simplicity constraint in Chern-Simons theory, we are able to nonperturbatively construct the new spinfoam amplitude with a cosmological constant for an arbitrary simplicial complex (with many 4-simplices). The semiclassical asymptotics of the amplitude is shown to correctly reproduce the four-dimensional (4d) Einstein-Regge action with a cosmological constant term.

In the classical Plebanski formulation, gravity in four dimensions is formulated by using the topological BF theory and implementing the simplicity constraint. The simplicity constraint restricts the bivector B field to be simple and related to the tetrad by $B^{IJ} = *(e^I \wedge e^J)$, which reduces the BF action to the Palatini action of gravity.

In the spinfoam formulation of covariant LQG, the simplicity constraint is quantized and imposed on the partition function of quantum BF theory. In the Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK) spinfoam model [15,16], a linear version of the simplicity constraint is imposed on the spinfoam amplitude. Given a simplicial complex, the linear simplicity constraint states that for each tetrahedron t, the bivector B-field smeared on its faces B_f^{IJ} shares the same time-like normal vector N^I. It is convenient to fix the time gauge so that, locally in each tetrahedron, the reference frame is chosen such that $N_I = (1,0,0,0)$. The time gauge breaks the local Lorentz symmetry to three-dimensional (3d) rotational symmetry. The simplicity constraint then implies that all bivectors B_f^{IJ} are spatial for each tetrahedron and are related to the spatial normals of tetrahedron faces.

The EPRL/FK spinfoam model is obtained by quantizing the above linear simplicity constraint and *weakly* imposing it on the BF partition function [17–20]. The reason for weakly imposing the constraint is that at the quantum level the components of the linear simplicity constraint are not commutative. Strongly imposing the constraint results in that the solution space does not have enough degrees of freedom. Similar phenomena also occur in the Gupta-Bleuler formalism of quantizing the electromagnetic field, and the covariant quantization of strings.

The quantum simplicity constraint of the EPRL/FK model guarantees that the boundary degrees of freedom of the spinfoam amplitude precisely match the quantum 3d geometry emerging from canonical LQG (namely, the boundary data of the EPRL/FK amplitude are SU(2)

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[1] See, e.g., Refs. [7–11] for reviews of loop quantum gravity, including both the canonical and covariant formalisms.

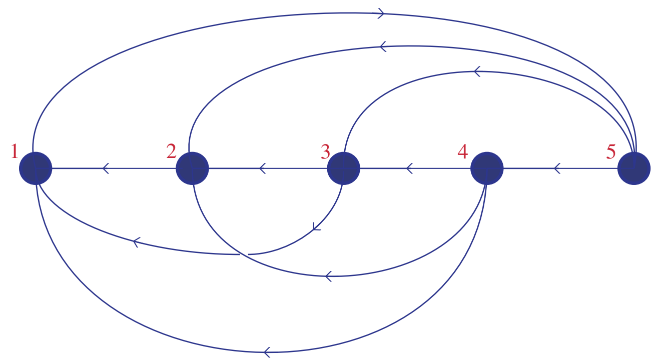

FIG. 1. Γ_5 graph embedded in S^3.

spin-network data). It also guarantees that the semiclassical large-spin asymptotics of the spinfoam amplitude correctly reproduces the Einstein-Regge action (without a cosmological constant term) evaluated at simplicial geometries with flat 4-simplices [21–25].

This work carries out the analysis of the simplicity constraint for the spinfoam model with a cosmological constant. The four-dimensional spinfoam amplitude with a cosmological constant is constructed by using $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory on a 3-manifold [1–3].[2] In this formalism, the local Lorentz symmetry of the 4d spinfoam model is translated to the $\mathrm{SL}(2,\mathbb{C})$ gauge symmetry of Chern-Simons theory. The bivectors B_f^{IJ} are naturally exponentiated and given by the holonomy of the flat connection traveling transversely around the Wilson line. The tetrahedron in the 4d spinfoam model corresponds to the neighborhood of the vertex where four Wilson lines join (see Fig. 1 for the Wilson graph used for constructing the 4-simplex amplitude).

It is explained in Sec. II that the simplicity constraint and time gauge correspond to requiring that on the four-holed sphere enclosing a four-valent vertex, the gauge group of Chern-Simons theory is broken from $\mathrm{SL}(2,\mathbb{C})$ to SU(2). In the classical limit, the flat connections on the four-holed sphere are restricted to be SU(2). It is known that SU(2) flat connections on a four-holed sphere are in one-to-one correspondence to tetrahedron geometries with constant curvature [6]. Thus, imposing the simplicity constraint ensures the geometricity of the tetrahedron at the classical level, similar to the situation in the EPRL/FK model (with flat tetrahedron geometries).

In Sec. III, we perform a quantization of the simplicity constraint, and define the constraint operators on the Hilbert space of $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons wave functions. Similar to the situation in the EPRL/FK model, we find that the constraint operators are noncommutative, which motivates us to instead impose a weaker version of the constraints. We propose to use the master constraint technique [26–28]. The master constraint effectively reduces the Hilbert space to a subspace, whose wave functions are equivalent to SU(2) Chern-Simons wave functions. We might view the master constraint as a Hamiltonian, for which the SU(2) Chern-Simons wave functions on the four-holed sphere are ground states; other $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons states are created as excitations similar to a harmonic oscillator.

In addition, we find that the weak simplicity constraint is not unique. Indeed, the solution of the master constraint is a coherent state peaked at the phase-space point which solves the classical simplicity constraint. We know that the coherent state which saturates the Heisenberg uncertainty is not unique, e.g., the squeezed coherent states. It turns out that different ways to define coherent states peaked at classical solutions of the simplicity constraint correspond to different ways of weakly imposing the simplicity constraint at the quantum level.

In Sec. IV, we consider the graph complement 3-manifold $S^3\backslash\Gamma_5$ similar to Refs. [2,3]. We impose the quantum simplicity constraint and project the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons wave function to the space of solutions. The resulting wave function $\mathcal{Z}$ is proposed as a spinfoam 4-simplex amplitude with a cosmological constant. We show that, thanks to the simplicity constraint, the amplitude ensures that the boundary degrees of freedom match precisely with discrete 3d geometry data on the boundary of the 4-simplex. The 3d geometry data is an analog of spin-network data (or semiclassically twisted geometry data) [5]. Also, the semiclassical asymptotics of the amplitude correctly reproduce the Einstein-Regge action with a cosmological constant term on a constant-curvature 4-simplex. The situation is a generalization of the EPRL/FK model to include the cosmological constant.

In Sec. V, we generalize the analysis to an arbitrary simplicial complex with many 4-simplices. The spinfoam amplitude on a 4d simplicial complex is a $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory on a 3-manifold $\mathcal{M}_3$ made by gluing copies of $S^3\backslash\Gamma_5$. We find that the implementation of the simplicity constraint corresponds to inserting 2d surface defects into $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory on the 3-manifold. The surface defects are inserted at the gluing interface (four-holed spheres) between pairs of $S^3\backslash\Gamma_5$, i.e., they divide the entire 3-manifold into copies of $S^3\backslash\Gamma_5$. Each surface defect restricts the Chern-Simons states—which travel from one $S^3\backslash\Gamma_5$ to another—to be solutions of the simplicity

[2]Following Ref. [1], $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory can be viewed to be equivalent to 4d BF theory with a cosmological constant term, when the 3d space where Chern-Simons lives is the boundary of the 4d space where BF theory lives. Schematically, the BF action with a cosmological constant reads $S_{BF} = \int_{\mathcal{M}_4} B^{IJ} \wedge *F_{IJ} + \frac{\Lambda}{6} B^{IJ} \wedge *B_{IJ}$. Integrating out the B field leads to $S \sim \frac{1}{\Lambda}\int_{\mathcal{M}_4} F^{IJ} \wedge *F_{IJ} \sim \frac{i}{\Lambda}\int_{\partial\mathcal{M}_4} \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A) +$ c.c., where $I, J = 0, \ldots, 3$ are Lorentz vector indices. $A = A^i\sigma_i$ is the $\mathfrak{sl}_2\mathbb{C}$-valued complex Chern-Simons connection. See Ref. [1] for details in the presence of the Barbero-Immirzi parameter.

constraint, i.e. to be equivalent to SU(2) Chern-Simons states. Due to our understanding of the simplicity constraint at the quantum level, we are able to formulate the spinfoam amplitude nonperturbatively on an arbitrary simplicial complex, which improves the result in Ref. [4].

Because surface defects impose the quantum simplicity constraint, the two key properties of the 4-simplex amplitude are generalized to the general spinfoam amplitude on a simplicial complex. The boundary data are always 3d geometry data, so the amplitude describes the quantum history of 3d geometries. The semiclassical asymptotics correctly reproduce the Einstein-Regge action with a cosmological constant term on the simplicial complex.

4d simplicial geometries emerge from critical points of the spinfoam amplitude, where locally each 4-simplex is of constant curvature. Interestingly, the 3-manifold $\mathcal{M}_3$ carrying the Chern-Simons theory has a number of nontrivial cycles, each of which is associated with a torus cusp defect. The longitude holonomy along the B-cycle of the torus cusp is noncontractible, since it is associated with a noncontractible cycle of 3-manifold. It turns out that each noncontractible cycle corresponds to a triangle in a 4d simplicial complex, and the noncontractible B-cycle holonomy corresponds to the nontrivial deficit angle hinged by the triangle. The 4d curvature is effectively created by the nontrivial cycles of the 3-manifold $\mathcal{M}_3$.

In Sec. VI, we consider the field-theoretic description of the surface defect. We can define an operator insertion in the Chern-Simons path integral in terms of the continuous field theory variable. The 2d "surface operator" inserted into the path integral effectively implements the quantum simplicity constraint. In general, the defect of topological quantum field theory has a certain dependence on the background metric, since it breaks the topological invariance to a certain extend. A typical example is the framing dependence of Wilson line operators. Here, the surface defect implementing the simplicity constraint also depends on the choice of the metric on the 2-surface. Different choices of metrics in the field-theoretic context may be viewed as analogs of choosing different squeezed coherent states mentioned above. Thus, different surface metrics for the surface defect correspond to different ways of weakly implementing the quantum simplicity constraint.

The semiclassical behavior is checked for the spinfoam amplitude in this field-theoretic formulation. The asymptotics again reproduce the Einstein-Regge action with a cosmological constant on the entire simplicial complex.

Although line defects have been widely studied in Chern-Simons theory, the results about surface defects (or domain walls) are insufficient in the literature (some results can be found in, e.g., Refs. [29–31]). The surface defect appearing here has not been studied before. In Sec. VII, we investigate the surface defect by studying the propagating physical degrees of freedom on the defect 2-surface. As mentioned above, the surface defect reduces $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons states to SU(2) in order to implement the simplicity constraint. On the defect where the gauge symmetry is broken, the previous gauge degrees of freedom become the physically propagating degrees of freedom. In other words, some additional propagating degrees of freedom have to be implemented in order to recover the original gauge symmetry on the defect. The standard example is the boundary of Chern-Simons theory, on which the Wess-Zumino-Witten model describes the propagating degree of freedom. We analyze the additional propagating degrees of freedom on the surface defect, which reinstall the $\mathrm{SL}(2,\mathbb{C})$ gauge invariance in the model. We show that, at least at the linearized level, the propagating field behaves as a 2d sigma model gauged by the Chern-Simons connection.

II. SIMPLICITY CONSTRAINT AND A CURVED TETRAHEDRON

In the spinfoam formulation without a cosmological constant, the classical linear simplicity constraint requires that, given a flat tetrahedron t, each of the four face bivectors B_f^{IJ} should be orthogonal to the time-like normal N^I of the tetrahedron,[3]

$$B_f^{IJ} N_I = 0, \quad \forall\ f \subset \partial t. \tag{2.1}$$

The time gauge may be chosen such that $N_I = (1,0,0,0)$, which is understood as a frame choice inside the tetrahedron. The frame can be located at any point inside the tetrahedron since the tetrahedron is flat.

The choice of time gauge breaks the local Lorentz symmetry down to spatial rotational symmetry. We have for each bivector $B_f^{IJ} = B_f^{ij}$, where i, j are 3d vector indices, and

$$\mathbf{a}_f \hat{n}_f^i = \frac{1}{2}\varepsilon^{ijk}(B_f)_{jk}, \tag{2.2}$$

where $\hat{n}$ is a unit space-like vector. Moreover, because of the closure constraint

$$0 = \sum_{f=1}^{4} B_f^{IJ} = \sum_{f=1}^{4} \mathbf{a}_f \hat{n}_f, \tag{2.3}$$

we know that the data B_f^{IJ} satisfying the simplicity constraint endow the tetrahedron t with the geometry, in which $\mathbf{a}_f$ is the face area and $\hat{n}_f$ is the unit face normal vector.

In the recent spinfoam models with a cosmological constant, the 4d spinfoam 4-simplex amplitude is formulated as a $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory on S^2 with Γ_5 Wilson graph defect (Fig. 1) [1–3]. In this formulation,

[3] $I, J = 0, \ldots, 3$ are vector indices of the Lorentz group.

each tetrahedron of the 4-simplex is related to a four-holed sphere $\mathcal{S}$ enclosing a vertex of the Γ_5 graph. By the Chern-Simons equation of motion (in the semiclassical limit), the $\mathrm{SL}(2,\mathbb{C})$ flat Chern-Simons connection on each four-holed sphere gives a holonomy version of the closure constraint,

$$H_4 H_3 H_2 H_1 = 1. \tag{2.4}$$

Fixing a base point on $\mathcal{S}$, H_f is the holonomy of the flat connection circling the fth hole (each hole is dual to a tetrahedron face). The above formula can be viewed as a closure constraint generalizing $\sum_f B_f^{IJ} = 0$ because each SL(2,ℂ) holonomy can be written as an exponential $H_f = \exp(\frac{\Lambda}{3} B_f^{IJ} \mathcal{J}_{IJ})$, where $\mathcal{J}_{IJ}$ are Lorentz generators. When the cosmological constant $\Lambda \to 0$, Eq. (2.4) implies the usual closure $\sum_f B_f^{IJ} = 0$ by linearization.

When we apply the simplicity constraint (2.1) and time gauge in this context, B_f^{IJ} is again restricted to be spatial; thus,

$$H_f = \exp\left(\frac{\Lambda}{3} B_f^{IJ} \mathcal{J}_{IJ}\right) = \exp\left(\frac{\Lambda}{3} \mathbf{a}_f \hat{n}_f \cdot \vec{\tau}\right) \in \mathrm{SU}(2),$$
$$\vec{\tau} = \frac{i}{2} \vec{\sigma}, \tag{2.5}$$

where $\vec{\sigma}$ are Pauli matrices. Therefore, the simplicity constraint and time gauge effectively reduce the structure group of Chern-Simons from $\mathrm{SL}(2,\mathbb{C})$ to SU(2) on each four-holed sphere. $\mathrm{SL}(2,\mathbb{C})$ flat connections are reduced to SU(2). Equation (2.4) becomes a product of SU(2) matrices.

It has been shown in Refs. [1,6] that the SU(2) flat connections on a four-holed sphere $\mathcal{S}$ are in one-to-one correspondence with the geometries of the constant-curvature tetrahedron, in which $\mathbf{a}_f$ in Eq. (2.5) is the face area and $\hat{n}_f$ is the unit face normal. However, since the tetrahedron is curved, a base point of the tetrahedron has to be chosen in order to make sense of the frame choice for the time gauge. Then, $\hat{n}_f$ is the unit face normal vector located at the tetrahedron base point.

The closure constraint (2.4) and the relation (2.5) suggest that in the presence of a cosmological constant, the flux variable used in LQG is naturally exponentiated. The exponentiated flux variable has been recently studied in, e.g., Refs. [5,32,33].

The moduli space of the flat SU(2) connection is of real dimension six, which parametrizes all degrees of freedom for constant-curvature tetrahedron geometries. The eigenvalues of SU(2) holonomies H_f around the four holes are related to the four triangle areas of the tetrahedron. It was shown in Ref. [6] that the shapes of a tetrahedron with fixed areas are parametrized by the flat connection coordinates $x, y \in \mathrm{U}(1)$. $x \in \mathrm{U}(1)$ is related to the diagonal length of a spherical four-sided polygon, while $y \in \mathrm{U}(1)$ is related to the "bending angle."

In the moduli space of $\mathrm{SL}(2,\mathbb{C})$ flat connections on $\mathcal{S}$, the coordinates x, y are known as Fenchel-Nielsen (FN) coordinates [5,6,34],[4] but now $x, y \in \mathbb{C}\backslash\{0\}$ since they parametrize $\mathrm{SL}(2,\mathbb{C})$ flat connections. The symplectic structure of the moduli space implies that x, y are canonically conjugate[5]:

$$\Omega = \frac{\mathrm{d}y^2}{y^2} \wedge \frac{\mathrm{d}x}{x}. \tag{2.6}$$

Recall that the simplicity constraint reduces the flat connection on $\mathcal{S}$ from $\mathrm{SL}(2,\mathbb{C})$ to SU(2). In terms of the coordinates, the simplicity constraint implies

$$\mathrm{Re}(\ln x) = 0, \qquad \mathrm{Re}(\ln y) = 0. \tag{2.7}$$

Namely, under the constraint, x, y become U(1) numbers parametrizing the shape of the tetrahedron.

For completeness, the simplicity constraint also restricts the eigenvalues of H_f to be U(1) numbers as well, since they are related to face areas. But it turns out that these restrictions can be easily imposed at the quantum level. The only nontrivial task is to quantize the constraint (2.7), which we focus on in the following. For convenience, we often denote $X = \ln x$ and $P = \ln y^2$ in the following discussion.

III. QUANTIZATION OF THE FLAT CONNECTION AND SIMPLICITY CONSTRAINT

We denote by $\mathcal{P}_\mathcal{S}$ the phase space of $\mathrm{SL}(2,\mathbb{C})$ flat connections on $\mathcal{S}$ with a fixed holonomy eigenvalue around each hole. $\mathcal{P}_\mathcal{S}$ is of complex dimension two. The coordinates on $\mathcal{P}_\mathcal{S}$ can be chosen to be (x, y^2). The symplectic structure of $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory reads

$$\begin{aligned}\omega_{k,s} &= \frac{1}{4\pi}(t\Omega + \bar{t}\bar{\Omega}) \qquad t = k + is, \quad \bar{t} = k - is \\ &= \frac{k}{2\pi}(\mathrm{dRe}P \wedge \mathrm{dRe}X - \mathrm{dIm}P \wedge \mathrm{dIm}X) \\ &\quad - \frac{s}{2\pi}(\mathrm{dRe}P \wedge \mathrm{dIm}X + \mathrm{dIm}P \wedge \mathrm{dRe}X).\end{aligned} \tag{3.1}$$

The quantization of the phase space $\mathcal{P}_\mathcal{S}$ can be carried out in a similar way as in Ref. [35]. $x = \exp X$ and

[4]The FN coordinates are defined by cutting the four-holed sphere $\mathcal{S}$ into two three-holed spheres. The flat connection on $\mathcal{S}$ gives a $\mathrm{SL}(2,\mathbb{C})$ holonomy h_x along the cut, whose eigenvalue is the FN complex length variable x. The FN twist variable y has a more technical definition. In nontechnical language, it comes from a holonomy h_y of the flat connection traveling from one three-holed sphere to the other, which transversely intersects h_x. The diagonalization of h_y gives the twist variable y. We refer the reader to, e.g., Refs. [3,5,34] for a mathematically precise definition.

[5]The square on y is conventional.

$y^2 = \exp P$ imply that ImX, ImP are periodic with period 2π. Weil's criterion of prequantization then requires $k \in \mathbb{Z}$. $s \in \mathbb{R}$ leads to $\omega_{k,s}$ being real, so that the Chern-Simons theory is unitary with respect to a standard Hermitian construction.[6]

As a convenient way to parametrize the complex Chern-Simons level, we write

$$is = k\frac{1-b^2}{1+b^2} \in i\mathbb{R} \tag{3.2}$$

with $|b| = 1$. We can parametrize x, y^2 and their complex conjugates by

$$x = \exp\frac{2\pi i}{k}(-ib\mu - m), \qquad \bar{x} = \exp\frac{2\pi i}{k}(-ib^{-1}\mu + m),$$
$$y^2 = \exp\frac{2\pi i}{k}(-ib\nu - n), \qquad \bar{y}^2 = \exp\frac{2\pi i}{k}(-ib^{-1}\nu + n), \tag{3.3}$$

where $m, n \in \mathbb{R}$ are periodic ($m \sim m + k$, $n \sim n + k$), and μ, ν are also real parameters. The Chern-Simons symplectic form $\omega_{k,s}$ can be rewritten in terms of new variables,

$$\omega_{k,s} = \frac{2\pi}{k}(\mathrm{d}\nu \wedge \mathrm{d}\mu - \mathrm{d}n \wedge \mathrm{d}m) \tag{3.4}$$

The quantization of $\mathcal{P}_{\mathcal{S}}$ promotes the parameters μ, ν, m, n to operators $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\mathbf{m}$, $\mathbf{n}$, whose nonvanishing commutation relation is

$$[\boldsymbol{\mu}, \boldsymbol{\nu}] = \frac{k}{2\pi i}, \qquad [\mathbf{m}, \mathbf{n}] = -\frac{k}{2\pi i}, \tag{3.5}$$

or, in terms of $\mathbf{x}$, $\mathbf{y}^2$,

$$\mathbf{x}\mathbf{y}^2 = q\mathbf{y}^2\mathbf{x}, \quad \bar{\mathbf{x}}\bar{\mathbf{y}}^2 = \tilde{q}\bar{\mathbf{y}}^2\bar{\mathbf{x}}, \quad q = \exp\frac{4\pi i}{t}, \quad \tilde{q} = \exp\frac{4\pi i}{\bar{t}}. \tag{3.6}$$

The operator algebra is represented on the space of wave functions $f(\mu, m)$ of two variables. Here, $\mu \in \mathbb{R}$ is continuous but $m \in \mathbb{Z}/k\mathbb{Z}$ is discrete. m only takes integer value because both of the canonically conjugate variables $\mathbf{m}$ and $\mathbf{n}$ are periodic. The operators $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\mathbf{m}$, $\mathbf{n}$ are represented by

$$\boldsymbol{\mu} f(\mu, m) = \mu f(\mu, m), \quad \boldsymbol{\nu} f(\mu, m) = -\frac{k}{2\pi i}\partial_\mu f(\mu, m)$$
$$e^{\frac{2\pi i}{k}\mathbf{m}} f(\mu, m) = e^{\frac{2\pi i}{k}m} f(\mu, m), \quad e^{\frac{2\pi i}{k}\mathbf{n}} f(\mu, m) = f(\mu, m+1). \tag{3.7}$$

The simplicity constraint E(2.7) leads to the condition $\mu = \nu = 0$ in the new parametrization. To quantize the constraint, one might naively impose the operator equations $\boldsymbol{\mu}\psi = \boldsymbol{\nu}\psi = 0$ on the wave functions. However, the naive operator equations trivialize the wave function since $[\boldsymbol{\mu}, \boldsymbol{\nu}] = \frac{k}{2\pi i}$. Therefore, to realize the simplicity constraint at the quantum level, we have to impose a weaker version of the constraint. This fact makes the quantum implementation of the simplicity constraint nontrivial. Here, we choose to impose the operator equation

$$(\boldsymbol{\mu} - i\boldsymbol{\nu})\psi = 0 \Rightarrow \psi_{\mathrm{sol}}(\mu, m) = \exp\left(-\frac{\pi\mu^2}{k}\right) f(m), \tag{3.8}$$

where $f(m)$ is an arbitrary function on $\mathbb{Z}/k\mathbb{Z}$. Here the solution space is simply a k-dimensional vector space $\mathbb{C}^k$, which is the Hilbert space of SU(2) Chern-Simons theory of level k. The simplicity constraint at the quantum level reduces the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons wave function to SU(2).

As an equivalent way to impose the constraint, one may also consider imposing the "master constraint" $(\boldsymbol{\mu}^2 + \boldsymbol{\nu}^2)\psi = 0$ up to the "zero-point" energy.[7] The solution (the dependence on μ) is simply the ground state of the harmonic oscillator, the same as above. In this sense the states (3.8) may be viewed as the ground states, while the full spectrum of $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons states are created by the action of the "creation operator" $(\boldsymbol{\mu} + i\boldsymbol{\nu})$.

As we have seen, the constraint $\mu = \nu = 0$ at the quantum level can only be satisfied weakly. The solution of the quantum constraint is a coherent state with its peak at $\mu = \nu = 0$. So $\mu = \nu = 0$ is satisfied only in the semi-classical limit. It is known that the coherent state peaks at a phase-space point that is not unique. We may choose other squeezed coherent states, which still minimize the Heisenberg uncertainty. We introduce a squeezing parameter $w \in \mathbb{R}$, and impose $(\boldsymbol{\mu} - iw^2\boldsymbol{\nu})\psi = 0$ instead of Eq. (3.8), whose solution is

$$\psi_{\mathrm{sol}}^{(w)}(\mu, m) = \exp\left(-\frac{\pi\mu^2}{w^2 k}\right) f(m). \tag{3.9}$$

[6]There is another unitary branch $s \in i\mathbb{R}$ via a nonstandard Hermitian structure [36].

[7]See Refs. [26–28] for the idea of the master constraint in canonical LQG. See Refs. [15,20] for the use of the master constraint in the spinfoam model to solve the simplicity constraint.

We may introduce a "metric" and define a "squeezed" master constraint $(w^{-2}\boldsymbol{\mu}^2 + w^2\boldsymbol{\nu}^2)$. The above squeezed coherent state satisfies $(w^{-2}\boldsymbol{\mu}^2 + w^2\boldsymbol{\nu}^2)\psi^{(w)}_{\rm sol} = 0$ up to the same zero-point energy as above (the zero-point energy is independent of w).

The squeezing parameter introduces an ambiguity in the solution of the simplicity constraint at each four-holed sphere $\mathcal{S}$. The essential reason for the ambiguity is the noncommutativity of the simplicity constraints $\boldsymbol{\mu} = 0$, $\boldsymbol{\nu} = 0$. In Secs. IV and V, we keep $w \neq 0$ as a free parameter, and focus on the construction of Chern-Simons theory with defects, as well as the geometrical reconstruction on $\mathcal{M}_4$. We come back to the issue of ambiguity in Sec. VI.

IV. SL$(2,\mathbb{C})$ CHERN-SIMONS THEORY ON $S^3\backslash\Gamma_5$

The partition function of SL$(2,\mathbb{C})$ Chern-Simons theory on $S^3\backslash\Gamma_5$ can be viewed as a wave function $Z_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, x_\mathcal{S}, \bar x_\mathcal{S})$ [2,3]. The phase space of flat connections $\mathcal{P}_{S^3\backslash\Gamma_5}$ associated to the boundary of $S^3\backslash\Gamma_5$ is of complex dimension 30. The boundary $\partial(S^3\backslash\Gamma_5)$ is a closed 2-surface made of five four-holed spheres $\mathcal{S}$ connected by ten annuli ℓ. For convenience, the complex FN coordinates λ_ℓ, τ_ℓ are used for each annulus, and the $x_\mathcal{S}, y^2_\mathcal{S}$ (or $\mu_\mathcal{S}$, $\nu_\mathcal{S}$, $m_\mathcal{S}$, $n_\mathcal{S}$) coordinates are used for each four-holed sphere $\mathcal{S}$. Here, λ_ℓ is the complex FN length, which is the eigenvalue of the meridian holonomy around the annulus ℓ.

The implementation of the simplicity constraint projects the Chern-Simons wave function to the above solution space. The projection is done by the inner product: $|Z_{S^3\backslash\Gamma_5}\rangle \to |\psi_{\rm sol}\rangle\langle\psi_{\rm sol}|Z_{S^3\backslash\Gamma_5}\rangle$. The resulting wave function reads

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, m_\mathcal{S}) = \int_{\mathbb{R}^5}\prod_\mathcal{S} \mathrm{d}\mu_\mathcal{S} Z_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, \mu_\mathcal{S}, m_\mathcal{S}) \times \prod_\mathcal{S}\exp\left(-\frac{\pi\mu_\mathcal{S}^2}{w^2 k}\right). \tag{4.1}$$

Here λ_ℓ is nothing but the eigenvalue of H_f in Eq. (2.4). The simplicity constraint about the eigenvalue of H_f can be easily implemented by restricting $\lambda_\ell \in U(1)$ in the wave function.

Now the wave function $\mathcal{Z}$ only depends on the data of SU(2) flat connections on the Riemann surface $\Sigma_6 = \partial S^3\backslash\Gamma_5$. It has been shown that the SU(2) flat connections on the Riemann surface parametrizes the twisted geometry on 3d discrete space [5]. Thus, $\mathcal{Z}$ is indeed qualified to be a quantum amplitude describing the evolution of 3d geometry. For the relation with spin-network data, we will explain shortly that λ_ℓ is related to the spins j_ℓ. $m_\mathcal{S} \in \mathbb{Z}/k\mathbb{Z}$ quantizes SU(2) flat connections on a four-holed sphere, and thus essentially is a label of the conformal blocks [or, equivalently, 4-valent intertwiners of the quantum group SU$(2)_q$, where q is a root of unity] [37].

We consider the semiclassical limit of the resulting $Z_{S^3\backslash\Gamma_5}(\lambda_\ell, m_\mathcal{S})$ as $k, s \to \infty$. Comparing the semiclassical limit to the commutators (3.5) motivates us to rescale $\mu_\mathcal{S}$, $\nu_\mathcal{S}$, $m_\mathcal{S}$, $n_\mathcal{S}$ by

$$\mu_\mathcal{S} \mapsto \frac{k}{2\pi}\mu_\mathcal{S}, \quad \nu_\mathcal{S} \mapsto \frac{k}{2\pi}\nu_\mathcal{S}, \quad m_\mathcal{S} \mapsto \frac{k}{2\pi}m_\mathcal{S}, \quad n_\mathcal{S} \mapsto \frac{k}{2\pi}n_\mathcal{S}. \tag{4.2}$$

After rescaling, $m_\mathcal{S}$, $n_\mathcal{S}$ become continuous periodic variables as $k \to \infty$.

The semiclassical behavior of $Z_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, \mu_\mathcal{S}, m_\mathcal{S})$ is known as [38,39] ($\cdots$ stands for the quantum corrections)

$$\begin{aligned} &Z_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, \mu_\mathcal{S}, m_\mathcal{S}) \\ &= \sum_\alpha \exp\bigg\{ i\int_{c\subset\mathcal{L}_\alpha}^{(\lambda_\ell,\bar\lambda_\ell,\mu_\mathcal{S},m_\mathcal{S})}\bigg[\frac{t}{4\pi}\sum_\ell \ln\tau_\ell \frac{\mathrm{d}\lambda'_\ell}{\lambda'_\ell} \\ &\quad + \frac{\bar t}{4\pi}\sum_\ell \ln\bar\tau_\ell\frac{\mathrm{d}\bar\lambda'_\ell}{\bar\lambda'_\ell} + \frac{k}{2\pi}\sum_\mathcal{S}(\nu_\mathcal{S}\mathrm{d}\mu'_\mathcal{S} + n_\mathcal{S}\mathrm{d}m'_\mathcal{S})\bigg] + \cdots\bigg\}. \end{aligned} \tag{4.3}$$

The moduli space of flat connections on $S^3\backslash\Gamma_5$, $\mathcal{M}_{flat}(S^3\backslash\Gamma_5, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}$ is understood as the Lagrangian submanifold of the phase space. $\mathcal{L}_\alpha$ is the branch of $\mathcal{L}$ associated to the flat connection α on $S^3\backslash\Gamma_5$. $\mathcal{L}$ can be represented as a set of polynomial equations in symplectic coordinates, whose expressions have been derived in Ref. [4]. The quantity in the exponential is an integral of the Liouville 1-form associated to $\omega_{k,s}$ along a contour c in $\mathcal{L}_\alpha$.

Inserting the asymptotic expression for $Z_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, \mu_\mathcal{S}, m_\mathcal{S})$ into the integral (4.1),

$$\begin{aligned} &\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar\lambda_\ell, m_\mathcal{S}) \\ &\quad = \sum_\alpha \int_{\mathbb{R}^5}\mathrm{d}\mu_\mathcal{S}\exp\left[S_\alpha(\lambda_\ell, \bar\lambda_\ell, \mu_\mathcal{S}, m_\mathcal{S}) + \cdots\right], \end{aligned} \tag{4.4}$$

where $S_\alpha(\lambda_\ell, \bar\lambda, \mu_\mathcal{S}, m_\mathcal{S})$ reads

$$\begin{aligned} S_\alpha = i\int_{c\subset\mathcal{L}_\alpha}^{(\lambda_\ell,\bar\lambda_\ell,\mu_\mathcal{S},m_\mathcal{S})}&\bigg[\frac{t}{4\pi}\sum_\ell \ln\tau_\ell\frac{\mathrm{d}\lambda'_\ell}{\lambda'_\ell} + \frac{\bar t}{4\pi}\sum_\ell \ln\bar\tau_\ell\frac{\mathrm{d}\bar\lambda'_\ell}{\bar\lambda'_\ell} \\ &+ \frac{k}{2\pi}\sum_\mathcal{S}(\nu_\mathcal{S}\mathrm{d}\mu'_\mathcal{S} + n_\mathcal{S}\mathrm{d}m'_\mathcal{S})\bigg] - \sum_\mathcal{S}\frac{k\mu_\mathcal{S}^2}{4\pi w^2}. \end{aligned} \tag{4.5}$$

In the semiclassical limit, the $\mu_\mathcal{S}$ integral (4.4) localizes asymptotically at the critical points, i.e., the solutions of critical equations $\partial S_\alpha/\partial\mu_\mathcal{S} = \mathrm{Re}(S_\alpha) = 0$. The critical equations are easy to derive:

$$iw^2\nu_{\mathcal{S}} - \mu_{\mathcal{S}} = \mu_{\mathcal{S}} = 0 \Rightarrow \mu_{\mathcal{S}} = \nu_{\mathcal{S}} = 0, \tag{4.6}$$

where we see that the critical equation $\partial S_\alpha/\partial\mu_{\mathcal{S}} = 0$ is a classical version of the quantum simplicity constraint (3.8). The critical equations imply the simplicity constraint, and thus require the flat connections on all $\mathcal{S}$ to be SU(2).

The condition $\mu_{\mathcal{S}} = \nu_{\mathcal{S}} = 0$ may not be simultaneously satisfied for generic branches $\mathcal{L}_\alpha$ of the Lagrangian submanifold. However, it has been shown in Ref. [3] that there exist exactly two branches $\mathcal{L}_{\alpha_{4d}}$ and $\mathcal{L}_{\tilde{\alpha}_{4d}}$, where $\nu_{\mathcal{S}}(\mu_{\mathcal{S}} = 0) = 0$ can be satisfied. The $\mathrm{SL}(2,\mathbb{C})$ flat connection α_{4d} on $S^3\backslash\Gamma_5$ equivalently describes the geometry of a nondegenerate 4-simplex with constant curvature. The other flat connection $\tilde{\alpha}_{4d}$ is referred to as the "parity partner," which corresponds to the same 4-simplex geometry as α_{4d}, but with opposite 4d orientation.

Those α's whose $\mathcal{L}_\alpha$ are not consistent with $\mu_{\mathcal{S}} = \nu_{\mathcal{S}} = 0$ only give exponentially suppressed contributions to $\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar{\lambda}_\ell, m_{\mathcal{S}})$ in Eq. (4.4). Therefore,

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar{\lambda}_\ell, m_{\mathcal{S}}) = e^{S_{\alpha_{4d}}(\lambda_\ell,\bar{\lambda}_\ell)+\cdots} + e^{S_{\tilde{\alpha}_{4d}}(\lambda_\ell,\bar{\lambda}_\ell)+\cdots}, \tag{4.7}$$

where $S_{\alpha_{4d}}$ reads[8]

$$S_{\alpha_{4d}} = i\int_{c\subset\mathcal{L}_{\alpha_{4d}}}^{(\lambda_\ell,\bar{\lambda}_\ell,m_{\mathcal{S}})}\left[\frac{t}{4\pi}\sum_\ell \ln\tau_\ell\frac{\mathrm{d}\lambda'_\ell}{\lambda'_\ell} + \frac{\bar{t}}{4\pi}\sum_\ell \ln\bar{\tau}_\ell\frac{\mathrm{d}\bar{\lambda}'_\ell}{\bar{\lambda}'_\ell} + \frac{k}{2\pi}\sum_{\mathcal{S}} n_{\mathcal{S}}\mathrm{d}m'_{\mathcal{S}}\right]. \tag{4.8}$$

The integral in $S_{\alpha_{4d}}$ has been reduced to be of the same form as the one treated in Refs. [2,3].

To compute $S_{\alpha_{4d}}$, we use the geometrical interpretation of flat connections and the FN coordinates in terms of constant-curvature 4-simplex geometries. This geometrical interpretation has been studied extensively in Refs. [1,3]. The ten annuli ℓ are in one-to-one correspondence to the ten triangles of the 4-simplex. By the correspondence between 4-simplex geometry and the flat connection on $S^3\backslash\Gamma_5$, the complex FN length λ_ℓ is related to the area of the triangle $\mathbf{a}(\mathfrak{f}_\ell)$. The dihedral angle $\Theta(\mathfrak{f}_\ell)$ hinged by the triangle $\mathfrak{f}_\ell$ corresponding to ℓ is related to the complex FN twist τ_ℓ. Explicitly,

$$\lambda_\ell = \exp\left[-\frac{i\Lambda}{6}\mathbf{a}(\mathfrak{f}_\ell) + \pi i\mathfrak{s}_\ell\right],$$
$$\tau_\ell = \exp\left[-\mathrm{sgn}(V_4)\Theta(\mathfrak{f}_\ell)\right], \tag{4.9}$$

where $\mathfrak{s}_\ell \in \{0,1\}$ parametrizes the lifts from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$. $\mathrm{sgn}(V_4)$ is the 4d orientation of the 4-simplex, which takes different values at α_{4d} and $\bar{\alpha}_{4d}$.

[8] We have choose the integration contour such that the flat connections on the contour all correspond to the 4d geometries. Therefore, the Schläfli identity can be used in the derivation (see Ref. [3] for details). The contour is in the plane with $\mu_{\mathcal{S}} = 0$.

Inserting Eq. (4.9) into the integral (4.8), the integrand becomes proportional to $\sum_{\ell=1}^{10}\Theta(\mathfrak{f}_\ell)\mathrm{d}\mathbf{a}(\mathfrak{f}_\ell)$ except for the last term in Eq. (4.8). Because all data $\Theta(\mathfrak{f}_\ell), \mathbf{a}(\mathfrak{f}_\ell)$ are associated to a geometrical 4-simplex, and satisfy the Schläfli identity [40,41]

$$\sum_{\ell=1}^{10}\mathbf{a}(\mathfrak{f}_\ell)\mathrm{d}\Theta(\mathfrak{f}_\ell) = \Lambda\mathrm{d}|V_4|, \tag{4.10}$$

where V_4 is the volume of the constant curvature 4-simplex, $\sum_{\ell=1}^{10}\Theta(\mathfrak{f}_\ell)\mathrm{d}\mathbf{a}(\mathfrak{f}_\ell)$ is a total derivative:

$$\sum_{\ell=1}^{10}\Theta(\mathfrak{f}_\ell)\mathrm{d}\mathbf{a}(\mathfrak{f}_\ell) = \mathrm{d}S_{\mathrm{Regge},\Lambda},$$
$$S_{\mathrm{Regge},\Lambda} = \sum_\ell \mathbf{a}(\mathfrak{f}_\ell)\Theta(\mathfrak{f}_\ell) - \Lambda|V_4|. \tag{4.11}$$

$S_{\mathrm{Regge},\Lambda}$ is the Regge action on a single 4-simplex with a cosmological constant term.

The last term in Eq. (4.8) contributes the same between α_{4d} and $\bar{\alpha}_{4d}$ [3]. To remove this overall term in the asymptotics, we may consider a coherent state peaked at the phase-space point $\mathring{m}_{\mathcal{S}}, \mathring{n}_{\mathcal{S}}$, which behaves as follows when $k\to\infty$:

$$\phi^{(k)}_{\mathring{m},\mathring{n}}(m_{\mathcal{S}}) \sim e^{-\frac{k}{4\pi}\sum_{\mathcal{S}}(m_{\mathcal{S}}-\mathring{m}_{\mathcal{S}})^2 - \frac{ik}{2\pi}\sum_{\mathcal{S}}\mathring{n}_{\mathcal{S}}m_{\mathcal{S}}}. \tag{4.12}$$

A candidate for $\phi^{(k)}$ can be chosen as a product of Jacobi theta functions [see Eq. (4.19) of Ref. [42]] to respect the periodicity of $m_{\mathcal{S}}$. As $k\to\infty$, the quantity

$$\sum_{m_{\mathcal{S}}}\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell, \bar{\lambda}_\ell, m_{\mathcal{S}})\phi^{(k)}_{\mathring{m},\mathring{n}}(m_{\mathcal{S}}) \tag{4.13}$$

gives the critical equation of $m_{\mathcal{S}}$:

$$m_{\mathcal{S}} = \mathring{m}_{\mathcal{S}}, \qquad n_{\mathcal{S}} = \mathring{n}_{\mathcal{S}}. \tag{4.14}$$

At the critical point, the last term in Eq. (4.8) cancels the second term in the exponential in $\phi^{(k)}$.

As a result, Eq. (4.13) behaves asymptotically as

$$e^{\frac{i}{\ell_P^2}S_{\mathrm{Regge},\Lambda}+\cdots} + e^{-\frac{i}{\ell_P^2}S_{\mathrm{Regge},\Lambda}+\cdots}, \tag{4.15}$$

where $\ell_P^2 = |\frac{12\pi}{s\Lambda}|$.

The above asymptotics reproduce the result in Refs. [1–3]. The previous asymptotic results were obtained either by semiclassically picking up the branches α_{4d} and $\bar{\alpha}_{4d}$, or by using a certain ansatz of the Wilson graph operator. However, here we obtain the result by a systematic study of the simplicity constraint at the quantum level,

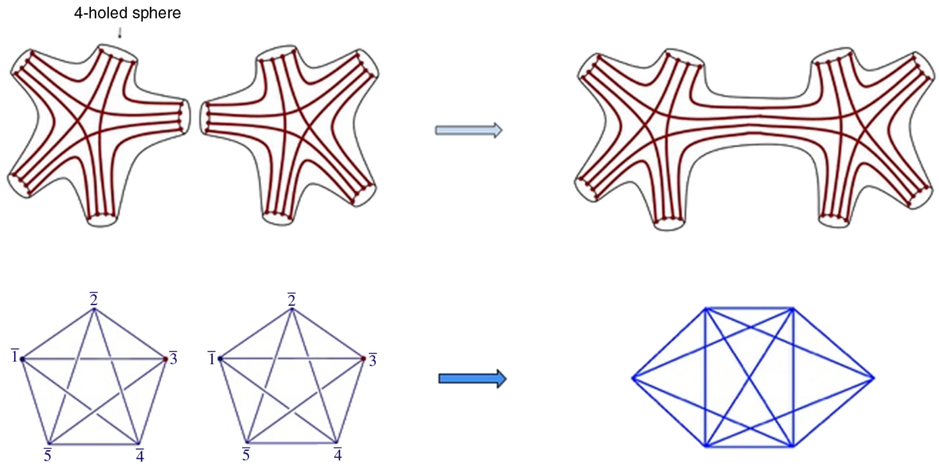

FIG. 2. $\mathcal{M}_3$ is obtained by gluing a number of $S^3\backslash\Gamma_5$'s, each of which corresponds to a 4-simplex in $\mathcal{M}_4$. The gluing of $S^3\backslash\Gamma_5$'s is deduced from the gluing of 4-simplices in $\mathcal{M}_4$. In drawing the 3-manifold $S^3\backslash\Gamma_5$ and $\mathcal{M}_3$, we imagine viewing $S^3\backslash\Gamma_5$ from 4d and suppress one dimension. The 3-manifold $S^3\backslash\Gamma_5$ has five geodesic boundary components as four-holed spheres, coming from removing the neighborhood of five vertices of Γ_5. It has ten cusp boundary components as ten annuli, coming from removing the neighborhood of ten edges of Γ_5. The red curves are the annuli connecting four-holed spheres. Two $S^3\backslash\Gamma_5$'s can be glued through a pair of four-holed spheres, via a certain identification of holes. Each four-holed sphere as the gluing interface corresponds to a tetrahedron shared by two 4-simplices in $\mathcal{M}_4$. Each hole of the four-holed sphere (or each tunnel traveling through the four-holed sphere) corresponds to a triangle in the shared tetrahedron.

and project the partition function onto the space of quantum solutions. The method used here is especially useful when generalizing the amplitude to many 4-simplices.

V. SL$(2,\mathbb{C})$ CHERN-SIMONS THEORY ON $\mathcal{M}_3$ WITH SURFACE DEFECT

The correspondence between the SL$(2,\mathbb{C})$ flat connection on a 3-manifold and 4d geometry can be generalized to an arbitrary 4d simplicial manifold $\mathcal{M}_4$. The corresponding 3-manifold $\mathcal{M}_3$ corresponding to $\mathcal{M}_4$ can be constructed by gluing copies of $S^3\backslash\Gamma_5$ (see Fig. 2). The number of glued $S^3\backslash\Gamma_5$'s coincides with the number of 4-simplices in $\mathcal{M}_4$. The gluing interface between a pair of $S^3\backslash\Gamma_5$'s is always a four-holed sphere $\mathcal{S}$.

To construct the partition function $\mathcal{Z}_{\mathcal{M}_3}$ on $\mathcal{M}_3$, we simply multiply the resulting partition functions $\mathcal{Z}_{S^3\backslash\Gamma_5}(\lambda_\ell,\bar{\lambda}_\ell,m_\mathcal{S})$ (reduced by the simplicity constraint), then identify and sum over the data $m_\mathcal{S}$ associated to the gluing interfaces $\mathcal{S}$. So we obtain a state-sum model,

$$\mathcal{Z}_{\mathcal{M}_3}(\lambda_\ell,\bar{\lambda}_\ell)=\sum_{m_\mathcal{S}\in\mathbb{Z}/k\mathbb{Z}}\prod_{S^3\backslash\Gamma_5}Z_{S^3\backslash\Gamma_5}(\lambda_\ell,\bar{\lambda}_\ell,m_\mathcal{S}). \qquad (5.1)$$

In this formula, $m_\mathcal{S}\in\mathbb{Z}/k\mathbb{Z}$ is the one before the rescaling (4.2) in the semiclassical analysis. In general, the resulting $\mathcal{Z}_{\mathcal{M}_3}$ may also depend on some leftover $m_\mathcal{S}$'s if $\mathcal{M}_3$ after gluing still has geodesic boundary components $\mathcal{S}$.

The simplicity constraint has been implemented at the gluing interfaces $\mathcal{S}$. The constraint projects the quantum states on $\mathcal{S}$ of SL$(2,\mathbb{C})$ Chern-Simons theory onto the space of solutions (3.8), which is essentially the state space of SU(2) Chern-Simons theory. Therefore, the simplicity constraint introduces the defects into SL$(2,\mathbb{C})$ Chern-Simons theory. The defects are localized at the interfaces $\mathcal{S}$ where a pair of $S^3\backslash\Gamma_5$'s are glued. The defects are supported on 2-surfaces $\mathcal{S}$ embedded in $\mathcal{M}_3$. The effect of the defect is that SL$(2,\mathbb{C})$ Chern-Simons theory reduces to SU(2) at the 2-surface.

Schematically, the surface defect may be understood via the insertions of certain "surface operators" in SL$(2,\mathbb{C})$ Chern-Simons theory, i.e., we write $\mathcal{Z}_{\mathcal{M}_3}(\lambda_\ell,\bar{\lambda}_\ell)$ as a functional integral,

$$\mathcal{Z}_{\mathcal{M}_3}(\lambda_\ell,\bar{\lambda}_\ell)=\int DAD\bar{A}e^{\frac{it}{8\pi}\int_{\mathcal{M}_3}\mathrm{tr}(AdA+\frac{2}{3}A^3)+\frac{i\bar{t}}{8\pi}\int_{\mathcal{M}_3}\mathrm{tr}(\bar{A}d\bar{A}+\frac{2}{3}\bar{A}^3)}\times\prod_\mathcal{S}\mathcal{O}_\mathcal{S}[A,\bar{A}]. \qquad (5.2)$$

The insertions $\mathcal{O}_\mathcal{S}$, located at the gluing interfaces $\mathcal{S}$ play the role of the projections $|\psi_{\mathrm{sol}}\rangle\langle\psi_{\mathrm{sol}}|$. We will discuss the operator $\mathcal{O}_\mathcal{S}[A,\bar{A}]$ further in Sec. VI.

We consider the semiclassical behavior of the state sum $\mathcal{Z}_{\mathcal{M}_3}$ as k, $s\to\infty$. We again perform the rescaling for $m_\mathcal{S}$ by Eq. (4.2). Then we see that as $k\to\infty$ the sum over $m_\mathcal{S}$ in Eq. (5.1) approximates an integral over S^1. The semiclassical asymptotics can again be studied using the stationary phase approximation, similar to the analysis of $Z_{S^3\backslash\Gamma_5}$. In addition to the critical equations (4.6), we have one more critical equation at each gluing interface $\mathcal{S}$, because of the integration of $m_{\mathcal{S}}$,

$$n_{\mathcal{S}} + n'_{\mathcal{S}} = 0, \tag{5.3}$$

where $n_{\mathcal{S}}$ comes from the flat connection on the $S^3\backslash\Gamma_5$ on the left of $\mathcal{S}$, and $n'_{\mathcal{S}}$ comes from the $S^3\backslash\Gamma_5$ on the right of $\mathcal{S}$.

Semiclassically, $n_{\mathcal{S}} = -n'_{\mathcal{S}}$ identifies the SU(2) flat connections on the interface $\mathcal{S}$ from the left and right $S^3\backslash\Gamma_5$'s ($m_{\mathcal{S}}$ has been identified). The minus sign reflects the opposite orientations on $\mathcal{S}$ in the gluing. Thus the $\mathrm{SL}(2,\mathbb{C})$ flat connections on the copies of $S^3\backslash\Gamma_5$ are glued to become a flat connection on the entire $\mathcal{M}_3$.

Let us first consider that $\mathcal{M}_3$ is obtained by gluing 2 copies of $S^3\backslash\Gamma_5$ through a pair of four-holed spheres $\mathcal{S}, \mathcal{S}'$, and the fundamental group $\pi_1(\mathcal{M}_3)$ is given by two copies of $\pi_1(S^3\backslash\Gamma_5)$ modulo the identification of generators on $\mathcal{S}$ and $\mathcal{S}'$ [$\pi_1(\mathcal{S}) \simeq \pi_1(\mathcal{S}')$ with the isomorphism denoted by $\mathcal{I}$]. $\pi_1(\mathcal{M}_3)$ is isomorphic to the fundamental group of a 1-skeleton $\pi_1(\mathrm{sk}(\mathcal{M}_4))$ from the 4d polyhedron $\mathcal{M}_4$ obtained by gluing a pair of 4-simplices. However, here the 1-skeleton $\mathrm{sk}(\mathcal{M}_4)$ includes the edges of the tetrahedron shared by the pair of 4-simplices (Fig. 2).

Given two flat connections A, A' as representations $\pi_1(S^3\backslash\Gamma_5) \to \mathrm{SL}(2,\mathbb{C})$ modulo conjugation, they are glued and give a flat connection $\mathcal{A}$ on $\mathcal{M}_3$ when they induce the same representation of $\pi_1(\mathcal{S})$ and $\pi_1(\mathcal{S}')$ (i.e., $A = A' \circ \mathcal{I}$). We reduce the flat connection on $\mathcal{S}, \mathcal{S}'$ to be SU(2), and consider that A, A' corresponds to two constant-curvature 4-simplices $\mathfrak{S}, \mathfrak{S}'$. When A, A' are glued to $\mathcal{A}$ on $\mathcal{M}_3$, they induce the same SU(2) representation (modulo conjugation) of $\pi_1(\mathcal{S})$ and $\pi_1(\mathcal{S}')$. The SU(2) flat connection reconstructs a unique geometrical tetrahedron of constant curvature. The constant-curvature tetrahedron belongs to both $\mathfrak{S}, \mathfrak{S}'$, and implies that $\mathfrak{S}, \mathfrak{S}'$ are of the same constant curvature. Therefore, the flat connection $\mathcal{A}$ on $\mathcal{M}_3$ effectively glues a pair of constant-curvature 4-simplices $\mathfrak{S}, \mathfrak{S}'$, and determines a four-dimensional simplicial geometry on $\mathcal{M}_4$. The procedure can be continued to arbitrary $\mathcal{M}_3 = \cup_{i=1}^N (S^3\backslash\Gamma_5)$. For each simplicial 4-manifold $\mathcal{M}_4$, the corresponding $\mathcal{M}_3$ can be constructed as in Fig. 2. A class of flat connections $\mathcal{A}$ on $\mathcal{M}_3$ can be obtained by gluing flat connections on $S^3\backslash\Gamma_5$. Each $\mathcal{A}$ determines a 4d simplicial geometry $(\mathcal{M}_4, g)$ obtained by gluing N 4-simplices with the same constant curvature. When the simplicial complex $\mathcal{M}_4$ is sufficiently refined, arbitrary smooth geometries can be approximated by the simplicial geometries.

The gluing of flat connections gives an extra constraint on A, A' as well as the boundary data λ_ℓ. It is possible that a set of λ_ℓ does not lead to any flat connection on $\mathcal{M}_3$ corresponding to 4d simplicial geometry. In that case, we say the areas relating to λ_ℓ are *non-Regge-like*, and otherwise we say the areas are *Regge-like*.

In general, $\mathcal{M}_3$ can be viewed as the complement of (the open neighborhood of) a certain graph Γ in an ambient

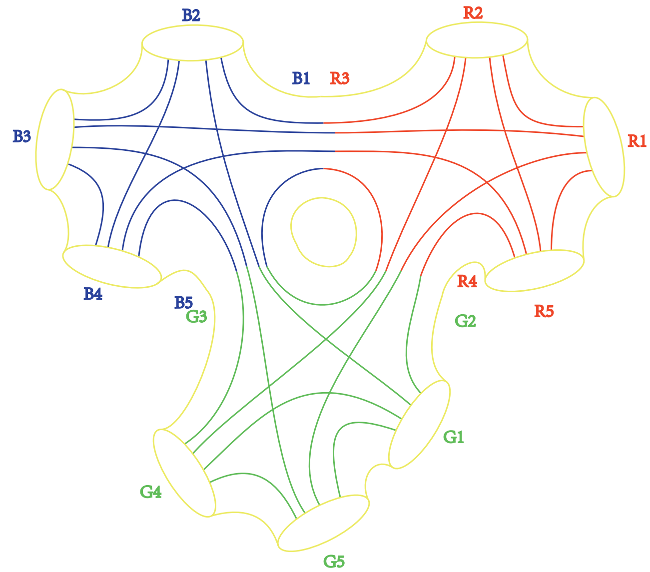

FIG. 3. This picture shows the result when we glue three $S^3\backslash\Gamma_5$'s. The yellow outer shell indicates the ambient 3-manifold $\mathfrak{X}_3$. The four-holed spheres $B1$ and $R3$ are a shared boundary between the blue $S^3\backslash\Gamma_5$ and red $S^3\backslash\Gamma_5$. Similarly, $(B5, G3)$ and $(R4, G2)$ are the blue-green shared boundary and red-green shared boundary, respectively. At the center of the picture, there is a noncontractible cycle, which makes $\pi_1(\mathfrak{X}_3)$ nontrivial. There is a closed tunnel with three different colors at the center which corresponds to an internal triangle shared by three 4-simplices in $\mathcal{M}_4$.

closed 3-manifold $\mathfrak{X}_3$. Generically $\mathfrak{X}_3$ is not S^3. It is shown as an example in Fig. 3, where we glue three $S^3\backslash\Gamma_5$'s. $\mathfrak{X}_3$ has a noncontractible cycle which is generated by the gluing procedure. In other words, the fundamental group $\pi_1(\mathfrak{X}_3)$ is nontrivial. In the case of Fig. 3, the noncontractible cycle of $\mathfrak{X}_3$ is associated with a closed tunnel which is made by connecting a number of annuli in $S^3\backslash\Gamma_5$. In general, each closed tunnel always goes along a noncontractible cycle in $\mathfrak{X}_3$. The tunnel gives a torus boundary T^2 of $\mathcal{M}_3$. Following the correspondence between $\mathcal{M}_3$ and $\mathcal{M}_4$, it is not hard to see that each torus boundary T^2 corresponds to an internal triangle shared by a number of 4-simplices.

The flat connection $\mathcal{A}$ gives the commutative meridian (A-cycle) and longitude (B-cycle) holonomies on each T^2. The commutativity implies that the two holonomies can be simultaneously diagonalized. The eigenvalue λ_{T^2} of the meridian holonomy is equal to the annulus meridian holonomy eigenvalue λ_ℓ for all λ_ℓ building T^2. From the correspondence between $\mathcal{A}$ and simplicial 4-geometry, it is not hard to see that λ_{T^2} is related to the area $\mathbf{a}(\mathfrak{f}_{T^2})$ of the internal triangle $\mathfrak{f}_{T^2}$,

$$\lambda_{T^2}^2 = \exp\left[-\frac{i\Lambda}{3}\mathbf{a}(\mathfrak{f}_{T^2})\right]. \tag{5.4}$$

The eigenvalue τ_{T^2} of the longitude holonomy can be obtained from the product of FN twists τ_ℓ for all λ_ℓ building T^2. This has been computed in Ref. [4]. When we connect a pair of annuli ℓ_1, ℓ_2 in $\mathfrak{X}_3$, the FN twist of the connected annuli $\ell_1 \cup \ell_2$ is a product of the FN twists of ℓ_1 and ℓ_2, and the same relation also holds for y_ℓ^2:

$$\tau_{\ell_1\cup\ell_2} = \tau_{\ell_1}\tau_{\ell_2}, \qquad y^2_{\ell_1\cup\ell_2} = y^2_{\ell_1}y^2_{\ell_2}. \tag{5.5}$$

When a number of annuli are connected to form a T^2, $\tau_{T^2} = y_{\ell_1\cup\cdots\cup\ell_n}$ is the eigenvalue of the longitude holonomy, which is a product of $y_{\ell_1},\ldots,y_{\ell_n}$. We also have $\tau^2_{T^2} = \tau_{\ell_1\cup\cdots\cup\ell_n}$. Because of the relation between y_ℓ and the 4-simplex hyper-dihedral angle in Eq. (4.9), τ_{T^2} is related to the deficit angle $\varepsilon(\mathfrak{f}_{T^2})$ hinged by the internal triangle $\mathfrak{f}_{T^2}$,

$$\tau_{T^2} = e^{-\frac{1}{2}\mathrm{sgn}(V_4)\varepsilon(\mathfrak{f}_{T^2}) - \frac{i}{2}\pi\eta(\mathfrak{f}_{T^2})}, \text{ where } \varepsilon(\mathfrak{f}_{T^2}) = \sum_{\mathfrak{S},\mathfrak{f}_{T^2}\subset\mathfrak{S}} \Theta_{\mathfrak{S}}(\mathfrak{f}_{T^2}), \tag{5.6}$$

when $\mathrm{sgn}(V_4)$ is a constant for all 4-simplices sharing $\mathfrak{f}_{T^2}$. It was shown in Ref. [4] that $\eta(\mathfrak{f}_{T^2})$ is an index taking values in $\{0,1\}$. The fact that $\eta(\mathfrak{f}_{T^2}) = 0$ everywhere on $\mathcal{M}_4$ means that the 4d spacetime is globally time oriented.

The gluing of 4-simplices via gluing $S^3\backslash\Gamma_5$ does not put any constraint on the orientation $\mathrm{sgn}(V_4)$ of each 4-simplex. There are 2^N flat connections on $\mathcal{M}_4$ corresponding to the same geometry on $\mathcal{M}_4$ but with different local orientations. Among them there are a pair of flat connections that give the globally oriented geometry on $\mathcal{M}_4$.

We again label by α_{4d} the flat connection on $\mathcal{M}_3$ which corresponds to 4d simplicial geometry, which is globally oriented [$\mathrm{sgn}(V_4)$ is constant] and globally time oriented [$\eta(\mathfrak{f}_{T^2})$ vanishes constantly]. The contribution from each α_{4d} to $Z_{\mathcal{M}_3}$ asymptotically behaves as $Z_{\mathcal{M}_3} \sim \exp S_{\alpha_{4d}}$, when $k, s \to \infty$, where

$$\begin{aligned} S_{\alpha_{4d}} = i\int_{c\subset\mathcal{L}_{\alpha_{4d}}}^{(\lambda,\bar{\lambda},m)} \Bigg[&\frac{t}{4\pi}\Bigg(\sum_{T^2\subset\partial\mathcal{M}_3} \ln\tau_{T^2}\frac{\mathrm{d}\lambda'^2_{T^2}}{\lambda'^2_{T^2}} + \sum_{\ell\subset\partial\mathcal{M}_3}\ln\tau_\ell\frac{\mathrm{d}\lambda'_\ell}{\lambda'_\ell}\Bigg) \\ &+\frac{\bar{t}}{4\pi}\Bigg(\sum_{T^2\subset\partial\mathcal{M}_3}\ln\bar{\tau}_{T^2}\frac{\mathrm{d}\bar{\lambda}'^2_{T^2}}{\bar{\lambda}'^2_{T^2}} + \sum_{\ell\subset\partial\mathcal{M}_3}\ln\bar{\tau}_\ell\frac{\mathrm{d}\bar{\lambda}'_\ell}{\bar{\lambda}'_\ell}\Bigg) \\ &+\frac{k}{2\pi}\sum_{\mathcal{S}\subset\partial\mathcal{M}_3} n_{\mathcal{S}}\mathrm{d}m'_{\mathcal{S}}\Bigg]. \end{aligned} \tag{5.7}$$

This type of integral has been computed in Ref. [4]. The method of computation is similar to Eq. (4.8). Key steps are again using the geometrical interpretations (5.6) and (5.4), as well as the Schläfli identity for each 4-simplex. The result gives the Einstein-Regge action on the simplicial complex $\mathcal{M}_4$, up to some additional boundary terms which correspond to the overall phase of the wave function (see Sec. VI in Ref. [4] for details):

$$\begin{aligned} S_{\alpha_{4d}} = &-\frac{is\Lambda\mathrm{sgn}(V_4)}{12\pi}\Bigg(\sum_{\mathfrak{f}}\mathbf{a}(\mathfrak{f})\varepsilon(\mathfrak{f}) - \Lambda\sum_{\mathfrak{S}}|V_4(\mathfrak{S})|\Bigg) \\ &+\frac{ik\Lambda}{3}\sum_{\mathfrak{f}}N(\mathfrak{f})\mathbf{a}(\mathfrak{f}) + \frac{ik}{2\pi}\int_{\mathcal{L}_{\alpha_{4d}}}^{(\lambda,\bar{\lambda},m)}\sum_{\mathcal{S}\subset\partial\mathcal{M}_3} n_{\mathcal{S}}\mathrm{d}m'_{\mathcal{S}}, \end{aligned} \tag{5.8}$$

where we have neglected the integration constant. To make the formula short, $\varepsilon(\mathfrak{f})$ here denotes the deficit angle for internal $\mathfrak{f}$ or the dihedral angle for boundary $\mathfrak{f}$. The coefficient in front of the Regge action is the (inverse) Planck scale in 4d,

$$\ell_P^2 = \left|\frac{12\pi}{s\Lambda}\right|. \tag{5.9}$$

$N(\mathfrak{f})$ indicates that the leading order of $S_{\alpha_{4d}}$ is a multivalued function since it comes from integrating the logarithmic function (see Ref. [42] for an interpretation). However, if there is a quantization of area,

$$\frac{\Lambda}{3}\sum_{\mathfrak{f}}N(\mathfrak{f})\mathbf{a}(\mathfrak{f}) \in 2\pi\mathbb{Z}. \tag{5.10}$$

The asymptotics of $\exp S_{\alpha_{4d}}$ does not depend on the choice of branches $N(\mathfrak{f})$. This area-quantization condition has been treated in Ref. [3]. It is fulfilled when the boundary condition λ_ℓ comes from the $\mathrm{SL}(2,\mathbb{C})$ Wilson lines in $\mathfrak{X}_3$ labeled by unitary irreps $(2j_\ell, 2\gamma j_\ell)$, where $j_\ell \in \mathbb{N}/2$ and $\gamma = s/k$ is a universal constant. The area relates the representation label by $\mathbf{a}(\mathfrak{f}) = \gamma j_\ell$ with the correspondence between $\mathfrak{f}$ and ℓ.

The last term in Eq. (5.8) is only related to the boundary of $\mathcal{M}_3$ or $\mathcal{M}_4$. If we fix the boundary data $\mathring{m}_{\mathcal{S}}$, $\mathring{n}_{\mathcal{S}}$ which parametrize the shapes of boundary tetrahedra, and consider the branches α on which the boundary data can be achieved, i.e., $n^{(\alpha)}_{\mathcal{S}}(\mathring{m}) = \mathring{n}_{\mathcal{S}}$, the last term in Eq. (5.8) on these branches α takes the same value, and thus corresponds to an overall phase in $Z_{\mathcal{M}_3}$ [3,4]. This overall phase can again be removed in the asymptotics by projecting the partition function on coherent states in $m_{\mathcal{S}}$, as in Eqs. (4.12) and (4.13).

The result (5.8) reproduces the earlier asymptotics result in Ref. [4], which is obtained by semiclassically picking up the branches α_{4d}. By picking up α_{4d} semiclassically, the amplitude is only defined perturbatively via a semiclassical expansion. However, here the result is achieved by a systematic quantization of the simplicity constraint and imposing the constraint quantum mechanically imposing the constraint on the amplitude. The resulting amplitude on the simplicial complex is a nonperturbative definition, which has not been achieved in earlier works. The above analysis shows that the branch α_{4d} stands out from the semiclassical approximation of the nonperturbative amplitude, which gives the correct semiclassical behavior.

VI. A FIELD-THEORETIC DESCRIPTION OF THE SURFACE DEFECT

Recall that the insertion $\mathcal{O}_{\mathcal{S}}$ in Eq. (5.2) projects Chern-Simons states on $\mathcal{S}$ onto the ground states $\psi_{\rm sol}$ of the "Hamiltonian" $H = \boldsymbol{\mu}^2 + \boldsymbol{\nu}^2$. It has been mentioned that we can also introduce a squeezed version $H^{(w)} = w^{-2}\boldsymbol{\mu}^2 + w^2\boldsymbol{\nu}^2$. The ground state $\psi^w_{\rm sol}$ of $H^{(w)}$ is a squeezed coherent state such that $\psi^{w=1}_{\rm sol} = \psi_{\rm sol}$. The squeezing parameter w introduces an ambiguity into the model at each four-holed sphere $\mathcal{S}$. [In the following, we equivalently understand $\mathcal{S}$ as a sphere with four marked points, which are the intersections with the Wilson lines; see Eq. (6.3).]

Here we would like to find a field-theoretic understanding of the surface defect $\mathcal{O}_{\mathcal{S}}$, as well as the associated ambiguity. The continuum counterparts of the conjugate variables μ, ν are $\phi^i_1 = \mathrm{Im}(A^i_1)$ and $\phi^i_2 = \mathrm{Im}(A^i_2)$ (the coordinates on $\mathcal{S}$ are chosen to be $x^{1,2}$),

$$[\phi^i_1(x), \phi^j_2(x')] = \frac{ik}{4\pi}\delta^{ij}\delta^{(3)}(x, x'). \tag{6.1}$$

The Hamiltonian $H = \boldsymbol{\mu}^2 + \boldsymbol{\nu}^2$ has the continuous counterpart $\int_{\mathcal{S}}(\phi^i_1\phi^i_1 + \phi^i_2\phi^i_2)$. However, in order to make it coordinate independent, we have to introduce a surface metric h^{ab} ($a, b = 1, 2$), and write $\int_{\mathcal{S}} \mathrm{d}^2x\sqrt{h}h^{ab}\phi^i_a\phi^i_b$. As a result, the following operator plays the role of the projector $|\psi_{\rm sol}\rangle\langle\psi_{\rm sol}|$:

$$\mathcal{O}_{\mathcal{S}}[A,\bar{A};h_{ab}] = \exp\left[-\frac{k}{4\pi}\int_{\mathcal{S}} \mathrm{d}^2x\sqrt{h}h^{ab}\phi^i_a\phi^i_b\right], \quad a,b=1,2. \tag{6.2}$$

The coupling constant has to be the same as the Chern-Simons level k. If we have chosen an independent coupling constant and scale it to be large, $\mathcal{O}_{\mathcal{S}}$ would have been the same as inserting delta functions $\delta(\phi^i_1)\delta(\phi^i_2)$ in the path integral. However, at the quantum level ϕ^i_1, ϕ^i_2 cannot be simultaneously constrained to zero by the uncertainty principle, since they are canonically conjugate variables. It is related to the zero-point energy of H. Letting the coupling constant be the same as k gives the sharpest projection.

Here we find that the surface metric h_{ab} is an analog or generalization of the above squeezing parameter w. Inserting $\mathcal{O}$ into the Chern-Simons theory breaks the topological invariance near the surface $\mathcal{S}$, and makes the path integral explicitly depend on the metric h_{ab} of each $\mathcal{S}$. This metric dependence is the ambiguity of imposing the simplicity constraint in the field-theoretic description.

It is standard that the defect in Chern-Simons theory has a certain metric dependence, by breaking the topological invariance of Chern-Simons theory. A standard example is the Wilson line defect, whose metric dependence is reflected as the framing dependence.

The defect might not depend on all of the metric degrees of freedom, similar to the situation of Wilson lines. At the classical level, $\mathcal{O}_{\mathcal{S}}$ is both conformal and reparametrization invariant on $\mathcal{S}$. The metric dependence of $\mathcal{O}_{\mathcal{S}}$ is essentially on the conformal equivalence classes of h_{ab}. Two metrics from different classes are not related by conformal transformation and reparametrization. On a sphere with four marked points, one can always use conformal transformation to move three marked points to 0,1, and ∞. The position of the last marked point on S^2, denoted by τ, labels the conformal equivalence classes of the metric. So the metric dependence of $\mathcal{O}_{\mathcal{S}}$ is essentially on a single complex parameter in classical theory. It is interesting to understand whether this type of metric dependence is preserved at the quantum level, or how this property receives quantum corrections. The study of this point is postponed to future research.

Explicitly, we write the spinfoam amplitude on $\mathcal{M}_4$ as a topological quantum field theory (TQFT) on a 3-manifold $\mathfrak{X}_3$ with both surface and line defects,

$$\mathcal{Z}_{\mathcal{M}_3} = \int DAD\bar{A}e^{-iCS[\mathfrak{X}_3|A,\bar{A}]}\prod_{\mathcal{S}}\mathcal{O}_{\mathcal{S}}[A,\bar{A};h_{\mu\nu}] \times \prod_{l} W_{(2j_l,2\gamma j_l)}[A,\bar{A}], \tag{6.3}$$

where the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons action reads

$$CS[\mathfrak{X}_3|A,\bar{A}] = \frac{t}{8\pi}\int_{\mathfrak{X}_3}\mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{\bar{t}}{8\pi}\int_{\mathfrak{X}_3}\mathrm{tr}\left(\bar{A}\wedge d\bar{A} + \frac{2}{3}\bar{A}\wedge\bar{A}\wedge\bar{A}\right). \tag{6.4}$$

Here, instead of defining the theory on $\mathcal{M}_3$, we write the theory on the ambient space $\mathfrak{X}_3$ and introduce a Wilson loop operator $W_{(j_l,\gamma j_l)}[A,\bar{A}]$ for each torus cusp. The Wilson loops are traces of holonomies in the unitary representation $(2j_l, 2\gamma j_l)$ of $\mathrm{SL}(2,\mathbb{C})$, where $\gamma = s/k$ is the Barbero-Immirzi parameter. In the case where $\mathcal{M}_4$ has a boundary, the Wilson line operators adjoint at vertices have to be introduced in $\mathfrak{X}_3$, corresponding to the annuli cusps adjoint at four-holed spheres as the boundary of $\mathcal{M}_3$. For the simplicity of the following discussion, we focus on the case where $\mathcal{M}_4$ has no boundary, so that $W_{(j_l,\gamma j_l)}[A,\bar{A}]$ are all Wilson loops.

Indeed, inserting Wilson loops into a TQFT on $\mathfrak{X}_3$ is equivalent to a TQFT on the complement $\mathcal{M}_3 = \mathfrak{X}_3\backslash\{l\}$. It is standard that the Wilson loop operator has a path integral expression [43],

$$\prod_{\ell} W_{(2j_l,2\gamma j_l)}[A,\bar{A}] = \int DYD\bar{Y} e^{\sum_l \frac{i}{l^2}\int_\ell \mathrm{tr}[(\nu_l+\kappa_l)Y^{-1}(d+A)Y+(\nu_l-\kappa_l)\bar{Y}^{-1}(d+\bar{A})\bar{Y}]}, \tag{6.5}$$

where ν, κ are related to the representation labels $\nu_l = -\gamma j_l \sigma_3$, $\kappa_l = i j_l \sigma_3$ (σ_3 is the third Pauli matrix). $Y\colon \times_l l \to \mathrm{SL}(2,\mathbb{C})$ is a group-valued field. In the tubular neighborhood $N(l)$ of each Wilson loop, the Chern-Simons action can be written as

$$\frac{t}{8\pi}\int_{N(l)} \mathrm{tr}(A_\perp \wedge \mathrm{d}A_\perp) + \frac{t}{4\pi}\int_{N(\ell)} \mathrm{tr}(F_\perp \wedge A_t) + \mathrm{c.c.}, \tag{6.6}$$

where A_t, $A_\perp$ are the components of A along and perpendicular to l. $F_\perp = \mathrm{d}A_\perp + A_\perp \wedge A_\perp$ is the curvature. The above Chern-Simons action on $N(l)$ is coupled with the path integral of the Wilson loop. The coupled action is linear in A_t and $\bar{A}_t$, while the other ingredients in $Z_{\mathcal{M}_3}$ do not depend on A_t or $\bar{A}_t$. A_t and $\bar{A}_t$ can be integrated to get two delta functions constraining $F_\perp$ and $\bar{F}_\perp$ [43]:

$$\frac{t}{4\pi}F_\perp^T = \frac{1}{2}\sum_l Y(\nu_l+\kappa_l)Y^{-1}\delta_l^{(2)}(x)\mathrm{d}x_1 \wedge \mathrm{d}x_2.$$
$$\frac{\bar{t}}{4\pi}\bar{F}_\perp^T = \frac{1}{2}\sum_l \bar{Y}(\nu_l-\kappa_l)\bar{Y}^{-1}\delta_l^{(2)}(x)\mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{6.7}$$

We have chosen local coordinates (x_1, x_2) on D so that the Wilson line goes through the origin. The constraints imply that the eigenvalue of median holonomy on each T^2 is

$$\lambda_l = \exp\left[\frac{2\pi i}{k} j_l\right], \tag{6.8}$$

which is the boundary condition imposed on the theory on the complement $\mathcal{M}_3 = \mathfrak{X}_3\backslash\{l\}$. λ_l is the same as λ_{T^2} in the last section. Equivalently, $\mathcal{Z}_{\mathcal{M}_3}$ can be written as a TQFT on $\mathcal{M}_3$ with surface defects and the above boundary condition,

$$\mathcal{Z}_{\mathcal{M}_3} = \int_{\lambda_l,\bar{\lambda}_l} DAD\bar{A} e^{-iCS[\mathcal{M}_3|A,\bar{A}]}\prod_{\mathcal{S}}\mathcal{O}_{\mathcal{S}}[A,\bar{A};h_{\mu\nu}]. \tag{6.9}$$

The above is the field-theoretic version of the wave function (5.1) defined in the previous sections.

As k, $s \to \infty$ and keeping λ_ℓ fixed, the leading contribution of $Z_{\mathcal{M}_3}$ comes from the solutions of critical equations $\delta S = \mathrm{Re}S = 0$ when the path integral is written as $\int e^S$. $\mathrm{Re}S = 0$ implies $\phi_a = 0$ on each interface $\mathcal{S}$, i.e., the connection reduces to SU(2) on $\mathcal{S}$. At the solution of $\mathrm{Re}S = 0$, the equation of motion $\delta S = 0$ is simply the same as the Chern-Simons theory without the surface defect, i.e., the connection is flat on $\mathcal{M}_3$,

$$F = \bar{F} = 0 \quad \text{on} \quad \mathcal{M}_3, \tag{6.10}$$

and satisfies the boundary condition. It was shown in the last section that all of the flat connections satisfying the critical equations correspond to the simplicial geometries on $\mathcal{M}_4$, although some flat connections may not give a uniform orientation $\mathrm{sgn}(V_4)$ on $\mathcal{M}_4$.

At the semiclassical limit k, $s \to \infty$, the leading contribution of each critical point is given by evaluating the action at the critical point. In Eq. (6.3), $\mathcal{O}_{\mathcal{S}} = 1$ at each critical point. The Chern-Simons action and the Wilson-loop action evaluated at a flat connection give [39,44]

$$-\frac{t}{2\pi}\int_{c\subset\mathcal{L}_\alpha} \ln\tau_l \frac{\mathrm{d}\lambda_l}{\lambda_l} - \frac{\bar{t}}{2\pi}\int_{c\subset\mathcal{L}_\alpha} \ln\bar{\tau}_l \frac{\mathrm{d}\bar{\lambda}_l}{\bar{\lambda}_l}, \tag{6.11}$$

where the integration is along a contour c in the Lagrangian submanifold $\mathcal{L} \simeq \mathcal{M}_{\mathrm{flat}}(\mathcal{M}_3, \mathrm{SL}(2,\mathbb{C}))$. α labels the branch of $\mathcal{L}$ where the flat connection is located. λ_l and τ_l are the eigenvalues of meridian and longitude holonomies on the T^2 boundary. As a result, the contribution of a critical point gives the same result as Eqs. (5.7) and (5.8) up to an overall constant [removing the boundary terms in Eqs. (5.7) and (5.8)]. The leading contribution of the flat connection α_{4d} gives the Regge action with a cosmological constant on $\mathcal{M}_4$,

$$\mathcal{Z}_{\mathcal{M}_4} \sim \exp\frac{i}{\ell_P^2}\left(\sum_{\mathfrak{f}}\mathbf{a}(\mathfrak{f})\varepsilon(\mathfrak{f}) - \Lambda\sum_{\mathfrak{S}}|V_4(\mathfrak{S})|\right). \tag{6.12}$$

VII. SURFACE DEGREE OF FREEDOM

The surface defect introduced in Eq. (6.2) explicitly breaks the $\mathrm{SL}(2,\mathbb{C})$ gauge invariance into SU(2) on the surface $\mathcal{S}$. Then from the field theory point of view, the gauge degree of freedom becomes the physically propagating degree of freedom on $\mathcal{S}$, similar to the case of the 2d Wess-Zumino-Witten model as the boundary field theory of Chern-Simons theory in the 3d bulk. In other words, introducing an additional degree of freedom on $\mathcal{S}$ recovers the $\mathrm{SL}(2,\mathbb{C})$ gauge invariance on $\mathcal{S}$.

We consider the infinitesimal gauge transformation of the $\mathrm{SL}(2,\mathbb{C})$ connection, which turns out to be sufficient for the present purpose:

$$\delta_\xi A_\mu^i = D_\mu\xi^i = \partial_\mu\xi^i + \varepsilon^{ijk}A_\mu^j\xi^k,$$
$$\delta_\xi\bar{A}_\mu^i = \bar{D}_\mu\bar{\varepsilon}^i = \partial_\mu\bar{\xi}^i + \varepsilon^{ijk}\bar{A}_\mu^j\bar{\xi}^k. \tag{7.1}$$

We consider that the background field $(A,\bar{A})$ is a critical point of the path integral, which satisfies $\phi_a^i = 0$ on $\mathcal{S}$. So we have A_a^i is a SU(2) connection, and $D_a = \bar{D}_a$ is a SU(2)-covariant derivative on $\mathcal{S}$ with respect to the background field. Therefore the gauge transformation of ϕ_a^i is

$$\delta_\xi \phi_a^i = \frac{1}{2i} D_a(\xi^i - \bar{\xi}^i) \equiv D_a \varphi^i, \qquad (7.2)$$

where $\varphi^i = \frac{1}{2i}(\xi^i - \bar{\xi}^i)$ is a scalar in the adjoint representation of SU(2).

The infinitesimal gauge transformation of $\mathcal{O}_\mathcal{S}$ in Eq. (6.2) gives

$$\delta_\varphi\left(-\frac{k}{4\pi}\int_\mathcal{S} \mathrm{d}^2x\sqrt{h}h^{ab}\phi_a^i\phi_b^i\right) = -\frac{k}{4\pi}\int_\mathcal{S} \mathrm{d}^2x\sqrt{h}h^{ab}D_a\varphi^i D_b\varphi^i + o(\varphi^3) \qquad (7.3)$$

at the critical background field with $\phi_a^i = 0$ on $\mathcal{S}$. If we add it to the exponent of Eq. (6.2) and redefine $\mathcal{O}_\mathcal{S}$ by

$$\mathcal{O}_\mathcal{S}[A,\bar{A},h_{ab}] := \int D\varphi^i \exp\left[-\frac{k}{4\pi}\int_\mathcal{S} \mathrm{d}^2x\sqrt{h}h^{ab}\phi_a^i\phi_b^i - \frac{k}{4\pi}\delta_\varphi\left(\int_\mathcal{S} \mathrm{d}^2x\sqrt{h}h^{ab}\phi_a^i\phi_b^i\right)\right], \qquad (7.4)$$

then $\mathcal{O}_\mathcal{S}[A,\bar{A},h_{ab}]$ is invariant under the $\mathrm{SL}(2,\mathbb{C})$ gauge transformation (any gauge transformation can be compensated by a shift of the gauge parameter φ). Expanding the term $\delta_\varphi(\int_\mathcal{S} \mathrm{d}^2x\sqrt{h}h^{ab}\phi_a^i\phi_b^i)$ at the critical background field gives Eq. (7.3) as the leading order for small φ. The additional term in φ looks like a (gauged) linear sigma model on $\mathcal{S}$.

When we insert the above complete operator $\mathcal{O}_\mathcal{S}[A,\bar{A},h_{ab}]$ into Eq. (6.3), the additional degree of freedom φ on $\mathcal{S}$ does not modify our previous semiclassical analysis. The Chern-Simons connections A that we are interested in are nontrivial on all four-holed spheres $\mathcal{S}$. Turning on a nontrivial background field $A_a \neq 0$ on $\mathcal{S}$ makes φ massive, whose mass term is given by

$$\varepsilon^{ijk}\varepsilon^{ilm}h^{ab}A_a^jA_b^l\varphi^k\varphi^m = h^{ab}(\delta^{jl}\delta^{\mathrm{km}} - \delta^{jm}\delta^{kl})A_a^jA_b^l\varphi^k\varphi^m = [h^{ab}(\delta^{\mathrm{km}}A_a^lA_b^l - A_a^mA_b^k)]\varphi^k\varphi^m. \qquad (7.5)$$

One can diagonalize the mass matrix in the square bracket by an orthogonal transformation N, i.e., $N^{-1}[\mathrm{tr}(A^ThA)\mathbf{1} - A^ThA]N = \mathrm{tr}(A^ThA)\mathbf{1} - \mathrm{diag}(x_1,x_2,x_3)$, where $x_{1,2,3} \geq 0$ and $\mathrm{tr}(A^ThA) = x_1 + x_2 + x_3 > 0$.[9] So the eigenvalues of the mass matrix are all positive. φ being massive motivates us to integrate out φ, which at the semiclassical level projects to the ground state $\varphi = 0$.

[9] A^ThA is a positive-semidefinite matrix when $A \neq 0$. x_i may vanish since A_a^i may not be a nondegenerate matrix. But if all $x_{1,2,3} = 0$, it would lead to $(AN)^Th(AN) = 0$, which implies $AN = 0$ and $A = 0$, since N is invertible.

The surface defect $\mathcal{O}_\mathcal{S}$ modifies the equations of motion by adding a singular term

$$F(A) = \delta(t)\mathrm{d}t \wedge J(h_{ab}, \phi_a^i, \varphi^i), \qquad (7.6)$$

where t is the coordinate transverse to $\mathcal{S}$, and the location of $\mathcal{S}$ corresponds to $t = 0$. The critical equations $\phi_a^i = 0$ and $\varphi = 0$ on $\mathcal{S}$ imply $J(h_{ab}, \phi_a^i, \varphi^i) = 0$ on the right-hand side of the equation of motion. Thus the equation of motion reduces to the flatness equation (6.10). So we conclude that all of the critical flat connections on $\mathcal{M}_3$ studied in the last section are still critical, even when we take into account the additional degree of freedom φ on the surface defect.

VIII. CONCLUSION AND OUTLOOK

In this paper we studied the quantization and implementation of the LQG simplicity constraint in the spinfoam model in the presence of a cosmological constant. Spinfoam amplitudes with a cosmological constant were formulated as complex Chern-Simons theories on a certain class of 3-manifolds. The implementation of the quantum simplicity constraint results in surface defects in the Chern-Simons theory. These surface defects guarantee that the amplitude has the correct semiclassical limit, which reproduces the Einstein-Regge action with a cosmological constant on the 4d simplicial complex.

This work relates the LQG simplicity constraint to surface defects in Chern-Simons theory. Although line defects have been widely studied in Chern-Simons theory, surface defects (or domain walls) have not been sufficiently studied in the literature. The surface defect appearing here has not been studied before. We have done some preliminary investigations of the surface defect by studying the propagating physical degrees of freedom on the defect surface. We have shown that at the linearized level, the propagating field behaves as a 2d sigma model gauged by the Chern-Simons connection.

The formalism in this paper makes it possible to rigorously define the spinfoam amplitude with a cosmological constant. The present definition of the amplitude either uses the infinite-dimensional path integral [1] or a semiclassical expansion [2]. However, it is known that the Chern-Simons partition function $Z_{S^3\setminus\Gamma_5}$ can be expressed as a finite-dimensional integral [4]. Now the spinfoam amplitude is constructed by projecting $Z_{S^3\setminus\Gamma_5}$ onto the solution of the simplicity constraint using Eq. (4.1), which is also a well-defined operation. So the entire spinfoam amplitude can be written as a finite-dimensional integral, whose finiteness is ready to be studied. Research on this aspect is currently in progress.

Further studies of the proposed surface defect are also postponed to future research. It would be interesting to understand the dynamics of the sigma model propagating on the defect, including its interaction with Chern-Simons theory. The metric dependence of the surface defect might be understood in more detail in the future. The defect may not depend on all of the metric degrees of freedom (like the situation of Wilson lines). Classically, the surface defect only depends on the complex structure of the four-holed sphere, since the defect is classically both reparametrization and conformally invariant. It would then be interesting to see whether this type of metric dependence is preserved at the quantum level, or how this property receives quantum corrections.

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