Gravity degrees of freedom on a null surface

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A canonical analysis for general relativity is performed on a null surface without fixing the diffeomorphism gauge, and the canonical pairs of configuration and momentum variables are derived. Next to the well-known spin-2 pair, also spin-1 and spin-0 pairs are identified. The boundary action for a null boundary segment of spacetime is obtained, including terms on codimension two corners.

I. INTRODUCTION

It has long been recognized that the canonical structure of gravity is especially simple on null surfaces. Sachs [1] was the first to realize that the initial data could be given in an unconstrained form using a double null sheet as an initial value surface. Ashtekar *et al.* studied the canonical quantization of radiative modes at null infinity [2]. Epp in [3] was the first one to write down a proposal for the null canonical pairs of configuration and momentum variables in the $2 + 2$ formalism (see [4]). An additional investigation using stretched horizon techniques was done by Parikh and Wilczek in [5], and investigations in special coordinates adapted to the null initial data focusing on the construction of the symplectic potential were done by Reisenberger in [6]. More recently Parattu *et al.* [7,8] have reconsidered this analysis, focusing on the construction of the analog of the Gibbons-Hawking boundary term for null boundaries, while Lehner *et al.* in [9] developed further the null boundary action formalism and included null corner terms that generalize Hayward's construction [10] (see also [11] and some preliminary work by Neiman on null corner terms [12]).

Our work improves on these latest developments and gives an independent derivation of some of the previous results. Our derivation identifies clearly what the boundary action and the null symplectic structure are without having to recourse to a choice of gauge fixing. In contrast, most approaches gauge fix parts of the diffeomorphism gauge at the beginning of the analysis. This is problematic in our setting for two reasons. Firstly, one has to make an arbitrary choice which clutters the covariance of the final expression. Secondly, we now understand that in the presence of boundaries we cannot fix diffeomorphisms without risking killing key boundary degrees of freedom as explained in [13]. Since we want to use our work in the future to shed some light on the issue of soft gravity modes which we expect to be related to such boundary degrees of freedom, we have to be careful in not making any assumption that

will negate the existence of such degrees of freedom. The final decision of whether a mode is physical *cannot* be decided beforehand; it is entirely decided by whether it enters nontrivially or not in the symplectic structure.

Let us emphasize that understanding the nature of the symplectic structure on a null surface is important for three separate reasons. First, since the initial data is constraint free on null surfaces, it can play a fundamental role in understanding the nature of quantum gravity degrees of freedom. Understanding what the canonical pairs are is a necessary step in this direction. Moreover, we can also understand the null symplectic potential as a term controlling the flow of information across a null surface [14], and as such it is a key element in understanding what could be an appropriate definition of informational horizons and in proving a generalization of the second law associated with finite regions.

Finally, the understanding of the gravitational null symplectic potential goes hand in hand with the construction of the boundary action and the corresponding corner terms. These action terms enter the Hamilton-Jacobi function and are the classical analog of the quantum gravity S matrix. They therefore play a key role in the interpretation of the physics of gravity inside a finite region when this region possesses null boundaries (see [15–17]).

It is thus of great interest to obtain the null canonical structure of gravity without introducing any gauge fixing. We accomplish this here, using a robust, physically intuitive framework, by evaluating the symplectic potential Θ_B and the corresponding boundary action A_B on a null surface B, restricted to variations that preserve the nullness of B.

Let us state our results. We recall that the symplectic potential is schematically of the form $\Theta_B = \int_B P\delta Q + \int_{\partial B} p\delta q + \delta A_B + \delta a_{\partial B}$, where (Q, q) are the bulk and boundary configuration variables, (P, p) are the corresponding momenta, A_B is the boundary action and $a_{\partial B}$ is the corner action. We assume that the null hypersurface B possesses a ruling of equal time slices $\phi_0 = \text{const}$ which define the $(D - 2)$-dimensional spacelike cross sections S of B (D is the spacetime dimension). The null geometry of

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B is encoded into the pair (q_{ab}, L^a), where $L = L^a \partial_a$ is a null vector tangential to B such that $L(\phi^0) = 1$, and q_{ab} is the metric induced on the surfaces S. It is convenient to decompose the metric q into the product of a conformal factor $\sqrt{q}^{2/(D-2)}$ and the conformal metric $\tilde{q}_{ab}$ with unit determinant.

One of the nontrivial features of this construction is that it is possible (see also [9]) to choose a boundary and corner action such that the configuration variables (Q, q) depend *only* on the null geometry (q_{ab}, L^a). In units where $8\pi G = 1$, we find that the null symplectic potential $\int_B P\delta Q$ is

$$\Theta_B^{\text{bulk}} = \int_B \left(\frac{1}{2} \tilde{\sigma}^{ab} \delta \tilde{q}_{ab} - \bar{\eta}_a \delta L^a \right.$$
$$\left. - \left(\kappa + \frac{D-3}{D-2} \theta \right) \delta \ln \sqrt{q} \right) \mathrm{d}B, \quad (1.1)$$

where $\mathrm{d}B = \mathrm{d}\phi^0 \wedge \mathrm{d}S$ is the volume element. The configuration variables $(\tilde{q}_{ab}, L^a, \sqrt{q})$ describe the induced geometry of the surface B, and their conjugate momenta are given by $(\frac{1}{2} \tilde{\sigma}^{ab}, -\bar{\eta}_a, (\kappa + \frac{D-3}{D-2}\theta))$. That is our central result.

We thus recover the well-known fact that the momentum of the conformal induced metric $\tilde{q}$ of a cross section S of B is the conformal shear $\tilde{\sigma}^{ab} = -\frac{1}{2} q^a{}_{a'} q^b{}_{b'} (\mathcal{L}_L \tilde{q}^{a'b'})$, a fact first established by Ashtekar *et al.* [2] in the context of asymptotic null infinity. However, we also see that this symplectic structure involves spin 1 (i.e., δL^a) and spin 0 modes (i.e., $\delta\sqrt{q}$), which are usually and unfortunately gauge fixed away in most treatments. These are exactly what are usually[1] called soft graviton modes. What our results show is that since the spin 0 and 1 modes enter the symplectic structure, they are physical degrees of freedom. They cannot be gauge fixed away, but only pushed to the corners of B by applying a diffeomorphism.

The momentum of the normal vector field L to B is the twist $\bar{\eta}$ defined as

$$\bar{\eta}_a = -q_a{}^b \nabla_L \bar{L}_b. \quad (1.2)$$

Here, $q_a{}^b$ is the projector onto S, and $\bar{L}$ is a null form that is orthogonal to S and normalized as $L^a \bar{L}_a = 1$. The twist is thus the parallel transport of the auxiliary null form $\bar{L}$ along L, and describes how the surfaces S twist inside B when one follows the integral curves of L. It is closely related to Damour's momentum [19], which appears in the study of stretched horizons. Lastly, the momentum of the volume element $\sqrt{q}$ of a cross section of B is a combination of the

expansion $\theta = q^{ab} \nabla_a L_b$ along L and the surface gravity κ which enters in $\nabla_L L^a = \kappa L^a$.

The form of the symplectic potential given above depends on what the precise action for gravity is when spacetime has a null boundary. When the cosmological constant vanishes, the on-shell action is a pure boundary integral, and its value is the Hamilton-Jacobi functional and thus of great importance. We give a null boundary action that includes corners and generalizes the result of [9]. One of the key elements entering the boundary action is the surface gravity κ already encountered, while the key element entering the corner term is the factor h, which is the (logarithmic) normal volume element: $e^h = \sqrt{|g|}/\sqrt{q}$. It thus measures the size of the normal geometry and can be physically identified as the redshift factor. As we will see in Sec. VA, it is proportional to the redshift experienced by light rays skimming along B as measured by geodesic observers crossing B. Our result for the boundary plus corner action is

$$A_B + a_{\partial B} = \int_B \kappa \mathrm{d}B + \frac{1}{2} \int_{\partial B} (1-h) L^a \mathrm{d}_a S, \quad (1.3)$$

where $\mathrm{d}_a S := i_{\partial_a} \mathrm{d}B$ is the directed volume element on ∂B.

Finally, we also determine the corner symplectic structure $\Theta_{\partial B} = \int_{\partial B} p\delta q$. Its general expression is given later [Eq. (4.32)], but if we assume that the boundary of B consists of an initial and a final sphere at constant ϕ^0: $\partial B = S_1 \cup S_0$, it simply reads

$$\Theta_{\partial B} = \frac{1}{2} \int_{S_0}^{S^1} (1+h) \delta \ln \sqrt{q} \mathrm{d}S. \quad (1.4)$$

This shows that the redshift factor h is a variable conjugate to the angular size $\sqrt{q}$. This is an interesting relationship especially in view of the Etherington reciprocity law relating the area distances to the redshift factor [20].

The remainder of this article is organized as follows. In Sec. II we briefly review the symplectic geometry of field space in order to fix our definitions and notation. Section III contains our geometrical setup. In Sec. IV we perform our central calculation, obtaining the null canonical pairs of gravity in Sec. V. Section VI contains our suggestion for a Lagrangian boundary term. We conclude in Sec. VII, and collect some of the more technical calculations in the appendixes.

II. THE SYMPLECTIC GEOMETRY OF FIELD SPACE

The presymplectic geometry of field space can be obtained in a covariant way (see, e.g., [13,21,22]), which we briefly review here. It is described by the presymplectic form Ω_B, which is a closed two-form on field space and an integral over a Cauchy hypersurface B in spacetime.

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[1]This denomination is very confusing since it suggests that soft graviton modes are spin-2 degrees of freedom, while they are in fact a combination of spin 1 and spin 0 modes. This fact, which is usually misunderstood, will be expanded on in [18].

The prefix "pre" refers to the fact that Ω_B on field space has degenerate directions, so it does not qualify as "symplectic." The degenerate directions are the gauge degrees of freedom, which have to be ultimately quotiented out to obtain the physical phase space. When the background fields are taken to be on shell, Ω_B is independent of the choice of B and only depends on its homology class. Here, we specialize to the case that the hypersurface B is null.

Schematically, Ω_B can be written as

$$\Omega_B = \int_B \delta P \wedge \delta Q + \int_{\partial B} \delta p \wedge \delta q. \qquad (2.1)$$

Here, δ is the exterior derivative on field space, and $\wedge$ is the wedge product on field space. The pairs (Q, P) of configuration and momentum variables are the canonical pairs. We have allowed for the presence of corner degrees of freedom (q, p). Note that in a Hamiltonian analysis, "corner" refers to the boundary of the hypersurface B, i.e., it is a codimension two surface. In the following and for gravity we will focus on the case where the configuration variables (Q, q) are linear functionals of the metric while the momenta include derivatives of the metric.

Ω_B is the field space exterior derivative $\Omega_B = \delta \Theta_B$ of the symplectic potential Θ_B. The symplectic potential Θ_B is the integral of the symplectic potential current Θ, which is a one-form on field space and a $(D-1)$-form on spacetime. Θ is obtained through the equation

$$\delta L =: d\Theta - E. \qquad (2.2)$$

Here L is the Lagrangian density, which is a D-form on spacetime. E are the equations of motion, which are a one-form on field space and a D-form on spacetime. By definition they do not contain derivatives of the variations of the fields, and they are uniquely determined by the Lagrangian. Here d is the spacetime exterior derivative. Eq. (2.2) determines Θ only up to the addition of a closed $(D-1)$-form on M. This ambiguity can be fixed by demanding the consistency of the variation for boundaries including corners (see [23]). Schematically, the symplectic potential is of the form

$$\Theta_B = \Theta_B^{\text{bulk}} + \Theta_{\partial B} + \delta A_B + \delta a_{\partial B}, \qquad (2.3)$$

where $\Theta_B^{\text{bulk}} = \int_B P \delta Q$ and $\Theta_{\partial B} = \int_{\partial B} p \delta q$. The total variation terms δA_B and $\delta a_{\partial B}$ do not contribute to the symplectic form, because $\delta \delta = 0$. These terms can be reabsorbed into a redefinition of the action $S \to S - A_B - a_{\partial B}$. The inclusion of these terms corresponds to a choice of polarization, and is necessary if one demands that the configuration variables do not include metric derivatives. Finally, an important point is that we assumed the field space exterior derivative δ and the integral $\int_B$

commute. That means the location of the hypersurface B must be specified in a field-independent way.

Let us specialize to our case of vacuum metric general relativity without cosmological constant. The Lagrangian density L, the equations of motion E and the symplectic potential current Θ are

$$L = \frac{1}{2} R\epsilon, \quad E = \frac{1}{2} G^{ab} \delta g_{ab} \epsilon, \quad \Theta = \frac{1}{2} \nabla_b (\delta g^{ab} - g^{ab} \delta g) \epsilon_a. \qquad (2.4)$$

$\delta g^{ab} = g^{ac} g^{bd} \delta g_{bd}$ denotes the metric variation with indices raised, not the variation of the inverse metric. $\delta g = g^{ab} \delta g_{ab}$ is its trace. We set $8\pi G = c = 1$, and introduced the volume D form and the directed volume $(D-1)$-form

$$\epsilon = *1 = \frac{\sqrt{|g|}}{D!} \epsilon_{a_1 \ldots a_D} dx^{a_1} \wedge \ldots \wedge dx^{a_D},$$

$$\epsilon_a = \iota_a \epsilon = \frac{\sqrt{|g|}}{(D-1)!} \epsilon_{aa_2 \ldots a_D} dx^{a_2} \wedge \ldots \wedge dx^{a_D}. \qquad (2.5)$$

We made the usual choice in fixing the spacetime closed ambiguity in Θ and used the standard expression (see, e.g., [24]). The analysis of the closed ambiguity will be part of future work.

We see from Eq. (2.4) that the covariant symplectic potential contains variations of metric derivatives. The challenge we face is to express it purely in terms of variations of the metric only, so that we can read off the proper canonical pairs of gravity. We therefore have to manipulate Θ_B to remove the derivatives of variations. The derivatives of variations will be of two kinds: derivatives tangential to B, and derivatives in directions transverse to B. The tangential derivatives can easily be taken care of by integrating by parts. The transverse derivatives are more subtle, but we will show that they can be eliminated through variation by parts, i.e., they can be absorbed into a total variation. Carefully carrying through this procedure and keeping all the terms is the first goal of this paper. It will give us an expression for Θ_B as the sum of a bulk term, a boundary term, a bulk total variation and a boundary total variation.

III. SETUP

In this section, we introduce the structures and notation we will use to evaluate the symplectic potential on the null hypersurface B. The setup is taken from [25,26]. Previously, similar formalisms were set up, e.g., in [4,27,28].

A. Foliations, Normal Forms and Coordinates

Let M be the D-dimensional spacetime. We are typically interested in a region R of a D-dimensional spacetime with boundary $B \cup \Sigma_0 \cup \Sigma_1$ where Σ_i are spacelike
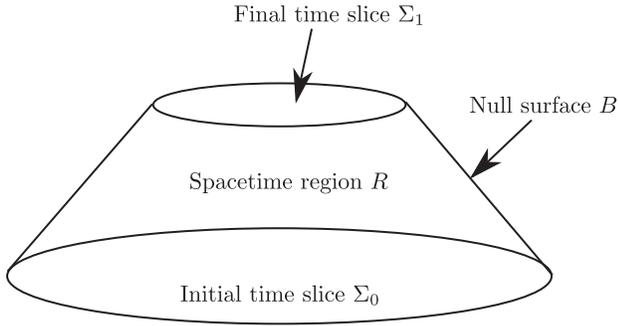
FIG. 1.   A typical situation where the symplectic structure on the null surface B is of interest is when B is part of the boundary of the spacetime region R under consideration. The other parts of the boundary are spacelike surfaces Σ_i.

hypersurfaces and B a null hypersurface (see Fig. 1). More generally, we want to understand the nature of the symplectic potential Θ_B on a null hypersurface B. The location of B is specified by the condition $\phi^1(x) = 0$, where ϕ^1 is a suitable scalar field on M that increases towards the past of B. B is a finite hypersurface with a boundary ∂B that we will call a "corner." It is a member of the foliation specified by $\phi^1 = \text{const}$ and located at $\phi_1 = 0$. We do not assume that every member of the foliation is a null hypersurface, but assume that ϕ^1 is a good foliation function in a neighborhood of B, i.e., $d\phi^1 \neq 0$ on B.

We introduce another foliation given by $\phi^0 = \text{const}$ of spacelike hypersurfaces, where ϕ^0 is a field that increases towards the future. We require that ϕ^0 is a good foliation function in a neighborhood of B, and that nowhere $d\phi^0$ is a multiple of $d\phi^1$. At the intersections of the two foliations lies a two-parameter family of spacelike codimension two surfaces S. Coordinates $\sigma^A(x)$ are also chosen on each surface S. They are not required to be constant on the null generators of B. Doing so would be a partial gauge fixing which we want to avoid, since the direction of the null generators is metric dependent.

Using also the foliation fields as coordinates, we introduce a frame $(X^a)(x) = (\phi^i, \sigma^A)(x)$ on M. This frame represents an invertible mapping $X : U \to M$, from a domain $U \in \mathbb{R}^D$ to M. The metric G on M can be represented as a metric on U via the pullback: $X^*G = g$. Here and in the following, $a \in \{0, \ldots, D-1\}, i \in \{0, 1\}$ and $A \in \{2, \ldots, D-1\}$. x represents a choice of coordinates while $X^a(x)$ represents points of M. We will refer to X^a as a foliation frame. We introduce the separation of foliation frame and coordinates also with an eye on future work in order to have full control of what we vary and what we do not. The setup also connects to the formalism developed in [13], where the frame fields become physical. In the foliation frame, the tangent vectors e_A to the surfaces S become

$$e_A = e_A^a \partial_a, \quad \text{where } e_A^a = \frac{\partial x^a}{\partial \sigma^A} = \delta_A^a, \quad (3.1)$$

while the metric in the foliation frame can be parametrized as

$$\begin{aligned} ds^2 = g_{ab} dx^a \otimes dx^b &= H_{ij} d\phi^i \otimes d\phi^j \\ &+ q_{AB}(d\sigma^A - A_i^A d\phi^i) \otimes (d\sigma^B - A_j^B d\phi^j). \end{aligned} \quad (3.2)$$

Here we have defined the shift connection $A^A := A_i^A d\phi^i$, which is a one-form in the normal plane to S valued into TS. We also defined the normal metric H_{ij}, which determines the geometry of the normal two-planes $(TS)^\perp$ to S, while q is the tangential metric which determines the geometry of the sphere S. The metric g contains $\frac{1}{2}D(D+1)$ parameters and this parametrization is completely general. No gauge fixing has taken place, and we have not yet specialized to the case of a null hypersurface B. Gauge fixing at this stage risks killing physical degrees of freedom. This important point is, in most previous approaches, completely neglected and leads to a deep source of confusion about what is physical and what is not.

The inverse metric is

$$\begin{aligned} g^{ab} \partial_a \otimes \partial_b = H^{ij}(\partial_i + A_i^A \partial_A) \otimes (\partial_j + A_j^B \partial_B) \\ + q^{AB} \partial_A \otimes \partial_B, \end{aligned} \quad (3.3)$$

where H^{ij} and q^{AB} are the inverses of H_{ij} and q_{AB}, respectively. We introduce the covariant normal derivatives

$$D_i := (\partial_i + A_i^A \partial_A). \quad (3.4)$$

They can be understood as normal derivatives covariant under the gauge group $\text{Diff}(S)$ of diffeomorphisms on S. That is because under an infinitesimal change of foliation frame $\delta_V \phi^i = 0$ and $\delta_V \sigma^A = V^A(x)$, the normal metric transforms as a scalar $\delta H_{ij} = V^C \partial_C H_{ij}$, and the tangential metric transforms as a tensor $\delta_V q_{AB} = \mathcal{L}_V q_{AB}$, while A_i^A transforms as a connection

$$\delta_V A_i^A = \partial_i V^A + [A_i, V]_S^A, \quad (3.5)$$

where $[.,.]_S$ is the Lie bracket on S. Then, the derivative D_i transforms covariantly as a scalar under the gauge group $\text{Diff}(S)$: $\delta_V(D_i f) = V^A \partial_A(D_i f)$ for a field f on M. The curvature of the normal connection is the vector field

$$[D_0, D_1]^A = \partial_0 A_1^A - \partial_1 A_0^A + [A_0, A_1]_S^A. \quad (3.6)$$

We introduce the logarithmic normal volume element h as

$$e^h = \sqrt{|H|}. \quad (3.7)$$

It will play a very important role in the symplectic structure and the boundary action and will be interpreted as the redshift factor in an adapted frame. The determinants of the normal metric H_{ij}, the induced metric q_{AB} and the full metric g_{ab} are therefore linked by

$$\sqrt{|g|} = e^h \sqrt{q}. \tag{3.8}$$

In order to write the symplectic potential using quantities intrinsic to the surfaces S, we need to be able to project along its two normal directions. We therefore have to choose a basis of one-forms normal to S. There is a simple choice of basis which is metric independent, and depends only on the choice of foliation. It is given by $(d\phi^0, d\phi^1) \in (TS)^\perp$. However, since the surfaces S are part of a null hypersurface, the most convenient choice is to use a null coframe $(\ell, \bar{\ell})$ consisting of two null forms normal to the family of surfaces S, one of which will be normal also to B. This is what we do here.

Let $\ell = \ell_a dx^a$ and $\bar{\ell} = \bar{\ell}_a dx^a$ be two smooth, null one-form fields normal to the surfaces S (here and in the following boldface letters denote one-forms). Let ℓ be such that at B, ℓ is normal to B, and $g^{-1}(\ell) = \ell^a \partial_a$ is future pointing. On the forms $(\ell, \bar{\ell})$, we impose the normalization condition that $g^{-1}(\ell, \bar{\ell}) = 1$. These conditions uniquely determine ℓ and $\bar{\ell}$ in a neighborhood of B, up to a rescaling $(\ell \to e^\epsilon \ell, \bar{\ell} \to e^{-\epsilon} \bar{\ell})$, where ϵ is an arbitrary function. Our choice of a null dyad diagonalizes the $SO(1,1)$ symmetry of the normal plane to S, and the rescaling is the action of a $SO(1,1)$ transformation. Since $(\ell, \bar{\ell})$ and $(d\phi^0, d\phi^1)$ both form a basis of $(TS)^\perp$, their relationships can be parametrized in terms of four fields $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ which form a set of generalized lapses. We set

$$\ell = e^\alpha (d\phi^1 - \beta d\phi^0),$$
$$\bar{\ell} = \frac{e^{\bar{\alpha}}}{1 + \beta\bar{\beta}} (d\phi^0 + \bar{\beta} d\phi^1). \tag{3.9}$$

The condition that the slices $\phi^1 = \text{const}$ are timelike or null and that the slices $\phi^0 = \text{const}$ are spacelike is encoded in the inequalities $\beta \geq 0, \bar{\beta} > 0$. The four functions $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ determine the inverse normal metric H through the conditions $H^{ij} = g^{-1}(d\phi^i, d\phi^j)$. We get

$$H^{ij} = \frac{e^{-h}}{1 + \beta\bar{\beta}} \begin{pmatrix} -2\bar{\beta} & 1 - \beta\bar{\beta} \\ 1 - \beta\bar{\beta} & 2\beta \end{pmatrix},$$
$$H_{ij} = \frac{e^h}{1 + \beta\bar{\beta}} \begin{pmatrix} -2\beta & 1 - \beta\bar{\beta} \\ 1 - \beta\bar{\beta} & 2\bar{\beta} \end{pmatrix}, \tag{3.10}$$

where the normal volume element h is

$$h = \alpha + \bar{\alpha}. \tag{3.11}$$

The quantity $\alpha - \bar{\alpha}$ does not enter the metric, and encodes the rescaling freedom in ℓ and $\bar{\ell}$ alluded to

above. $\alpha - \bar{\alpha}$ is therefore not physical, it is pure gauge freedom. We will refer to it as the boost gauge, because a boost transformation in the normal plane to S will change $\alpha - \bar{\alpha}$, keeping h and the directions of $(\ell, \bar{\ell})$ fixed. A boost transformation $(\ell, \bar{\ell}) \to (e^\epsilon \ell, e^{-\epsilon} \bar{\ell})$ acts as $(\alpha, \bar{\alpha}) \to (\alpha + \epsilon, \bar{\alpha} - \epsilon)$.

Even though it is pure gauge, we will not fix $\alpha - \bar{\alpha}$ for now. In the literature different choices are made, and the generality of our boost gauge allows us to connect them. For instance, [8] and the BMS literature work in the gauge $\alpha = 0$ while [9] works in the gauge $\bar{\alpha} = 0$. We will see that it is more convenient for the problem at hand to choose $\bar{\alpha} = 0$ such that $\alpha = h$. Note that the boost gauge can be fixed only with reference to the foliation functions ϕ^0, ϕ^1, and a boost gauge fixing thus depends on how we parametrize the family of surfaces S.

While the forms are denoted by bold letters, we denote the corresponding vectors with unbolded letters as $\ell = g^{-1}(\ell)$ and $\bar{\ell} = g^{-1}(\bar{\ell})$. They are obtained by raising the index on ℓ and $\bar{\ell}$ and are given by

$$\ell = \ell^a \partial_a = e^{-\bar{\alpha}}(D_0 + \beta D_1),$$
$$\bar{\ell} = \bar{\ell}^a \partial_a = \frac{e^{-\alpha}}{1 + \beta\bar{\beta}}(D_1 - \bar{\beta}D_0). \tag{3.12}$$

Note that the forms $(\ell, \bar{\ell})$ as well as the vectors $(\ell, \bar{\ell}, D_i)$ contain metric parameters and are thus metric dependent.

For notational convenience, we will mostly work with tensors that have D-dimensional indices, even if they are intrinsic to S. Vectors v^A and contravariant tensors on S are pushed forward into M along the inclusion, yielding in the foliation frame $v^a = e^a_A v^A = \delta^a_A v^A$. Covectors and covariant tensors like q_{AB} are pushed forward using the forms

$$e^A := e^A_a dx^a = (q^{AB} g_{ab} e^b_B) dx^a = (\delta^A_a - A^A_a) dx^a, \tag{3.13}$$

yielding, e.g., $q_{ab} = (\delta^A_a - A^A_a)(\delta^B_b - A^B_b) q_{AB}$.

Using that notation, we can write the components of the shifted derivative in foliation coordinates as $D^a_i = \delta^a_i + A^a_i$. It can be checked that $q_{ab} D^b_i = q_{ab} \ell^a = q_{ab} \bar{\ell}^a = 0$. The two vectors (D_i) span the same space as the vectors $(\ell, \bar{\ell})$, and all four are indeed orthogonal to S. It can easily be checked that the induced metric q on S satisfies the completeness relation

$$q_{ab} + \ell_a \bar{\ell}_b + \bar{\ell}_a \ell_b = g_{ab}. \tag{3.14}$$

q also satisfies $q_a{}^b q_{bc} = q_{ac}$.

We now have a variety of ways to repackage the information contained in the metric g. The basic variables are the matrices (H_{ij}, q_{AB}, A^A_i) in the parametrization (3.2). Using Eq. (3.10), H_{ij} can be rewritten as $(h, \beta, \bar{\beta})$, which contains the same number of independent components. After introducing the quantity $\alpha - \bar{\alpha}$, we can rewrite
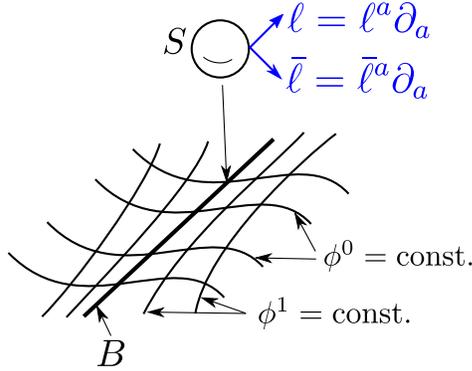
FIG. 2. The geometry of our setup is depicted. The null hypersurface B is a member of the foliation $\phi^1 = $ const that need not be null everywhere. It is ruled into codimension two surfaces S by a second foliation $\phi^0 = $ const. The vectors ℓ and $\bar{\ell}$ are null and normal to S. ℓ is normal also to B, and since B is null, it is at the same time tangential to B. $\bar{\ell}$ is transverse to B, and the vectors are normalized as $\ell^a \bar{\ell}^b g_{ab} = 1$.

$(\alpha - \bar{\alpha}, H)$ as the one-forms $(\ell, \bar{\ell})$ using (3.9). The one-forms have two independent components each because they are constrained to be orthogonal to S. Finally, the variables $(\ell, \bar{\ell}, A_i^A)$ can be rewritten as the vectors $(\ell, \bar{\ell})$ using Eq. (3.12). The vectors $(\ell, \bar{\ell})$ contain the shifts A_i^A, and determine $2D$ independent variables, one of which is $(\alpha - \bar{\alpha})$. The variables $(\ell, \bar{\ell}, q_{AB})$ thus fully determine the metric, and the quantity $(\alpha - \bar{\alpha})$. They combine covariance with an intuitive picture adapted to null structures, and we will use them in the following.[2]

So far, the setup we described works for any two foliations (ϕ^0, ϕ^1) with spacelike intersections. Let us now specialize to the case that $\phi^1 = 0$ describes a null hypersurface B. For an illustration of the null geometry, see Fig. 2. The nullness condition reads $g^{ab}(\nabla_a \phi^1)(\nabla_b \phi^1) = H^{11} \overset{B}{=} 0$, and from Eq. (3.10) we see that this is equivalent to the condition $\beta \overset{B}{=} 0$. So we get

$$\beta \overset{B}{=} 0, \qquad \ell \overset{B}{=} e^\alpha \mathrm{d}\phi^1, \qquad \ell \overset{B}{=} e^{-\bar{\alpha}} D_0. \qquad (3.15)$$

Note that also the derivatives $\nabla_\ell \beta$ and $q_a{}^b \nabla_b \beta$ vanish on B. We see that as expected, the vector ℓ is parallel to B since on B it does not contain a transverse derivative ∂_{ϕ^1}. Its integral curves are the null generators of B. If we had chosen the coordinates σ to be constant along the null generators of B, then the shift A_0^a would vanish on B yielding $\ell^a = e^{\bar{\alpha}} \partial_{\phi^0}$. The induced metric on B is

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[2]Except for the ambiguity $\alpha - \bar{\alpha}$, the situation is analogous to the spacelike case, where the metric is parametrized in terms of induced metric, lapse and shift. These variables can be repackaged into the induced metric and the unit normal vector n.

$$\mathrm{d}s^2|_B = q_{AB}(\mathrm{d}\sigma^A - A_0^A \mathrm{d}\phi^0) \otimes (\mathrm{d}\sigma^B - A_0^B \mathrm{d}\phi^0), \quad (3.16)$$

where we have used that $h_{00} \overset{B}{=} 0$. Its parameters are q_{AB} and A_0^A, and the number of parameters is $\frac{1}{2}D(D-1) - 1$, as expected for the induced metric of a codimension one hypersurface that satisfies one condition. The directed volume element on B can be written in terms of the volume form $\mathrm{d}S$ on S: We set $\mathrm{d}S := \sqrt{q}\mathrm{d}^{D-2}\sigma$ and obtain

$$\epsilon_a|_B = \iota_a \epsilon|_B = -\ell_a e^{\bar{\alpha}} \mathrm{d}\phi^0 \wedge \mathrm{d}S. \qquad (3.17)$$

The equality can be seen by writing $\iota_a \epsilon$ from Eq. (2.5) in the foliation frame, where $\epsilon = e^h \mathrm{d}\phi^0 \wedge \mathrm{d}\phi^1 \wedge \mathrm{d}S$, setting $\mathrm{d}\phi^1 = 0$, substituting $\iota_a \mathrm{d}\phi^1 = \partial_a \phi^1 \overset{B}{=} e^{-\alpha} \ell_a$, and using $h = \alpha + \bar{\alpha}$. The combination $L_a = e^{\bar{\alpha}} \ell_a$ that enters the induced volume form will play a special role in our construction, as we will see.

The introduction of the auxiliary foliation ϕ^0 on S should be thought of as a choice of reference frame on B. It avoids dealing with the degenerate induced metric on B and makes calculations more straightforward, but comes at the cost of introducing some additional structure into the setup: the decomposition of B into spheres S. Note, however, that we need an auxiliary foliation ϕ^0 in order to locate the position of the corner ∂B, so we cannot avoid introducing such extra data, at least near the boundary of B. It has been understood recently that when B is a part of the boundary of a spacetime region, then the foliation fields (ϕ^0, ϕ^1) which provide a frame around B acquire direct physical meaning, as a label of boundary degrees of freedom [13].

B. Decomposition of Metric Variations

The symplectic potential contains the variation of the spacetime metric, δg_{ab}. For now, we will consider a completely general metric variation, but later we will specialize to the case that the metric variations leave the hypersurface B null. We view the foliations (ϕ^0, ϕ^1) and the coordinates σ as fixed, so they do not vary: $\delta\phi^i = \delta\sigma^A = 0$. Since the position of B is described using the foliations, this also ensures that B does not move, while its geometry varies, so that integral signs and variations commute. As before, we write $\delta g = g^{ab}\delta g_{ab}$ for the trace of the metric variation, and $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$ is the variation of the metric with the indices raised.

The variation of the metric will be decomposed into tensors intrinsic to S, using the structure of the two foliations. We then express it using the variations of q, ℓ and $\bar{\ell}$. Note that since the forms $(\ell, \bar{\ell})$ are linear combinations of the $\mathrm{d}\phi^i$ which do not vary, their variations stay orthogonal to the surfaces S, i.e., $q^{ab}\delta\ell_b = 0$ (and similarly for $\bar{\ell}$). The relationships among ℓ and $\bar{\ell}$, which are implemented by the definition of the metric dependent

coefficients $(\alpha, \bar\alpha, \beta, \bar\beta)$, are also preserved under variations: We have $\delta(\ell_a \ell^a) = 0$, $\delta(\ell^a \bar\ell_a) = 0$, etc.

Our first variation quantity,

$$\delta q_{ab} := q_a{}^c q_b{}^d \delta g_{ab} = \delta q_{AB} e_a^A e_b^B, \qquad (3.18)$$

is the variation of the induced metric, pushed forward into M. Its trace $\delta q := q^{ab} \delta q_{ab} = q^{AB} \delta q_{AB}$ is related to the change of the area element on S as $\delta\sqrt{q} = \frac{1}{2}\sqrt{q}\delta q$. Note that $\delta q_{ab} \neq \delta(q_{ab})$ because the latter expression contains the variation of the tensors e_a^A.

The vector ℓ is null and normal to S by definition, but both of these properties are metric dependent. When the metric varies, ℓ will therefore change to restore the properties. The change in ℓ parallel to S is $q^a{}_b \delta\ell^b$. It can be written as

$$q^a{}_b \delta\ell^b = -q^{ab}\ell^c \delta g_{bc} = e^{-\bar\alpha}(\delta A_0^a + \beta \delta A_1^a) \overset{B}{=} e^{-\bar\alpha}\delta A_0^a. \qquad (3.19)$$

For the first identity, we have used $\ell^a \delta g_{ab} = \delta\ell_b - g_{ab}\delta\ell^a$, and that the variation $q_a{}^b \delta\ell_b = 0$, since ℓ is fixed to be normal to S. For the second identity, we varied the expression Eq. (3.12), and used that $\delta D_i^a = \delta A_i^a$ and that $D_i^a q_{ab} = 0$. Similarly, we get

$$q^a{}_b \delta\bar\ell^b = -q^{ab}\bar\ell^c \delta g_{bc} = \frac{e^{-\alpha}}{1 + \beta\bar\beta}(\delta A_1^a - \bar\beta \delta A_0^a). \qquad (3.20)$$

The change of the normal volume element e^h is given by $\ell^a \bar\ell^b \delta g_{ab}$:

$$\ell^a \bar\ell^b \delta g_{ab} = -\bar\ell_a \delta\ell^a - \ell_a \delta\bar\ell^a = \delta(\alpha + \bar\alpha) = \delta h. \qquad (3.21)$$

The second equality can be checked explicitly using the expressions Eq. (3.9) and varying them. Remembering that $\sqrt{|g|} = \sqrt{q}e^h$, and noting that $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\delta g$, we get

$$\delta g = g^{ab}\delta g_{ab} = \delta q - 2(\bar\ell_a \delta\ell^a + \ell_a \delta\bar\ell^a). \qquad (3.22)$$

The part of the change in ℓ that lies in the normal plane to S and is not parallel to ℓ is given by $\ell_a \delta\ell^a$. We obtain

$$\ell_a \delta\ell^a = -\frac{1}{2}\ell^a \ell^b \delta g_{ab} = e^{\alpha - \bar\alpha}\delta\beta, \qquad (3.23)$$

so on B, $\ell_a \delta\ell^a$ encodes how much B changes away from being null. We will later fix this quantity to zero on B. Similarly, we get

$$\bar\ell_a \delta\bar\ell^a = -\frac{1}{2}\bar\ell^a \bar\ell^b \delta g_{ab} = -\frac{e^{\bar\alpha - \alpha}}{(1 + \beta\bar\beta)^2}\delta\bar\beta. \qquad (3.24)$$

In Eqs. (3.18) through (3.24), we have listed all possible projections of δg_{ab} with ℓ, $\bar\ell$ and q, and expressed them

using the variations $\delta\ell^a, \delta\bar\ell^a$ and δq_{AB}. We have also given them as variations of the parameters $(\alpha, \bar\alpha, \beta, \bar\beta, q_{AB}, A_i^A)$. The variations $(\delta q_{ab}, q^a{}_b \delta\ell^b, q^a{}_b \delta\bar\ell^b, \bar\ell_a \delta\ell^a, \ell_a \delta\bar\ell^a, \ell_a \delta\ell^a, \bar\ell_a \delta\bar\ell^a)$ are linearly independent, as can be seen from their expressions in metric parameters. Using the completeness relation Eq. (3.14), the metric variation can be expressed fully in terms of the variations we have given as

$$\begin{aligned}
\delta g_{ab} = {} & \delta q_{ab} - (\ell_a q_{bc} \delta\bar\ell^c + \ell_b q_{ac} \delta\bar\ell^c) \\
& - (\bar\ell_a q_{bc} \delta\ell^c + \bar\ell_b q_{ac} \delta\ell^c) \\
& - (\ell_a \bar\ell_b + \bar\ell_a \ell_b)(\bar\ell_c \delta\ell^c + \ell_c \delta\bar\ell^c) \\
& - 2\ell_a \ell_b(\bar\ell_c \delta\bar\ell^c) - 2\bar\ell_a \bar\ell_b(\ell_c \delta\ell^c).
\end{aligned} \qquad (3.25)$$

The change of normalization of ℓ is $\bar\ell_a \delta\ell^a$, and the change of the normalization of $\bar\ell$ is $\ell_a \delta\bar\ell^a$. They enter the metric variation only through the symmetric combination $\bar\ell_a \delta\ell^a + \ell_a \delta\bar\ell^a$. This is the variational expression of the fact that the boost gauge of ℓ and $\bar\ell$ is indeed pure gauge.

C. Extrinsic Geometry

The momenta conjugate to the metric are the extrinsic geometry of S. As it was the case with our variations, all of the extrinsic geometry is expressed in tensors intrinsic to S, which we push forward onto M. We will not give a complete list here, but just define the ones that will appear in our calculations.

The extrinsic curvature associated with ℓ is

$$\theta_\ell^{ab} := q^{ac} q^{bd} \nabla_c \ell_d = \frac{1}{2} q^{ac} q^{bd} \pounds_\ell q_{cd}. \qquad (3.26)$$

It describes how the induced two-metric changes along the vectors ℓ and is symmetric by Frobenius's theorem because ℓ is surface orthogonal to S. Its trace $\theta_\ell = q_{ab}\theta_\ell^{ab} = q^{ab}\nabla_a \ell_b$ is the expansion. It measures how the area element $\sqrt{q}$ on S changes along ℓ, corrected for the divergence of the coordinate lines $\sigma = $ const relative to ℓ and the normalization of ℓ, and can be written as

$$\sqrt{q}e^{\bar\alpha}\theta_\ell = \partial_a(\sqrt{q}D_0^a) - \beta\partial_a(\sqrt{q}D_1^a) \overset{B}{=} \partial_a(\sqrt{q}D_0^a) \qquad (3.27)$$

(see Appendix A). If the shift A_0^A is set zero, and the boost gauge $\bar\alpha = 0$ is chosen such that $\ell \overset{B}{=} \partial/\partial_{\phi^0}$, the last expression reduces on B to the usual $\sqrt{q}\theta = \partial_\ell \sqrt{q}$. The barred expansion is analogously defined as $\bar\theta_{\bar\ell} = q^{ab}\nabla_a \bar\ell_b$.

The tangential acceleration a_a is defined as

$$a_a := q_a{}^b \nabla_\ell \ell_b. \qquad (3.28)$$

It vanishes on B. That can be seen by writing $a_a = q_a{}^c \ell^b (\mathrm{d}\ell)_{bc}$ and then employing Frobenius's theorem or explicitly evaluating $a_a = e^{-\bar{\alpha}} q_a{}^b \nabla_b \beta \overset{B}{=} 0$. Since $a_a \overset{B}{=} 0$ and also $\ell^a \nabla_\ell \ell_a = 0$ because ℓ has constant modulus, we obtain that $\nabla_\ell \ell_a$ on B must be parallel to ℓ_a: ℓ is geodesic. The proportionality factor is the normal acceleration

$$\gamma := \bar{\ell}^a \nabla_\ell \ell_a. \tag{3.29}$$

Although $\bar{\ell}$ is, in general, not geodesic, we introduce the "barred" normal acceleration

$$\bar{\gamma} := \ell^a \nabla_{\bar{\ell}} \bar{\ell}_a. \tag{3.30}$$

Lastly, we introduce the twists η_a and $\bar{\eta}_a$, and the normal connection ω_a

$$\eta_a := -q_a{}^b \nabla_{\bar{\ell}} \ell_b,$$
$$\bar{\eta}_a := -q_a{}^b \nabla_\ell \bar{\ell}_b,$$
$$\omega_a := q_a{}^b \bar{\ell}^c \nabla_b \ell_c. \tag{3.31}$$

The combination $\eta - \bar{\eta}$ which computes the commutator $q_{ab}[\ell, \bar{\ell}]^b$ is essentially the curvature of the Diff(S)-connection,

$$\eta^a - \bar{\eta}^a = e^{-h}[D_0, D_1]^a. \tag{3.32}$$

This completes our geometrical setup. Let us note that under the boost transformations $(\ell, \bar{\ell}) \to (e^\epsilon \ell, e^{-\epsilon} \bar{\ell})$, the tensors $(\eta, \bar{\eta}, \theta_\ell, \bar{\theta}_{\bar{\ell}})$ transform covariantly to become $(\eta, \bar{\eta}, e^\epsilon \theta_\ell, e^{-\epsilon} \bar{\theta}_{\bar{\ell}})$, while the coefficients $(\gamma, \bar{\gamma}, \omega_a)$ transform inhomogeneously as connections and become $(e^\epsilon(\gamma + \nabla_\ell \epsilon), e^{-\epsilon}(\bar{\gamma} - \nabla_{\bar{\ell}} \epsilon), (\omega_a + q_a{}^b \nabla_b \epsilon))$. We now turn to our main task of evaluating the symplectic potential on a null hypersurface.

IV. THE SYMPLECTIC POTENTIAL ON A NULL HYPERSURFACE

The symplectic potential current integrated on B is

$$\Theta_B = -\int_B (e^{\bar{\alpha}} \Theta^a \ell_a) \mathrm{d}B = \frac{1}{2} \int_B (e^{\bar{\alpha}}(\nabla_\ell \delta g - \ell_a \nabla_b \delta g^{ab})) \mathrm{d}B, \tag{4.1}$$

where we have used the expression Eq. (2.4) for the symplectic current and our expression Eq. (3.17) for the pullback of the volume $(D-1)$-form. We also introduced the abbreviation $\mathrm{d}B = \sqrt{q} \mathrm{d}\phi^0 \wedge \mathrm{d}\sigma^2 \wedge \ldots \wedge \mathrm{d}\sigma^{D-1} = \mathrm{d}\phi^0 \wedge \mathrm{d}S$ for the volume form on B. $\mathrm{d}B$ depends on q and its variation is given by

$$\delta \mathrm{d}B = \frac{1}{2} \delta q \mathrm{d}B. \tag{4.2}$$

Let us first evaluate

$$-\Theta^a \ell_a = \frac{1}{2}(\nabla_\ell \delta g - \ell_a \nabla_b \delta g^{ab}) \tag{4.3}$$

using the decomposition of variations and the extrinsic geometry introduced in Sec. III.

A. Evaluation of $\Theta^a \ell_a$

The second term $-\frac{1}{2} \ell_a \nabla_b \delta g^{ab}$ of the last equation requires some work. We integrate it by parts, and using that $\delta g_{ab} \ell^b = \delta \ell_a - g_{ab} \delta \ell^b$ obtain

$$-\frac{1}{2} \ell_a \nabla_b \delta g^{ab} = \frac{1}{2}(\delta g_{ab} \nabla^a \ell^b + \nabla_a(\delta \ell^a - g^{ab} \delta \ell_b)). \tag{4.4}$$

Let us consider the last term of the last equation, and insert the completeness relation Eq. (3.14) inside the derivative.

$$\frac{1}{2} \nabla_a(\delta \ell^a - g^{ab} \delta \ell_b)$$
$$= \frac{1}{2} \nabla_a(q^a{}_b \delta \ell^b + \ell^a(\bar{\ell}_b \delta \ell^b + \ell_b \delta \bar{\ell}^b) + 2\bar{\ell}^a(\ell_b \delta \ell^b))$$
$$= \frac{1}{2}(\nabla_a(q^a{}_b \delta \ell^b) + (\bar{\ell}_b \delta \ell^b + \ell_b \delta \bar{\ell}^b)(\theta_\ell + \gamma)$$
$$\quad + 2(\ell_b \delta \ell^b)(\bar{\theta}_{\bar{\ell}} + \bar{\gamma})$$
$$\quad + \nabla_\ell(\bar{\ell}_b \delta \ell^b + \ell_b \delta \bar{\ell}^b) + 2\nabla_{\bar{\ell}}(\ell_b \delta \ell^b)). \tag{4.5}$$

In the first line we have used that the variations stays orthogonal to the surfaces S, i.e., $q^{ab} \delta \ell_b = 0$, for the second line, we have used $\nabla_a \ell^a = \theta_\ell + \gamma$ and $\nabla_a \bar{\ell}^a = \bar{\theta}_{\bar{\ell}} + \bar{\gamma}$.

The first term in Eq. (4.4) is $\delta g_{ab} \nabla^a \ell^b$. It is already of the form $P \delta Q$. To evaluate it, we insert the decomposition of the metric twice. Comparing with the projected variations and the definitions of extrinsic geometry from Sec. III, it becomes the sum of six terms which are not identically zero,

$$\delta g_{ab} q^{ac} q^{bd} \nabla_c \ell_d = \delta q_{ab} \theta_\ell^{ab}$$
$$\delta g_{ab} q^{ac} \ell^b \bar{\ell}^d \nabla_c \ell_d = -q^a{}_b \delta \ell^b \omega_a$$
$$\delta g_{ab} \ell^a \bar{\ell}^c q^{bd} \nabla_c \ell_d = q^a{}_b \delta \ell^b \eta_a$$
$$\delta g_{ab} \ell^a \bar{\ell}^c \ell^b \bar{\ell}^d \nabla_c \ell_d = 2\ell_a \delta \ell^a \bar{\gamma}$$
$$\delta g_{ab} \bar{\ell}^a \ell^c q^{bd} \nabla_c \ell_d = -q^a{}_b \delta \bar{\ell}^b a_a$$
$$\delta g_{ab} \bar{\ell}^a \ell^c \ell^b \bar{\ell}^d \nabla_c \ell_d = -(\bar{\ell}_a \delta \ell^a + \ell_a \delta \bar{\ell}^a)\gamma. \tag{4.6}$$

We have used that $q^{ab} \delta \ell_b = 0$ and that the remaining terms are zero because $\ell^a \nabla_b \ell_a = 0$. Adding this up yields

$$\frac{1}{2}\delta g_{ab}\nabla^a\ell^b = \frac{1}{2}(\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) - \delta\bar\ell^a a_a$$
$$- (\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)\gamma - 2\ell_a\delta\ell^a\bar\gamma). \quad (4.7)$$

We have dropped some projectors $q_a{}^b$ where they are unnecessary. Now all that is left to evaluate is the term $\frac{1}{2}\nabla_\ell\delta g$ in Eq. (4.3). Using Eq. (3.22), it becomes just

$$\frac{1}{2}\nabla_\ell\delta g = \frac{1}{2}\nabla_\ell(\delta q - 2(\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)). \quad (4.8)$$

We add everything up to obtain

$$-\Theta^a\ell_a = \frac{1}{2}(\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) + \theta_\ell(\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a)$$
$$+ \nabla_\ell(\delta q - (\bar\ell_a\delta\ell^a + \ell_a\delta\bar\ell^a))$$
$$+ \nabla_a(q^a{}_b\delta\ell^b) + 2\nabla_{\bar\ell}(\ell_a\delta\ell^a)$$
$$+ 2(\ell_a\delta\ell^a)(\bar\theta_{\bar\ell} + 2\bar\gamma) - \delta\bar\ell^a a_a). \quad (4.9)$$

Remembering that $\Theta_B = -\int_B(e^{\bar\alpha}\Theta^a\ell_a)dB$, this is our first version of the symplectic potential on a null hypersurface. It is obtained in a straightforward way by integrating by parts, inserting the completeness relation, and substituting the variations and extrinsic geometry we defined. We have not assumed $\delta\beta\overset{B}{=}0$ or $\beta\overset{B}{=}0$. This general form for $\Theta^a\ell_a$ was first derived in [26] and is new in this form in the published literature. It would be the starting point if we wanted to obtain the presymplectic form $\Omega_B = \delta\Theta_B$, without restricting to variations that keep B null,[3] and will be useful also in other contexts.[4] In this form, the result is not suited yet to read off the canonical pairs of gravity, since it still contains derivatives of variations. It is not of the form $P\delta Q$ with the configuration variable Q depending only on the geometry of B.

B. Splitting the Symplectic Potential into Bulk, Boundary and Total Variation

In the following, we will again restrict attention to metrics for which B is null and metric variations that keep the hypersurface B null, i.e., we set

$$\beta\overset{B}{=}0, \qquad \delta\beta\overset{B}{=}0. \quad (4.10)$$

This nullness condition means that we restrict attention to the submanifold of field space on which B is null, and consider only metric variations which are tangential to this submanifold. In other words, what is calculated in the

present paper is the pullback of the presymplectic potential Θ_B to the submanifold of field space where B is null. If B is part of the boundary of spacetime, fixing β and its variation is a partial boundary condition. For example, the asymptotic boundary conditions of asymptotically flat null infinity, or the condition that B is a horizon, contain the nullness condition.

The nullness condition is the only condition we impose on the metric and its variation. We believe that it does not remove a physical degree of freedom. The reason is that so far we have frozen the frame variations, and we expect the nullness condition to be implemented when we make ϕ^1 dynamical, allowing nonzero $\delta\phi^1$. We leave this to future work. Which other conditions on the metric are admissible is discussed in Sec. V.

Using those conditions, the expressions a_a and $\ell^a\delta\ell_a$ vanish on B (but not the transverse derivative $\nabla_{\bar\ell}(\ell^a\delta\ell_a)$ of the latter expression). The vanishing of a_a implies that the variation $q_{ab}\delta\bar\ell^b$ drops out of the symplectic structure. We see that since $q_{ab}\delta\bar\ell^b$ dropped out, the symplectic potential does not contain δA_1, see Eq. (3.20), hinting that A_1 is a gauge degree of freedom. We will comment on this later. These conditions therefore lead to a simpler expression for the symplectic potential,

$$\Theta_B = \frac{1}{2}\int_B e^{\bar\alpha}(\delta q_{ab}\theta_\ell^{ab} + \delta\ell^a(\eta_a - \omega_a) - \theta_\ell\delta h$$
$$+ \nabla_\ell(\delta q + \delta h) + \nabla_a(q^a{}_b\delta\ell^b) + 2\nabla_{\bar\ell}(\ell_a\delta\ell^a))dB. \quad (4.11)$$

We have written δh for $(-\bar\ell^a\delta\ell_a - \ell^a\delta\bar\ell_a)$.

This expression, while correct and expressed in terms of the extrinsic tensors, is not fully satisfactory yet for two reasons: It is not manifestly boost gauge invariant, and it still contains derivatives of variations. From Eq. (4.1) we see that the integrand of Θ_B is boost invariant because the combination $e^{\bar\alpha}\ell_a$ is, but in the equation above the invariance is not insured term by term. In order to achieve this it is worthwhile to notice that the combination $e^{\bar\alpha}\ell$ enters the symplectic potential in many instances. We therefore introduce the boost-invariant combination

$$L^a := e^{\bar\alpha}\ell^a, \qquad L^a\partial_a = D_0 + \beta D_1\overset{B}{=}D_0,$$
$$\delta L^a\overset{B}{=}q^a{}_b\delta L^b = \delta A_0^a. \quad (4.12)$$

We also denote its extrinsic curvature θ_L^{ab} simply by θ^{ab}, which is equal to $\theta^{ab} = e^{\bar\alpha}\theta_\ell^{ab}$. Now using the identity $\bar\eta_a - \omega_a = q_a{}^b\nabla_b\bar\alpha$, we can evaluate

$$e^{\bar\alpha}\nabla_a(q^a{}_b\delta\ell^b) = \nabla_a\delta L^a - (\bar\eta_a - \omega_a)\delta L^a. \quad (4.13)$$

We can also use that $e^{\bar\alpha}\nabla_{\bar\ell}(\ell_a\delta\ell^a)\overset{B}{=}\nabla_{\bar\ell}(e^{\bar\alpha}\ell_a\delta\ell^a)$ since $\ell_a\delta\ell^a\overset{B}{=}0$. The symplectic potential can then be written as

$$\Theta_B = \frac{1}{2} \int_B (\delta q_{ab} \theta^{ab} + \delta L^a (\eta_a - \bar{\eta}_a) - \theta \delta h$$
$$+ \nabla_L (\delta q + \delta h) + \nabla_a \delta L^a + 2\nabla_{\bar{\ell}} (e^{\bar{\alpha}} \ell_a \delta \ell^a)) dB. \tag{4.14}$$

In this form all the terms are now individually boost invariant. For the last term this is due to the fact that $\ell^a \delta \ell_a \overset{B}{=} 0$. We have also discovered that the most convenient boost gauge for the symplectic structure is $\bar{\alpha} = 0$ since it identifies $\ell = L$. Note that the induced metric Eq. (3.16) on B is determined by (q_{AB}, L^a).

The last term $\nabla_{\bar{\ell}} (e^{\bar{\alpha}} \ell_a \delta \ell^a)$ is still problematic though. Indeed even if $\ell_a \delta \ell^a$ vanishes on B, its derivative $\nabla_{\bar{\ell}} (\ell_a \delta \ell^a)$ does not, since the derivative is in a direction transverse to B. The challenge we are facing is to find a way to eliminate this transverse derivative. In the case where the boundary is spacelike a similar issue arises. In [23] it is shown that it is possible to eliminate the transverse derivative by including it in the variation of the densitized extrinsic curvature, which leads to the Gibbons-Hawking term. This is therefore exactly the strategy we are now going to follow: We show that it is possible to absorb the transverse derivative $\nabla_{\bar{\ell}} (e^{\bar{\alpha}} (\ell_a \delta \ell^a))$ into a total variation.

Using that $\delta \beta \overset{B}{=} 0$, we can evaluate that $\nabla_{\bar{\ell}} [e^{\bar{\alpha}} (\ell_a \delta \ell^a)] \overset{B}{=} D_1 (\delta \beta)$. Note that (even outside of B) the normal acceleration can be written as $\gamma = e^{-h} [D_0 e^{\alpha} + D_1 (e^{\alpha} \beta)]$. This suggests that we can extract from its variation the transverse derivative up to tangential derivatives. Before doing so, one has to remember that the normal acceleration transforms as a connection under boosts, while we want to keep boost invariance manifest. Under the rescaling $(\ell, \bar{\ell}) \rightarrow (e^{\epsilon} \ell, e^{-\epsilon} \bar{\ell})$, γ transforms as

$$\gamma \rightarrow e^{\epsilon} (\gamma + \nabla_{\ell} \epsilon). \tag{4.15}$$

This suggests to introduce the surface gravity which is the boost-invariant combination

$$\kappa := e^{\bar{\alpha}} (\gamma + \nabla_{\ell} \bar{\alpha}). \tag{4.16}$$

It is boost invariant since the transformation of $\bar{\alpha}$ and $\nabla_{\ell} \bar{\alpha}$ cancels the noninvariant terms in γ. It corresponds to the normal acceleration $\kappa = \bar{L}_a \nabla_L L^a$ of the vector $L = D_0 + \beta D_1$. Using metric parameters, the surface gravity κ can be written as

$$\kappa \overset{B}{=} D_0 h + D_1 \beta, \tag{4.17}$$

and is manifestly boost gauge invariant (see Appendix A).

In Appendix B, we calculate the total variation of the surface gravity on B for variations that preserve the nullness of B, i.e., such that $\delta \beta \overset{B}{=} 0$. It is given by

$$\delta \kappa \overset{B}{=} \delta L^a (\eta_a + \bar{\eta}_a) + \nabla_L \delta h + \nabla_{\bar{\ell}} (e^{\bar{\alpha}} \ell_a \delta \ell^a). \tag{4.18}$$

By using these results, we can now write down the symplectic potential in a form intrinsic to B which does not involve any transverse derivatives. It reads

$$\Theta_B = \frac{1}{2} \int_B (\delta q_{ab} \theta^{ab} - \delta L^a (\eta_a + 3\bar{\eta}_a) - \theta \delta h$$
$$+ \nabla_L (\delta q - \delta h) + \nabla_a \delta L^a + 2\delta \kappa) dB. \tag{4.19}$$

In order to finalize the expression we first need to integrate by parts the derivative along L, producing a total derivative. We use that for any ρ,

$$\sqrt{q} \nabla_L \rho \overset{B}{=} \partial_a [\sqrt{q} D_0^a \rho] - \sqrt{q} \theta \rho, \tag{4.20}$$

where we used that $L^a \overset{B}{=} D_0^a$ and that $\partial_a (\sqrt{q} D_0^a) = \sqrt{q} \theta$. We can also express divergences of vectors tangential to S as

$$\sqrt{q} \nabla_a \delta L^a = \partial_a (\sqrt{q} \delta L^a) + \delta L^a (\eta_a + \bar{\eta}_a). \tag{4.21}$$

These identities are proven in Appendix C. We also need to convert the last term into a total variation, using that $\delta(2\kappa dB) = (2\delta\kappa + \kappa \delta q) dB$. This gives us

$$\Theta_B = \frac{1}{2} \int_B (\delta q_{ab} \theta^{ab} - 2\delta L^a \bar{\eta}_a - (\kappa + \theta) \delta q) dB$$
$$+ \frac{1}{2} \int_{\partial B} (L^a (\delta q - \delta h) + \delta L^a) d_a S + \delta \left(\int_B \kappa dB \right). \tag{4.22}$$

Here we defined the directed volume element on ∂B as $d_a S := \iota_a dB$. In particular, in the foliation frame we have $d_0 S = dS$.

This expression is the sum of three terms, a bulk symplectic potential, a boundary symplectic potential and a total variation. The variational terms in the bulk symplectic potential only involve δq_{ab} and δL^a, which form the intrinsic geometry of B. In particular, we see that the term involving the variation δh has canceled from the bulk part. This term is still present in the boundary contribution of the symplectic potential. In order to remove it we introduce another total variation

$$-\delta h L^a d_a S = -\delta(h L^a d_a S) + \left[\delta L^a h + \frac{1}{2} h \delta q L^a \right] d_a S, \tag{4.23}$$

where we have used $\delta d_a S = \frac{1}{2} \delta q d_a S$. We get

$$\Theta_B = \frac{1}{2}\int_B (\delta q_{ab}\theta^{ab} - 2\delta L^a\bar{\eta}_a - (\kappa+\theta)\delta q)dB$$
$$+ \frac{1}{2}\int_{\partial B}\left(\left(1+\frac{1}{2}h\right)L^a\delta q + (1+h)\delta L^a\right)d_a S$$
$$+ \delta\left(\int_B \kappa dB - \frac{1}{2}\int_{\partial B} hL^a d_a S\right). \tag{4.24}$$

It turns out that the boundary and the total variation part of the symplectic potential can be written in a variety of different ways, all keeping with our principle that Θ_B should just contain variations of induced geometry and total variations. First, using $\sqrt{q}\theta = \partial_a(\sqrt{q}D_0^a)$ and $D_0 \stackrel{B}{=} L$, it is important to note that the expansion θ is a boundary term on B,

$$\int_B \theta dB = \int_{\partial B} L^a d_a S. \tag{4.25}$$

The variation of the last equation becomes

$$\delta\left(\int_B \theta dB\right) = \delta\left(\int_{\partial B} L^a d_a S\right) = \int_{\partial B}\left(\delta L^a + \frac{1}{2}\delta q L^a\right)d_a S. \tag{4.26}$$

We can thus rewrite the last expression as

$$\Theta_B = \frac{1}{2}\int_B (\delta q_{ab}\theta^{ab} - 2\delta L^a\bar{\eta}_a - (\kappa+\theta)\delta q)dB$$
$$+ \frac{1}{2}\int_{\partial B}\left(\frac{1}{2}hL^a\delta q + (h-1)\delta L^a\right)d_a S$$
$$+ \delta\left(\int_B (\theta+\kappa)dB - \frac{1}{2}\int_{\partial B} hL^a d_a S\right). \tag{4.27}$$

Also noting that $\nabla_a L^a = \theta + \kappa$, we see that in this form, the bulk total variation $A_B = \int_B(\theta+\kappa)dB$ is a close null analog of the Gibbons-Hawking term which features the divergence $K = \nabla_a n^a$ of the unit normal to the hypersurface. A boundary action of this form is given in [7].

Below, we write the boundary part in a different way that we expect to be more adapted to the study of boundary degrees of freedom, and separate the trace and the traceless part of δq_{ab}. Firstly, we again use $\delta(L^a d_a S) = [\delta L^a + \frac{1}{2}\delta q L^a]d_a S$. Secondly, note that the momenta for the trace $\delta q = \delta q_{ab}q^{ab}$ and the traceless part $\delta q_{\langle ab\rangle}$ of the variation of the induced metric on S have

different forms. It is therefore natural to split the induced metric into a conformal part and the volume element. We define the conformal induced metric on S, which has unit determinant

$$\tilde{q}_{ab} := |q|^{-\frac{1}{D-2}}q_{ab}. \tag{4.28}$$

Its variation $\delta\tilde{q}_{ab} = |q|^{-\frac{1}{D-2}}(\delta q_{ab} - \frac{1}{D-2}\delta q q_{ab})$ is traceless. Its momentum is the conformal shear

$$\tilde{\sigma}^{ab} = |q|^{\frac{1}{D-2}}\left(\theta^{ab} - \frac{1}{D-2}q^{ab}\theta\right) = -\frac{1}{2}q^a{}_{a'}q^b{}_{b'}\pounds_L\tilde{q}^{a'b'}, \tag{4.29}$$

which is also traceless, and captures the change of the conformal inverse metric $\tilde{q}^{ab} = |q|^{\frac{1}{D-2}}q^{ab}$ along L. Splitting the term $\delta q_{ab}\theta^{ab}$ into its trace and traceless parts then yields

$$\delta q_{ab}\theta^{ab} = \delta\tilde{q}_{ab}\tilde{\sigma}^{ab} + \frac{1}{D-2}\delta q\theta. \tag{4.30}$$

Lastly, we will substitute $\delta q = 2\delta\ln\sqrt{q}$ to produce an exact variation. These replacements give the symplectic potential as the sum of a bulk term, a boundary term, and the variations of a boundary action and a corner action,

$$\Theta_B = \Theta_B^{\text{bulk}} + \Theta_{\partial B} + \delta A_B + \delta a_{\partial B}, \tag{4.31}$$

where

$$\Theta_B^{\text{bulk}} = \int_B\left(\frac{1}{2}\delta\tilde{q}_{ab}\tilde{\sigma}^{ab} - \left(\frac{D-3}{D-2}\theta+\kappa\right)\delta\ln\sqrt{q} - \delta L^a\bar{\eta}_a\right)dB,$$

$$\Theta_{\partial B} = \frac{1}{2}\int_{\partial B}((1+h)L^a\delta\ln\sqrt{q} + h\delta L^a)d_a S,$$

$$A_B = \int_B \kappa dB,$$

$$a_{\partial B} = \frac{1}{2}\int_{\partial B}(1-h)L^a d_a S. \tag{4.32}$$

This is the final expression we are looking for. We analyze it in the next sections.

V. CANONICAL PAIRS

We now read off the null canonical pairs of gravity from Eq. (4.32), comparing with the schematic expression Eq. (2.3).

Bulk configuration		Bulk momentum	
Conformal metric:	$\tilde{q}_{ab}$	$\frac{1}{2}\tilde{\sigma}^{ab}$	Conformal shear
Normal vector:	L^a	$-\bar{\eta}_a$	Twist
Volume element:	$\ln\sqrt{q}$	$-(\kappa+\frac{D-3}{D-2}\theta)$	Expansion, surface gravity

$$\tag{5.1}$$

Note that what we call momenta are B densities PdB. The boundary canonical pairs can also be read off from Eq. (4.32) and are

$$\left(L^a, \frac{1}{2} h q_a{}^b d_b S\right) \quad \text{and} \quad \left(\ln \sqrt{q}, \frac{1}{2}(1+h)L^a d_a S\right).$$
(5.2)

We postpone their detailed analysis to a future publication.

We recover the surprising fact that the configuration variables in the bulk of B contain only variations of the induced metric Eq. (3.16) on B and no variations of the normal metric. That is analogous to the spacelike and timelike symplectic structure, and was not obvious from the outset. In the null case, also the configuration variables on the corner ∂B are a subset of the induced geometry. It is well known that the momentum conjugate to the conformal metric is the shear. More interesting and surprising are the spin-1 and spin-0 momenta $-\bar{\eta}_a$ and $-(\kappa + \frac{D-3}{D-2}\theta)$. Let us analyze them.

The spin-1 configuration variable is L. Since its ∂_{ϕ^0} component is fixed, its variation is $\delta L^a = \delta A_0^a$ and is purely tangential to S. The momentum $\bar{\eta}$ conjugate to L is given by $\bar{\eta}_a = -q_a{}^b \nabla_\ell \bar{\ell}_b$. Since $\bar{\ell}$ determines the orientation of S within B, $\bar{\eta}_a$ describes how the cross sections S of B tilt and twist when parallel-transporting them along ℓ. It can be expressed as the sum of two terms (see Appendix A),

$$\bar{\eta}_A = \frac{1}{2}(\partial_A h - F_A),$$
$$F_A := q_{AB} e^{-h}(\partial_0 A_1^B - \partial_1 A_0^B + [A_0, A_1]^B) \quad (5.3)$$

Here F_A measures the nonintegrability of the normal two-planes. Indeed it is equal to $q \cdot [L, \bar{L}]$, which vanishes only if the normal two-planes are integrable. The other term measures the rate of change of the redshift factor h along the cross section S.

Using the boost gauge $\bar{\alpha} = 0$, Damour [19] first interpreted $\omega_a = q_a{}^b \bar{\ell}^c \nabla_b \ell_c$ as a momentum density. He was motivated by the fact that for a cylindrically symmetric black hole, an integral of ω_a is the total angular momentum, and that in the Navier-Stokes–like equation $q_a{}^b L^c R_{bc} = 0$, ω_a plays the role of a linear momentum. However, ω_a is not boost gauge invariant and transforms as a connection under the boost gauge. The twist $\bar{\eta}$ is boost gauge invariant and coincides with ω in the boost gauge $\bar{\alpha} = 0$ since $\bar{\eta}_a - \omega_a = q_a{}^b \nabla_b \bar{\alpha}$. The twist $\bar{\eta}$ is thus the proper boost-gauge-invariant generalization of ω. In the light of the fluid interpretation of null surfaces, it is thus very natural that we found $\bar{\eta}$ as the momentum conjugate to the "displacement" A_0. We have confirmed Damour's interpretation of ω from a symplectic analysis.

The spin-0 momentum $-(\kappa + \frac{D-3}{D-2}\theta)$ conjugate to $\ln \sqrt{q}$ is a dimension dependent combination of the expansion θ

and the surface gravity κ. The surface gravity κ, which can be defined through $\nabla_L L_a = \kappa L_a$, is given as the sum of two terms,

$$\kappa = D_0 h + D_1 \beta.$$
(5.4)

We will see in more detail that this acceleration is the sum of an inertial acceleration term $D_0 h$ with h playing the role of a velocity, and a Newtonian acceleration term $D_1 \beta$ with β playing the role of the Newtonian potential. In the case of a nonexpanding null surface ($\theta = 0$) we recover the pair from black hole thermodynamics: the volume element $\sqrt{q}$ is conjugate to the surface gravity κ.

In Appendix A, expressions for the momenta are derived in terms of the metric parameters $(A_i^A, q_{AB}, h, \beta, \bar{\beta})$. The bulk momentum conjugated to the conformal metric is

$$\tilde{\sigma}^{AB} = \tilde{\theta}^{\langle AB \rangle}, \quad \text{with}$$
$$\tilde{\theta}^{AB} = -\frac{1}{2}\partial_0 \tilde{q}^{AB} + \frac{1}{2}(\tilde{q}^{AC}\partial_C A_0^B + \tilde{q}^{CB}\partial_C A_0^A - A_0^C \partial_C \tilde{q}^{AB})$$
(5.5)

where $\langle AB \rangle$ denotes the traceless part. Note that if one introduces δ_A to be the two-dimensional covariant derivative compatible with q_{AB} and denotes $\tilde{\delta}^A = \tilde{q}^{AB}\delta_B$, this can be also simply expressed as

$$\tilde{\theta}^{AB} = -\frac{1}{2}\partial_0 \tilde{q}^{AB} + \frac{1}{2}(\tilde{\delta}^A A_0^B + \tilde{\delta}^B A_0^A).$$
(5.6)

The first contribution comes from the time dependence of q_{AB} while the second contribution is analogous to the rate of strain $\partial^{(A} v^{B)}$ familiar in hydrodynamics if we identify A_0^A as the "velocity" $\dot{\sigma}^A$ of nonrotating observers on S which follow the integral curves of ℓ.

From the expressions for the momenta in terms of metric parameters, two transformations are apparent that are gauge in the sense that they are in the kernel of the symplectic form. Firstly, once we fix $\beta \overset{B}{=} 0$, the parameter A_1 enters the symplectic potential only through the curvature F_A [see also discussion after Eq. (4.10)]. More precisely, the only term in Θ_B that contains A_1 is the term $F_A \delta A_0^A$. It can be easily seen that this term is invariant under the gauge transformations that we introduced in Eq. (3.5),

$$\delta_V A_0^A = \partial_0 V^A + [A_0, V]^A, \qquad \delta_V A_1^A = \partial_1 V^A + [A_1, V]^A.$$
(5.7)

Since the transformation δ_V does not affect any other term in the symplectic potential, it is pure gauge. The transformation can be used to control A_1^A without affecting the rest of the analysis. Similarly, we see that the metric parameter $\bar{\beta}$, which was introduced in Eq. (3.9) and depends on the choice of the observers ϕ^0, does not appear

in the symplectic potential at all, neither in the configuration variation nor in the momentum variables. This is also a consequence of the fact that we have restricted to variations that keep B null, i.e., obey $\delta\beta = 0$. We can therefore choose $\bar{\beta}$ arbitrarily without affecting the analysis (as long as $\bar{\beta} > 0$), a fact we will exploit in the next section.

A. Normal frames, redshift factor and surface gravity

As we have just seen, the parameter $\bar{\beta}$ is at our disposal. Changing $\bar{\beta}$ can be achieved by a change of foliation that affects only the time foliation ϕ^0. This can be shown explicitly by considering a diffeomorphism parallel to D_0, associated with the foliation transformation

$$\delta_Y\phi^1 = 0, \qquad \delta_Y\phi^0 = e^{-h}Y, \qquad \delta_Y\sigma^A = e^{-h}YA_0^A. \quad (5.8)$$

We can check that h, β, and A_1 are unchanged on B if Y is chosen to vanish on B, which we now do. The only change of the normal metric then comes from a transformation of $\bar{\beta}$, given by $\delta_Y\bar{\beta} \overset{B}{=} e^{-h}D_1Y$. And we see that fixing $\bar{\beta}$ can be achieved by chosing the ϕ_0 foliation appropriately. There exist several special values for the parameter $\bar{\beta}$ that are of physical interest, because they give back generalizations of various coordinate systems for the Schwarzschild metric: Schwarzschild coordinates, Eddington-Finkelstein coordinates or Painlevé-Gullstrand coordinates.

Schwarzschild type coordinates are obtained by chosing $\beta^{-1} = \bar{\beta}$. In this case the normal metric is diagonal, and reads

$$H_{ij}d\phi^id\phi^j = e^h[-\beta(d\phi^0)^2 + \beta^{-1}(d\phi^1)^2]. \quad (5.9)$$

We can also choose Eddington-Finkelstein type coordinates[5] which are characterized by choosing $\bar{\beta} = a$, where a is a constant. In this case the normal metric reads

$$H_{ij}d\phi^id\phi^j$$
$$= e^h\left[-\frac{2\beta}{1+a\beta}d(\phi^0 + a\phi^1)^2 + d\phi^1 \otimes d(\phi^0 + a\phi^1)\right]. \quad (5.10)$$

In analogy to Eddington-Finkelstein coordinates, the vector D_1 is null for that metric, and the surfaces $\phi^0 + a\phi^1 = $ const are null.

Finally, Painlevé-Gullstrand type coordinates are obtained when one chooses $\bar{\beta}$ such that $g^{-1}(d\phi^0, d\phi^0) = -1$, i.e.,

$$\bar{\beta} = \frac{e^h}{(2 - \beta e^h)}. \quad (5.11)$$

For this choice the normal metric[6] corresponds to

$$H_{ij}d\phi^id\phi^j = -(d\phi^0)^2 + e^{2h}(d\phi^1 - (\beta - e^{-h})d\phi^0)^2. \quad (5.12)$$

It is easy to see that as for Painlevé-Gullstrand coordinates, the family of observers $v = -g^{-1}(d\phi^0)$ follow affinely parametrized geodesics, with proper time ϕ^0. Let us analyze this frame in more detail.

The velocity of the free-falling observers v is explicitly given by

$$v = D_0 + (\beta - e^{-h})D_1. \quad (5.13)$$

In other words, restricting to the normal plane of S the velocity of free-falling observers is given by $v_{FF} = \dot{\phi}^1 = \beta - e^{-h}$. We see that all the observers that have $\beta < e^{-h}$ are radially moving inwards. In this frame, light orthogonal to S travels along outgoing or ingoing curves respectively given by

$$v_+ = \dot{\phi}^1 = \beta, \qquad v_- = \dot{\phi}^1 = (\beta - 2e^{-h}), \quad (5.14)$$

where v_+ denotes the outgoing velocity and v_- the ingoing light rays respectively. This means that the speed of light as measured by the freely falling observers v in this geodesic frame is given by $\pm e^{-h}$. In particular we have that

$$v_\pm = \frac{1}{2}(v_{FF} \pm c), \qquad c = e^{-h}. \quad (5.15)$$

We can thus interpret the speed of light $c = e^{-h}$ as encoding the redshift. More precisely, let us fix the remaining freedom in the foliation ϕ^1 to set $\beta = 0$ everywhere. Then, the surfaces $\phi^1 = $ const are null, and ϕ^0 is the proper time of the geodesic observers $v = -g^{-i}(d\phi^0)$. This frame is called the geodesic light cone frame, and is used in cosmology in order to define cosmological averaging [29,30]. The normal metric then reads

$$H_{ij}d\phi^id\phi^j = e^hd\phi^0 \otimes d\phi^1 + e^{2h}(d\phi^1)^2. \quad (5.16)$$

In this frame, the static observers are moving at the speed of outgoing light. In this frame, the frequency of light the observer measures is given by the (negative) scalar product between the affinely parametrized null generator $\ell \overset{B}{=} g^{-1}(d\phi^1)$ and the velocity v of geodesic observers, up to a normalization that is constant on each light ray,

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[5]In order to get the Schwarzschild black hole metric in Eddington-Finkelstein coordinates we need to choose $v = \phi^0 + a\phi^1$ as an ingoing null coordinate, $r = \phi^1$ as a radial coordinate, and set $h = 0$ and β such that $(2\beta)/(1 + a\beta) = 1 - 2GM/r$.

[6]To recover Schwarzschild with $\phi^1 = r$ we need to impose $h = 0$ and $\beta = 1 - \sqrt{2GM/r}$.
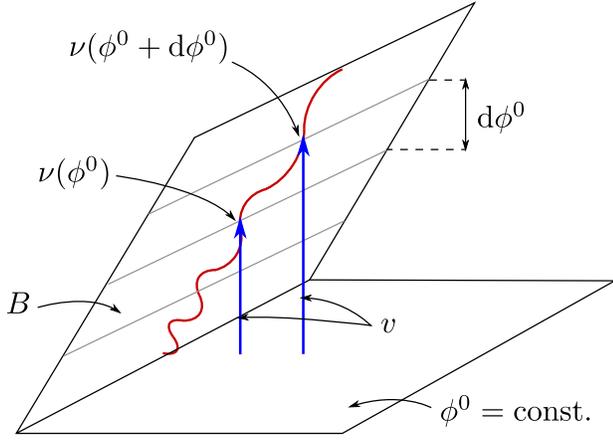
FIG. 3. The observer $v = -g^{-1}(\mathrm{d}\phi^0)$ crosses the null surface B, and measures the frequency ν of light rays propagating along B. The redshift, i.e., the relative change of frequency, per unit time ϕ^0 is given in the geodesic light cone frame by $z = (\nu(\phi^0) - \nu(\phi^0 + \mathrm{d}\phi^0))/\nu(\phi^0) = D_0 h \cdot \mathrm{d}\phi^0$

$$\nu = -\ell^a v_a = e^{-h}. \tag{5.17}$$

The redshift between source and observer is then given by

$$(1 + z) := \frac{\nu_s}{\nu_o} = e^{h_0 - h_s}. \tag{5.18}$$

We see that e^{-h} indeed encodes the redshift. That justifies the term "redshift factor" for h.

The surface gravity κ, which appears in the spin-0 momentum and the boundary action, is the sum of two terms,

$$\kappa = D_0 h + D_1 \beta. \tag{5.19}$$

By a Newtonian analogy we can see this as the sum of an inertial acceleration term $D_0 h$ and a Newtonian acceleration term $D_1 \beta$, which we will interpret in different frames. To interpret the term $D_0 h$, we again use the geodesic light cone frame. $D_0 h$ is the relative change of frequency per unit time, or the infinitesimal redshift, along a light ray (see Fig. 3). Loosely speaking, it is thus the acceleration of a lightlike observer following lightrays perpendicular to S, relative to the geodesic observer v. We can then interpret it as the inertial acceleration term for that observer.

To interpret the term $D_1 \beta$, let us start from Painlevé-Gullstrand–type coordinates, and use the freedom in the foliation field ϕ^1 to set $h = 0$. Then, from Eq. (5.15) we see that the speed of light is 1 in that frame. We call this frame the Galilean light frame. On B, the acceleration of the geodesic observer v relative to the static observer in the frame (ϕ^0, ϕ^1) is then given by

$$v^a \partial_a v^1 = -D_1 \beta. \tag{5.20}$$

This is the expression for a radial acceleration with the radial coordinate ϕ^1, with β taking the role of the Newtonian potential. It is consistent with the standard Newtonian limit of general relativity because $\beta = -\frac{1}{2} g_{00}$ in the Galilean light frame.

VI. LAGRANGIAN BOUNDARY TERMS

In the expression Eq. (4.31) for the symplectic potential, we have extracted a total variation from the symplectic potential. That corresponds to a choice of polarization: It tells us which are the configuration and which the momentum variables. This can be seen most clearly by noting that such a total variation can be used to interchange to roles of configuration and momentum variables,

$$\Theta = P\delta Q = -Q\delta P + \delta(PQ). \tag{6.1}$$

The choice of polarization we have made is that the configuration variables Q should not contain derivatives of the metric. From Eq. (2.2) one sees that the addition of a boundary term in the Lagrangian will modify the symplectic potential Θ by a total variation (up to the spacetime closed ambiguity),

$$L \to L + \mathrm{d}A \Rightarrow \Theta \to \Theta + \delta A. \tag{6.2}$$

The total variation in $\Theta_B = \Theta_B^{\text{bulk}} + \Theta_{\partial B} + \delta(A_B + a_{\partial B})$ can thus be canceled by adding a boundary term and a corner term to the action,

$$S = \int_M L - A_B - a_{\partial B} \tag{6.3}$$

From Eq. (4.32), we therefore make the following suggestion for the action of a spacetime region with null boundaries which may possess corners:

$$S = \frac{1}{2}\int_M R\epsilon - \int_B \kappa \mathrm{d}B - \frac{1}{2}\int_{\partial B}(1 - h)L^a \mathrm{d}_a S. \tag{6.4}$$

Note that the corner term vanishes for segments of ∂B that contain the null direction L since in that case $L^a \mathrm{d}_a S = 0$.

A similar line of reasoning to that of this section was followed in the recent paper by Lehner *et al.* [9], and a proposal for the boundary and corner action for null boundaries was given. We have thus reproduced one of the results there with a different calculation.[7] There, the null surface B is taken to have cylindrical topology, with past and future boundaries that are members of the family of surfaces S. The boost gauge $\bar{\alpha} = 0$ is chosen, and the coordinates σ are chosen to be constant along the null generators of B, fixing $A_0 = 0$ and $\delta A_0 = 0$ from the outset.

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[7]Up to the summand 1 in the corner term, which is a choice of corner polarization, see discussion after Eq. (4.24).

Our calculations confirm that $\bar{\alpha} = 0$ is the most convenient gauge because in that gauge the boundary action κ is indeed the inaffinity of the null generators. Furthermore, we have disentangled the normalizations of the configuration variable L and the null generators ℓ, and allowed for general topologies of B.

Parattu *et al.* [8] also gave a suggestion for the boundary action and the canonical structure. They mostly work in the boost gauge $\alpha = 0$, and extract a total variation containing the normal acceleration γ rather than the surface gravity κ. As can be seen from Eq. (4.16), γ and κ are inequivalent unless $\bar{\alpha} = 0$. For that reason Parattu *et al.* obtain an extra canonical pair on B, which contains a piece of normal geometry as a configuration variable. As we have seen, that pair can be removed by choosing κ rather than γ in the total variation extracted.

VII. CONCLUSION

We have set up a robust and intuitive framework for dealing with the geometry of null hypersurfaces and variational quantities on them. We used it to find the null canonical pairs of gravity from the symplectic potential current Θ, without introducing any gauge fixing, and gave definitive answers especially for the spin-1 and spin-0 degrees of freedom. Included in our analysis are degrees of freedom on the boundary of the null hypersurface, about which we will have more to say in a future publication [18]. Our calculations also yielded a boundary action that includes corner terms.

Two areas come to mind where the technology and results we found can be applied. This includes, firstly,

understanding the "soft graviton modes" of the BMS group at asymptotically flat null infinity (see, e.g., [17]), which we expect to be related to the spin-0 and spin-1 degrees of freedom and boundary degrees of freedom. Secondly, the symplectic potential controls the flow of information. We will use this intuition to define a symplectic notion of informational horizon. That will involve going on shell and comparing to other notions of horizons, such as isolated horizons [31], and may provide insight on informational quantities such as the Bousso bound [14].

APPENDIX A: EXTRINSIC GEOMETRY EXPRESSED IN METRIC PARAMETERS

We relate the extrinsic geometry to the derivatives of metric parameters. The versions of these identities which hold true on B where used in Sec. IV. We start with the normal acceleration which is the most involved expression,

$$\gamma = \bar{\ell}^a \nabla_\ell \ell_b = \bar{\ell}^a (\pounds_\ell \ell)_a = \nabla_\ell \alpha + e^{\alpha - \bar{\alpha}} \nabla_{\bar{\ell}} \beta + \frac{\bar{\beta}}{1 + \beta\bar{\beta}} \nabla_\ell \beta$$

$$= e^{-\bar{\alpha}}[(D_0 + \beta D_1)\alpha + D_1\beta] = e^{-\bar{\alpha}-\alpha}[D_0 e^\alpha + D_1(e^\alpha \beta)] \stackrel{B}{=} \nabla_\ell \alpha + e^{\alpha - \bar{\alpha}} \nabla_{\bar{\ell}} \beta. \tag{A1}$$

From this we can evaluate the surface gravity

$$\kappa := e^{\bar{\alpha}}(\gamma + \nabla_\ell \bar{\alpha}) = (D_0 + \beta D_1)h + D_1\beta. \tag{A2}$$

The tangential acceleration a, the twists $(\eta, \bar{\eta})$ and normal connection ω are given by

$$a_a = q_a{}^b \nabla_\ell \ell_b = q_a{}^b (\pounds_\ell \ell)_b = e^{-\bar{\alpha}} q_a{}^b \nabla_b \beta \stackrel{B}{=} 0$$

$$\eta_a + \omega_a = -q_a{}^b \nabla_{\bar{\ell}} \ell_b + q_a{}^b \bar{\ell}^c \nabla_b \ell_c = -q_a{}^b (\pounds_{\bar{\ell}} \ell)_c = q_a{}^b \left(\nabla_b \alpha + \frac{\bar{\beta}}{1 + \beta\bar{\beta}} \nabla_b \beta \right) \stackrel{B}{=} q_a{}^b \nabla_b \alpha$$

$$\bar{\eta}_a - \omega_a = -q_a{}^b \nabla_\ell \bar{\ell}_b + q_a{}^b \ell^c \nabla_\ell \bar{\ell}_c = -q_a{}^b (\pounds_\ell \bar{\ell})_b = q_a{}^b \left(\nabla_b \bar{\alpha} - \frac{\bar{\beta}}{1 + \beta\bar{\beta}} \nabla_a \beta \right) \stackrel{B}{=} q_a{}^b \nabla_b \bar{\alpha}$$

$$\eta_a - \bar{\eta}_a = -q_a{}^b \nabla_{\bar{\ell}} \ell_b + q_a{}^b \nabla_\ell \bar{\ell}_b = q_{ab}[\ell, \bar{\ell}]^b = e^{-h} q_{ab}[D_0, D_1]^a. \tag{A3}$$

These identities are proven by inserting the parametrizations Eqs. (3.9) and (3.12), and executing the Lie derivatives. Linear combinations of the last three identities yield

$$\eta_a + \bar{\eta}_a = q_a{}^b \nabla_b h,$$

$$\bar{\eta}_a = \frac{1}{2}(q_a{}^b \nabla_b h - q_{ab} e^h [D_0, D_1]^b). \quad (A4)$$

In Sec. IV, we used the identity $\partial_a(\sqrt{q} D_0^a) = \sqrt{q}\theta$. Let us prove it. First, evaluate

$$\nabla_a L^a = (q^{ab} + \bar{\ell}^a \ell^b + \ell^a \bar{\ell}^b)\nabla_a L_b$$
$$= \theta_L + \bar{\ell}^a \nabla_\ell L_a = \theta + \kappa. \quad (A5)$$

Let us evaluate the same object now using the relationship between covariant and regular derivative and that $L^a = D_0 + \beta D_1$,

$$\nabla_a L^a = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}L^a) = \frac{e^{-h}}{\sqrt{q}}\partial_a(\sqrt{q}e^{\alpha+\bar{\alpha}}L^a)$$

$$= \frac{1}{\sqrt{q}}\partial_a(\sqrt{q}L^a) + \nabla_L h$$

$$= \frac{1}{\sqrt{q}}\partial_a(\sqrt{q}(D_0^a + \beta D_1^a)) + \kappa - D_1\beta$$

$$= \frac{1}{\sqrt{q}}\partial_a(\sqrt{q}D_0^a) + \frac{\beta}{\sqrt{q}}\partial_a(\sqrt{q}D_1^a) + \kappa. \quad (A6)$$

Comparing Eqs. (A6) and (A5) gives what we wanted to show,

$$\theta = \frac{1}{\sqrt{q}}\partial_a(\sqrt{q}D_0^a) + \frac{\beta}{\sqrt{q}}\partial_a(\sqrt{q}D_1^a) \overset{B}{=} \frac{1}{\sqrt{q}}\partial_a(\sqrt{q}D_0^a).$$
$$(A7)$$

Lastly, the bulk momentum for the conformal metric $\tilde{q}_{ab}$ is the conformal shear, which is the traceless part of the expansion of the conformal metric,

$$\tilde{\sigma}^{AB} = \tilde{\theta}^{\langle AB \rangle}, \quad \text{where}$$

$$\tilde{\theta}^{AB} = -\frac{1}{2}\partial_0\tilde{q}^{AB} + \frac{1}{2}(\tilde{q}^{AC}\partial_C A_0^B + \tilde{q}^{CB}\partial_C A_0^A - A_0^C \partial_C \tilde{q}^{AB}).$$
$$(A8)$$

The shear can also be written as $\tilde{\sigma}^{AB} = |q|^{1/(D-2)}\theta^{\langle AB \rangle}$, with $\theta^{AB} = -\frac{1}{2}\partial_0 q^{AB} + \frac{1}{2}(q^{AC}\partial_C A_0^B + q^{CB}\partial_C A_0^A - A_0^C \partial_C q^{AB})$.

APPENDIX B: CALCULATION OF THE VARIATION OF THE SURFACE GRAVITY

Let us evaluate the total variation $\delta\kappa$ that was used in Eq. (4.18). Using the coordinate expression for κ

given in the previous appendix and assuming $\delta\beta \overset{B}{=} 0$, we obtain

$$\delta\kappa \overset{B}{=} \delta(D_0 h + D_1\beta). \quad (B1)$$

We distribute the variation, and use that $\delta\beta \overset{B}{=} 0$, that the variations $\delta D_i^a = \delta A_i^a$ are purely tangential to S, that $q_a{}^b \nabla_b h = \eta_a + \bar{\eta}_a$ and that $q_a{}^b \nabla_b \beta = 0$,

$$\delta\kappa \overset{B}{=} D_0 \delta h + \delta A_0^a(\eta_a + \bar{\eta}_a) + D_1\delta\beta. \quad (B2)$$

Substituting $\delta L^a \overset{B}{=} \delta A_0^a$, $\ell_a \delta\ell^a = e^{\alpha-\bar{\alpha}}\delta\beta$ and the coordinate expressions for L and $\bar{\ell}$ yields

$$\delta\kappa \overset{B}{=} \nabla_L \delta h + \delta L^a(\eta_a + \bar{\eta}_a) + \nabla_{\bar{\ell}}(e^{\bar{\alpha}}\ell_a \delta\ell^a). \quad (B3)$$

That is the expression we used.

APPENDIX C: CALCULATION OF INTEGRATION BY PARTS

We prove identities that we used in Sec. IV B to integrate by parts in Θ_B, producing boundary terms on ∂B. We first use that for any vector V

$$\sqrt{q}\nabla_a V^a = \sqrt{|g|}e^{-h}\nabla_a V^a = e^{-h}\partial_a(\sqrt{|g|}V^a)$$
$$= e^{-h}\partial_a(\sqrt{q}e^h V^a)$$
$$= \partial_a(\sqrt{q}V^a) + \sqrt{q}V^a\partial_a h. \quad (C1)$$

If $V^a = q^a{}_b V^b$ is a tangential vector to S, this means that

$$\sqrt{q}\nabla_a V^a = \partial_a(\sqrt{q}V^a) + \sqrt{q}V^a(\eta_a + \bar{\eta}_a). \quad (C2)$$

If on the other hand we take $V^a = \rho L^a$, we obtain the identity

$$\sqrt{q}\nabla_L \rho = \sqrt{q}L^a\nabla_a\rho \overset{B}{=} \sqrt{q}D_0^a\nabla_a\rho$$
$$= \sqrt{q}\nabla_a(D_0^a\rho) - \rho\sqrt{q}\nabla_a D_0^a]$$
$$= \partial_a(\sqrt{q}D_0^a\rho) - \rho\partial_a(\sqrt{q}D_0^a)$$
$$= \partial_a(\sqrt{q}D_0^a\rho) - \sqrt{q}\theta, \quad (C3)$$

where we used that $L^a \overset{B}{=} D_0^a$ and that $\partial_a(\sqrt{q}D_0^a) = \sqrt{q}\theta$.

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