Consistent cosmic bubble embeddings

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The Raychaudhuri equation for null rays is a powerful tool for finding consistent embeddings of cosmological bubbles in a background spacetime in a way that is largely independent of the matter content. We find that spatially flat or positively curved thin wall bubbles surrounded by a cosmological background must have a Hubble expansion that is either contracting or expanding slower than the background, which is a more stringent constraint than those obtained by the usual Israel thin-wall formalism. Similarly, a cosmological bubble surrounded by Schwarzschild space, occasionally used as a simple "swiss cheese" model of inhomogenities in an expanding universe, must be contracting (for spatially flat and positively curved bubbles) and bounded in size by the apparent horizon.

I. INTRODUCTION

Cosmic bubbles—spherically symmetric homogeneous and isotropic spacetimes surrounded by a spherically symmetric background spacetime—can arise in many interesting applications of general relativity, including local inflation, eternal inflation, cosmological phase transitions, and gravitational collapse. In these scenarios, spacetime is separated into two separate manifolds $\mathcal{M}^+$ and $\mathcal{M}^-$, with their own distinct metrics, which are typically joined across a "thin wall" hypersurface[1] using the Israel junction conditions [2].

The dynamics of such bubbles can be extremely complicated, depending on the matter content of the background and interior of the bubble as well as the tension on the bubble wall and how it interacts with the surrounding matter. The most well-understood solutions involve simple assumptions, such as the dust-collapse model of Oppenhemier-Snyder [3] or false vacuum de Sitter bubbles in empty space [4,5]. Some results have also been obtained for interior and exterior spacetimes described by homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) spaces [6–10]. See also [11–17] for other investigations of cosmic bubble dynamics.

More recently, there has been renewed interest in the onset of local inflation arising from inhomogeneous initial conditions, in which inflation starts in a small patch surrounded by a noninflating background. Earlier analytical arguments and numerical work suggested that the inflationary patch will not grow unless the initial size of the homogeneous patch is larger than the Hubble length (see e.g. [18–21] and the recent review [22]). These results have been revisited by recent works [23,24], which use modern numerical relativity codes to study the conditions under which local inflation begins. While these numerical analyses are not necessarily spherically symmetric, inflation will act to homogenize and isotropize the spacetime inside the bubble, so we can view these models as cosmic bubbles embedded into a larger spacetime.

The dynamics of cosmic bubbles appear to be strongly model dependent, so it has been difficult to make general statements about their behavior. In this paper, we will consider cosmic bubbles from a different angle, by using the null Raychaudhuri equation to study the consistency of cosmic bubble embeddings. While we will not be able to derive dynamical equations of motion for the bubble wall—which arise from the Israel boundary conditions—we will nonetheless find interesting constraints from the consistency of the null Raychaudhuri equation for the propagation of null rays across the bubble wall. The Raychaudhuri equation is particularly powerful because it is independent of specific solutions to the Einstein equations, and it has been used before [25,26] to study the consistency of embeddings of local inflation.

In particular, we will be interested in the behavior of the expansion θ of radial, inwardly directed, future-oriented null rays N^α. For an affinely parametrized null tangent vector $N^\alpha \nabla_\alpha N^\beta = 0$, the expansion $\theta = \nabla_\alpha N^\alpha$ satisfies the null Raychaudhuri equation,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - |\sigma|^2 - R_{\alpha\beta}N^\alpha N^\beta, \qquad (1)$$

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[1]The assumption of an infinitesimally thin bubble wall does not apply to phase transition bubbles formed in new inflation [1].

where λ is an affine parameter along the geodesic and σ is the shear tensor. The Raychaudhuri equation (1) arises from the geometric properties of null vectors, and as such is independent of the Einstein equations and their solutions. Imposing the Einstein equations on the last term of (1), we have

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - |\sigma|^2 - T_{\alpha\beta}N^\alpha N^\beta. \qquad (2)$$

If we assume that the matter inside and outside of the bubble as well as on the bubble wall obeys the null energy condition (NEC) $T_{\alpha\beta}N^\alpha N^\beta \geq 0$, the Raychaudhuri equation (2) implies that the expansion must be nonincreasing,

$$\frac{d\theta}{d\lambda} \leq 0. \qquad (3)$$

As a null ray traverses the bubble wall boundary Σ from the background spacetime into the bubble, as shown in Fig. 1, the value of θ will, in principle, change. In order for the bubble embedding to be consistent with the Raychaudhuri equation (3), the value of θ must not increase across the wall,

$$\Delta\theta = \theta^- - \theta^+ \leq 0. \qquad (4)$$

It is common to study cosmic bubbles in the "thin wall" limit in which the boundary is infinitesimally thin, for which (4) must be true when evaluated at the (singular) boundary; however, (4) must also be satisfied for "thick wall" bubbles as well, as a null ray leaves the background and enters the bubble.

The Raychaudhuri equation (2) is independent of specific solutions to the Einstein equations, and thus (4) must be satisfied for any inwardly directed radial null ray, independent of the details of the matter content of the background and bubble spacetimes as well as the bubble
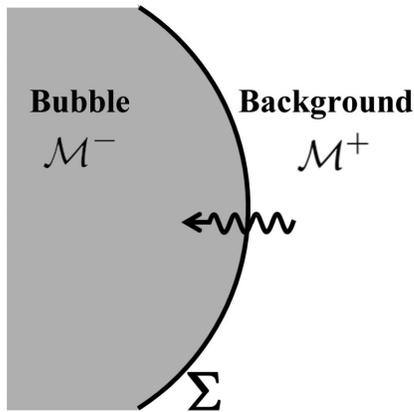

FIG. 1. We will be considering radially ingoing null rays as they cross the boundary Σ from a spherically symmetric background into an embedded cosmological bubble.

wall, as long as such matter obeys the NEC. In order to evaluate the constraints implied by (4), we need to evaluate the expansion θ on the bubble and background spacetimes. There are a number of possible choices for these spacetimes, depending on the specific details of the bubble wall and matter content as well as the bubble embedding structure.

In order to make progress, we will consider in this paper two simple choices for the bubble and background spacetimes: first, we will consider both bubble and background spacetimes to be described by distinct homogeneous and isotropic FLRW metrics. Then we will consider the case in which the bubble interior is a FLRW spacetime, while the background is Schwarzschild spacetime, as a "swiss cheese" model of the bubble inhomogeneity. In both cases, the Raychaudhuri equation in the form (4) will allow us to constrain the allowed bubble embeddings.

II. A BUBBLE IN A COSMOLOGICAL BACKGROUND

We will begin by considering a spatially flat homogeneous and isotropic cosmological bubble embedded at the origin of a homogeneous and isotropic cosmological background, as in [6–10]; nonspatially flat bubble and background spacetimes will be considered in Sec. II B. Both spacetimes are described by the FLRW metric,

$$ds_\pm^2 = -dt_\pm^2 + a_\pm(t_\pm)^2 dr_\pm^2 + a_\pm(t_\pm)^2 r_\pm^2 d\Omega^2, \qquad (5)$$

where $\pm$ refers to the background/bubble spacetimes, respectively. The choice of a homogeneous FLRW background represents an assumption of vanishing (or small) gravitational backreaction of the bubble, as in the spontaneous formation of a bubble through a phase transition [27] or the "detonation wave" approximation in which the energy density of the background is converted into energy of the bubble interior on the bubble wall, leading to vanishing surface energy density [7,8]. However, we will also allow for the presence of a thick wall, which can presumably capture some of the local backreaction of the bubble on the background for more general bubbles, and derive constraints on the required thickness of such a backreacted interpolating region.

The (timelike) boundary upon which these spacetime regions are joined is the wall of the bubble Σ. The metric must be continuous across the bubble wall,[2] which implies that the coefficient of $d\Omega^2$ in the metric must be continuous across the boundary $a_+(t_+)r_+|_\Sigma = a_-(t_-)r_-|_\Sigma \equiv R(\tau)$, where τ is the proper time of the wall. Since the bubble and background spacetimes have (in principle) different cosmological evolutions, the comoving radial coordinates $r_\pm$ cannot be continuous across the wall. The "areal radius"

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[2]See [9,10] for an analysis of the Israel junction conditions across the boundary.

$\tilde{r} \equiv a(t)r$, however, will be continuous across the wall, and we will set $\tilde{r}_+|_\Sigma = \tilde{r}_-|_\Sigma = R$ on the bubble wall.

Radial, inwardly directed, future-oriented null rays in the cosmological spacetimes (5) are described by the (affine) null tangent vector

$$N_\pm^\alpha = (a_\pm^{-1}, -a_\pm^{-2}, 0, 0) \qquad (6)$$

in either spacetime. The expansion of these rays $\theta^\pm = \nabla_a N_\pm^\alpha$ is then

$$\theta^\pm = \frac{2}{a_\pm}\left(H_\pm - \frac{1}{\tilde{r}}\right). \qquad (7)$$

Following [28,29], (7) defines an *apparent horizon* for the bubble and background spacetimes where $\theta^\pm = 0$,

$$\tilde{r}_{AH}^\pm = H_\pm^{-1}. \qquad (8)$$

Null rays that are at an areal distance from the origin that is smaller than the radius of the apparent horizon $\tilde{r} < \tilde{r}_{AH}^\pm$ have negative expansion, as is expected for converging rays. However, null rays that are at an areal distance larger than the radius of the apparent horizon $\tilde{r} > \tilde{r}_{AH}^\pm$ have a positive divergence, indicating that the expansion of space is overcoming the expected geometric convergence of the rays.

A. Crossing the bubble wall

In the limit of an infinitesimally thin bubble wall, as a null ray crosses the boundary the radial coordinate is continuous $\tilde{r}_\pm|_\Sigma = R$ but the scale factor $a_\pm$ and Hubble expansion rate $H_\pm$ are not. Thus, the requirement that θ must decrease (4) becomes

$$\Delta\theta = \frac{2}{a_-}\left(H_- - \frac{1}{R}\right) - \frac{2}{a_+}\left(H_+ - \frac{1}{R}\right) \le 0. \qquad (9)$$

The metrics (5) contain an ambiguity: it is always possible to make a simultaneous rescaling of the scale factor and comoving radial coordinate by a constant $a_\pm \to \lambda_\pm a_\pm, r_\pm \to \lambda_\pm^{-1} r_\pm$ that leaves the metric and physical distances invariant. We will use this freedom to fix the bubble and background scale factors to be equal to one at the time of the null ray wall crossing $t_{\pm,\text{cross}}$ for the respective spacetimes only, e.g. $a_+(t_{+,\text{cross}}) = a_-(t_{-,\text{cross}}) = 1$. Since the matter content of the background and bubble generically differs, this implies that the scale factors should typically not be the same at any other time. Utilizing this freedom, we can simplify (9) to

$$\Delta\theta = \theta^- - \theta^+ = 2(H_- - H_+) \le 0. \qquad (10)$$

This constraint has important implications for bubble embeddings described by (5).

If the background is expanding but the bubble spacetime is collapsing $H_- < 0$, then we are able to satisfy 10)) without any difficulty, indicating that a collapsing bubble inside an expanding background is always an allowed solution. Alternatively, if the background spacetime is collapsing, then (10) indicates that the bubble spacetime cannot be expanding while still satisfying the Raychaudhuri equation. Since this is of little practical cosmological interest for us, however, we will not consider this situation further.

If both the bubble and the background spacetimes are expanding $H_\pm > 0$, then (10) is only satisfied for a bubble that has a lower Hubble rate than the background $H_- < H_+$, indicating that the energy density of the bubble is smaller than the background $\rho_- < \rho_+$. Some simple examples include a true vacuum bubble embedded inside a false vacuum background, or a low temperature bubble inside a higher temperature background. However, (10) fails for a bubble that has a higher Hubble rate than the background $H_- > H_+$, including false vacuum or high temperature bubbles. This further extends the results of [9,10], which found that the Israel junction conditions do not allow for a spatially flat bubble embedded in the spacetime (5) with $H_- > H_+$ unless the bubble is superhorizon sized. We find a stronger result here, which is that a spatially flat bubble with $H_- > H_+$ is not allowed for any size bubble. It is interesting that this result is independent of the details of the matter content inside and outside of the bubble as well as the tension of the bubble wall, as long as the matter obeys the NEC and the embedding is as described by (5).

A limited form of this result was found in [25], which considered an inflationary bubble embedded inside a non-inflationary cosmological background with $H_- > H_+$. The authors of [25] noticed that for inflationary bubbles that are larger than their own apparent horizon but smaller than the apparent horizon of the background $\tilde{r}_{AH}^- < R < \tilde{r}_{AH}^+$, an ingoing light ray traversing the bubble wall that starts in the background will begin with negative expansion $\theta^+ < 0$. However, after traversing the bubble wall boundary, the expansion is now positive $\theta^- > 0$ and therefore θ is nondecreasing, in violation of the Raychaudhuri equation (4). This change in sign of θ can be avoided if the inflationary bubble is larger than both the bubble and background apparent horizons $R > \tilde{r}_{AH}^+, \tilde{r}_{AH}^-$, so that $\theta^\pm$ is positive in both spacetimes [25]. Putting aside the difficulty of establishing such a configuration in a causal way, our result (10) indicates that it is not sufficient for $\theta^\pm$ to simply be positive in both spacetimes, as θ^- for the inflationary bubble is still larger than θ^+ for the background when $H_- > H_+$ for any size bubble.

The challenge presented by (10) to bubbles with $H_- > H_+$ is quite basic, and it is compelling to view the failure of (10) as due to the unrealistic assumption that the bubble and background cosmological spacetimes (5) are glued together on an infinitesimally thin wall with no backreaction on the
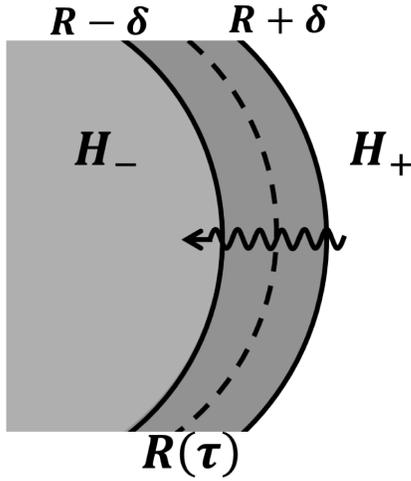
FIG. 2. A null ray traveling across a thick wall boundary between a cosmological background H_+ and bubble H_- leaves the background at $\tilde{r} = R + \delta$ and enters the bubble at $\tilde{r} = R - \delta$.

background spacetime. We would expect that for a generic bubble the energy density of the bubble should locally backreact on the background metric so that it deviates from the pure cosmological form. Indeed, for a generic de Sitter bubble embedded in a background that is empty [5] or dominated by a positive cosmological constant [13], the spacetime is approximately Schwarzschild in the vicinity of the bubble wall. Unfortunately, analytic forms[3] for the backreaction of the bubble energy density are not known when the bubble and background are filled with a generic cosmological fluid.

Nevertheless, we can move beyond our approximation of an infinitesimally thin wall in a generic way by assuming that the bubble wall now has a "thickness" of 2δ, as in Fig. 2. We will leave the spacetime geometry inside the thick wall unspecified, as its form likely will include the unknown backreaction of the bubble and tension of the wall. Since we do not know the details of the metric inside of the thick wall, we cannot compute θ inside the wall. Nevertheless, it must still be true that (4) is satisfied for a null ray as it enters and exits the thick wall.

An ingoing null ray enters the thick wall from the background at $\tilde{r} = R + \delta$ and leaves the wall in the bubble at $\tilde{r} = R - \delta$, so we have[4]

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[3]A notable exception is the McVittie metric [30], which is a specific solution representing a central inhomogeneity in an asymptotically FLRW background with a cosmological fluid that does not accrete onto the central object. We will attempt to include the effects of backreaction in a less specific way through the inclusion of an unknown thick wall, though it would be interesting to study the constraints from the Raychaudhuri equation for the McVittie metric in a future work.

[4]We have again used the rescaling freedom in the scale factor to set $a_+ = a_- = 1$ when the ray crosses the respective exterior and interior boundaries.

$$\Delta\theta = 2(H_- - H_+) - \frac{4\delta}{R^2 - \delta^2} \leq 0. \qquad (11)$$

In contrast to (10), it is now possible to solve (11) for an expanding bubble with $H_- > H_+$.

In particular, (11) is satisfied if the thickness of the wall is larger than

$$\delta \geq \frac{1}{2}R^2\Delta H, \qquad (12)$$

where $\Delta H = H_- - H_+$, and we assumed[5] $R|\Delta H| \ll 1$.

It is interesting to see how the presence of a thick wall satisfying (12) also evades the argument of [25] described above. In the presence of a thick wall of size $\tilde{r}_{AH}^- < R < \tilde{r}_{AH}^+$ satisfying (12), a null ray exits the wall and enters the bubble cosmology at the inner boundary of the wall $\tilde{r} = R - \delta$. Since $H_-(R - \delta) \leq H_- R(1 - \frac{1}{2}H_- R(1 - \frac{H_+}{H_-})) \leq 1 - \frac{1}{2}(1 - \frac{H_+}{H_-}) < 1$, the inner boundary of the wall is smaller than the bubble apparent horizon $R - \delta < H_-^{-1} = \tilde{r}_{AH}^-$. Thus, the expansion of the null ray is negative when it enters the cosmological part of the bubble, and Raychaudhuri's equation (11) can be satisfied.

It is important to note that the condition (12) is a necessary condition for the thickness of a wall surrounding a cosmological bubble with $H_- > H_+$, but it is not sufficient. Our approach has avoided specifying the behavior of θ inside the thick bubble wall, and it must be the case that θ is decreasing inside the thick bubble wall in a way that interpolates between the expansions θ of the exterior and interior spacetimes to satisfy the Raychaudhuri equation. In Sec. III we consider a cosmological bubble surrounded by a Schwarzschild spacetime, which can serve as a swiss cheese model for the backreaction of the bubble on the background, in which the empty Schwarzschild spacetime serves as a model for the thick wall. In the next subsection, we generalize our argument to include nonzero spatial curvature for both the bubble and the background, finding that the main results of this section hold for a larger range of signs of the spatial curvature of the bubble and background.

B. Nonzero spatial curvature for bubble and background

The constraint (10), while independent of the matter content of the bubble and background, applies only to bubble and background spacetimes that are spatially flat. In general, however, the bubble and background can have their own distinct spatial curvature. Indeed, [26]

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[5]If $R\Delta H \sim \mathcal{O}(1)$, then the wall must be approximately the same size as the bubble itself $\delta \geq \Delta H^{-1} \sim R$ in order to solve (11), which implies that the bubble spacetime is almost certainly not described by a homogeneous and isotropic cosmological metric.

generalized the argument of [25] to nonflat spatial sections and found that it is possible to avoid changes of sign of $\theta^\pm$ when crossing the boundary if the bubble is initially negatively curved. In this section, we will generalize our results from Sec. II A to nonflat spatial geometries by requiring that θ be nonincreasing.

Including spatial curvature in a homogeneous and isotropic spacetime, we start with the metrics

$$ds_\pm^2 = -dt_\pm^2 + a_\pm(t)^2\left(\frac{dr_\pm^2}{1 - k_\pm r_\pm^2} + r_\pm^2 d\Omega^2\right), \quad (13)$$

where $k_\pm > 0$ ($k_\pm < 0$) corresponds to positive (negative) spatial curvature, and $k_\pm = 0$ is flat space. Note that (13) still allows for an independent constant rescaling of the scale factors $a_\pm \to \lambda_\pm a_\pm$ as long as the comoving radial coordinates and spatial curvatures are rescaled as well as $r_\pm \to \lambda_\pm^{-1} r_\pm, k_\pm \to \lambda_\pm^2 k_\pm$. This is consistent only if one does not choose $k_\pm = \pm 1$, as is common in the presence of spatial curvature, which we will avoid.

Radially inward affine null rays in the spacetimes (13) take the form

$$N_\pm^\alpha = \frac{1}{a_\pm}\left(1, -\frac{\sqrt{1 - k_\pm r_\pm^2}}{a_\pm}, 0, 0\right), \quad (14)$$

with corresponding expansion $\theta^\pm = \nabla_\alpha N_\pm^\alpha$,

$$\theta^\pm = \frac{2}{a_\pm}\left(H_\pm - \frac{\sqrt{1 - k_\pm r_\pm^2}}{\tilde{r}}\right), \quad (15)$$

where again we have used the areal radius $\tilde{r} = a_- r_- = a_+ r_+$, which is continuous across the boundary, even in the presence of spatial curvature. As in flat space, a vanishing expansion for ingoing null rays $\theta^\pm = 0$ defines an apparent horizon for nonspatially flat FLRW space,

$$\tilde{r}_{AH}^\pm = \frac{1}{\sqrt{H_\pm^2 + k_\pm/a_\pm^2}}. \quad (16)$$

For $\tilde{r} > r_{AH}^\pm$, the expansion is positive due to the expansion of space.

As the null ray crosses the boundary $\tilde{r}|_\Sigma = R$, the requirement from the Raychaudhuri equation (4) that θ must be nonincreasing $\Delta\theta = \theta^- - \theta^+ \leq 0$ becomes

$$\Delta\theta = \frac{2}{a_-}\left(H_- - \frac{\sqrt{1 - k_- r_-^2}}{R}\right) - \frac{2}{a_+}\left(H_+ - \frac{\sqrt{1 - k_+ r_+^2}}{R}\right)$$
$$= 2(H_- - H_+) - \frac{2}{R}\left(\sqrt{1 - k_- r_-^2} - \sqrt{1 - k_+ r_+^2}\right) \leq 0, \quad (17)$$

where we again used our rescaling freedom to set $a_\pm = 1$ at the wall crossing. It does not seem possible to make general statements about the implications of (17), so we will examine the constraints imposed by (17) for specific cases. The results are summarized in Table I.

In particular, for a background that is either flat or negatively curved $k_+ \leq 0$, we have $\sqrt{1 - k_+ r_+^2} \geq 1$, while a bubble that is either flat or positively curved $k_- \geq 0$ has $\sqrt{1 - k_- r_-^2} \leq 1$. Combining these two inequalities with (17), we find

$$H_- - H_+ \leq 0. \quad (18)$$

This is an analogous constraint as (10) from Sec. II A: for an expanding bubble and background, only bubbles with a smaller Hubble rate than that of the background are consistent with the Raychaudhuri equation. (Similar arguments as given in Sec. II A hold for a nonexpanding bubble or background.)

For a background with positive spatial curvature $k_+ > 0$ and positive or flat spatial curvature for the bubble $k_- \geq 0$, we can rearrange (17) into the form

$$H_- - H_+ \leq \frac{1}{R}\left(\sqrt{1 - k_- r_-^2} - \sqrt{1 - k_+ r_+^2}\right) \leq \frac{1}{R}, \quad (19)$$

where the last inequality follows because the square roots are bounded for non-negative $k_\pm$. This constraint (19) is considerably less stringent than (18) since while an expanding bubble with a Hubble rate smaller than that of the background still satisfies (19), we can now have an expanding bubble with a Hubble rate *larger* than that of the background, provided that the size of the bubble is smaller than $R \leq (H_- - H_+)^{-1}$. If the bubble and the background expansion rates are not too different, this does not amount to too stringent of a constraint. However, if the bubble is expanding much faster than the background $H_- \gg H_+$, then the bubble size is bounded above by the bubble's inverse Hubble length $R < H_-^{-1}$. Similarly, if the background itself is collapsing $H_+ < 0$, the size of the bubble is again constrained by a combination of the expansion rates. Alternatively, if the bubble is collapsing, (19) is automatically satisfied, as in the flat space case.

TABLE I. Requiring that the expansion θ of a null ray be nonincreasing as the ray traverses the boundary between a background spacetime and bubble spacetime when including spatial curvature leads to different constraints depending on the relative signs of the spatial curvatures of the spaces.

Bubble	Background		
	$k_+ > 0$	$k_+ = 0$	$k_+ < 0$
$k_- > 0$ $k_- = 0$	$H_- - H_+ \leq R^{-1}$		$H_- < H_+$
$k_- < 0$	$R(H_- - H_+) \leq \sqrt{1 + \|k_-\|r_-^2} - \sqrt{1 - k_+ r_+^2}$		

Finally, to enumerate all possibilities we must consider the cases when the bubble is negatively curved $k_- < 0$. In all of these cases, (17) implies a bound on the size of the bubble,

$$R \leq \frac{\sqrt{1 + |k_-|r_-^2} - \sqrt{1 - k_+ r_+^2}}{H_- - H_+} \qquad (20)$$

(where we took the absolute value of the bubble spatial curvature for clarity).

It is interesting to compare the results of Table I with the constraints obtained from the Israel boundary conditions on FLRW spacetimes embedded in FLRW backgrounds from [9,10]. In particular, the authors of [10] finds that there are no restrictions on the bubble embedding when the difference in energy density between the background and bubble is positive $\rho^+ > \rho^-$ and larger than the surface energy density S of the bubble wall $\rho^+ - \rho^- > 6\pi GS$. However, when this difference is either negative, as when $\rho^- > \rho^+$, or smaller than the surface energy density, then it is not possible to satisfy the Israel junction conditions for flat or negatively curved spacetimes $k_+ \leq 0$ (for any value of the bubble spatial curvature) unless the bubble is larger than super-Hubble size $R > H_+^{-1}$. In contrast, we find a much stronger result since if the background is flat or negatively curved and the bubble is flat or positively curved, the embedding is inconsistent with the NEC and the Raychaudhuri equation unless the bubble expansion rate is smaller than the background $H_- < H_+$, for any sized bubble. On the other hand, if the bubble is negatively curved, our constraint (20) appears to put an upper bound on the size of an allowed bubble, rather than the lower bound of [10]. Thus, we see that the constraints imposed by the Raychaudhuri equation are complementary, and in some cases much stronger than constraints obtained by the Israel boundary conditions.

As in Sec. II A, it is tempting to see the failure of the Raychaudhuri equation for nonspatially flat bubbles as due to a possibly unrealistic assumption that the size of the bubble wall is infinitesimally thin. We can generalize our argument here to include a wall with a thickness of 2δ as in Fig. 2 with unspecified geometry, so that a null ray leaves the background and enters the wall at $\tilde{r} = R + \delta$ and leaves the wall and enters the bubble at $\tilde{r} = R - \delta$. The expansion θ must be nonincreasing from when the null ray leaves the background and enters the bubble wall,

$$\Delta\theta = \theta^-|_{\tilde{r}=R-\delta} - \theta^+|_{\tilde{r}=R+\delta} \qquad (21)$$

leading to the constraint

$$\frac{2}{a_-(t_-^*)} \left(H_- - \frac{\sqrt{1 - k_-(R-\delta)^2/a_-^2}}{R - \delta} \right)$$

$$- \frac{2}{a_+(t_+^*)} \left(H_+ - \frac{\sqrt{1 - k_+(R+\delta)^2/a_+^2}}{R + \delta} \right) \leq 0, \qquad (22)$$

where $t_\pm^*$ are the times when the null ray crosses the boundary out of (into) the background (bubble) spacetimes. It is difficult to draw generic conclusions from (22). Specific constraints can be obtained in the $H_+ \to 0$ limit, though since in this limit the background is empty, so we should consider a Schwarzschild background spacetime instead.

III. BUBBLE IN A SCHWARZSCHILD BACKGROUND

We will now consider our cosmological bubble to be surrounded by a background which is vacuum Schwarzschild spacetime. This can serve as a model of a false vacuum bubble in flat space, as in [4,5,31], or as a Schwarzschild vacuole or swiss cheese model of an inhomogeneity embedded in a larger cosmological background, as in Fig. 3. Indeed, the authors of [16,17] have argued that vacuum bubbles will develop just such a Schwarzschild layer as they evolve in a postinflationary universe. This Schwarzschild layer can thus serve as a simple model for the backreaction inside the thick wall introduced in Sec. II, or as a background for the bubble in its own right.

We will thus take the background spacetime to be spatially flat Schwarzschild space,

$$ds_S^2 = -\left(1 - \frac{2GM}{\tilde{r}_S}\right)dt_S^2 + \left(1 - \frac{2GM}{\tilde{r}_S}\right)^{-1}d\tilde{r}_S^2 + \tilde{r}_S^2 d\Omega^2, \qquad (23)$$
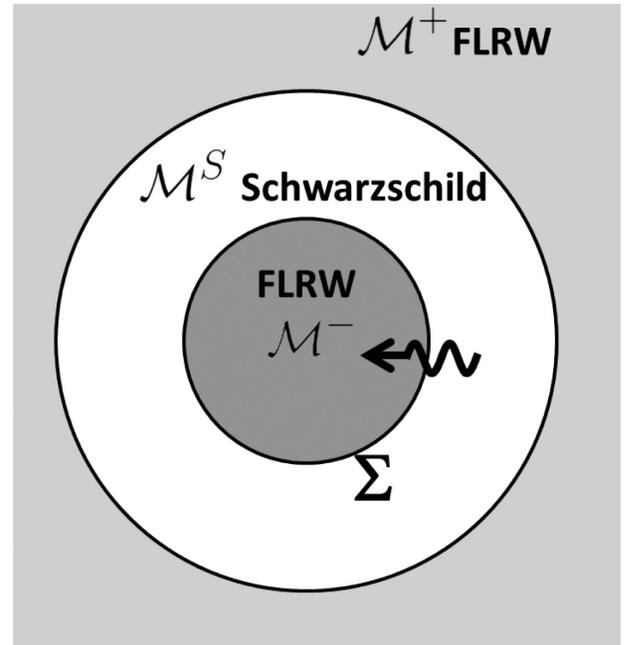

FIG. 3. A FLRW bubble surrounded by a Schwarzschild background can serve as a model for a "vacuole" or swiss cheese inhomogeneity embedded in a larger expanding background. We study the behavior of radial null rays as they traverse from the Schwarzschild background into the bubble.

where we will use a subscript S for the background to avoid confusion with the FLRW background coordinates from the previous section. A radially ingoing (affine) null ray in this spacetime,

$$N_S^\alpha = \left(\left(1 - \frac{2GM}{\tilde{r}_S}\right)^{-1}, -1, 0, 0 \right), \qquad (24)$$

has the expansion

$$\theta^S = \nabla_\alpha N_S^\alpha = -\frac{2}{\tilde{r}_S}, \qquad (25)$$

which is what we would expect to get by setting $H_+ = 0$ in the FLRW background of the previous section.

As the null ray traverses the bubble wall, the spacetime changes from the exterior Schwarzschild space to the interior FLRW cosmology (we will consider $k_- \neq 0$ here),

$$ds^2 = -dt_-^2 + \frac{a_-(t_-)^2}{1 - k_- r_-^2} dr_-^2 + a_-(t_-)^2 r_-^2 d\Omega^2. \qquad (26)$$

As before, we will find it convenient to work with the areal radius $\tilde{r}_- = a_-(t_-)r_-$, which is continuous across the boundary of the bubble. A radially ingoing affine null ray for this nonspatially flat FLRW spacetime takes the form

$$N_-^\alpha = \frac{1}{a_-}\left(1, -a_-^{-1}\sqrt{1 - k_- r_-^2}, 0, 0\right), \qquad (27)$$

which has the expansion

$$\theta^- = \frac{2}{a_-}\left(H_- - \frac{\sqrt{1 - k_- \tilde{r}_-^2/a_-^2}}{\tilde{r}_-}\right). \qquad (28)$$

In order for the Raychaudhuri equation (4) to be satisfied as the null ray crosses the thin wall boundary $\tilde{r}_S|_\Sigma = \tilde{r}_-|_\Sigma = R$, we must have

$$\Delta\theta = \theta^- - \theta^S$$
$$= \frac{2}{a_-}H_- + \frac{2}{R}\left(1 - \frac{\sqrt{1 - k_- R^2/a_-^2}}{a_-}\right) \leq 0. \qquad (29)$$

Note that since θ^- is positive when the size of an expanding bubble is larger than its apparent horizon $R > \tilde{r}_{AH}^-$, while θ^S is always negative, the requirement from the Raychaudhuri equation that θ be nonincreasing implies that the bubble may never be larger than its own apparent horizon. Thus, an arbitrarily large expanding bubble cannot develop when surrounded by a Schwarzschild spacetime. It is interesting to compare this result to that of [31], which found that an expanding bubble larger than its apparent horizon must have begun in an initial singularity. We find a complementary result here, that such a bubble embedding would not be consistent with the Raychaudhuri equation to begin with.

Utilizing our freedom to rescale the scale factor of the bubble to $a(t_*^-) = 1$ at the time of crossing t_*^-, the condition (29) becomes

$$2H_- + \frac{2}{R}\left(1 - \sqrt{1 - k_- R^2}\right) \leq 0. \qquad (30)$$

As in the previous section, we can impose stronger constraints from (30) for specific assumptions about the spatial curvature of the bubble. Assuming a bubble that is either spatially flat or positively curved $k_- \geq 0$, the second term of (30) is always positive; thus (30) requires that the bubble spacetime be contracting

$$H_- \leq 0. \qquad (31)$$

This implies that a flat or positively curved cosmic bubble embedded inside of a Schwarzschild background must be collapsing in order to satisfy the Raychaudhuri equation. This is consistent with the $k_- \geq 0$ Oppenheimer-Snyder solutions [3] for a ball of collapsing dust, some of the first and most famous solutions of cosmological bubbles embedded in a Schwarzschild spacetime background. While the Oppenheimer-Snyder solution is specifically for pressureless dust and zero surface tension, our result (31) does not make any assumptions about the matter content of the bubble or surface tension (as long as it satisfies the NEC), thus generalizing the conditions under which the bubble collapses.

One motivation for considering a background described by a Schwarzschild spacetime was that it could serve as a simple model of a backreacted thick wall between a cosmological FLRW background and the bubble. Because of this, we have assumed the thin wall approximation for the bubble wall. However, it can be the case that the boundary between the Schwarzschild and bubble spacetimes is also itself thick; we will therefore generalize our results for a thick wall of size 2δ, as in Sec. II.

First, note that even in the presence of a thick wall, the inner boundary of the wall $R - \delta$ must be smaller than the apparent horizon of the bubble, for the same reasons as described below (29), again indicating that the bubble size may not be larger than its apparent horizon for any value of the bubble spatial curvature. Constraints on the minimum thickness of a subhorizon bubble wall can be obtained from (22) by setting $H_+ = 0$ and $k_+ = 0$ (since θ^S and θ^+ agree in this limit). For a spatially flat bubble, the thickness of the bubble wall must be larger than

$$\delta > H_-^{-1}\left(-1 + \sqrt{1 + R^2 H_-^2}\right), \qquad (32)$$

in order to satisfy the Raychaudhuri equation. For bubbles much smaller than their apparent horizon size $RH_- \ll 1$,

the required thickness $\delta_{\min}$ is small compared to the size of the bubble $\delta_{\min}/R = \frac{1}{2}H_-R \ll 1$. However, for bubbles that are comparable in size to their apparent horizon $H_-R \sim 1$, the wall thickness is comparable to the size of the bubble itself $\delta \sim R$, necessitating a different spacetime structure for the entire bubble. Thus, even with a thick wall with an unspecified metric, large expanding bubbles embedded into Schwarzschild space are inconsistent with the Raychaudhuri equation.

IV. DISCUSSION

The Raychaudhuri equation requires that the expansion of radially inward null rays must be nonincreasing as long as matter obeys the NEC, independent of the Einstein equations. We have used this to study allowed spherically symmetric embeddings of FLRW cosmological bubbles into various background spacetimes.

We found that when the background is FLRW space, as in the cases of a spontaneous phase transition or a "detonation wave" bubble wall, with nonpositive spatial curvature $k_+ \leq 0$ and the bubble has non-negative spatial curvature $k_- \geq 0$, the Raychaudhuri equation constrains the bubble's Hubble expansion rate to be smaller than that of the background's $H_- < H_+$ for any size bubble. (For other combinations of the spatial curvature of the bubble and background, see Table I.) This result has important implications for cosmological bubbles of this type. In particular, it rules out embeddings of false vacuum bubbles into true vacuum backgrounds (with vanishing local backreaction), irrespective of the details of the matter content as long as the matter obeys the NEC, extending a more limited earlier result [25].

We also considered bubbles surrounded by empty Schwarzschild space, which could serve as a model of a vacuole embedded in a larger FLRW background. We found that expanding bubbles must be smaller than their own apparent horizon in order to satisfy the Raychaudhuri equation, extending the result [31] that a vacuum dominated bubble must start from an initial singularity. Further, we show that a bubble of any size with flat or positive spatial curvature cannot be expanding but rather must be contracting, generalizing the collapsing behavior of the Oppenheimer-Snyder dust ball solution [3] to any matter content satisfying the NEC.

We also considered the relaxation of the infinitesimal thin wall approximation for the bubble wall as a model for including local gravitational backreaction of the bubble on the background spacetime. Without specifying the metric within the thick wall of the bubble, the Raychaudhuri equation requires that the expansion of the null rays be nonincreasing between the outer and inner boundaries of the bubble wall. For a cosmological background, we derived a lower bound on the required thickness of the bubble wall to satisfy the Raychaudhuri equation, and we showed that for a spatially flat background and bubble the

inner boundary of the bubble is smaller than the apparent horizon of the bubble spacetime, so that again the bubble may not be larger than its own apparent horizon. For a Schwarzschild background the bubble must be much smaller than its own apparent horizon or else the required thickness of the wall is comparable to the size of the bubble itself.

It is possible to avoid the constraints we have derived here by embedding the cosmological bubble in such a way that the bubble is casually disconnected from the background spacetime. In particular, an expanding cosmological bubble tucked behind a wormhole will be casually disconnected from radially ingoing null rays from the background, as in Fig. 4. Solutions of this type have been described previously in [4–6] for a de Sitter bubble and Schwarzschild background, in [8] for some limited FLRW spacetimes, in [11,13] for de Sitter bubble and background spacetimes, and in [16,17] for false-vacuum and domain-wall bubbles embedded in FLRW backgrounds. It is important to note, however, that the conclusions of [31] still apply, so that bubbles that are bigger than their apparent horizons still must begin in an initial singularity, even if they are tucked behind a wormhole.

Finally, it would be useful to see how robust these arguments are to deviations from homogeneous spherical symmetry in the bubble and the background (see [32] for a discussion of an inhomogeneous background). We have also been assuming that the matter content of the background, bubble interior, and bubble wall obeys the NEC; it is possible to consider cosmological models based on violations of the NEC, through quantum gravity effects [33], NEC-violating matter fields (see e.g. [34,35] for some common models), or nonminimal coupling [36,37]. We will leave a detailed study of bubbles with these effects for future work.
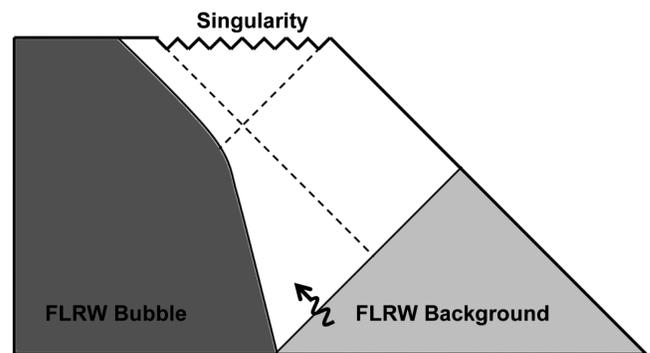

FIG. 4.   One way to evade the constraints on bubble embeddings from the Raychaudhuri equation is for the bubble to become disconnected from the background spacetime by the creation of a wormhole, as shown in this Penrose diagram adapted from [16]. Radially ingoing null rays starting in the background either see the bubble contracting or are casually disconnected from the bubble, while the bubble continues to expand in a "baby universe."

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