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Dynamical analysis of an interacting dark energy model in the framework of a particle creation mechanism

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In this work we present the cosmological dynamics of interacting dark energy models in the framework of the particle-creation mechanism. The particle-creation mechanism presented here describes the true nonequilibrium thermodynamics of the Universe. In the spatially flat Friedmann-Lemaître-Robertson-Walker universe considered here, the dissipative bulk viscous pressure is due to the nonconservation of particle number. For simplicity, we assume that the creation of perfect-fluid particles is isentropic (adiabatic) and consequently the viscous pressure obeys a linear relationship with the particle-creation rate. Due to the complicated nature of Einstein's field equations, dynamical systems analysis is performed to understand the cosmological dynamics. We find some interesting cosmological scenarios, like a late-time evolution of the universe dominated by dark energy which could mimic quintessence, a cosmological constant, and a phantom field through a dark-matter-dominated era. We also find a possibility of crossing the phantom divide line which is favored by observations.

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I. INTRODUCTION

Various observations suggest that our Universe is currently undergoing a phase of accelerated expansion [1–5]. This is a challenging issue in standard cosmology which shows a new imbalance in the governing Friedmann equations. People have addressed such imbalances either by introducing new sources or by altering the governing equations. In the frame of standard cosmology, the first one is called dark energy with a huge negative pressure, and the second one involves the introduction of some modifications to the gravity sector commonly known as modified gravity theories. The simplest dark energy (DE) candidate is the cosmological constant Λ, which together with cold dark matter provides the simplest cosmological model known as the Λ cold dark matter model (ΛCDM), which according to a large number of observations is the best cosmological model at present. However, ΛCDM suffers from severe problems at the interface of cosmology and particle physics, such as the cosmological constant problem [6–8] and the cosmic coincidence problem [9].

In order to address these issues related to Λ cosmology, extensive analyses have been performed, ranging from various DE models to modified gravity theories [10,11]. Among them, the cosmological models where dark matter (DM) and DE interact with each other have gained significant attention with the growing amount of observational data. Although the latest observations indicate a nonvanishing interaction in the dark sector [12–14], this interaction is very compatible with zero within the 1σ confidence region. In any case, the interaction between DE and DM could be a major issue in studying the physics of DE. However, since the nature of these two dark components (DE and DM) remains unknown, the precise form of the interaction is unknown as well, and as such there is no fundamental theory for choosing a specific coupling. So, the choice of coupling is purely phenomenological. Further, in the framework of field theory, it is natural to consider the inevitable interaction between the dark components. Interacting dark sector models have been extensively studied in several works [15–27]. In fact, an appropriate interaction between DE and DM can provide a mechanism to alleviate the coincidence problem [28] and cosmic age problem. Furthermore, it also provides a possibility of crossing the phantom divide line [29,30] and explains the transient nature of the deceleration parameter. It should be noted that there are other options apart from the above-mentioned choices for explaining the cosmic coincidence and other cosmological conundrums. In particular, there are the ΛXCDM-type of models (for detailed studies, see Refs. [31,32]), where there exists an interaction between the vacuum energy and another DE component (X). In this case, matter can be conserved and the ratio between DE and DM remains bounded throughout

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the entire cosmic history. Further, in this context one can find effective quintessence and phantom-like behaviors in Refs. [33–35].

Therefore, interacting DE models provide richer cosmological dynamics than noninteracting ones by allowing energy exchange between dark sectors. This might provide a similar energy density in dark sectors which can be achieved by the accelerated scaling attractor solution [20,36,37] with

$$\frac{\Omega_{\mathrm{DE}}}{\Omega_{\mathrm{DM}}} \approx \mathcal{O}(1) \quad \text{and} \quad \omega_{\mathrm{eff}} < -\frac{1}{3}. \tag{1}$$

Thus, the proper choice of parameters without fine-tuning of the initial conditions is required in order to match the ratio of the energy densities of the dark sectors with observations.

Present observations [4,5,38–41] also favor the possibility that our Universe is entering the phantom era with an effective equation of state $\omega_{\mathrm{eff}} < -1$. To obtain this scenario, a scalar field with a negative kinetic term is usually introduced [42]. However, this leads to some instabilities at both classical and quantum levels [43,44], and also induces some other theoretical problems [45–47].

Another choice to explain this present acceleration is the particle-creation mechanism. This model can successfully mimic the ΛCDM cosmology [48–52]. Historically, in 1939 Schrodinger [53] introduced a microscopic description of particle production in an expanding universe where gravity plays a crucial role. Following his idea, Parker *et al.* [54] and Zel'dovich *et al.* [55] started investigating the possible physical scenarios arising from the production of particles. Since the evolution of the Universe can be understood from Einstein's field equations, Prigogine *et al.* [56] studied the evolution of the Universe after introducing the particle-creation mechanism in Einstein's field equations by changing the usual balance equation for the number density of particles.

In cosmological dynamics, the only dissipative phenomenon in the homogeneous and isotropic flat Friedmann-Lemaître-Robertson-Walker (FLRW) model may be in the form of bulk viscous pressure either due to the coupling of different components of the cosmic substratum [57–61] or the nonconservation of (quantum) particle number. Thus, for an open thermodynamical system where the number of fluid particles is not preserved ($N^{\mu}_{;\mu} \neq 0$) [62–64], the particle conservation equation gets modified as

$$N^{\mu}_{;\mu} \equiv n_{,\mu}u^{\mu} + \Theta n = n\Gamma \Leftrightarrow N_{,\mu}u^{\mu} = \Gamma N,$$
$$\text{i.e.,}\ \ \dot{N} = \Gamma N. \tag{2}$$

This equation is also known as the balance equation for the particle flux. Also, the implied relation states that the rate of change of total particle number is proportional to the total number of particles. Here, Γ stands for the rate of change of particle number in a comoving volume V containing N particles, $N^{\mu} = nu^{\mu}$, the particle flow vector u^{μ} is the four-velocity vector, $n = N/V$ is the particle number density, and $\Theta = u^{\mu}_{;\mu}$ is the fluid expansion. The quantity Γ is unknown in nature, but the validity of the second law of thermodynamics implies the positivity of Γ. In the present work, a dissipative effect due to the second alternative is chosen. However, for simplicity, adiabatic (i.e., isentropic) production [56,65] of perfect fluid particles is considered and as a result viscous pressure obeys a linear relationship with the particle production rate.

The particle creation scenario can successfully describe the accelerated expansion model of the Universe without introducing DE. Also, many interesting results with this mechanism (such as the possibility of future deceleration) have been proposed in Refs. [66,67]; consequently, the existence of an emergent universe was proposed in Refs. [68,69] and the complete cosmic scenario was subsequently studied in Ref. [70]. Further, in the framework of the particle creation mechanism, the evolution of the Universe from the big bang scenario to a late-time de Sitter phase was studied in Ref. [71] and the accelerated expansion of the Universe at early and present times were studied in Ref. [72]. Furthermore, the possibility of a phantom universe without invoking any phantom fields has recently been realized in a similar context [73,74]. So it is worth studying interacting DE models using the particle creation mechanism.

In the present work, considering our Universe as an open thermodynamical system in the framework of flat FLRW spacetime, an interacting dynamics between dark energy and dark matter is proposed where the dark matter particles are assumed to be created from the gravitational field. This is achieved by rewriting the Friedmann equation and Raychaudhuri equation in the context of the matter-creation mechanism, and assuming that the particle production rate is proportional to the Hubble parameter and uniform throughout the Universe. The main scope of this work is to analyze the cosmological dynamics of interacting DE models in the framework of adiabatic particle creation using dynamical system techniques. Dynamical system tools have been extensively used to study the asymptotic behavior of various cosmological models where exact solutions of evolution equations cannot be obtained (see, e.g., Refs. [75–82]). We obtain some interesting critical points which describe many interesting results from the phase-space analysis of linear interactions. These include the early matter-dominated Universe, and late-time DE-dominated attractors in some parameter region, where DE is associated with quintessence, a cosmological constant, or a phantom field, respectively.

The organization of the paper is as follows. In Sec. II, we present the basic equations of the present particle creation model, and the evolution equations are transformed to an autonomous system by suitable transformation of the dynamical variables. In Sec. III, critical points are shown for various choices for the interaction term and the cosmological parameters are evaluated. Section IV shows the phase-space analysis and stability criteria for the critical points. In Sec. V, the cosmological implications of critical points for several interaction models are given. The paper ends with a short discussion in Sec. VI.

II. THE BASIC EQUATIONS OF PARTICLE CREATION AND AUTONOMOUS SYSTEMS

In accordance with inflation and the cosmic microwave background radiation, the Universe is well described by the spatially flat FLRW spacetime,

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \tag{3}$$

where $a(t)$ is the scale factor of the Universe and the spherical line element $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit 2-sphere. For a comoving observer, $u^\mu = \delta^\mu_t$ is the velocity vector so that $u^\mu u_\mu = -1$. With the line element (3) the fluid expansion Θ will become $\Theta = 3H$, where H is the Hubble parameter. Hence, the particle conservation Eq. (2) is reduced to

$$N^\mu_{;\mu} \equiv n_{,\mu} u^\mu + 3Hn = n\Gamma \tag{4}$$

for the present open thermodynamical model. Further, using the above conservation Eq. (4), Gibb's relation [56,83,84]

$$Tds = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{5}$$

gives the variation of the entropy per particle as [56,71]

$$nT\dot{s} = \dot{\rho} + 3H\left(1 - \frac{\Gamma}{3H}\right)(\rho + p), \tag{6}$$

where T represents the fluid temperature, s is the entropy per particle, i.e., the specific entropy (the specific entropy of a system is the entropy of the unit mass of the system), ρ is the total energy density, and p denotes the total thermodynamic pressure. We consider our thermodynamical system to be an ideal one—i.e., isentropic (or adiabatic) (see, for instance, Refs. [65,85])—and consequently we have the production of perfect fluid particles with a constant entropy (i.e., $\dot{s} = 0$). However, there is entropy production due to the enlargement of the phase space of the system since the number of perfect fluid particles increases. Hence, from Eq. (6) one can obtain the conservation equation as

$$\dot{\rho} + 3H(\rho + p) = \Gamma(\rho + p), \tag{7}$$

which, however, can also be written as [72]

$$\dot{\rho} + 3H(\rho + p + p_c) = 0. \tag{8}$$

So, comparing Eqs. (7) and (8), one can easily obtain the creation pressure as follows [70,71,86–88]:

$$p_c = -\frac{\Gamma}{3H}(\rho + p), \tag{9}$$

where p_c is called the creation pressure, and Γ is the particle production rate (number of particles created per unit time), which is assumed to be uniform throughout the Universe. Now, we assume that the main constituents of our Universe are DM in the form of dust with energy density ρ_m and the dark energy fluid with an equation state $\omega_d = p_d/\rho_d$ (where ρ_d and p_d are the energy density and thermodynamic pressure of the cosmic fluid, respectively). The Friedmann and Raychaudhuri equations $(8\pi G = c = 1)$ can now be written as

$$H^2 = \frac{1}{3}(\rho_m + \rho_d), \quad \text{(Friedmann's equation)} \tag{10}$$

$$\dot{H} = -\frac{1}{2}(\rho_m + \rho_d + p_d + p_c), \quad \text{(Raychaudhuri's equation)} \tag{11}$$

where $H = \dot{a}/a$ is the Hubble parameter and an overdot represents differentiation with respect to cosmic time "t." If we assume that the created particles are pressure-less DM in the thermodynamically open model of the Universe under the adiabatic condition, then the creation pressure p_c in Eq. (9) becomes [73]

$$p_c = -\frac{\Gamma}{3H}(\rho_m).$$

In standard cosmology, the dynamic interactions between the homogeneously distributed DE in the Universe and the DM component (clumping around the ordinary particles) are extremely weak or even negligible. As a result, the energy conservation equations for the two matter components are

$$\dot{\rho}_m + 3H(\rho_m + p_c) = 0 \tag{12}$$

or, using the above expression for p_c,

$$\dot{\rho}_m + 3H\rho_m = \Gamma\rho_m, \tag{13}$$

and

$$\dot{\rho}_d + 3H(\rho_d + p_d) = 0. \tag{14}$$

In order to alleviate the cosmological coincidence problem, it has been found that a nongravitational interaction between these dark sectors could be a viable alternative. So the interacting DM and DE models of the Universe are becoming of great interest and are widely used in the literature [23]. Thus the above energy conservation equations are modified as

$$\dot{\rho}_m + 3H\rho_m\left(1 - \frac{\Gamma}{3H}\right) = -Q, \tag{15}$$

$$\dot{\rho}_d + 3H(\rho_d + p_d) = Q, \tag{16}$$

where Q indicates the rate of energy exchange between the dark sectors.

In particular, $Q > 0$ indicates the conversion of DM into DE, while $Q < 0$ represents the opposite. A complete study of the interaction of dynamical vacuum energy with matter can be found in Refs. [89,90] (for an extension, see Refs. [91,92]). Further, it should be noted that the running vacuum models [93,94] give an overall fit to the observational data that is better than ΛCDM. These studies were based on the general expectations of the effective action of quantum field theory in curved spacetime and provide an interaction of the dynamical vacuum and matter [95].

In the present work, we describe the background dynamics using several different interactions: (i) $Q \propto H\rho_m$ [20,22], (ii) $Q \propto H\rho_d$ [96,97], (iii) $Q \propto H(\rho_m + \rho_d)$ [16,98], (iv) $Q \propto \frac{\rho_m \rho_d}{H}$ [30], and (v) $Q \propto \rho_m$ [20].

One may note that in the above interactions, the dimensionless parameters (proportionality constants) should not be chosen in an *ad hoc* manner. These parameters can actually be fitted to the overall observational data and one finds that they are typically of the order of 10^{-3} to 10^{-2} [92–94,99] depending on the normalization of the parameters involved. Further, in these references the justifications for such small values of these parameters were shown from two perspectives: theoretically, these coefficients represent the beta function of the running vacuum energy [94,99] and hence are expected to be very small; experimentally, the fitted values of these coefficients to the recent SNIa + BAO + LSS + BBN + CMB data (in which WMAP9 and Planck-13 and -15 data were taken into account) [5] were found to be of the same order as the theoretically expected values. Further, if we compare our Eq. (15) with Eq. (4) of the Q_m model in Ref. [99] and choose $\Gamma = \Gamma_0 H$ (where Γ_0 is a constant), we see that the effective interaction term will be $(\Gamma_0 - \alpha_m)H\rho_m$ in the first case. Thus, comparing with Eq. (7) of Ref. [99], we have

$$\Gamma_0 - \alpha_m = 3\nu_{dm}. \tag{17}$$

Moreover, recent observationally estimated values of the parameters ν_{dm} and ν_Λ [in Eqs. (7) and (8) of Ref. [99]] similar to our interaction models 1 and 2 are given by (see Table II of Ref. [99])

$$\nu_{dm} = 0.00618 \pm 0.00159,$$
$$\nu_\Lambda = 0.01890 \pm 0.00744.$$

Thus, Γ_0 and α_m are not arbitrary: their difference has an observational estimate. Note that because ν_{dm} is positive, from Eq. (17) Γ_0 is always greater than α_m and the effective interaction term has the same sign convention as in Ref. [99].

Due to the complicated nonlinear forms on the evolution Eqs. (10), (11), (15), and (16), we convert these evolution equations to an autonomous system of first-order differential equations. To do this we consider the dimensionless variables [17]

$$x = \frac{\rho_d}{3H^2}, \qquad y = \frac{p_d}{3H^2}, \tag{18}$$

which are normalized over the Hubble scale.

Then, the autonomous system of ordinary differential equations is

$$\frac{dx}{dN} = \frac{Q}{3H^3} - (1 - x)(3y + \Gamma_0 x), \tag{19}$$

$$\frac{dy}{dN} = \frac{Qy}{3xH^3} - (1 - x)\left(\frac{3y^2}{x} + \Gamma_0 y\right). \tag{20}$$

Here, the independent variable is chosen as the lapse time $N = \ln a$ (which is called the e-folding number), and the particle production rate Γ is a function of the Hubble parameter [66,70] [Γ has dimensions $(\text{time})^{-1}$] and is chosen as above: $\Gamma = \Gamma_0 H$ (Γ_0 is a constant). The value of the parameter Γ_0 is assumed to be non-negative as only the creation of particles is considered in this study.

Now, in terms of the new dimensionless quantities, the cosmological parameters can be written as follows: the energy density parameter for dark matter is

$$\Omega_m = 1 - x, \tag{21}$$

and the energy density parameter for the dark energy is

$$\Omega_d = x. \tag{22}$$

It may be noted that in the case of noninteracting DE models, the energy density is usually considered to be non-negative. However, in this case of interacting DE models, the energy density can be taken to be negative [100]. This would imply that there is no constrain for dimensionless variables, making the phase space that is analyzed here to be not compact.

So, there might be a possibility of critical points at infinity. In general, the analysis of fixed points at infinity is done by compactifying the phase space using Poincaré compactification. However, from a phenomenological point of view, in the present work we shall only determine the dynamics in the neighborhood of finite fixed points. This is enough, since our aim is to find physically viable solutions, namely, trajectories connecting DM to DE domination.

Also, the equation of state parameter for the DE can be expressed as

$$\omega_d = \frac{p_d}{\rho_d} = \frac{y}{x}, \tag{23}$$

and the effective equation of state parameter will be of the form

$$\omega_{\rm eff} = y - \frac{\Gamma_0}{3}(1-x). \tag{24}$$

Moreover, we have the evolution equation of the Hubble function as

$$\frac{1}{H}\frac{dH}{dN} = -\frac{3}{2}\left(1 + y - \frac{\Gamma_0}{3}(1-x)\right). \tag{25}$$

We now determine the critical points of the above autonomous system for different choices of Q, and then we perturb the equations up to first order about the critical points in order to determine their stability.

III. CRITICAL POINTS OF THE AUTONOMOUS SYSTEM (19)–(20) FOR VARIOUS CHOICES OF THE INTERACTION TERM AND COSMOLOGICAL PARAMETERS

In this section, we discuss the existence of the critical points and the corresponding physical parameters for various interaction models. These are presented in detail in tabular form.

A. Interaction model 1

First, we choose the interaction as

$$Q = \alpha_m H \rho_m, \tag{26}$$

where the coupling parameter α_m is a dimensionless constant. The indefiniteness in the sign of α_m indicates that the energy transfer takes place in either direction: DE or DM. This interaction is well motivated due to mathematical simplicity as the dimensions of the autonomous system (19)–(20) remain the same because H can be eliminated from the equations. Now, using this interaction in the system (19)–(20), the autonomous system for this interaction model will be

$$\frac{dx}{dN} = (-1+x)(\Gamma_0 x - \alpha_m + 3y), \tag{27}$$

TABLE I. The existence of critical points and the corresponding physical parameters for the interaction model $Q_1 = \alpha_m H \rho_m$.

Critical Points	Existence	ω_d	$\omega_{\rm eff}$	Ω_m	Ω_d
$A_1:(1, y_c)$	Always	y_c	y_c	0	1
$B_1:(\frac{\alpha_m}{\Gamma_0}, 0)$	$\Gamma_0 \neq 0$	0	$-\frac{\Gamma_0}{3}(1-\frac{\alpha_m}{\Gamma_0})$	$1-\frac{\alpha_m}{\Gamma_0}$	$\frac{\alpha_m}{\Gamma_0}$
$C_1:(1,0)$	Always	0	0	0	1
$D_1:(1,-1)$	Always	-1	-1	0	1
$E_1:(x_c, \frac{\alpha_m}{3} - \frac{\Gamma_0 x_c}{3})$	Always	$\frac{\alpha_m - \Gamma_0 x_c}{3x_c}$	$\frac{\alpha_m}{3} - \frac{\Gamma_0}{3}$	$1-x_c$	x_c

$$\frac{dy}{dN} = \frac{y}{x}(-1+x)(\Gamma_0 x - \alpha_m + 3y). \tag{28}$$

The critical points for the system (27)–(28) are as follows.

(i) Set of critical points: $A_1 = (1, y_c)$, where y_c takes any real value.
(ii) Critical point: $B_1 = (\frac{\alpha_m}{\Gamma_0}, 0)$.
(iii) Critical point: $C_1 = (1, 0)$.
(iv) Critical point: $D_1 = (1, -1)$.
(v) Set of critical points: $E_1 = (x_c, \frac{\alpha_m}{3} - \frac{\Gamma_0 x_c}{3})$.

The existence of critical points and their cosmological parameters are displayed in Table I. It is observed that point B_1 is a point in the set E_1, and points C_1 and D_1 are points in the set A_1. So, in the next section we shall analyze only the stability of sets A_1 and E_1. However, the critical points B_1, C_1, and D_1 show some interesting cosmological features, which we shall discuss in Sec. V.

B. Interaction model 2

We consider another choice of interaction as

$$Q = \alpha_d H \rho_d, \tag{29}$$

where α_d is the coupling parameter. Using this interaction in the system (19)–(20), we have the following autonomous system:

$$\frac{dx}{dN} = (\Gamma_0 x + 3y)(x-1) + \alpha_d x, \tag{30}$$

$$\frac{dy}{dN} = \frac{y}{x}((\Gamma_0 x + 3y)(x-1) + \alpha_d x). \tag{31}$$

The autonomous system (30)–(31) admits the following critical points.

(1) Critical point: $A_2 = (\frac{\Gamma_0 - \alpha_d}{\Gamma_0}, 0)$.
(2) Set of critical points: $B_2 = (x_c, \frac{x_c(\Gamma_0 x_c - \Gamma_0 + \alpha_d)}{3(1-x_c)})$.

The existence criteria and the cosmological parameter related to the critical points are shown in Table II. It is again noted that point A_2 is a point in a set B_2. So, in the next section we shall analyze only the stability of set B_2. However, depending on the choice of the coupling

TABLE II. The existence of critical points and the corresponding physical parameters for the interaction model $Q_2 = \alpha_d H\rho_d$.

Critical Points	Existence	ω_d	$\omega_{\rm eff}$	Ω_m	Ω_d
$A_2: \left(\frac{\Gamma_0-\alpha_d}{\Gamma_0}, 0\right)$	$\Gamma_0 \neq 0$	0	$-\frac{\alpha_d}{3}$	$\frac{\alpha_d}{\Gamma_0}$	$1-\frac{\alpha_d}{\Gamma_0}$
$B_2: \left(x_c, \frac{x_c(\Gamma_0 x_c-\Gamma_0+\alpha_d)}{3(1-x_c)}\right)$	$x_c \neq 1$	$\frac{(\Gamma_0 x_c-\Gamma_0+\alpha_d)}{3(1-x_c)}$	$\frac{(\Gamma_0 x_c-\Gamma_0+\alpha_d x_c)}{3(1-x_c)}$	$1-x_c$	x_c

TABLE III. The existence of critical points and the corresponding physical parameters for the interaction model $Q_3 = \alpha H(\rho_m+\rho_d)$.

Critical Points	Existence	ω_d	$\omega_{\rm eff}$	Ω_m	Ω_d
$A_3: \left(\frac{1}{2}\left(1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right), 0\right)$	$\frac{\alpha}{\Gamma_0} < \frac{1}{4}$	0	$\frac{\Gamma_0}{6}\left(-1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$	$\frac{1}{2}\left(1-\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$	$\frac{1}{2}\left(1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$
$B_3: \left(\frac{1}{2}\left(1-\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right), 0\right)$	$\frac{\alpha}{\Gamma_0} < \frac{1}{4}$	0	$-\frac{\Gamma_0}{6}\left(1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$	$\frac{1}{2}\left(1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$	$\frac{1}{2}\left(1-\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right)$
$C_3: \left(x_c, \frac{\Gamma_0 x_c^2-\Gamma_0 x_c+\alpha}{3(1-x_c)}\right)$	$x_c \neq 1$	$\frac{\Gamma_0 x_c^2-\Gamma_0 x_c+\alpha}{3x_c(1-x_c)}$	$\frac{\Gamma_0 x_c-\Gamma_0+\alpha}{3(1-x_c)}$	$1-x_c$	x_c

parameter α_d, the critical point A_2 shows some interesting cosmological features, which we shall discuss in Sec. V.

C. Interaction model 3

Now, we consider the linear interaction as

$$Q = \alpha H(\rho_m+\rho_d). \tag{32}$$

For this interaction model, the system (19)–(20) will take the form

$$\frac{dx}{dN} = \alpha + (\Gamma_0 x + 3y)(x-1), \tag{33}$$

$$\frac{dy}{dN} = \frac{y}{x}(\alpha + (\Gamma_0 x + 3y)(x-1)). \tag{34}$$

The critical points are as follows:

(1) Critical point: $A_3: (\frac{1}{2}\left(1+\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right), 0)$, where $\Gamma_0 \geq 4\alpha$.
(2) Critical point: $B_3: (\frac{1}{2}\left(1-\sqrt{1-\frac{4\alpha}{\Gamma_0}}\right), 0)$, where $\Gamma_0 \geq 4\alpha$.
(3) Set of critical points: $C_3: (x_c, \frac{\Gamma_0 x_c^2-\Gamma_0 x_c+\alpha}{3(1-x_c)})$.

The cosmological parameters related to the critical points are shown in Table III. It is again noted that points A_3 and B_3 are points in the set C_3. So, in the next section we shall analyze only the stability of set C_3.

D. Interaction model 4

Considering the nonlinear interaction

$$Q = \frac{\beta}{H}\rho_m\rho_d \tag{35}$$

in Eqs. (19)–(20), the autonomous system will be of the form

$$\frac{dx}{dN} = (x-1)(\Gamma_0 x - 3\beta x + 3y), \tag{36}$$

$$\frac{dy}{dN} = \frac{y}{x}(x-1)(\Gamma_0 x - 3\beta x + 3y). \tag{37}$$

The critical points for this case are as follows.
(1) Set of critical points: $A_4 = (1, y_c)$.
(2) Critical point: $B_4 = (1, 0)$.
(3) Critical point: $C_4 = (1, -1)$.
(4) Set of critical points: $D_4 = (x_c, \beta x_c - \frac{\Gamma_0 x_c}{3})$.

The condition for the existence of critical points and the corresponding physical parameters are presented in Table IV. It is again noted that points B_4 and C_4 are points in the set A_4. So, in the next section we shall analyze only the stability of sets A_4 and D_4.

E. Interaction model 5

We are now going to discuss another type of interaction in the dark sectors which is completely based on the local properties of the Universe, and hence it is different from the other interaction models discussed in the previous subsections. Here, we replace the nonlocal transfer rate (discussed in the previous subsections) by the local rate η, and the interaction [20] has the following form:

$$Q = \eta\rho_m, \tag{38}$$

TABLE IV. The existence of critical points and the corresponding physical parameters for the interaction model $Q_4 = \frac{\beta}{H}\rho_m\rho_d$.

Critical Points	Existence	ω_d	$\omega_{\rm eff}$	Ω_m	Ω_d
$A_4: (1, y_c)$	Always	y_c	y_c	0	1
$B_4: (1, 0)$	Always	0	0	0	1
$C_4: (1, -1)$	Always	-1	-1	0	1
$D_4: (x_c, \beta x_c - \frac{\Gamma_0 x_c}{3})$	Always	$\beta - \frac{\Gamma_0}{3}$	$\beta x_c - \frac{\Gamma_0}{3}$	$1-x_c$	x_c

TABLE V. The existence of critical points and the corresponding physical parameters for the interaction model $Q_5 = \eta\rho_m$.

Critical Points	Existence	ω_d	$\omega_{\rm eff}$	Ω_m	Ω_d
$A_5:(1, y_c, 0)$	Always	y_c	y_c	0	1
$B_5:(1, -1, v)$	$0 \le v \le 1$	-1	-1	0	1
$C_5:(x_c, -\frac{\Gamma_0 x_c}{3}, 0)$	Always	$-\frac{\Gamma_0}{3}$	$-\frac{\Gamma_0}{3}$	$1 - x_c$	x_c
$D_5:\left(x_c, -1 + \frac{\Gamma_0}{3}(1 - x_c), \frac{-3+\Gamma_0}{-3+\Gamma_0+\gamma}\right)$	$\Gamma_0 + \gamma \neq 3$	$-\frac{1}{3x_c}(3 - \Gamma_0 + \Gamma_0 x_c)$	-1	$1 - x_c$	x_c

where the coefficient η related to the local rate is assumed to be constant. When $\eta > 0$ (i.e., $Q > 0$), the energy decays from DM to DE which reveals the possibility of a vanishing DE field in the primordial Universe, so that DE condenses as a result of the slow decay of DM. This interaction was studied in Ref. [22] describing the phase space analysis when the dark energy equation of state has the phantom behavior. Moreover, for $\eta > 0$ the interaction is used to describe the decay of DM into radiation [101], the decay of a curvaton field into radiation [102], and the decay of superheavy DM particles into a quintessence scalar field [103]. On the other hand, for $\eta < 0$ the energy is transferred in the opposite way. Further, in order to close the dynamical system (19)–(20), one has to introduce the new variable v given by

$$v = \frac{H_0}{H + H_0}, \tag{39}$$

where H_0 is constant and hence $0 \le v \le 1$. Introducing a dimensionless coupling constant

$$\gamma = \frac{\eta}{H_0}, \tag{40}$$

the autonomous system of equations can be written as

$$\frac{dx}{dN} = \frac{(-1 + x)}{(-1 + v)}((\Gamma_0 x + 3y)(v - 1) + \gamma v), \tag{41}$$

$$\frac{dy}{dN} = \frac{y(-1 + x)}{x(-1 + v)}((\Gamma_0 x + 3y)(v - 1) + \gamma v), \tag{42}$$

$$\frac{dv}{dN} = \frac{1}{2} v(v - 1)(3 + 3y - \Gamma_0(1 - x)). \tag{43}$$

The critical points for this system are as follows.

(1) Set of critical points: $A_5 = (1, y_c, 0)$.
(2) Set of critical points: $B_5 = (1, -1, v)$.
(3) Set of critical points: $C_5 = (x_c, -\frac{\Gamma_0 x_c}{3}, 0)$.
(4) Set of critical points: $D_5 = (x_c, -1 + \frac{\Gamma_0}{3}(1 - x_c), \frac{-3+\Gamma_0}{-3+\Gamma_0+\gamma})$.

The set of critical points and the corresponding cosmological parameters are presented in Table V. We note here that sets A_5 and B_5 have the common point $(1, -1, 0)$.

IV. PHASE-SPACE ANALYSIS AND STABILITY CRITERIA OF CRITICAL POINTS

We shall now discuss the phase-space analysis of critical points and their stability by analyzing the eigenvalues of the linearized Jacobian matrix evaluated at the critical points presented in Tables VI and VII. It can be seen from

TABLE VI. The eigenvalues of the critical points for different interaction models.

Interaction	Critical points	λ_1	λ_2
1. $Q = \alpha_m H \rho_m$	$A_1:(1, y_c)$	0	$3y_c + \Gamma_0 - \alpha_m$
,,	$B_1:(\frac{\alpha_m}{\Gamma_0}, 0)$	0	$-\Gamma_0 + \alpha_m$
,,	$C_1:(1, 0)$	0	$\Gamma_0 - \alpha_m$
,,	$D_1:(1, -1)$	0	$-3 + \Gamma_0 - \alpha_m$
,,	$E_1:(x_c, \frac{\alpha_m}{3} - \frac{\Gamma_0 x_c}{3})$	0	$\frac{\alpha_m(-1+x_c)}{x_c}$
2. $Q = \alpha_d H \rho_d$	$A_2:\left(\frac{\Gamma_0 - \alpha_d}{\Gamma_0}, 0\right)$	0	$\Gamma_0 - \alpha_d$
,,	$B_2:\left(x_c, \frac{x_c(\Gamma_0 x_c - \Gamma_0 + \alpha_d)}{3(1-x_c)}\right)$	0	$\frac{\alpha_d x_c}{1-x_c}$
3. $Q = \alpha H(\rho_m + \rho_d)$	$A_3:(\frac{1}{2}\left(1 + \sqrt{1 - \frac{4\alpha}{\Gamma_0}}\right), 0)$	0	$\Gamma_0\sqrt{1 - \frac{4\alpha}{\Gamma_0}}$
,,	$B_3:\left(\frac{1}{2}\left(1 - \sqrt{1 - \frac{4\alpha}{\Gamma_0}}\right), 0\right)$	0	$-\Gamma_0\sqrt{1 - \frac{4\alpha}{\Gamma_0}}$
,,	$C_3:(x_c, \frac{\Gamma_0 x_c^2 - \Gamma_0 x_c + \alpha}{3(1-x_c)})$	0	$\frac{\alpha(2x_c - 1)}{x_c(1-x_c)}$
4. $Q = \frac{\beta}{H}\rho_m \rho_d$	$A_4:(1, y_c)$	0	$3y_c + \Gamma_0 - 3\beta$
,,	$B_4:(1, 0)$	0	$\Gamma_0 - 3\beta$
,,	$C_4:(1, -1)$	0	$-3 + \Gamma_0 - 3\beta$
,,	$D_4:(x_c, \beta x_c - \frac{\Gamma_0 x_c}{3})$	0	0

TABLE VII. The eigenvalues of the critical points for interaction model 5, $Q = \gamma H_0 \rho_m$.

Critical points	λ_1	λ_2	λ_3
$A_5: (1, y_c, 0)$	0	$\frac{3}{2}(1 + y_c)$	$3y_c + \Gamma_0$
$B_5: (1, -1, v)$	0	0	$-3 + \Gamma_0 - \frac{\gamma v}{1-v}$
$C_5: \left(x_c, -\frac{\Gamma_0 x_c}{3}, 0\right)$	0	0	$\frac{1}{2}(3 - \Gamma_0)$
$D_5: \left(x_c, -1 + \frac{\Gamma_0}{3}(1 - x_c), \frac{-3+\Gamma_0}{-3+\Gamma_0+\gamma}\right)$	0	$\frac{(x_c-1+\sqrt{1-x_c^2})(-3+\Gamma_0)}{2x_c}$	$-\frac{(-x_c+1+\sqrt{1-x_c^2})(-3+\Gamma_0)}{2x_c}$

Tables VI and VII that all critical points are actually a nonisolated set of critical points. It is also noted that Eq. (20) is $\frac{y}{x}$ times Eq. (19), but this does not imply that Eq. (20) is obtained from Eq. (19) as the equation-of-state parameter ($\omega_d = \frac{y}{x}$) is not a constant. As a result, all critical points obtained are nonisolated sets. By definition, a nonisolated set contains at least one vanishing eigenvalue, so it is nonhyperbolic in nature [104]. The type of nonisolated set with exactly one vanishing eigenvalue is called a normally hyperbolic set. Its stability condition is similar to the linear stability analysis and can be determined simply by looking for the signature of the remaining nonvanishing eigenvalues [104]. In this work, all sets of points are normally hyperbolic except sets B_5, C_5, and D_4 (see Tables VI and VII).

A. Interaction 1

The system (27)–(28) admits two sets of critical points: A_1 and E_1. As mentioned earlier, point B_1 is in the set E_1, whereas points C_1 and D_1 are in the set A_1. In what follows, we therefore analyze the stability of sets A_1 and E_1 only.

(i) The solution associated with the set of critical points $A_1(1, y_c)$ (where y_c takes any real value) always exists. They are completely DE-dominated solutions ($\Omega_d = 1$), where DE corresponds to an exotic-type fluid with equation of state $\omega_d = y_c$. For this case, DE can describe quintessence, a cosmological constant, or a phantom field, or any other perfect fluid according to the choice of y_c. So, the critical points may have different features in their cosmic evolutions. Points in this set correspond to an accelerating universe (i.e., $\omega_{\rm eff} < -\frac{1}{3}$) for $y_c < -\frac{1}{3}$ (see Table I), and there exists an expanding universe if the evolution of the Hubble function satisfies $\omega_{\rm eff} < -1$ [see Eq. (25)] (i.e., the Hubble parameter increases gradually) for $y_c < -1$, i.e., in the phantom region. This set is normally hyperbolic and hence corresponds to a late-time attractor for $y_c < \frac{\alpha_m - \Gamma_0}{3}$ (see Table VI). This is also confirmed numerically in Fig. 1(a). The one-dimensional center subspace spanned by the eigenvector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(which corresponds to a vanishing eigenvalue) identifies the direction of the set A_1. The one-dimensional stable subspace near this set is spanned by the eigenvector

$$\begin{pmatrix} 1/y_c \\ 1 \end{pmatrix},$$

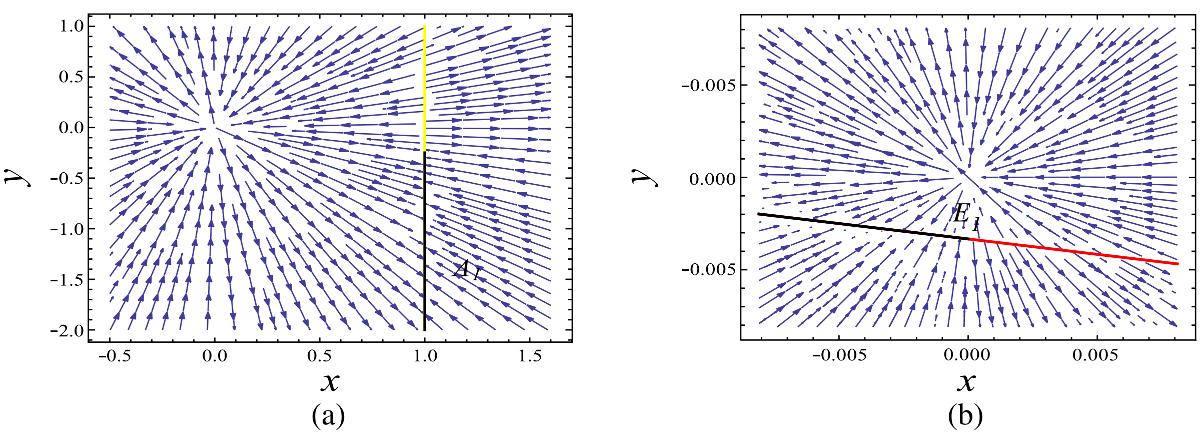


FIG. 1. The vector field of the autonomous system (27)–(28) for the interaction model 1 with the parameter values $\alpha_m = -0.01$ and $\Gamma_0 = 0.5$. In panel (a), the colored line represents the line A_1, and the black line is the stable portion of set A_1. In panel (b), the colored line represents the line E_1, and the black line is the stable portion of set E_1.

which corresponds to a nonvanishing eigenvalue with $y_c < \frac{\alpha_m - \Gamma_0}{3}$. Since there is no unstable subspace near A_1 when $y_c < \frac{\alpha_m - \Gamma_0}{3}$, trajectories will approach some points in set A_1. While the set of critical points A_1 represents a stable attractor in the quintessence region for $y_c < \min\{\frac{\alpha_m - \Gamma_0}{3}, -\frac{1}{3}\}$, they are stable solutions with cosmological-constant behavior for $\Gamma_0 < 3 + \alpha_m$. On the other hand, stable solutions are obtained in the phantom region for $y_c < \min\{-1, \frac{\alpha_m - \Gamma_0}{3}\}$. Hence, the set of critical points represents the solutions of an accelerated stable attractor in some parameter region where DE behaves as quintessence, a cosmological constant, or a phantom field. This is one of the important results in this context of interacting DE since in this scenario DE can mimic three distinct phases of the cosmic evolution. It should be noted that in Fig. 1 the origin (0,0) of the phase space acts as a critical point. As the dynamical system is singular at this point, its stability cannot be determined directly; only numerical investigation can infer its behavior. So, we can say that (0,0) acts as a (nonlinear) critical point of the system.

(ii) The set E_1 exists for all model parameters. It represents a scaling solution where DM and DE scale as $\Omega_m/\Omega_d = (1 - x_c)/x_c$. The DE describes any perfect fluid with equation-of-state parameter $\omega_d = \frac{\alpha_m - \Gamma_0 x_c}{3x_c}$. This set is normally hyperbolic, and hence it is stable when $0 < x_c < 1$, $\alpha_m > 0$; $x_c < 0$, $\alpha_m < 0$; $x_c > 1$, $\alpha_m < 0$. This is confirmed numerically in Fig. 1(b). The one-dimensional stable subspace near this set is spanned by the eigenvector

$$\begin{pmatrix} \frac{3x}{\alpha_m - \Gamma_0 x} \\ 1 \end{pmatrix},$$

with x_c and α_m satisfying the above stability condition. The eigenvector

$$\begin{pmatrix} -\frac{3}{\Gamma_0} \\ 1 \end{pmatrix}$$

(which corresponds to a vanishing eigenvalue) determines the direction of the set. This set describes an accelerated quintessence behavior for $1 < \Gamma_0 - \alpha_m < 3$ $(-1 < \omega_{\mathrm{eff}} < -\frac{1}{3})$, while it represents a cosmological-constant behavior for $\Gamma_0 = 3 + \alpha_m$ $(\omega_{\mathrm{eff}} = -1)$, and phantom behavior for $\Gamma_0 > 3 + \alpha_m$ ($\omega_{\mathrm{eff}} < -1$; see Table I). This set is interesting from the cosmological point of view as it describes a late-time attractor in quintessence, the cosmological constant or, the phantom region for certain choices of α_m and x_c. Interestingly, from Fig. 1 the origin (0,0) behaves as a (nonlinear) critical point of the system. However, at this point the system is singular and hence its stability cannot be determined analytically, but numerically the system behaves as if the origin is not stable.

B. Interaction 2

(i) The autonomous system (30)–(31) only has one set of critical points: B_2. As mentioned earlier, it is to be noted that point A_2 is in the set B_2. So, we shall analyze the stability of set B_2 only. Set B_2 corresponds to a scaling solution and it always exists, except at $x_c = 1$. For this solution, DM and DE scale in a constant fraction as $\Omega_m/\Omega_d = (1 - x_c)/x_c$, where DE behaves as a perfect fluid with the barotropic equation of state $\omega_d = \frac{\Gamma_0 x_c - \Gamma_0 + \alpha_d}{3(1 - x_c)}$ (see Table II). This set corresponds to an accelerated universe if $\frac{\alpha_d x_c}{1 - x_c} < \Gamma_0 - 1$. This set is normally hyperbolic and hence it is stable if $\alpha_d < 0$, $0 < x_c < 1$; $x_c < 0$, $\alpha_d > 0$; $x_c > 1$, $\alpha_d > 0$. This is confirmed numerically from Fig. 2. The eigenvector

$$\begin{pmatrix} -\frac{3(x_c - 1)^2}{\Gamma_0 (x - 1)^2 - \alpha_d} \\ 1 \end{pmatrix}$$

(which corresponds to a vanishing eigenvalue) determines the direction of the tangent at each point of the set. The one-dimensional stable subspace near this set is spanned by the eigenvector

$$\begin{pmatrix} \frac{3(1 - x)}{\Gamma_0 (x - 1) + \alpha_d} \\ 1 \end{pmatrix},$$

with α_d and x_c satisfying the above stability condition. So, depending on the choice of α_d and x_c this set can explain the late-time behavior of our Universe.

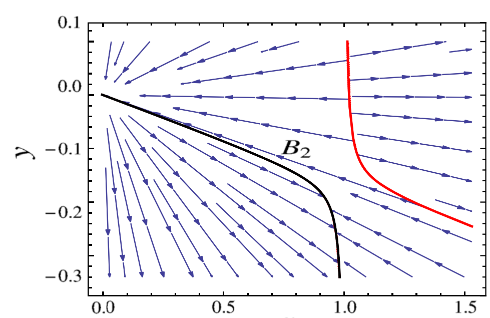


FIG. 2. The vector field of the autonomous system (30)–(31) for the interaction model 2 with the parameter values $\alpha_d = -0.01$ and $\Gamma_0 = 0.5$. The colored curve represents the set B_2, and the black curve is the stable portion of set B_2.

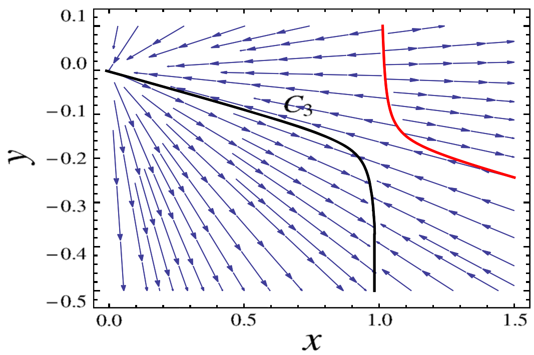


FIG. 3. The vector field of the autonomous system (33)–(34) for the interaction model 3 with the parameter values $\alpha = -0.01$ and $\Gamma_0 = 0.5$. The colored curve represents the set C_3, and the black curve is the stable portion of set C_3.

C. Interaction 3

(i) The system (33)–(34) admits only one critical set of points C_3. As mentioned earlier, we see that the points A_3 and B_3 are points in the set C_3. So, we shall analyze the stability of set C_3 only. One interesting point for this solution C_3 is that it is a combination of DM and DE with the ratio $\frac{\Omega_m}{\Omega_d} = \frac{1-x_c}{x_c}$, and will exist for all model parameters except $x_c = 1$. An accelerating universe is predicted by the set when $\frac{\alpha}{1-x_c} < \Gamma_0 - 1$. This set is normally hyperbolic and it is stable if $0 < x_c < \frac{1}{2}$, $\alpha > 0$; $\frac{1}{2} < x_c < 1$, $\alpha < 0$; $x_c < 0$, $\alpha < 0$; $x_c > 1$, $\alpha > 0$. Hence, this set provides some interesting features for both positive and negative couplings of the interaction. Its stability is confirmed numerically in Fig. 3. The one-dimensional stable subspace near this set is spanned by the eigenvector

$$\begin{pmatrix} \frac{3x_c(1-x_c)}{\Gamma_0 x_c(x_c-1)+\alpha} \\ 1 \end{pmatrix}$$

(which corresponds to a nonvanishing eigenvalue), with α and x_c satisfying the above stability condition. The direction of the tangent at each point on the set is along the eigenvector

$$\begin{pmatrix} \frac{3(x_c-1)^2}{\alpha-\Gamma_0(x_c-1)^2} \\ 1 \end{pmatrix},$$

corresponding to a vanishing eigenvalue. So, depending on the choice of parameters and fine-tuning of initial conditions, trajectories near this set approach points of this set. Hence, some critical points on this set correspond to a late-time accelerated universe.

D. Interaction 4

There are two sets of critical points arising from the interaction model 4 (A_4 and D_4). As mentioned earlier, points B_4 and C_4 are points in the set A_4. So, in what follows we shall analyze the stability of sets A_4 and D_4 only.

(i) The set of critical points A_4 exists for all model parameters involved. It represents a DE-dominated solution ($\Omega_d = 1$). This DE-dominated solution describes the late-time acceleration of the Universe when DE behaves as quintessence, a cosmological constant, or a phantom or any other exotic fluid for $y_c < -\frac{1}{3}$. This set is again normally hyperbolic and it is stable when $y_c < \frac{\alpha_m - \Gamma_0}{3}$. The stability of A_4 is confirmed numerically in Fig. 4. The one-dimensional stable subspace near this set is spanned by the eigenvector

$$\begin{pmatrix} 1/y_c \\ 1 \end{pmatrix},$$

where $y_c < \frac{\alpha_m - \Gamma_0}{3}$. The eigenvector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

corresponds to a vanishing eigenvalue and determines the direction of the set. There is no unstable subspace near this set for $y_c < \frac{\alpha_m - \Gamma_0}{3}$. This means that depending on the choice of α_m, Γ_0, and y_c trajectories approach some points on this set.

(ii) The solution represented by the point D_4 is a combination of both DE and DM with the constant ratio $\frac{\Omega_m}{\Omega_d} = \frac{1-x_c}{x_c}$, where DE describes any perfect fluid with equation of state $\omega_d = \beta - \frac{\Gamma_0}{3}$. The set exists for all model parameters. Depending on some parameter restrictions, an acceleration will occur for the set, but since both eigenvalues vanish we do not obtain any

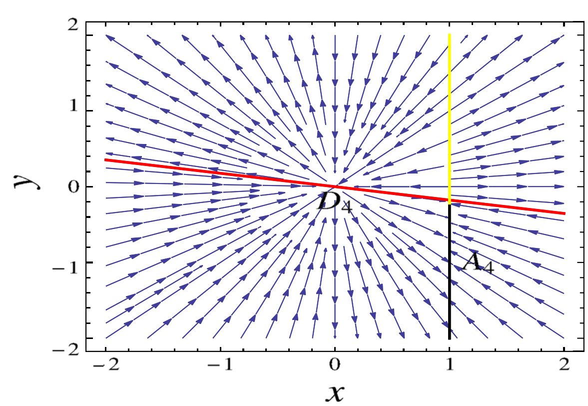


FIG. 4. The vector field of the autonomous system (36)–(37) for the interaction model 4 with the parameter values $\beta = -0.01$ and $\Gamma_0 = 0.5$. The colored lines represent A_4 and D_4, and the black curve is the stable portion of set A_4.

information regarding the stability of D_4 (see Fig. 4). It behaves as a neutral line.

E. Interaction 5

(i) From the local interaction model 5, we get four sets of critical points presented in Table V. The set A_5 (similarly to A_1 and A_4) is a completely DE-dominated solution. It corresponds to a phantom universe for $y_c < -1$, and it corresponds to a quintessence-dominated phase for $-1 < y_c < -\frac{1}{3}$. The DE associated with this set can mimic any kind of fluid for different choices of y_c. It is a normally hyperbolic set and hence it is stable if (a) $\Gamma_0 \le 3$ and $y_c < -1$, or (b) $\Gamma_0 > 3$ and $y_c < -\frac{\Gamma_0}{3}$. This means that the set will be stable only in the phantom regime. Furthermore, in this region the set becomes physically relevant, describing the late-time accelerated expansion of the Universe. The two-dimensional stable subspace is spanned by the eigenvectors

$$\begin{pmatrix} 1/y_c \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(corresponding to two nonvanishing eigenvalues), where y_c satisfies the above stability condition. The one-dimensional center subspace is spanned by the eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(corresponding to a vanishing eigenvalue) and determines the direction of set A_5. Hence, this set can describe the late-time behavior of our Universe.

(ii) The set of points B_5 exists for all values of model parameters. This solution is completely DE dominated, where DE behaves as a cosmological constant. There always exists an accelerating universe ($\omega_{\rm eff} = -1$; see Table V). It is a nonisolated set of critical points where all points are nonhyperbolic, but it is not a normally hyperbolic set since it contains two vanishing eigenvalues. A numerical projection plot of the system (41)–(43) shows that this set cannot be stable. It can be seen that in (x, v) phase space, trajectories are attracted to the set B_5 [see Fig. 5(a)]; however, trajectories in (y, v) phase space are not attracted to the set B_5 [see Fig. 5(b)]. We have checked that this actually happens for different choices of model parameters. This implies that points of this set are saddle points.

(iii) The set of critical points C_5 exists for all values of model parameters. The set corresponds to a solution with both DE and DM in the phase space where DE behaves as any perfect fluid model with an equation-of-state parameter $\omega_d = -\Gamma_0/3$. Hence, it is clear that the DE may have different features during its evolution, such as quintessence in the parameter region $1 < \Gamma_0 < 3$, a cosmological constant for $\Gamma_0 = 3$, and the phantom regime for $\Gamma_0 > 3$. Now the expansion of the Universe is accelerated for $\Gamma_0 > 1$ ($\omega_{\rm eff} < -\frac{1}{3}$). This set is again nonhyperbolic but not normally hyperbolic. Numerically, by plotting the projection of trajectories in the (x, y) plane (see Fig. 6), we observe that points in this set are saddle points.

(iv) The set of points D_5 is the combination of both DE and DM. This set behaves as a cosmological constant (i.e., $\omega_{\rm eff} = -1$; see Table V), and hence there is always an accelerating universe near this set. Also, the set of critical points under consideration is a normally hyperbolic set. Hence, it is a stable spiral if $\Gamma_0 < 3$, $x_c > 1$ or $\Gamma_0 < 3$, $x_c < -1$, and it is a stable node if $-1 < x_c < 0$, $\Gamma_0 < 3$ or $0 < x_c < 1$, $\Gamma_0 > 3$.

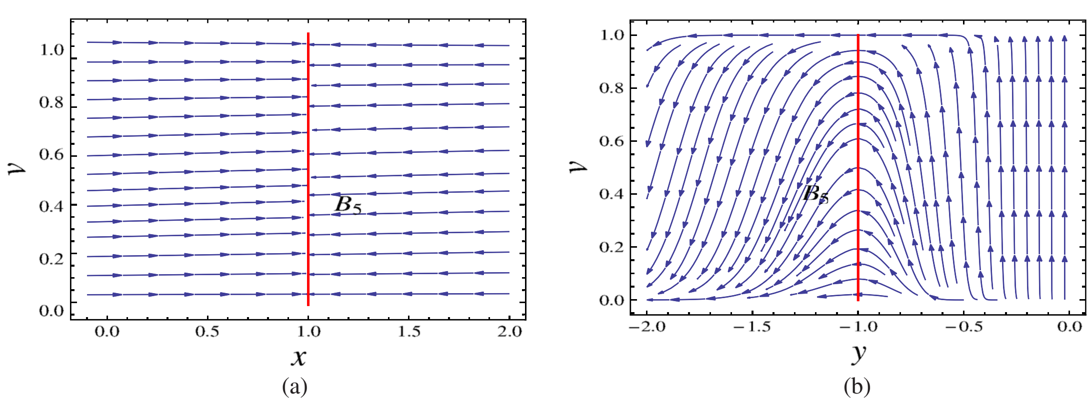


FIG. 5. The vector field projection in (a) xv phase plane and (b) yv phase plane of the autonomous system (41)–(43) for the interaction model 5 with the parameter values $\gamma = 0.5$ and $\Gamma_0 = 0.001$.

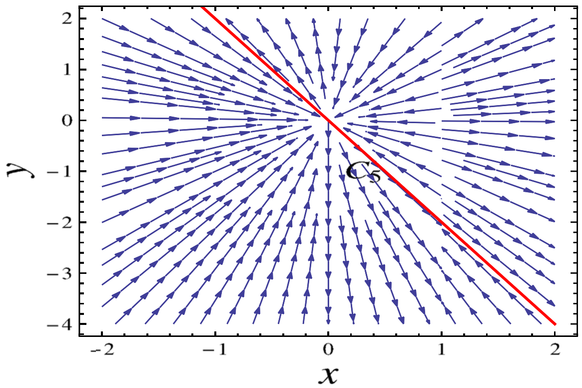


FIG. 6. The vector field projection in xy phase plane of the autonomous system (41)–(43) for the interaction model 5 with the parameter values $\gamma = 1$ and $\Gamma_0 = 6$. It may be noted that we take Γ_0 to be very large for this particular plot simply to check its instability, since for $\Gamma_0 < 3$, the eigenvalue $\lambda_3 > 0$ and it will surely be unstable.

V. COSMOLOGICAL IMPLICATIONS

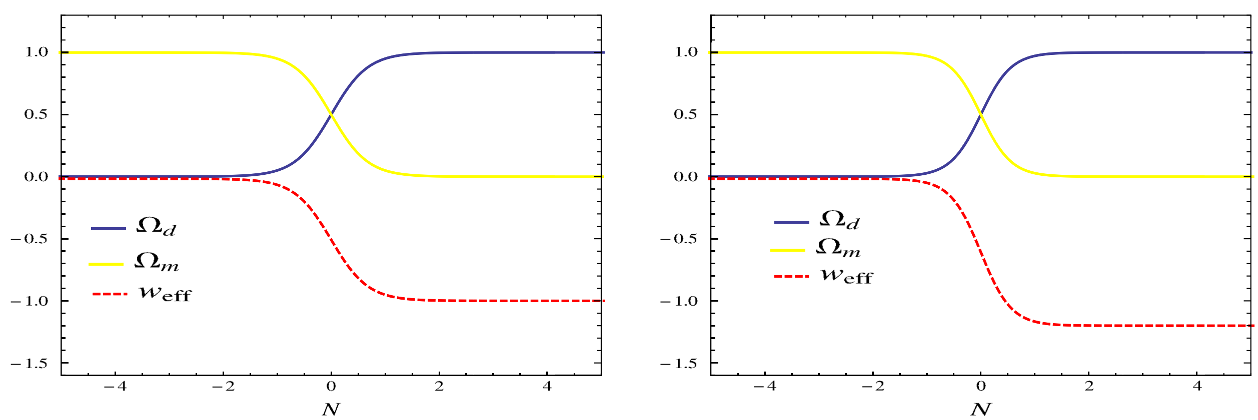
In this section, we shall describe the main cosmological features extracted from the interacting dark energy models in the presence of gravitational particle production. In the following subsections, we shall describe the physics of the critical points for each interacting model in this framework along with their viability to describe different cosmic phases. An interesting feature is that in all of the interaction models, the evolutions of Ω_m, Ω_d, and $\omega_{\rm eff}$ (see Fig. 7) are similar and so we have not plotted them for each interaction model.

A. Interaction model 1

In this model we obtained two set of critical points: A_1 and E_1. Set A_1 represents a de Sitter universe for $y_c = -1$ (point D_1), it represents a stiff matter-dominated universe for $y_c = 1$ and for $y_c = 0$ (i.e., point B_1); we actually get a DE-dominated universe ($\Omega_d = 1$), but it appears as if it is a matter-dominated solution ($\omega_{\rm eff} = 0$). We also note that the critical point B_1 is a special case of set E_1 and it corresponds to a matter-dominated universe for $\alpha_m = 0$ when no interaction between DE and DM is considered. Moreover, set E_1 also represents a matter-dominated universe for $x_c = 0$ [i.e., the point $(0, \frac{\alpha_m}{3})$]. From the analysis performed in Sec. IV, we see that—depending on the choice of model parameters and the fine-tuning of the initial conditions—the Universe evolves from a matter-dominated phase (set E_1) to a DE-dominated phase (set A_1), for either a quintessence regime for $-1 < y_c < -\frac{1}{3}$, a cosmological constant for $y_c = -1$, or a phantom regime for $y_c < -1$ (see, e.g., Fig. 7). Hence, we observe that the background dynamics of this model can possibly mimic the ΛCDM model [see Fig. 7(a)]. Moreover, there is a possibility of crossing the phantom barrier [$\omega_{\rm eff} = -1$; see Fig. 7(b)], which is slightly favored by observations and cannot be achieved in the case of noninteracting DE models. Hence, this model can well describe the late-time transition from a DM- to a DE-dominated phase of the Universe.

B. Interaction model 2

In this model, there is only one set of critical points: B_2. This set represents a DM-dominated universe when $x_c = 0$, i.e., when we consider the origin (0,0) as a critical point. Unfortunately, we do not obtain any information regarding the stability of the point (0,0) as both eigenvalues vanish for this particular point. However, from Fig. 2 it looks like the point (0,0) is not stable. The critical point A_2 is a special case of set B_2, and it represents a DE-dominated but decelerated universe ($\Omega_d = 1$, $\omega_{\rm eff} = 0$) for $\alpha_d = 0$ when



(a) (b)

FIG. 7. The evolution of the DE energy density parameter Ω_d, the DM energy density parameter Ω_m, and the effective equation of state $\omega_{\rm eff}$ of the system (27)–(28) for the interaction model 1 with the parameter values $\alpha_m = -0.001$ and $\Gamma_0 = 0.05$ with different choices of initial conditions. Panel (a) shows the cosmological constant as a late-time attractor, and panel (b) shows the phantom regime as a late-time attractor.

there is no coupling between DE and DM. Set B_2 can represent a late-time accelerated scaling solution for $\alpha_d < 0$ and $0 < x_c < 1$. However, viable trajectories are attracted to B_2 near the limit $x_c = 1$, i.e., it is a DE-dominated universe (see Fig. 2). So, for this model the Universe evolves from a matter-dominated solution (set B_2 for $x_c = 0$) towards a DE-dominated solution (set B_2 for limit $x_c \to 1$) (a similar scenario is obtained for this model as in Fig. 7).

C. Interaction model 3

The background cosmological behavior of this model is similar to interaction model 2. In this model, there is only one set of critical points: C_3. This set represents a DM-dominated universe when $x_c = 0$ for which no information is obtained regarding its stability, as both eigenvalues vanish for this particular case. However, numerically it can be seen from Fig. 3 that this set is not stable. The critical points A_3 and B_3 are special cases of set C_3 and correspond to scaling solutions. The point A_3 corresponds to a DE-dominated but decelerated universe ($\Omega_d = 1$, $\omega_{\rm eff} = 0$) for $\alpha = 0$. So, when no interaction is considered this point corresponds to a DE-dominated universe, but the universe expands as if it was matter dominated. The point B_3 corresponds to a DM-dominated universe for $\alpha = 0$. Set C_3 can represent a late-time accelerated scaling solution for some choices of α and x_c. It corresponds to a DM-dominated universe for $x_c = 0$. Moreover, viable trajectories are attracted to C_3 near the limit $x_c = 1$ (see Fig. 3). So, for this model the Universe evolves from a matter-dominated solution (set C_3 for $x_c = 0$) towards a DE-dominated solution (set C_3 for $x_c \to 1$).

D. Interaction model 4

In this model we obtained two sets of critical points: A_4 and D_4. Set A_4 represents a stiff matter-dominated universe for $y_c = 1$. It also represents a de Sitter universe for $y_c = -1$ (i.e., the point C_4). For $y_c = 0$ (i.e., the point B_4), this set corresponds to a DE-dominated universe ($\Omega_d = 1$), but the universe appears as if it was matter dominated ($\omega_{\rm eff} = 0$). The set of critical points D_4 behaves as a neutral line, but its stability cannot be determined as all of its eigenvalues vanish. However, for $x_c = 0$ [i.e., the point (0,0)] it corresponds to a matter-dominated universe. Even though its stability cannot be determined analytically, numerically we can see that the origin is not stable (see Fig. 4). Hence, we see that depending on the choice of model parameters and the fine-tuning of the initial conditions, the Universe evolves from a matter -dominated phase (set D_4) to a DE-dominated phase (set A_4), for either a quintessence regime for $-1 < y_c < -\frac{1}{3}$, a cosmological constant for $y_c = -1$, or a phantom regime for $y_c < -1$.

E. Interaction model 5

In this model we obtained four sets of critical points: A_5, B_5, C_5, and D_5. Set A_5 corresponds to a late-time attractor where DE dominates only in the phantom regime. The critical points in sets B_5 and C_5 behave as saddle points, and interestingly set C_5 corresponds to a matter-dominated universe for $x_c = 0$. Set D_5 also corresponds to a late-time accelerated solution for some choices of model parameters. Hence, depending on the initial conditions and the choices for the model parameters, we see that the universe can evolve from a matter-dominated phase (set B_5 or C_5) to a DE-dominated phase (set D_5). Physically, this means that there is a transition from DM to DE domination in the late universe.

VI. SHORT DISCUSSION

In the present work, we have performed a dynamical system analysis for the scenarios of interacting dark matter and dark energy, where additionally gravitational particle production is also allowed. The particle-creation mechanism describes many interesting results, such as the possibility of a phantom universe without invoking any phantom field, the formation of an emergent universe, a complete cosmic scenario, etc. Here, we have considered the dark matter fluid as dust and the dark energy as a perfect fluid with equation of state ω_d. Moreover, the particles created by the gravitational field have been considered to be dark matter particles (equivalently, dust particles) in agreement with the local gravity constraints, and the production rate was taken to be varying linearly with the Hubble function (i.e., $\Gamma \propto H$). We have considered five interacting models which correspond to five distinct forms of interaction Q. The objective for choosing such a complex system was to examine whether there is any model (interacting) which could explain the overall evolution of the universe. In particular, a complete description of evolution at late times can be obtained in quintessence, ΛCDM, or the phantom era connected through a DM-dominated era. Critical points, their existence, and their corresponding cosmological parameters are shown in Tables I–V for the respective models. Additionally, we have presented the eigenvalues for different interaction models in Tables VI and VII. A detailed stability analysis was carried out in Sec. IV. It was also noted that all sets of points except D_4, B_5, and C_5 are normally hyperbolic, where stability is confirmed by the signature of the remaining nonvanishing eigenvalues.

We found that the sets of critical points A_1, A_4, and A_5 correspond to a DE-dominated universe where DE could mimic a quintessence era, a cosmological constant, a phantom phase, sometimes dust, or even any other exotic fluid. However, it was found that some of the critical points in the above set of critical points representing the above cosmic phases (i.e., quintessence, cosmological constant, or phantom phase) could describe the late-time expansion of the universe but they cannot alleviate the coincidence problem. On the other hand, some sets of critical points (E_1, B_2, C_3, D_5) can possibly represent scaling solutions for $0 < x_c < 1$ with an accelerated expansion of the universe. However, we observed that trajectories are attracted to a portion of sets where $x_c \approx 1$ and hence the critical points of these sets cannot alleviate the coincidence problem (since $\Omega_d \approx 1$). Moreover, critical points in the sets E_1, B_2, C_3, and D_5 with $x_c = 0$ represent a DM-dominated universe. Our stability analysis (in Sec. IV) showed that for some choices of model parameters and fine-tuning of initial conditions, one can connect these DM-dominated solutions to DE-dominated solutions (A_1, A_4, A_5 or B_2, C_3 for the limit $x_c \to 1$) which can possibly mimic quintessence, a cosmological constant, or a phantom phase. It may be noted that phenomenologically interesting solutions depend on the strong fine-tuning of the initial conditions, since all the critical points lie in different nonisolated sets and only a few points describe the correct observed cosmological dynamics.

Thus, in summary, one may conclude that the present interacting DE model in the framework of the particle-creation mechanism may describe different evolutionary phases of the universe. These interacting models can possibly allow the crossing of the phantom divide line [see Fig. 7(b) which shows the clear cosmic evolution of the physical quantities $\omega_{\rm eff}$, Ω_d, Ω_m], which is not possible in the case of uncoupled standard cosmology. The present particle-creation mechanism describes the true nonequilibrium thermodynamics of the universe compared to other standard DE models. As a result, the present model shows stable critical points representing various cosmological scenarios. Moreover, the background dynamics of these interacting models can possibly mimic the ΛCDM model but only for $y_c = -1$, so there might be some differences at the level of perturbations. However, cosmological perturbation analysis lies beyond the scope of our present study. This can be left for future works.

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