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Action principle for action-dependent Lagrangians toward nonconservative gravity: Accelerating universe without dark energy

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In the present work, we propose an action principle for action-dependent Lagrangians by generalizing the Herglotz variational problem for several independent variables. This action principle enables us to formulate Lagrangian densities for nonconservative fields. In particular, from a Lagrangian depending linearly on the action, we obtain generalized Einstein field equations for nonconservative gravity and analyze some consequences of their solutions for cosmology and gravitational waves. We show that the nonconservative part of the field equations depends on a constant cosmological four-vector. Depending on this four-vector, the theory displays damped/amplified gravitational waves and an accelerating Universe without dark energy.

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The action principle was introduced in its mature formulation by Euler, Hamilton, and Lagrange and, since then, it has become a fundamental principle for the construction of all physical theories. In order to obtain the dynamical equations of any theory, the Lagrangian defining the action is constructed from the scalars of the theory. In this case, the action itself is a scalar. Consequently, we might ask: what would happen if the Lagrangian itself is a function of the action? The answer to this question can be given by the action principle proposed by Herglotz [1–3]. The Herglotz variational calculus consists in the problem of determining the path $x(t)$ that extremizes (minimizes or maximizes) $S(b)$, where $S(t)$ is a solution of

$$\dot{S}(t) = L(t, x(t), \dot{x}(t), S(t)), \qquad t \in [a, b],$$
$$S(a) = s_a, x(a) = x_a, x(b) = x_b, \qquad s_a, x_a, x_b \in \mathbb{R}. \quad (1)$$

It is easy to note that Eq. (1) represents a family of differential equations since for each function $x(t)$ a different differential equation arises. Therefore, $S(t)$ is a functional. The problem reduces to the classical fundamental problem of the calculus of variations if the Lagrangian function L does not depend on $S(t)$. In this case we have $\dot{S}(t) = L(t, x(t), \dot{x}(t))$, and by integrating we obtain the classical variational problem

$$S(b) = \int_a^b \tilde{L}(t, x(t), \dot{x}(t))dt \to \text{extremum}, \quad (2)$$

where $x(a) = x_a$, $x(b) = x_b$, and

$$\tilde{L}(t, x(t), \dot{x}(t)) = L(t, x(t), \dot{x}(t)) + \frac{s_a}{b-a}. \quad (3)$$

It is important to notice from Eq. (2) that for a given fixed function $x(t)$ the functional S reduces to a function of the domain boundary a, b. Herglotz proved [1,2] that a necessary condition for a path $x(t)$ to be an extremizer of the variational problem (1) is given by the generalized Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial S}\frac{\partial L}{\partial \dot{x}} = 0. \quad (4)$$

In the simplest case where the dependence of the Lagrangian function on the action is linear, the Lagrangian describes a dissipative system and, from Eq. (4), the resulting equation of motion includes the well-known dissipative term proportional to $\dot{x}$. It should also be noticed that in the case of the classical problem of the calculus of variations (2) one has $\frac{\partial L}{\partial S} = 0$, and the differential equation (4) reduces to the classical Euler-Lagrange equation.

In what follows we will be interested in a more general problem where the Lagrangian function depends on several independent variables $x^1, x^2, \ldots, x^d$ $(d = 1, 2, 3, \ldots)$. Besides (as we are especially interested in the problem of gravity), we will consider a curved space with metric $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2, \ldots, x^d)$ defined on a domain $\Omega \subset \mathbb{R}^d$. Thus, the classical problem of the calculus of variations deals with the problem of finding $g_{\alpha\beta}$ that extremize the functional

$$S(\delta\Omega) = \int_\Omega \mathcal{L}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})\sqrt{\ } d^d x, \quad (5)$$

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where $g_{\alpha\beta,\mu} = \partial_\mu g_{\alpha\beta}$, $\sqrt{} = \sqrt{|g|}$, $\delta\Omega$ is the boundary of Ω, and $g_{\alpha\beta}$ satisfy the boundary condition $g_{\alpha\beta}(\delta\Omega) = g_{\alpha\beta}^{\delta\Omega}$ with $g_{\alpha\beta}^{\delta\Omega}$: $\delta\Omega \to \mathbb{R}$. Unfortunately, despite the fact that the Herglotz problem was introduced in 1930, a covariant generalization of Eq. (1) for several independent variables is not direct and is lacking up to now. In order to generalize the Herglotz problem for fields, let us first note that, as in Eq. (2), for a given fixed $g_{\alpha\beta}$ the functional S defined in Eq. (5) reduces to a function of the boundary $\delta\Omega$. Let us now consider that $\delta\Omega$ is an orientable Jordan surface with normal n^μ. If there is a differentiable vector field s^μ such that

$$S(\delta\Omega) = \int_{\delta\Omega} n_\nu s^\nu \sqrt{|h|} d^{d-1}x, \tag{6}$$

where $\sqrt{|h|}$ is the induced metric over $\delta\Omega$, then we obtain

$$\begin{aligned} S(\delta\Omega) &= \int_{\delta\Omega} n_\nu s^\nu \sqrt{|h|} d^{d-1}x = \int_\Omega \nabla_\nu s^\nu \sqrt{} d^d x \\ &= \int_\Omega \mathcal{L}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}) \sqrt{} d^d x, \end{aligned} \tag{7}$$

where we used Stokes' theorem and ∇_ν stands for a covariant derivative. Consequently, we can generalize the action principle by stating that the space-time metric $g_{\mu\nu}$ is that which extremizes the action $S(\delta\Omega)$ given by

$$\begin{aligned} &\nabla_\nu s^\nu = \mathcal{L}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu), \qquad x^\mu \in \Omega, \\ &S(\delta\Omega) = \int_{\delta\Omega} n_\nu s^\nu \sqrt{|h|} d^{d-1}x, \qquad g_{\alpha\beta}(\delta\Omega) = g_{\alpha\beta}^{\delta\Omega}, \end{aligned} \tag{8}$$

where $g_{\alpha\beta}^{\delta\Omega}$ is fixed. It is important to notice that our action principle (8) [that generalizes Eq. (1) for fields] reduces to the classical action principle if the Lagrangian is independent of s^μ. Furthermore, for the case where $s^\nu = (s^0, 0, 0, 0)$ and $\Omega = [t_a, t_b] \otimes \mathbb{R}^3$, Eq. (8) contains as a particular case the noncovariant problem introduced in Ref. [3]. Moreover, in this last situation Eq. (8) can be easily solved for Lagrangians linear in s^0, giving a s^0 expressed as a history-dependent function of the source.

For the gravity field, the Lagrangian we propose is given by $\mathcal{L} = \mathcal{L}_m + \mathcal{L}_g$, where $\mathcal{L}_m$ is the Lagrangian for matter and

$$\mathcal{L}_g(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu) = R - \lambda_\nu s^\nu, \tag{9}$$

where λ_ν is a constant cosmological four-vector. In Eq. (9), $R = \tilde{L} - L$ is the Ricci scalar with $\tilde{L} = g^{\mu\nu}(\Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma})$ and $L = g^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho})$. Since the second-order derivatives in Eq. (9) occur only linearly in the Lagrangian, the field equations can be obtained by an effective Lagrangian $\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{ef}$ with

$$\mathcal{L}_{ef}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu) = L - \lambda_\nu s^\nu, \tag{10}$$

instead of Eq. (9), because $\int_\Omega \tilde{L}\sqrt{} d^d x = 2\int_\Omega L\sqrt{} d^d x +$ constant (see Ref. [4]).

In order to obtain the generalized field equations, let us define a family of metrics $g_{\alpha\beta}$ such that

$$g_{\alpha\beta}(x^\mu) = g^*_{\alpha\beta}(x^\mu) + \delta_\epsilon(g_{\alpha\beta})(x^\mu), \tag{11}$$

where $g^*_{\alpha\beta}$ is the metric that extremizes $S(\delta\Omega)$ in Eq. (8), $\epsilon \in \mathbb{R}$, and $\delta_\epsilon(g_{\alpha\beta})$ satisfies the boundary condition $\delta_\epsilon(g_{\alpha\beta})(\delta\Omega) = 0$ and $\lim_{\epsilon\to 0}\delta_\epsilon(g_{\alpha\beta})(x^\mu) = 0$ (weak variations). Since $S(\delta\Omega)$ attains an extremum at $g^*_{\alpha\beta}$, we have

$$\lim_{\epsilon\to 0} \frac{\delta_\epsilon(S)(\delta\Omega)}{\epsilon} = 0. \tag{12}$$

From Eq. (6), we get

$$\lim_{\epsilon\to 0} \frac{\delta_\epsilon(S)(\delta\Omega)}{\epsilon} = \int_{\delta\Omega} n_\nu \lim_{\epsilon\to 0} \frac{\delta_\epsilon(s^\nu)}{\epsilon} \sqrt{|h|} d^{d-1}x = 0 \tag{13}$$

since the surface $\delta\Omega$, and consequently $\sqrt{|h|}$, is independent of ϵ. A sufficient condition to satisfy Eq. (13) for an arbitrary boundary $\delta\Omega$ is

$$\lim_{\epsilon\to 0} \frac{\delta_\epsilon(s^\nu)(\delta\Omega)}{\epsilon} = 0. \tag{14}$$

On the other hand, by integrating over Ω both sides of the differential equation in Eq. (8) we obtain

$$S(\delta\Omega) = \int_\Omega \mathcal{L}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu) \sqrt{} d^d x, \tag{15}$$

and by taking the variation of Eq. (15) we get

$$\begin{aligned} \delta_\epsilon(S) &= \int_\Omega \delta_\epsilon(\mathcal{L}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu)\sqrt{}) d^d x \\ &= \int_\Omega [\delta_\epsilon(L\sqrt{}) + \delta_\epsilon(\mathcal{L}_m\sqrt{}) - \lambda_\nu \delta_\epsilon(s^\nu\sqrt{})] d^d x. \end{aligned} \tag{16}$$

We also have from Eq. (7), by using $\nabla_\nu(\cdot)\sqrt{} = \partial_\nu(\cdot\sqrt{})$,

$$\delta_\epsilon(S) = \delta_\epsilon \int_\Omega \nabla_\nu s^\nu \sqrt{} d^d x = \int_\Omega \partial_\nu \delta_\epsilon(s^\nu \sqrt{}) d^d x. \tag{17}$$

From Eqs. (16) and (17) we obtain

$$\int_\Omega [\partial_\nu \delta_\epsilon(s^\nu\sqrt{}) - \delta_\epsilon((L + \mathcal{L}_m)\sqrt{}) + \lambda_\nu \delta_\epsilon(s^\nu\sqrt{})] d^d x = 0. \tag{18}$$

Since Eq. (18) should be satisfied for any domain Ω, we have

$$\partial_\nu \zeta^\nu = \delta_\epsilon(L\sqrt{\ }) + \delta_\epsilon(\mathcal{L}_m\sqrt{\ }) - \lambda_\nu \zeta^\nu, \tag{19}$$

where $\zeta^\nu = \delta_\epsilon(s^\nu \sqrt{\ })$. Due to the fact that λ_ν is a constant four-vector, Eq. (19) implies that ζ^ν can be written as

$$\zeta^\nu(\epsilon) = A^\nu(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, s^\mu)e^{-\lambda_\gamma x^\gamma}, \tag{20}$$

where

$$\partial_\nu A^\nu = (\delta_\epsilon(L\sqrt{\ }) + \delta_\epsilon(\mathcal{L}_m\sqrt{\ }))e^{\lambda_\gamma x^\gamma}. \tag{21}$$

From Eq. (14) we should have, since $\delta_\epsilon(g_{\mu\nu})(\delta\Omega) = 0$,

$$\zeta^\nu(0) = A^\nu|_{\epsilon=0}e^{-\lambda_\gamma x^\gamma} = 0 \tag{22}$$

for all $x^\mu \in \delta\Omega$. As a consequence, A^ν is identically zero over $\delta\Omega$. In this case, we obtain from Stokes' theorem

$$\int_{\delta\Omega} n_\nu \frac{A^\nu}{\sqrt{\ }} \sqrt{|h|} d^{d-1}x = \int_\Omega \partial_\nu A^\nu d^d x = 0. \tag{23}$$

Thus,

$$\begin{aligned} &\int_\Omega \delta_\epsilon(L\sqrt{\ } + \mathcal{L}_m\sqrt{\ })e^{\lambda_\gamma x^\gamma} d^d x \\ &= \int_\Omega [\Gamma^\alpha_{\mu\nu}\delta_\epsilon(g^{\mu\nu}\sqrt{\ })_{,\alpha} - \Gamma^\alpha_{\mu\alpha}\delta_\epsilon(g^{\mu\nu}\sqrt{\ })_{,\nu} \\ &\quad + (\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{\mu\nu})\delta_\epsilon(g^{\mu\nu}\sqrt{\ }) \\ &\quad + 8\pi G T_{\mu\nu}\delta_\epsilon(g^{\mu\nu})\sqrt{\ }]e^{\lambda_\gamma x^\gamma} d^d x \\ &= -\int_\Omega \delta_\epsilon(g^{\mu\nu})\sqrt{\ }\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + K_{\mu\nu} \right. \\ &\quad \left. - \frac{1}{2}g_{\mu\nu}K - 8\pi G T_{\mu\nu}\right] e^{\lambda_\gamma x^\gamma} d^d x \\ &\quad + \int_\Omega [(\Gamma^\alpha_{\mu\nu}\delta_\epsilon(g^{\mu\nu}\sqrt{\ }) \\ &\quad - \Gamma^\nu_{\mu\nu}\delta_\epsilon(g^{\mu\alpha}\sqrt{\ }))e^{\lambda_\gamma x^\gamma}]_{,\alpha} d^d x = 0, \end{aligned} \tag{24}$$

where we define the symmetric tensor $K_{\mu\nu} = \lambda_\alpha\Gamma^\alpha_{\mu\nu} - \frac{1}{2}(\lambda_\nu\Gamma^\alpha_{\mu\alpha} + \lambda_\mu\Gamma^\alpha_{\nu\alpha})$, and $\delta_\epsilon(\mathcal{L}_m\sqrt{\ }) = \frac{8\pi G}{c^4}T_{\mu\nu}\delta_\epsilon(g^{\mu\nu})\sqrt{\ }$, where $T_{\mu\nu}$ is the energy-momentum tensor. The last integral in Eq. (24) is zero since $\delta_\epsilon(g_{\mu\nu})(\delta\Omega) = 0$. Thus, from the fundamental lemma of the calculus of variations we obtain from Eq. (24) the generalized gravitational field equation

$$R_{\mu\nu} + K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + K) = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{25}$$

It is important to remark that the generalized gravity field (25) depending on the cosmological four-vector λ_μ can be used to describe nonconservative phenomena, since the covariant divergence $\nabla_\mu(K^\mu_\nu - \frac{1}{2}g^\mu_\nu K)$ is in general different from zero for $\lambda_\mu \neq 0$. A notable consequence of this nonconservation is that the space-time manifold behaves similarly to an imperfectly elastic rubber sheet. In order to shed light on the effects of the nonconservation on the geometrical side of the field equation (25) when $\lambda_\mu \neq 0$, it is interesting to investigate the behavior of gravitational waves. We suppose the metric to be close to the Minkowski one [5], i.e., $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. To first order in h, by choosing the modified harmonic gauge $\eta^{\mu\nu}(h_{\mu\rho,\nu} - \frac{1}{2}h_{\mu\nu,\rho} + \lambda_\mu h_{\nu\rho} - \frac{1}{2}\lambda_\rho h_{\mu\nu}) = 0$, we obtain from the field equations

$$\Box^2 h_{\mu\nu} + \lambda^\rho h_{\mu\nu,\rho} = -\frac{16\pi G}{c^4}S_{\mu\nu}, \tag{26}$$

where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda$. For simplicity, let us consider only the homogeneous case $S_{\mu\nu} = 0$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and a gravitational wave traveling in the $x^3 = z$ direction. In this case $h_{\mu\nu}$ is a function of t and z, and we also have $h_{0\mu} = h_{3\mu} = 0$. From the wave equation (26) we obtain three possible solutions for $h_{\mu\nu}$:

$$h_{\mu\nu}(t,z) = \begin{cases} h^{(\pm)}_{\mu\nu}e^{-\frac{\lambda_0 \pm \lambda'}{2}ct}e^{ikz} & \text{if } \lambda_0^2 > 4k^2, \\ (h^{(+)}_{\mu\nu} + h^{(-)}_{\mu\nu}ct)e^{-\frac{\lambda_0}{2}ct}e^{ikz} & \text{if } \lambda_0^2 = 4k^2, \\ h^{(\pm)}_{\mu\nu}e^{-\frac{\lambda_0 \pm i\lambda'}{2}ct}e^{ikz} & \text{if } \lambda_0^2 < 4k^2, \end{cases}$$

where $\lambda' \equiv \sqrt{|\lambda_0^2 - 4k^2|}$, and $h^{(\pm)}_{\mu\nu}$ are constant symmetric tensors with non-null components $h^{(\pm)}_{11}$, $h^{(\pm)}_{22} = -h^{(\pm)}_{11}$, and $h^{(\pm)}_{12}$. When $\lambda_0 > 0$ ($\lambda_0 < 0$) we observe three cases of damped (amplified) waves and, in any of these cases, the amplitude of the gravitational waves decreases (increases) with time. It is important to notice that both $\lambda_0^2 > 4k^2$ and $\lambda_0^2 = 4k^2$ solutions correspond to stationary waves and occur for small spatial frequencies ($k \leq |\lambda_0|/2$). On the other hand, the solution when $\lambda_0^2 < 4k^2$ corresponds to traveling waves with velocity $v = \frac{\lambda'}{2k}c$, smaller than the speed of light c. Furthermore, the dispersion relation $\omega = \frac{\lambda'}{2}c$ relating time and space frequencies gives us an experimental test for the existence of the cosmological four-vector λ_μ.

Despite the nonconservation on the geometrical side of the field equation (25), there are two simple possibilities for enabling solutions where we have energy-momentum conservation (which implies $T^\mu_{\nu;\mu} = 0$). The first is to change how mass-energy generates curvature by considering that the gravity constant G is actually a function of x^μ. In this approach, the function G equalizes the conserved matter side in Eq. (25) with a nonconservative geometry. The second possibility is to introduce a cosmological constant Λ in the theory that is actually a function of x^μ. The cosmological constant can easily be included by adding -2Λ to the Lagrangians (9) and (10). For simplicity, in the present work we consider only the first case where, by taking the covariant derivative of Eq. (25) with $\nabla_\mu T^\mu_\nu = 0$, we have the conservation condition

$$\nabla_\mu \left(K^\mu_\nu - \frac{1}{2} g^\mu_\nu K \right) = 8\pi G_{,\mu} T^\mu_\nu. \quad (27)$$

Finally, in order to investigate the cosmological consequences of the constant four-vector λ_μ, we analyze the dynamics of a Bianchi I universe filled with a perfect fluid. The metric we consider is given by [6]

$$ds^2 = dt^2 - a_1^2(t)dx^2 - a_2^2(t)dy^2 - a_3^2(t)dz^2, \quad (28)$$

where we set $c = 1$ for simplicity. From the field equation (25) and from Eq. (27) we get

$$\frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} + \frac{\dot a_1}{a_1}\frac{\dot a_3}{a_3} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} = 8\pi G\rho = -\frac{4\pi}{\lambda_0}\dot G\rho,$$
$$\frac{\ddot a_i}{a_i} + \frac{\ddot a_j}{a_j} + \frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j} + \lambda_0\left(\frac{\dot a_i}{a_i} + \frac{\dot a_j}{a_j}\right) = -8\pi G p, \quad i \neq j, \quad (29)$$

where we consider $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and $T_{\mu\nu} = (\rho + p)U_\mu U_\nu - p g_{\mu\nu}$ for the perfect fluid (where ρ is the matter density, p is the pressure, and U_μ is the fluid velocity), with the pressure p and density ρ obeying the equation of state $p = \gamma\rho (0 \leq \gamma \leq 1)$ [7]. From the first equation in Eq. (29) we obtain

$$G(t) = G_0 e^{-2\lambda_0 t}, \quad (30)$$

where G_0 is a constant. It is important to notice from Eq. (30) that for $\lambda_0 < 0$ ($\lambda_0 > 0$) the coupling G between geometry and matter is strengthened (weakened) as a consequence of the nonconservation on the geometrical side of the field equation (25). In Fig. 1 we display the isotropic solution of the scale factor $a_1(t) = a_2(t) = a_3(t) = R(t)$, in the cases where γ takes the values 0 and 1, respectively, corresponding to matter- and strong radiation-dominated eras. In both cases, one can see that the most important consequence of the constant cosmological four-vector is the arising of a universe with an accelerated expansion rate when $\lambda_0 < 0$ without the necessity of introducing dark energy. The accelerated expansion rate is evident from the concavity inversion for $R(t)$ when $\lambda_0 < 0$. For an isotropic matter-dominated era this concavity inversion occurs at a time $t^* = \frac{1}{|c\lambda_0|}\ln(\frac{3}{2})$. Consequently, from observational evidence [8] we should have $|\lambda_0|c$ of order 10^{-10} yr^{-1}. Despite the fact that we consider a very simple Bianchi I cosmological model, this result is in good agreement with observational and experimental bounds on the temporal rate of variation for G [9]. Furthermore, although Fig. 1 only shows the isotropic case,

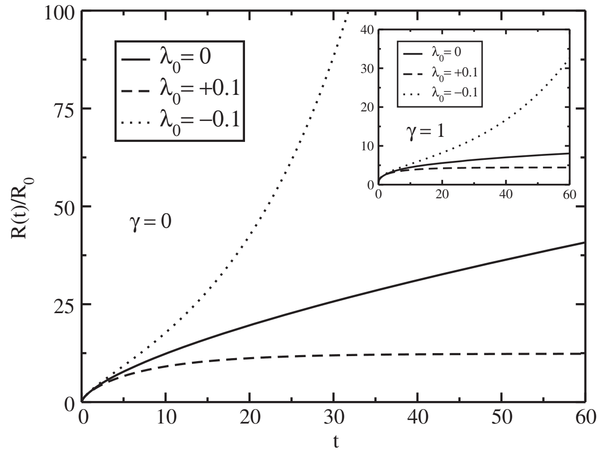

FIG. 1. The isotropic scale factor $R(t)$ versus t (for a cosmological time scale) in a matter-dominated era ($\gamma = 0$), with $G_0 = 1$, $p = \gamma\rho$, and where R_0 is a constant. The inset shows the strong radiation-dominated era ($\gamma = 1$).

we have checked that the same behavior is obtained in the more general anisotropic case. Actually, we expect that the same phenomenon will be present in more realistic models since the main mechanism behind the accelerated expansion is the nonconservation on the geometrical side of the field equation (25). Finally, from Fig. 1 it is also evident that when $\lambda_0 > 0$ the universe quickly reaches a stationary state. Furthermore, in this case, the weakening of the coupling (30) for $\lambda_0 > 0$ results in the asymptotic decoupling between matter and geometry.

Finally, due to its smallness, the effects of λ_0 in the Solar System for noncosmological time scales is very small. For a short time interval, it is easy to verify from Eq. (26) that a spherically symmetric mass distribution reproduces Newtonian gravity for weak fields since, in this case, we get $h_{00} = \frac{2\phi}{c^2}$, where ϕ is the Newtonian gravitational potential. Furthermore, as the metric should be a smooth function of time, we can estimate an upper limit of only $|\Delta\theta_\lambda - \Delta\theta_0| \lesssim 10^{-7}$ seconds of arc per century for the difference between the Mercury precession $\Delta\theta_\lambda$ in our theory (with $|\lambda_0|c \approx 10^{-10}$) and the precession of $\Delta\theta_0 = 43.03''$ per century in classical gravity [5].

In conclusion, in this work we presented a generalization of the action principle for action-dependent Lagrangians and considered it on a curved space with metric $g_{\mu\nu}(x^\mu)$. From this action principle, we obtained a generalized gravitational field equation, which can be used in the description of nonconservative phenomena. An interesting feature of this theory is that the gravitational field depends on a constant cosmological four-vector. The potential importance of this new gravitational theory is evident when applied to the problem of gravitational waves and to cosmology. Depending on the cosmological four-vector, we have shown that gravitational waves propagate with velocity smaller than the speed of the light, and with amplitudes which decrease (or increase) with time. Moreover, applying this generalization to cosmology led to another remarkable result: a universe (here considered as filled with a perfect fluid) displaying an accelerated expansion rate with no need to introduce dark energy. Finally, there are many directions of investigation left to explore related to developments of our former results. In particular, we outlined the post-Newtonian limit for a spherically symmetric mass distribution, enabling the investigation of the stability of planetary orbits on cosmological time scales, and the effects on galaxy rotations. Furthermore, although we only considered the gravitational problem, the action principle we propose is general and can be easily extended to any physical field.

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