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Relativistic corrections to the form factors of B_c into S-wave charmonium

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We investigate the form factors of B_c meson into S-wave charmonium within the nonrelativistic QCD effective theory and obtain the next-to-leading order relativistic corrections to the form factors, where both the B_c meson and the charmonium are treated as the nonrelativistic bound states. Treating the charm quark as a light quark in the limit $m_c/m_b \to 0$, some form factors are identical at the maximum recoil point, which are consistent with the predictions in the heavy-quark effective theory and the large-energy effective theory. Considering that the branching ratios of $B_c^+ \to J/\psi D_s^+$ and $B_c^+ \to J/\psi D_s^{*+}$ have been measured by the LHCb and ATLAS Collaborations recently, we employ the form factors of B_c meson into S-wave charmonium at the next-to-leading order accuracy to these two decay channels and obtain more precise predictions of their decay rates. Numerical results indicate that the factorizable diagrams dominate the contribution in these two channels, while the color-suppressed and the annihilation diagrams contribute less than 10 percent. Our results are consistent with the LHCb and ATLAS data.

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I. INTRODUCTION

The Large Hadron Collider (LHC) provides a solid platform to test the consistency and the correctness of the quantum chromodynamics (QCD) as the fundamental theory of the strong interaction. On heavy flavor side, nonrelativistic QCD (NRQCD) effective theory is a powerful framework to calculate the production cross section and the decay rate of heavy quarkonium [1]. Because the heavy quark relative velocity v is small in the rest frame of heavy quarkonium, the cross sections and the decay rates can be expanded as the series of the NRQCD long-distance matrix elements (LDMEs) with the corresponding short-distance coefficients.

The B_c^- meson is composed of two different heavy flavors and has three kinds of decay modes: (i) the bottom quark decays through $b \to c, u$; (ii) the charm quark decays through $\bar{c} \to \bar{s}, \bar{d}$; and (iii) the weak annihilation. The contributions to the total decay width of the B_c^- meson are found to be around 20, 70, and 10 percent for these three categories of decay modes, respectively [2]. Therein the transition of the bottom quark into the charm quark, where the antiquark $\bar{c}$ is the spectator, has attracted a lot of attention in both theoretical and experimental communities [3].

In the theoretical side, the form factors of B_c meson into S-wave charmonium have been investigated in different methods. These methods can be assigned according to the following four types. (i) The perturbative QCD (PQCD) approach in which the form factors can be computed in terms of the perturbative hard kernels and the nonperturbative meson wave functions with the harmonic oscillator form [4] or the transverse momentum dependent form [5–7]. (ii) QCD and light-cone sum rules. In QCD sum rules (QCD SR) the form factors are related to the three-point Green functions [8–10], while in light-cone sum rules (LCSR) the form factors depend mainly on the leading twist light cone distribution amplitudes of the mesons [11]. (iii) The relativistic, nonrelativistic and the light-front quark models. In relativistic quark model (RQM), the bottom and charm quarks in mesons are treated as relativistic objects [12–14]. In the nonrelativistic limit, the form factors have been calculated in a nonrelativistic constituent quark model (NCQM) [15]. While in the light-front quark model (LFQM), the form factors can be extracted from the plus component of the current operator matrix elements [16,17]. (iV) The NRQCD approach. In this effective theory, the leading order results of the form factors have been given in Refs. [18,19]. The next-to-leading order (NLO) corrections have been presented in Refs. [19–21], in which the dependence of the form factors on the momentum transfer squared q^2 are also obtained. Besides, the optimal renormalization scale of form factors has also been discussed by using the principle of maximum conformality (PMC) in Ref. [22]. But the form factors have the singularity at the minimum recoil point, which makes the prediction of form factors in the minimum recoil region invalid [21,23].

On the other hand, some hot topics are studied in heavy flavor field along with continuous accumulation of experimental data at LHC, which are very helpful to investigate the properties of form factors. According to the naive factorization scheme, the decay rates of B_c exclusive decays to a charmonium and a light meson such as $B_c \to J/\psi + \pi$ and $B_c \to J/\psi + \rho$ depend mainly on the form factors at the maximum recoil point. While the decay rates of B_c semileptonic decays to a charmonium such as $B_c \to J/\psi + \ell + \bar{\nu}_\ell$ will depend on the form factors in

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different momentum recoil region, and the exclusive two-body decays to a charmonium and a heavy meson such as $B_c \to J/\psi + D_s^{(*)}$ will depend on the form factors far from the maximum recoil point. Therein the decay channels $B_c^+ \to J/\psi + D_s^+$ and $B_c^+ \to J/\psi + D_s^{*+}$ were first observed by the LHCb Collaboration in 2013 [24]. These two channels have been studied recently by the ATLAS Collaboration using a data set corresponding to integrated luminosities of 4.9 fb^{-1} and 20.6 fb^{-1} of pp collisions collected at center-of-mass energies $\sqrt{s} = 7$ TeV and 8 TeV, respectively [25]. The data indicate some discrepancies from some theoretical predictions [25]. The studies of these channels shall test the form factors' predictions in different approaches, and improve our understanding of the nonperturbative properties of QCD.

The paper is organized as the following. In Sec. II, we will give the results of the relativistic corrections to the form factors of B_c into S-wave charmonium within NRQCD approach. The form factors will be investigated in the limit of heavy bottom quark, i.e. $m_c/m_b \to 0$. In Sec. III, we are going to calculate the decay rates of $B_c \to J/\psi + D_s^{(*)}$. In this section, the contributions to the branching ratios of $B_c \to J/\psi + D_s^{(*)}$ from the factorizable diagrams, the colour-suppressed and the annihilation diagrams will be considered, respectively. We summarize and conclude in the end.

II. RELATIVISTIC CORRECTIONS TO THE FORM FACTORS

A. NRQCD approach

The heavy quark pair inside the heavy quarkonium is nonrelativistic in the rest frame of heavy quarkonium, since the heavy quark's mass is much larger than the QCD binding energy. The quark relative velocity squared is estimated as $v^2 \approx 0.3$ for J/ψ and $v^2 \approx 0.1$ for Υ [1]. If a heavy quarkonium is produced in a hard-scattering process or the heavy quark decays in a heavy quarkonium, the cross sections and the decay rates can be ordered in powers of both the strong coupling constant α_s and the quark relative velocity v, which have been investigated in NRQCD effective theory by Bodwin, Braaten, and Lepage [1].

The NRQCD Lagrangian can be written into the following terms [1]

$$\begin{aligned}\mathcal{L}_{\rm NRQCD} &= \psi^\dagger\left(iD_t + \frac{\mathbf{D}^2}{2m}\right)\psi + \frac{c_F}{2m}\psi^\dagger\boldsymbol{\sigma}\cdot g_s\mathbf{B}\psi \\ &\quad + \psi^\dagger\frac{\mathbf{D}^4}{8m^3}\psi + \frac{c_D}{8m^2}\psi^\dagger(\mathbf{D}\cdot g_s\mathbf{E} - g_s\mathbf{E}\cdot\mathbf{D})\psi \\ &\quad + \frac{ic_S}{8m^2}\psi^\dagger\boldsymbol{\sigma}\cdot(\mathbf{D}\times g_s\mathbf{E} - g_s\mathbf{E}\times\mathbf{D})\psi \\ &\quad + (\psi \to i\sigma^2\chi^*, A_\mu \to -A_\mu^T) + \mathcal{L}_{\rm light}, \qquad (1)\end{aligned}$$

where $\mathcal{L}_{\rm light}$ represents the Lagrangian for the light quarks and gluons. ψ and χ denote the Pauli spinor field that annihilates a heavy quark and creates a heavy antiquark, respectively. The short-distance coefficients c_D, c_F, and c_S can be perturbatively expanded in powers of α_s, which can be expressed as $c_i = 1 + \mathcal{O}(\alpha_s)$.

The inclusive annihilation decay width of a heavy quarkonium can be factorized as [1]

$$\Gamma(H) = \sum_n \frac{2{\rm Im}f_n(\mu_\Lambda)}{m_Q^{d_n-4}}\langle H|\mathcal{O}_n(\mu_\Lambda)|H\rangle, \qquad (2)$$

where $\langle H|\mathcal{O}_n(\mu_\Lambda)|H\rangle$ are NRQCD annihilation LDMEs, which involve nonperturbative information and are ordered by the relative velocity v between the heavy quark and antiquark inside the heavy quarkonium H. The heavy quark's mass is denoted as m_Q. The imaginary part of the short-distance coefficients $f_n(\mu_\Lambda)$ can be calculated order by order in the perturbative theory.

The leading order NRQCD operators for the decay of S-wave heavy quarkonium are

$$\mathcal{O}(^1S_0^{[1]}) = \psi^\dagger\chi\chi^\dagger\psi, \qquad (3)$$

$$\mathcal{O}(^3S_0^{[1]}) = \psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi. \qquad (4)$$

The NLO relativistic correction operators for S-wave heavy quarkonium are

$$\mathcal{P}(^1S_0^{[1]}) = \frac{1}{2}\left[\psi^\dagger\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi + {\rm H.c.}\right], \qquad (5)$$

$$\mathcal{P}(^3S_1^{[1]}) = \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi + {\rm H.c.}\right], \qquad (6)$$

where the H.c. denotes the corresponding complex conjugate term. Using the vacuum-saturation approximation, the NRQCD LDMEs can be estimated as $\langle H|\mathcal{O}_n|H\rangle \simeq \langle H|\psi^\dagger\mathcal{K}'_n\chi|0\rangle\langle 0|\chi^\dagger\mathcal{K}_n\psi|H\rangle$ with $\mathcal{O}_n = \psi^\dagger\mathcal{K}'_n\chi\chi^\dagger\mathcal{K}_n\psi$. Furthermore, the vacuum expectations of production operators $\mathcal{O}_n^H$ are related to the decay matrix elements as $\langle 0|\mathcal{O}_n^H|0\rangle \simeq (2J+1)\langle H|\mathcal{O}_n|H\rangle$ with heavy quarkonium angular momentum J.

B. Covariant projection method

Instead of the traditional matching method where both of the QCD and NRQCD calculations are required in order to extract the short distance coefficients, we will use an equivalent method, i.e., the covariant projection method. In order to get the coefficients of the relativistic correction operators, the quark relative momentum should be kept. Let p_1 and p_2 represent the momenta of the heavy quark Q and antiquark $\bar{Q}'$, respectively. Without loss of generality, one may decompose the momenta as the following

$$p_1 = \alpha P_H - k, \qquad (7)$$

$$p_2 = \beta P_H + k, \qquad (8)$$

where P_H is the momentum of the heavy quarkonium, and k is a half of the relative momentum between the quark pair with $P_H \cdot k = 0$. The energy fractions for Q and $\bar{Q}'$ in heavy quarkonium are denoted as α and β, respectively. The explicit expressions for all the momenta in the rest frame of the heavy quarkonium are given by

$$P_H^\mu = (E_1 + E_2, 0), \tag{9}$$

$$k^\mu = (0, \mathbf{k}), \tag{10}$$

$$p_1^\mu = (E_1, -\mathbf{k}), \tag{11}$$

$$p_2^\mu = (E_2, \mathbf{k}). \tag{12}$$

The heavy quarkonium momentum becomes purely timelike while the relative momentum is spacelike in the rest frame. $\alpha = E_1/(E_1 + E_2)$ and $\beta = 1 - \alpha$ with the on-shell conditions $E_1 = \sqrt{m_1^2 - k^2}$, $E_2 = \sqrt{m_2^2 - k^2}$, and $k^2 = -\mathbf{k}^2$. m_1 and m_2 denote the masses of the heavy quark Q and antiquark $\bar{Q}'$, respectively.

The Dirac spinors for the heavy quark Q and antiquark $\bar{Q}'$ can be written as

$$u_1(p_1, \lambda) = \sqrt{\frac{E_1 + m_1}{2E_1}} \begin{pmatrix} \xi_\lambda \\ \frac{\vec{\sigma}\cdot\vec{p_1}}{E_1 + m_1} \xi_\lambda \end{pmatrix}, \tag{13}$$

$$v_2(p_2, \lambda) = \sqrt{\frac{E_2 + m_2}{2E_2}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p_2}}{E_2 + m_2} \xi_\lambda \\ \xi_\lambda \end{pmatrix}, \tag{14}$$

where ξ_λ is the two-component Pauli spinors and λ is the polarization quantum number.

One can easily get the covariant expressions for the spin-singlet and spin-triplet combinations of spinor bilinearities. The corresponding projection operators are

$$\Pi_S(k) = -i \sum_{\lambda_1, \lambda_2} u_b(p_1, \lambda_1) \bar{v}_c(p_2, \lambda_2) \left\langle \frac{1}{2}\lambda_1 \frac{1}{2}\lambda_2 | S S_z \right\rangle \otimes \frac{\mathbf{1}_c}{\sqrt{N_c}}$$
$$= \frac{i}{4\sqrt{2E_1 E_2}\,\omega} (\alpha \not{P}_H - \not{k} + m_1) \frac{\not{P}_H + E_1 + E_2}{E_1 + E_2} \Gamma_S (\beta \not{P}_H + \not{k} - m_2) \otimes \frac{\mathbf{1}_c}{\sqrt{N_c}}, \tag{15}$$

where the auxiliary parameter $\omega = \sqrt{E_1 + m_1}\sqrt{E_2 + m_2}$ and $\mathbf{1}_c$ is the unit matrix in the fundamental representation of the color SU(3) group. $\Gamma_S = \gamma^5$ for spin-singlet combination with spin $S = 0$, while $\Gamma_S = \not{\varepsilon}_H = \varepsilon_\mu(p_H)\gamma^\mu$ for spin-triplet combination with spin $S = 1$.

To get the relativistic corrections to the form factors of B_c into S-wave charmonium, one may perform the Taylor expansion of the amplitudes in powers of k^μ

$$\mathcal{A}(k) = \mathcal{A}(0) + \left.\frac{\partial \mathcal{A}(0)}{\partial k^\mu}\right|_{k=0} k^\mu + \frac{1}{2!} \left.\frac{\partial^2 \mathcal{A}(0)}{\partial k^\mu \partial k^\nu}\right|_{k=0} k^\mu k^\nu + \ldots, \tag{16}$$

where the terms linear in k should be dropped since they do not contribute to the matching coefficients. In this paper, we will consider the contributions at the $\mathcal{O}(|\mathbf{k}|^2)$ level. One can use the following replacement to simplify the calculation [26]

$$k^\mu k^\nu \to \frac{|\mathbf{k}|^2}{D-1} \left(-g^{\mu\nu} + \frac{P_H^\mu P_H^\nu}{P_H^2} \right). \tag{17}$$

The treatment of the final state phase space integrations at the $\mathcal{O}(|\mathbf{k}|^2)$ level is slightly different from the leading order calculation. We will adopt the following rescaling transformation for the external momenta in order to get their relativistic corrections [26]

$$p_H = p'_H \frac{E_1 + E_2}{m_1 + m_2}. \tag{18}$$

C. Form factors

The form factors of B_c into S-wave charmonium, i.e., f_+, f_0, V, A_0, A_1, and A_2 are defined in common [27]

$$\langle \eta_c(p) | \bar{c}\gamma^\mu b | B_c(P) \rangle = f_+(q^2) \left(P^\mu + p^\mu - \frac{m_{B_c}^2 - m_{\eta_c}^2}{q^2} q^\mu \right) + f_0(q^2) \frac{m_{B_c}^2 - m_{\eta_c}^2}{q^2} q^\mu, \tag{19}$$

$$\langle J/\psi(p, \varepsilon^*) | \bar{c}\gamma^\mu b | B_c(P) \rangle = \frac{2iV(q^2)}{m_{B_c} + m_{J/\psi}} \epsilon^{\mu\nu\rho\sigma} \varepsilon_\nu^* p_\rho P_\sigma,$$
$$\langle J/\psi(p, \varepsilon^*) | \bar{c}\gamma^\mu \gamma_5 b | B_c(P) \rangle = 2m_{J/\psi} A_0(q^2) \frac{\varepsilon^* \cdot q}{q^2} q^\mu - A_2(q^2) \frac{\varepsilon^* \cdot q}{m_{B_c} + m_{J/\psi}} \left(P^\mu + p^\mu - \frac{m_{B_c}^2 - m_{J/\psi}^2}{q^2} q^\mu \right)$$
$$+ (m_{B_c} + m_{J/\psi}) A_1(q^2) \left(\varepsilon^{*\mu} - \frac{\varepsilon^* \cdot q}{q^2} q^\mu \right), \tag{20}$$

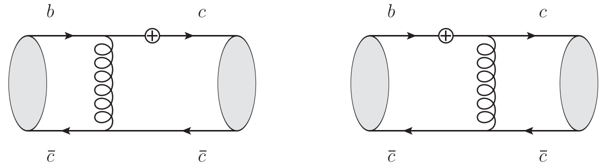

FIG. 1. Feynman diagrams for the form factors of B_c into S-wave charmonium, where "⊕" denotes certain current operators.

where the momentum transfer is defined as $q = P - p$ with the B_c meson momentum P and the charmonium momentum p.

The leading order results at $\mathcal{O}(\alpha_s)$ and the NLO QCD corrections at $\mathcal{O}(\alpha_s^2)$ of the form factors can be found in Refs. [19–21]. The leading order results are obtained from the Feynman diagrams in Fig. 1. For completeness, we list the leading order results of the form factors here

$$f_+^{LO}(q^2) = 8\sqrt{2}C_A C_F \pi \sqrt{z+1}\alpha_s \psi(0)_{B_c}\psi(0)_{\eta_c} \times \frac{(-y^2+3z^2+2z+3)}{((1-z)^2-y^2)^2 z^{3/2} m_b^3 N_c}, \tag{21}$$

$$f_0^{LO}(q^2) = 8\sqrt{2}C_A C_F \pi \sqrt{z+1}\alpha_s \psi(0)_{B_c}\psi(0)_{\eta_c} \times \frac{(9z^3+9z^2+11z-y^2(5z+3)+3)}{((1-z)^2-y^2)^2 z^{3/2}(3z+1) m_b^3 N_c}, \tag{22}$$

$$V^{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi (3z+1)\alpha_s \psi(0)_{B_c}\psi(0)_{J/\Psi}}{((1-z)^2-y^2)^2 (\frac{z}{z+1})^{3/2} m_b^3 N_c}, \tag{23}$$

$$A_0^{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi (z+1)^{5/2}\alpha_s \psi(0)_{B_c}\psi(0)_{J/\Psi}}{((1-z)^2-y^2)^2 z^{3/2} m_b^3 N_c}, \tag{24}$$

$$A_1^{LO}(q^2) = 16\sqrt{2}C_A C_F \pi \sqrt{z+1}\alpha_s \psi(0)_{B_c}\psi(0)_{J/\Psi} \times \frac{(4z^3+5z^2+6z-y^2(2z+1)+1)}{((1-z)^2-y^2)^2 z^{3/2}(3z+1) m_b^3 N_c}, \tag{25}$$

$$A_2^{LO}(q^2) = 16\sqrt{2}C_A C_F \pi \sqrt{z+1}\alpha_s \psi(0)_{B_c}\psi(0)_{J/\Psi} \times \frac{(3z+1)}{((1-z)^2-y^2)^2 z^{3/2} m_b^3 N_c}, \tag{26}$$

where $z = m_c/m_b$ and $y = \sqrt{q^2/m_b^2}$. $C_A = 3$ and $C_F = 4/3$, which are the SU(3) color group parameters. The wave functions at the origin of charmonium and B_c meson are defined through the nonperturbative operator matrix elements

$$\psi(0)_{\eta_c} = \frac{1}{\sqrt{2N_c}}\langle \eta_c|\psi_c^\dagger \chi_c|0\rangle, \tag{27}$$

$$\psi(0)_{B_c} = \frac{1}{\sqrt{2N_c}}\langle 0|\chi_b^\dagger \psi_c|B_c\rangle, \tag{28}$$

$$\psi(0)_{J/\Psi} = \frac{1}{\sqrt{2N_c}}\langle J/\Psi|\psi_c^\dagger \boldsymbol{\sigma}\chi_c|0\rangle. \tag{29}$$

D. Relativistic corrections

In the following let us calculate the relativistic corrections to the form factors. The Feynman diagrams are plotted in Fig. 1. Using the covariant projection method, one can get the corresponding amplitudes of Fig. 1. Performing the Taylor expansion of the amplitudes in powers of k^μ and extracting the quadratic terms in the series, one can obtain the relativistic corrections at the $\mathcal{O}(|\mathbf{k}|^2)$ level.

The relativistic corrections to the form factors shall be separated into two parts, since there are two bound states, i.e. a charmonium and the B_c meson which are composed of heavy quark and heavy antiquark. In the following, let us assign v as the quark relative velocity inside the charmonium and v' as the equivalent quark relative velocity inside the B_c meson. Then a half of the quark relative momentum is defined as $k = m_c v/2$ inside the charmonium and a half of the quark relative momentum is defined as $k' = m_{\rm red} v' = m_b m_c v'/(m_b + m_c)$ inside the B_c meson. The masses of the bound states can be written as $m_{\eta_c} = 2\sqrt{m_c^2 - k^2}$ and $m_{B_c} = \sqrt{m_c^2 - k'^2} + \sqrt{m_b^2 - k'^2}$. Using the heavy quark spin symmetry, one can also assume $m_{J/\psi} = 2\sqrt{m_c^2 - k^2}$.

If we expand the amplitudes in powers of k^μ, the relativistic corrections from the charm quark-antiquark pair interactions inside the charmonium can be obtained. Analogously, the relativistic corrections to the form factors from the charm and bottom quarks interaction inside the B_c meson can be obtained when we expand the amplitudes in powers of k'^μ.

In order to get the accurate relativistic corrections, one should keep the relative momentum dependence in the expressions of the masses of charmonium and B_c meson. The rescaling transformation in Eq. (18) for the external charmonium and B_c meson's momenta should be employed in order to get the relativistic corrections from the phase space integration.

To estimate the magnitude of the relativistic correction operator matrix elements, we have

$$\langle \eta_c|\psi_c^\dagger \left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2 \chi_c|0\rangle \simeq |\mathbf{k}|^2 \langle \eta_c|\psi_c^\dagger \chi_c|0\rangle, \tag{30}$$

$$\langle 0|\chi_b^\dagger \left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2 \psi_c|B_c\rangle \simeq |\mathbf{k}'|^2 \langle 0|\chi_b^\dagger \psi_c|B_c\rangle, \tag{31}$$

$$\langle J/\Psi|\psi_c^\dagger \left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2 \chi_c|0\rangle \simeq |\mathbf{k}|^2 \langle J/\Psi|\psi_c^\dagger \chi_c|0\rangle, \tag{32}$$

where $|\mathbf{k}|^2$ and $|\mathbf{k}'|^2$ can be described by the heavy quark relative velocities, i.e. $|\mathbf{k}|^2 = m_c^2|\mathbf{v}|^2/4$ and $|\mathbf{k}'|^2 = m_{\rm red}^2|\mathbf{v}'|^2 = m_b^2 m_c^2|\mathbf{v}'|^2/(m_b+m_c)^2$.

First let us keep the charm quark relative momentum k^μ nonzero in the charmonium and set the quark relative momentum $k'^\mu = 0$ in the B_c meson. Expanding the amplitudes in powers of k^μ, we can obtain the related relativistic corrections. The relativistic corrections can be expressed as the product of the relativistic operator's matrix elements and the related short-distance coefficients. According to Eqs. (30)–(32), the relativistic operator's matrix elements can be estimated by the wave functions at the origin of heavy quarkonium. The results of relativistic corrections from the charmonium can be written as the following simple forms

$$f_+^{RC}(q^2) = |\mathbf{k}|^2 f_+^{LO}(q^2)\frac{-3y^4+8y^2z(2z-3)-53z^4+4z^3-6z^2+52z+3}{4m_b^2z^2((z-1)^2-y^2)(-y^2+3z^2+2z+3)}, \tag{33}$$

$$f_0^{RC}(q^2) = |\mathbf{k}|^2 f_0^{LO}(q^2)\Bigg(-\frac{(-3z^2+2z+1)^2(53z^3+49z^2+55z+3)}{4m_b^2(z-1)z^2(3z+1)((z-1)^2-y^2)(-y^2(5z+3)+9z^3+9z^2+11z+3)} + \frac{y^2(y^4(3z+1)-y^2(83z^3+55z^2+17z+5)+365z^5-173z^4-14z^3+246z^2+81z+7)}{4m_b^2(z-1)z^2(3z+1)((z-1)^2-y^2)(-y^2(5z+3)+9z^3+9z^2+11z+3)}\Bigg), \tag{34}$$

$$V^{RC}(q^2) = |\mathbf{k}|^2 V^{LO}(q^2)\frac{-y^2(24z^2+27z+5)-12z^4+87z^3+171z^2+69z+5}{6m_b^2z^2(z+1)(3z+1)((z-1)^2-y^2)}, \tag{35}$$

$$A_0^{RC}(q^2) = |\mathbf{k}|^2 A_0^{LO}(q^2)\frac{-3y^4-2y^2(14z^3+5z^2-3)-4z^5+85z^4+348z^3+214z^2-3}{24m_b^2z^3(z+1)^2((z-1)^2-y^2)}, \tag{36}$$

$$A_1^{RC}(q^2) = |\mathbf{k}|^2 A_1^{LO}(q^2)\Bigg(\frac{-45z^6+721z^5+1554z^4+1954z^3+807z^2+125z+4}{12m_b^2z^2(3z+1)((z-1)^2-y^2)(-y^2(2z+1)+4z^3+5z^2+6z+1)} - \frac{y^2(y^2(3z^2-7z-4)+48z^4+580z^3+512z^2+132z+8)}{12m_b^2z^2(3z+1)((z-1)^2-y^2)(-y^2(2z+1)+4z^3+5z^2+6z+1)}\Bigg), \tag{37}$$

$$A_2^{RC}(q^2) = |\mathbf{k}|^2 A_2^{LO}(q^2)\frac{-y^2(39z^2+55z+10)+15z^4-79z^3+187z^2+123z+10}{12m_b^2z^2(3z+1)((z-1)^2-y^2)}. \tag{38}$$

According to the leading order results, we have learned that the form factors have singularity at the minimum recoil point $y \to 1-z$, which exist also in the relativistic corrections from Eqs. (33)–(38).

Because $z = m_c/m_b \approx 0.3$ is small, the form factors can be expanded in powers of z. In the heavy quark limit $m_b \to \infty$, one can get more information among form factors. In the $m_b \to \infty$ limit, the form factors become

$$V(q^2)_{m_b\to\infty} = \frac{16\sqrt{2}C_AC_F\pi\alpha_s\psi(0)_{B_c}\psi(0)_{J/\Psi}}{(1-y^2)^2z^{3/2}m_b^3N_c} \times\left(1+\frac{5|\mathbf{k}|^2}{6m_b^2z^2}\right), \tag{39}$$

$$A_2(q^2)_{m_b\to\infty} = V(q^2)_{m_b\to\infty}, \tag{40}$$

$$A_0(q^2)^{LO}_{m_b\to\infty} = V(q^2)^{LO}_{m_b\to\infty}. \tag{41}$$

At the maximum recoil point with $q^2 = 0$, some form factors turn to be identical

$$f_0(0) = f_+(0), \tag{42}$$

$$V(0)_{m_b\to\infty} = A_2(0)_{m_b\to\infty}, \tag{43}$$

which are consistent with the predictions of the heavy quark effect theory [28] and the large energy effective theory [29].

Next let us keep the quark relative momentum k'^μ nonzero in the B_c meson and set the charm quark relative momentum $k^\mu = 0$ in the charmonium. We will expand the amplitudes in powers of k'^μ, and the relativistic corrections to the form factors at the $\mathcal{O}(|\mathbf{k}'|^2)$ level are

$$f_+^{RC'}(q^2) = |\mathbf{k}'|^2 f_+^{LO}(q^2)\left(\frac{-9z^6+264z^5+285z^4+408z^3+241z^2+112z-21}{24z^2m_b^2((z-1)^2-y^2)(-y^2+3z^2+2z+3)} - \frac{y^2(y^2(3z^2-4z+11)+4(-3z^4+32z^3+23z^2+16z-8))}{24z^2m_b^2((z-1)^2-y^2)(-y^2+3z^2+2z+3)}\right), \quad (44)$$

$$f_0^{RC'}(q^2) = |\mathbf{k}'|^2 f_0^{LO}(q^2)\left(-\frac{(3z+1)(9z^7-273z^6-21z^5-123z^4+167z^3+129z^2+133z-21)}{24(z-1)z^2m_b^2((z-1)^2-y^2)(-y^2(5z+3)+9z^3+9z^2+11z+3)} + \frac{y^2(63z^7-1110z^6-762z^5-189z^4+307z^3+216z^2-40z-21)}{12(z-1)z^2(3z+1)m_b^2((z-1)^2-y^2)(-y^2(5z+3)+9z^3+9z^2+11z+3)} + \frac{y^4((32-6y^2)z^2+(31-2y^2)z-15z^5+161z^4+104z^3+7)}{8(z-1)z^2(3z+1)m_b^2((z-1)^2-y^2)(-y^2(5z+3)+9z^3+9z^2+11z+3)}\right), \quad (45)$$

$$V^{RC'}(q^2) = |\mathbf{k}'|^2 V^{LO}(q^2)\frac{y^2(9z^3-105z^2-23z+7)-9z^5+315z^4+108z^3+180z^2+53z-7}{24z^2(3z+1)m_b^2((z-1)^2-y^2)}, \quad (46)$$

$$A_0^{RC'}(q^2) = |\mathbf{k}'|^2 A_0^{LO}(q^2)\left(\frac{-12z^7+277z^6+916z^5+605z^4+596z^3+207z^2-28z-1}{96z^3(z+1)^2m_b^2((z-1)^2-y^2)} + \frac{2y^2(6z^5+5z^4-140z^3-22z^2+22z+1)-y^4(39z^2+16z+1)}{96z^3(z+1)^2m_b^2((z-1)^2-y^2)}\right), \quad (47)$$

$$A_1^{RC'}(q^2) = |\mathbf{k}'|^2 A_1^{LO}(q^2)\left(\frac{-36z^8+807z^7+2047z^6+2807z^5+2503z^4+1693z^3+421z^2+5z-7}{24z^2(3z+1)m_b^2((z-1)^2-y^2)(-y^2(2z+1)+4z^3+5z^2+6z+1)} - \frac{y^2(18z^4-33z^3-9z^2+25z+7)+2(-27z^6+258z^5+539z^4+504z^3+151z^2-10z-7)}{24z^2y^{-2}(3z+1)m_b^2((z-1)^2-y^2)(-y^2(2z+1)+4z^3+5z^2+6z+1)}\right), \quad (48)$$

$$A_2^{RC'}(q^2) = |\mathbf{k}'|^2 A_2^{LO}(q^2)\frac{-y^2(12z^3+67z^2+16z-3)-6z^5+175z^4+102z^3+92z^2+24z-3}{12z^2(3z+1)m_b^2((z-1)^2-y^2)}. \quad (49)$$

In the heavy quark limit $m_b \to \infty$, however, the relations among the relativistic corrections to the form factors from the B_c meson have not been found.

In order to estimate the relativistic corrections, one should first evaluate the heavy quark relative velocities. Using the heavy quark kinetic and potential energy approximation [1], one has

$$|\mathbf{v}| \simeq \alpha_s(2m_{\rm red}|\mathbf{v}|). \quad (50)$$

We adopt the values which have been evaluated in Ref. [30]

$$|\mathbf{v}|^2_{J/\psi} \approx 0.267, \qquad |\mathbf{v}'|^2_{B_c} \approx 0.186. \quad (51)$$

In the literatures, there already exist a lot of studies on the form factors of B_c into S-wave charmonium at the maximum recoil point, so we give the results with different approaches in Table I. For the NRQCD predictions in Table I, the heavy quark masses $m_c = 1.4$ and $m_b = 4.9$ are adopted. From the table, the NRQCD predictions of the form factors are larger than that from the approaches of QCD SR, RQM, NCQM, and LFQM. The predictions of some of form factors are consistent with each other among PQCD, LCSR, PMC, and the LO NRQCD results. The relativistic corrections from both the S-wave charmonium and B_c cannot be ignored, which will bring about 15 to 25 percent enhancements to the NLO predictions of the form factors.

III. DECAY RATIOS OF $B_c \to J/\psi + D_s^{(*)}$

The form factors of B_c into S-wave charmonium can be employed into the calculation of decay widths of a lot of decay channels, some of which have been studied by the LHCb and ATLAS Collaborations. The channels of B_c exclusive decays to a charmonium and a light meson can determine the form factors at the maximum recoil point, while the exclusive decays to a charmonium and a heavy meson can give the information of the form factors far from the maximum recoil point.

The ATLAS Collaboration have used a data set of integrated luminosities of 4.9 fb^{-1} and 20.6 fb^{-1} at $\sqrt{s} =$ 7 TeV and 8 TeV and measured the branching ratios of $B_c^+ \to J/\psi + D_s^{+(*)}$ recently [25]. Furthermore, the

TABLE I. B_c into S-wave charmonium form factors at $q^2 = 0$ evaluated in the literatures.

Approaches		$f_+^{B_c\eta_c}(0) = f_0^{B_c\eta_c}(0)$	$A_0^{B_cJ/\psi}(0)$	$A_1^{B_cJ/\psi}(0)$	$A_2^{B_cJ/\psi}(0)$	$V^{B_cJ/\psi}(0)$
PQCD	DW [4][a]	0.420	0.408	0.416	0.431	0.296
	SDY [5]	0.87	0.27	0.75	1.69	0.85
	WFX [6]	0.48	0.59	0.46	0.64	0.42
	ZLWX [31]	1.06	0.78	0.96	1.36	1.59
QCD SR	CNP [8]	0.20	0.26	0.27	0.28	0.19
	KT [9]	0.23	0.21	0.21	0.23	0.17
	KLO [10]	0.66	0.60	0.63	0.69	0.52
LCSR	HZ [11]	0.87	0.27	0.75	1.69	1.69
RQM	NW [12]	0.5359	0.532	0.524	0.509	0.368
	EFG [13]	0.47	0.40	0.50	0.73	0.25
	IKS2 [14]	0.61	0.57	0.56	0.54	0.42
NCQM	HNV [15]	0.49	0.45	0.49	0.56	0.31
LFQM	WSL [16]	0.61	0.53	0.50	0.44	0.37
	KLL [17]	–	0.502	0.467	0.398	0.369
PMC	SWMW [22]	1.65	0.87	1.07	1.15	1.47
NRQCD	LO [19–21]	0.96	0.84	0.87	0.94	1.21
	NLO [19–21]	1.43	1.09	1.19	1.27	1.63
	NLO + RC (This work)	1.67	1.43	1.57	1.73	2.24

[a]We quote the results with $\omega = 0.6$ GeV.

ATLAS Collaboration have analyzed three helicity dependent amplitudes in the channel of $B_c^+ \to J/\psi + D_s^{+(*)}$, i.e. A_{++}, A_{--}, and A_{00}, where the subscripts correspond to the helicities of J/ψ and D_s^* mesons. Therein A_{++} and A_{--} denote the amplitudes where J/ψ and D_s^* are transversely polarized. We will employ the form factors to analyze these channels, and compare our results with data and other theoretical predictions.

The typical Feynman diagrams of $B_c^+ \to J/\psi + D_s^{+(*)}$ are plotted in Fig. 2. There are four types of topologies: (a) factorizable diagrams which are determined by the form factors; (b) nonfactorizable diagrams which cannot be factorized into the product of form factors and the decay constant of the $D_s^{(*)}$ meson; (c) color-suppressed diagrams where the spectator quark generates the $D_s^{(*)}$ meson with a strange quark; (d) annihilation diagrams where both the bottom and anticharm quarks in the B_c meson are annihilated.

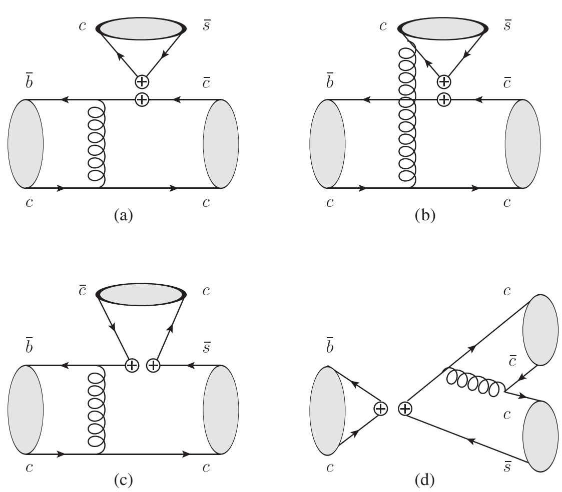

FIG. 2. Typical Feynman diagrams for $B_c^+ \to J/\psi + D_s^{+(*)}$, where two "$\oplus$" denote four-fermion weak interaction operators. There are four types of topologies: (a) factorizable diagrams; (b)nonfactorizable diagrams; (c) color-suppressed diagrams; (d) annihilation diagrams.

The form factors dependence on q^2 are obtained using the NRQCD approach, however, a divergence exists at the minimum recoil point where $q^2 = (m_{B_c} - m_{J/\psi})^2 \approx (m_b - m_c)^2$ according to the LO results [19] and the NLO QCD corrections [19–21] and relativistic corrections. This singularity problem leads to the predictions of the form factors invalid near the minimum recoil point. Besides, the NRQCD prediction of the branching ratio of $B_c^+ \to J/\psi + \pi^+ + \pi^- + \pi^+$ is slightly larger than the cental value of the measurement by the LHCb Collaboration [23,32], where the channel $B_c^+ \to J/\psi a_1^+(1260)$ dominates the contribution. This indicates the applications of the NRQCD predictions of the form factors far from the maximum recoil point should be careful. For the channels $B_c^+ \to J/\psi + D_s^{+(*)}$ where $q^2 = m^2_{D_s^{(*)}} \approx 4\ \mathrm{GeV}^2$, the direct NRQCD predictions of form factors maybe invalid, since $q^2 \approx 4\ \mathrm{GeV}^2$ is far from the maximum recoil point.

To extrapolate the form factors to the minimum recoil region, the pole mass dependence model are generally adopted in many literatures [10,16], where each form factor $F(q^2)$ is parametrized as

$$F(q^2) = \frac{F(0)}{1 - \frac{q^2}{m_{\mathrm{pole}}^2} - \beta \frac{q^4}{m_{\mathrm{pole}}^4}}, \tag{52}$$

with the effective pole mass $m_{\rm pole}$ and a free parameter β which is set to be zero in our calculation. And $F(q^2)$ can be any one of the form factors of B_c into S-wave charmonium, i.e. $f_+(q^2)$, $f_0(q^2)$, $V(q^2)$, $A_0(q^2)$, $A_1(q^2)$, and $A_2(q^2)$.

In the calculation, the heavy quark mass are adopted as $m_c = 1.4 \pm 0.1$ GeV, $m_b = 4.9 \pm 0.1$ GeV [23,33]. The masses of $D_s^{(*)}$ are adopted as $m_{D_s} = 1.968$ GeV and $m_{D_s^*} = 2.112$ GeV [3]. The decay constants are adopted as $f_\pi = 130.4$ MeV [23], $f_{D_s} = 257.5$ MeV [7]. The decay constant of D_s^* can be obtained by the heavy quark symmetry $f_{D_s^*} = f_{D_s}(m_{D_s}/m_{D_s^*})^{1/2}$. The effective pole mass $m_{\rm pole}$ in Eq. (52) is set to be near the bottom quark mass, i.e., 5 GeV. The Schrödinger wave function at the origin for J/ψ is determined through its leptonic decay width $\Gamma_{ee}^{\psi} = 5.55$ keV. Numerically we can obtain $|\psi_\Psi^{LO}(0)|^2 = 0.0447\,(\mathrm{GeV})^3$ and $|\psi_\Psi^{\rm NLO}(0)|^2 = 0.0801\,(\mathrm{GeV})^3$. For that of B_c, we shall determine its value to be: $|\psi_{B_c}(0)|^2 = 0.1307$ (GeV)^3, which is derived under the Buchmüller-Tye potential [34].

Based on the NRQCD framework, we can calculate the amplitudes in Fig. 2 and numerical results indicate that the factorizable diagrams dominate the contribution of the decay widths of $B_c^+ \to J/\psi + D_s^{+(*)}$, but color-suppressed and annihilation topologies diagrams contribute less than 10 percent. The factorizable diagrams can be factorized into the form factor part and the $D_s^{(*)}$ decay constant part. Thus we can employ the results of NLO QCD and relativistic corrections to the form factors, and obtain more precise predictions.

In order to compare with data, the auxiliary parameters are written as

$$R_{D_s^+/\pi^+} = \frac{\Gamma(B_c^+ \to J/\psi + D_s^+)}{\Gamma(B_c^+ \to J/\psi + \pi^+)}, \tag{53}$$

$$R_{D_s^{*+}/\pi^+} = \frac{\Gamma(B_c^+ \to J/\psi + D_s^{*+})}{\Gamma(B_c^+ \to J/\psi + \pi^+)}, \tag{54}$$

$$R_{D_s^{*+}/D_s^+} = \frac{\Gamma(B_c^+ \to J/\psi + D_s^{*+})}{\Gamma(B_c^+ \to J/\psi + D_s^+)}, \tag{55}$$

$$\Gamma_{\pm\pm}/\Gamma = \frac{\Gamma_{\pm\pm}(B_c^+ \to J/\psi + D_s^{*+})}{\Gamma(B_c^+ \to J/\psi + D_s^{*+})}. \tag{56}$$

The decay width can be written as

$$\Gamma(B_c \to J/\psi D_s^{(*)}) = \frac{|\mathbf{p}|}{8\pi m_{B_c}^2} |\mathcal{A}(B_c \to J/\psi D_s^{(*)})|^2, \tag{57}$$

with the final meson momentum $|\mathbf{p}| = (m_{B_c}^4 - 2m_{B_c}^2(m_{D_s^{(*)}}^2 + m_\psi^2) + (m_{D_s^{(*)}}^2 - m_\psi^2)^2)^{1/2}/(2m_{B_c})$ in the B_c meson rest frame.

Ignoring the small contributions from nonfactorizable, color-suppressed, and annihilation diagrams, we reach the naive factorization. In naive factorization, the decay amplitudes can be factorized as

$$\begin{aligned}
&\mathcal{A}(B_c^+ \to J/\psi D_s^+) \\
&\quad\approx \frac{G_F}{\sqrt{2}} V_{cb}^* V_{cs} C_0(\mu) \langle J/\psi D_s^+ | \mathcal{O}_0 | B_c^+ \rangle \\
&\quad\approx \frac{G_F}{\sqrt{2}} V_{cb}^* V_{cs} C_0(\mu) \langle J/\psi | \bar{b}\gamma^\mu (1-\gamma_5) c | B_c^+ \rangle \\
&\quad\quad \times \langle D_s^+ | \bar{c}\gamma_\mu (1-\gamma_5) s | 0 \rangle \\
&\quad= i\frac{2G_F}{\sqrt{2}} V_{cb}^* V_{cs} C_0(\mu) f_{D_s} A_0(m_{D_s}^2) m_{J/\psi} \varepsilon_{J/\psi}^* \cdot q,
\end{aligned} \tag{58}$$

where G_F is the Fermi constant. V_{ud} and V_{cb} are the Cabibbo-Kobayashi-Maskawa (CKM) matrix-elements. $\mathcal{O}_0$ is the effective four-quark operator, and $C_0(\mu)$ is the perturbatively calculable Wilson coefficient. $\varepsilon_{J/\psi}$ is the polarization vector of J/ψ. Because $\varepsilon_{J/\psi}^* \cdot q = \varepsilon_{J/\psi}^* \cdot P_{B_c}$ is nonzero only when $\varepsilon_{J/\psi}^\mu$ is longitudinal, J/ψ is longitudinally polarized in the decay channels of B_c exclusive decays to J/ψ and a pseudoscalar meson. In order to reliably predict the decay rate of B_c exclusive decays to J/ψ and a heavy meson, we will adopt the pole mass dependence model and get $A_0(m_{D_s}^2) = A_0^{\rm NLO+RC}(0)/(1 - \frac{m_{D_s}^2}{m_{\rm pole}^2})$.

Analogously, we can obtain the expression of the amplitude $\mathcal{A}(B_c^+ \to J/\psi\pi^+)$ by replacing the meson decay constant $f_{D_s} \to f_\pi$, the form factors $A_0(m_{D_s}^2) \to A_0(m_\pi^2)$ and the CKM matrix-element $V_{cs} \to V_{ud}$ in Eq. (58). Because the pion's mass is much less than the heavy quark mass, the pion's mass can be ignored in the decay of $B_c^+ \to J/\psi\pi^+$.

The amplitude of $B_c^+ \to J/\psi D_s^{*+}$ can be estimated as

$$\begin{aligned}
&\mathcal{A}(B_c^+ \to J/\psi D_s^{*+}) \\
&\quad\approx \frac{G_F}{\sqrt{2}} V_{cb}^* V_{cs} C_0(\mu) \langle J/\psi D_s^{*+} | \mathcal{O}_0 | B_c^+ \rangle \\
&\quad= i\frac{G_F}{\sqrt{2}} V_{cb}^* V_{cs} C_0(\mu) f_{D_s^*} \varepsilon_{J/\psi}^{*\alpha} \varepsilon_{D_s^*}^{*\beta} \\
&\quad\quad \times \left(S_1 g_{\alpha\beta} - S_2 \frac{P_{B_c\alpha} P_{B_c\beta}}{m_{B_c}^2} + iS_3 \epsilon_{\alpha\beta\gamma\sigma} \frac{P_{J/\psi}^\gamma P_{D_s}^\sigma}{P_{J/\psi} \cdot P_{D_s}} \right),
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
S_1 &= -m_{D_s^*} A_1(m_{D_s^*}^2)(m_{B_c} + m_{J/\psi}), \\
S_2 &= -\frac{2m_{D_s^*} A_2(m_{D_s^*}^2) m_{B_c}^2}{m_{B_c} + m_{J/\psi}}, \\
S_3 &= \frac{2m_{D_s^*} V(m_{D_s^*}^2) P_{J/\psi} \cdot P_{D_s}}{m_{B_c} + m_{J/\psi}}.
\end{aligned} \tag{60}$$

TABLE II. Comparisons of the results of the decay ratios of $B_c \to J/\psi + D_s^{(*)}$ with data and other theoretical predictions.

$R_{D_s^+/\pi^+}$	$R_{D_s^{*+}/\pi^+}$	$R_{D_s^{*+}/D_s^+}$	$\Gamma_{\pm\pm}/\Gamma$	Refs.
3.8 ± 1.2	10.4 ± 3.5	$2.8^{+1.2}_{-0.9}$	0.38 ± 0.24	ATLAS [25]
2.90 ± 0.62	–	2.37 ± 0.57	0.52 ± 0.20	LHCb [24].
2.6	4.5	1.7	–	Potential model [35]
1.3	5.2	3.9	–	QCD SR [36]
2.0	5.7	2.9	–	RCQM [37]
2.2	–	–	–	BSW [38]
2.06 ± 0.86	–	3.01 ± 1.23	–	LFQM [17]
$3.45^{+0.49}_{-0.17}$	–	$2.54^{+0.07}_{-0.21}$	0.48 ± 0.04	PQCD [7]
–	–	–	0.410	RIQM [2]
$3.07^{+0.21+0.14}_{-0.38-0.13}$	$11.8^{+1.0+2.3}_{-1.4-0}$	$3.85^{+0.04+0.54}_{-0.02-0}$	$0.601^{+0.001+0.033}_{-0.001-0.040}$	NRQCD NLO + RC

In the B_c meson rest frame, it is convenient to choose the momentum $\mathbf{P}_{D_s^*}$ to be directed in positive z-direction. The transverse polarization vectors can be defined as $\epsilon^{\mu}_{D_s^*,\pm} = \epsilon^{\mu}_{J/\psi,\mp} = (0, \pm 1, i, 0)/\sqrt{2}$. The longitudinal polarization vectors can be defined as $\epsilon^{\mu}_{D_s^*,0} = (\mathbf{P}_{D_s^*}, 0, 0, P^0_{D_s^*})/m_{D_s^*}$ and $\epsilon^{\mu}_{J/\psi,0} = (-\mathbf{P}_{D_s^*}, 0, 0, P^0_{J/\psi})/m_{J/\psi}$. We then can get the results for the polarized decay widths $\Gamma_{\pm\pm}(B_c^+ \to J/\psi + D_s^{*+})$.

Considering the NLO QCD corrections and the relativistic corrections, our results are given in the end line of Table II. For convenience, we also list the data and other theoretical predictions in Table II. For our results, the first column of the uncertainties is from the choice of the scale $\mu = 4.9 \pm 1$ GeV, while the second error is from the uncertainty of the heavy quark mass with $m_c = 1.4 \pm 0.1$ GeV and $m_b = 4.9 \pm 0.1$ GeV.

From Table II, our results of $R_{D_s^+/\pi^+}$, $R_{D_s^{*+}/\pi^+}$, and $\Gamma_{\pm\pm}/\Gamma$ are consistent with the LHCb and ATLAS data when considering the experiment uncertainties.

IV. CONCLUSION

In this paper, we calculated the relativistic corrections to the form factors of B_c into S-wave charmonium at the $\mathcal{O}(|\mathbf{k}|^2)$ and $\mathcal{O}(|\mathbf{k}'|^2)$ level, where k and k' are a half of quark relative momentum inside the charmonium and B_c meson, respectively. The corresponding analytic expression are given. In the heavy bottom quark limit, the properties of form factors are studied. We found that the relativistic corrections bring about 15 to 25 percent enhancements to the form factors.

Based on the NRQCD approach, we studied the decay channels of $B_c \to J/\psi D_s^{(*)}$. Employing the form factors of B_c meson into S-wave charmonium up to the next-to-leading order in both α_s and the quark relative velocity squared v^2 and v'^2, the decay rates of $B_c^+ \to J/\psi D_s^{(*)+}$ are studied. Numerical results indicate that the factorizable diagrams dominate the decay rates of the considered $B_c \to J/\psi D_s^{(*)}$ decay modes. The ratios of $R_{D_s^+/\pi^+}$, $R_{D_s^{*+}/\pi^+}$, $R_{D_s^{*+}/D_s^+}$ and $\Gamma_{\pm\pm}/\Gamma$ provide a precise platform to test the form factors. Our results of $R_{D_s^+/\pi^+}$, $R_{D_s^{*+}/\pi^+}$, and $\Gamma_{\pm\pm}/\Gamma$ are consistent with the LHCb and ATLAS data, however, the result of $R_{D_s^{*+}/D_s^+}$ only support the ATLAS data. Thus more studies are needed to investigate the inner properties of D_s^*. This work is also helpful to understand the non-perturbative properties of heavy quarkonium.

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