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Simple way to calculate a UV-finite one-loop quantum energy in the Randall-Sundrum model

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The surprising simplicity of Barvinsky-Nesterov or equivalently Gelfand-Yaglom methods of calculation of quantum determinants permits us to obtain compact expressions for a UV-finite difference of one-loop quantum energies for two arbitrary values of the parameter of the double-trace asymptotic boundary conditions. This result generalizes the Gubser and Mitra calculation for the particular case of difference of "regular" and "irregular" one-loop energies in the one-brane Randall-Sundrum model. The approach developed in the paper also allows us to get "in one line" the one-loop quantum energies in the two-brane Randall-Sundrum model. The relationship between "one-loop" expressions corresponding to the mixed Robin and to double-trace asymptotic boundary conditions is traced.

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I. INTRODUCTION

In 2001, Witten [1] showed that in frames of the AdS/CFT correspondence, multi-trace deformation $W(\hat{O})$ of the boundary quantum field theory may be equivalent to the boundary condition,

$$\alpha = \frac{\partial W(\beta)}{\partial \beta}, \tag{1}$$

imposed upon the regular (α) and irregular (β) asymptotics at the anti-de Sitter (AdS) horizon ($z \to 0$) of the bulk scalar field ϕ:

$$\phi = \alpha z^{\frac{d}{2}+\nu} + \beta z^{\frac{d}{2}-\nu}, \tag{2}$$

where α corresponds to the source of single-trace operator $\hat{O}$ whereas β corresponds to its quantum average.

In case of the double-trace deformation $W = (1/2) f \hat{O}^2$ (1) comes to

$$\alpha = f\beta. \tag{3}$$

Here, the Euclidean metric of $(d+1)$-dimensional AdS space of the Randall-Sundrum (RS) model is taken in a form

$$ds^2 = \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{(kz)^2}, \tag{4}$$

and $\epsilon < z < L$ ($z = \epsilon$, L are positions of UV and IR branes), $\mu, \nu = 0, 1, \ldots (d-1)$, $\eta_{\mu\nu} = \delta_{\mu\nu}$ in the Euclidean signature, k is the AdS curvature scale, and $\phi = \phi(\vec{p}, z)$ satisfies the equation ($\vec{p}$ is momentum in Euclidean d-space, $p = |\vec{p}|$):

$$\hat{D}(p)\phi = \left[-z^2 \frac{\partial^2}{\partial z^2} + (d-1) z \frac{\partial}{\partial z} + \left(\nu^2 - \frac{d^2}{4}\right) + z^2 p^2\right]\phi = 0, \tag{5}$$

$\nu = \sqrt{d^2/4 + m^2/k^2}$ for minimal action of the bulk scalar field of mass m.

Gubser and Mitra showed in [2] (see also [3] and [4]) that the difference of bulk Green functions satisfying asymptotic boundary condition (3) for two values of double-trace parameter f is UV finite at coinciding arguments:

$$\int [G_{f_2}(p; z, z) - G_{f_1}(p; z, z)] d^d p < \infty, \tag{6}$$

where the Green function $G_f(p; z, z')$ is taken in the Euclidean signature and is given by formula (32) of [2]:

$$G_f(p; z, z') = -\frac{k^{d-1}(zz')^{d/2} K_\nu(pz')}{1 + \bar{f}} \times \{[I_{-\nu}(pz) + \bar{f} I_\nu(pz)]\theta(z'-z) + (z \leftrightarrow z')\}$$

$$\bar{f} = f\left(\frac{2}{p}\right)^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}, \tag{7}$$

$I_{\pm\nu}$, K_ν are Bessel functions of the imaginary argument, here $L = \infty$, and an expression for $\bar{f}$ is obtained from a comparison of the asymptotic of $(I_{-\nu} + \bar{f} I_\nu)$ at $z \to 0$ with (2) and (3) [2].

However, as it was pointed out in [2], it is hard to calculate for general values of f the one-loop vacuum energy corresponding to a difference of Green functions in (6).

In the present paper, which is development of [5], calculation of this one-loop energy is performed with a simple "boundary operator" formula, proposed by Barvinsky and Nesterov (BN) [6–9] for ratio of determinants of one and the same differential operator in the one-dimensional

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problem for two different boundary conditions imposed upon eigenfunctions of $\hat{D}$. It will be shown also that, in this case, the BN approach is equivalent to the Gelfand-Yaglom (GY) method [10–13].

Let us describe in short BN and GY approaches which application is crucial for this paper.

As it was demonstrated in [6–9] the ratio of determinants of the differential operator for two different boundary conditions is equal to the ratio of determinants of certain "boundary operators" given by the corresponding Green functions with their arguments taken at the boundary. The idea behind it is seemingly simple although proves to be very effective: the Gauss functional integral, which gives the desired determinant, is a product of the functional integral over the bulk field with fixed values at the boundaries (that is when Dirichlet boundary conditions are imposed) and of the functional integral over the boundary values of the field weighted by the boundary operator depending on boundary conditions under consideration; thus, in the ratio of determinants the bulk functional integrals reduce. In the one-dimensional problem, the boundary is a dot and boundary operator is just a number equal, as is shown in [6–9], to the value of the corresponding Green function at the boundary. Finally, the ratio of determinants comes to the product of ratios of boundary operators of the one-dimensional problem over the quantity parametrizing one-dimensional problem (momentum $\vec{p}$ in transverse d-space in this paper).

The GY approach [10–13] says that the product of eigenvalues (determinant) of a differential operator of the one-dimensional problem $\hat{D}\phi_n(z) = \lambda_n\phi_n(z)$ defined on interval $a < z < b$ and determined by boundary conditions $A[\phi(a)] = 0$ and $B[\phi(b)] = 0$ ($A[\phi]$, $B[\phi]$ are some linear combinations of ϕ and its derivative ϕ' taken at corresponding points) may be expressed through solution $v(z)$ of the homogeneous equation $\hat{D}v(z) = 0$ which obeys a given boundary condition at one boundary, say at $z = a$, that is $A[v(a)] = 0$; then the GY method gives $\mathrm{Det}\hat{D} \sim B[v(b)]$. The logic of the proof of this quite effective formula is double step: (1) for solution $\phi(z|\lambda)$ of the following equation: $\hat{D}\phi = \lambda\phi$, which obeys boundary condition $A[\phi(a|\lambda)] = 0$ and which is considered as a function of λ; function $B(\lambda) \equiv B[\phi(b|\lambda)]$ has zeroes at $\lambda = \lambda_n$. (2) Since the logarithmic derivative of $B(\lambda)$ $[d \ln B(\lambda)/d\lambda]$ has poles in a complex λ plane exactly at $\lambda = \lambda_n$ it is possible to express the ζ function $[\zeta(s) = \sum \lambda_n{}^{-s}]$ with a contour integral over this logarithmic derivative and finally, after a number of rather conventional steps, to get the looked for GY formula $e^{-\zeta'(0)} = \mathrm{Det}\hat{D} \sim B(\lambda = 0) = B[\phi(b|0)] = B[v(b)]$ [since $\phi(z|0)$ is nothing but a homogeneous solution $v(z)$ introduced above in this paragraph].

As to our knowledge, the correspondence of BN and GY methods was not considered in the literature so far. The bulk of the paper consists of the examples of the application of the BN method with certain parallels with the GY approach. In the Appendix, the power of the GY method is demonstrated by a number of physical problems where GY formulas immediately give well-known values of Casimir potential calculated conventionally in a rather complex way.

In the standard approach applied in [2–5], the calculation of one-loop energy $V^{(d)}$ in the $(d+1)$-dimensional RS model is performed with three integrations: over p like in (6), over z between its endpoints, and over mass squared parameter α according to the well-known identity:

$$V = \frac{1}{2}\ln \mathrm{Det}\hat{D} = \int d\tilde{\alpha}\frac{\partial V}{\partial\tilde{\alpha}} = \frac{1}{2}\int^{\alpha} d\tilde{\alpha}\mathrm{Tr}G(x, z; x, z; \tilde{\alpha}). \tag{8}$$

The BN or GY methods permit us to "jump over" integrations over z and α, and immediately give an answer for the ratio of determinants of the differential operator of the one-dimensional problem parametrized in our case by $\vec{p}$ [see (5)]. Then corresponding difference of one-loop quantum energies in d dimensions is given by an integral over $\vec{p}$:

$$V_2^{(d)} - V_1^{(d)} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\ln\left[\frac{\mathrm{Det}_2\hat{D}(p)}{\mathrm{Det}_1\hat{D}(p)}\right]. \tag{9}$$

It is shown in the paper that the integral in (9) is UV-convergent if indexes 2,1 in (9) refer to two values f_2, f_1 of the double-trace parameter in asymptotic boundary condition (3). And on the other hand, the integral in (9) is UV divergent if these indexes refer to two fixed Robin parameters of the mixed boundary condition imposed at $z = \epsilon$.

The "strange discrepancy" (see Sec. III in [5]) of expressions for the difference of "regular" and "irregular" one-loop energies ($V_+ - V_- = V_{f=\infty} - V_{f=0}$) calculated with a different choice of parameter α in (8) ($\sqrt{\alpha} = m$ in [2–4], and $\sqrt{\alpha}$ is auxiliary mass introduced in [5]) perhaps is resolved in this paper. In any case, the formula for $V_+^{(d)} - V_-^{(d)}$ obtained in Sec. III differs from both competing expressions of [2] and [5].

The structure of the paper is as follows. In Sec. II the work of BN and GY methods, their equivalence, and the correspondence of Robin boundary conditions and asymptotic boundary conditions are demonstrated by an elementary dynamical example. Section III presents the results of the calculation of UV-finite one-loop quantum energy for the double-trace asymptotic boundary condition in one-brane ($L = \infty$) and two-brane ($L < \infty$) RS models. The Conclusion outlines the possible ways of future work. The Appendix presents a number of striking examples of the power of the GY method.

II. ELEMENTARY EXAMPLE: IDENTITY OF BN AND GY METHODS

In this section we demonstrate the identity of BN and GY methods of the calculation of the ratio of quantum determinants determined by different Robin or asymptotic boundary conditions by an elementary example of the differential operator $\hat{D}_0$ of the massless scalar field ϕ in flat $(d+1)$ dimensions:

$$\hat{D}_0(p)\phi(p,z) = \left[-\frac{\partial^2}{\partial z^2} + p^2\right]\phi(p,z). \tag{10}$$

$\hat{D}_0$ is defined on interval $\epsilon < z < L$ (here notations of the Introduction are used).

We consider two spectra of eigenvalues $\lambda_n^{(1)}$, $\lambda_n^{(2)}$ of the following equation: $\hat{D}_0\phi_{1,2}(z) = \lambda\phi_{1,2}(z)$, determined by the one and the same Neumann boundary condition at $z = L$ and two mixed Robin boundary conditions at $z = \epsilon$ (prime means derivative over z throughout the paper):

$$\phi'_{1,2}(L) = 0; \qquad \phi'_{1,2}(\epsilon) + r_{1,2}\phi_{1,2}(\epsilon) = 0. \tag{11}$$

Then, according to the BN boundary operator approach, the ratio of corresponding determinants of $\hat{D}_0$ is given by the ratio of corresponding Green functions with both arguments taken at the boundary where different boundary conditions are imposed (that is at $z = \epsilon$ in our example):

$$\frac{\prod_n \lambda_n^{(2)}}{\prod_n \lambda_n^{(1)}} = \frac{\mathrm{Det}_{r_2-N}\hat{D}_0(p)}{\mathrm{Det}_{r_1-N}\hat{D}_0(p)} = \frac{G^{(0)}_{r_1-N}(p;z,z')}{G^{(0)}_{r_2-N}(p;z,z')}\bigg|_{z=z'=\epsilon} \equiv Q_0(p). \tag{12}$$

Green functions in (12) obeying equation $\hat{D}_0(p)G^{(0)}(p;z,z') = \delta(z-z')$ and boundary conditions (11) are given by the following standard expression:

$$G^{(0)}_{r-N}(p;z,z') = \frac{u_r(z)v(z')\theta(z'-z) + (z \leftrightarrow z')}{u'_r v - u_r v'}, \tag{13}$$

where $v(z) = \cosh p(z-L)$ and $u_{r_{1,2}}(z) = p\cosh p(z-\epsilon) - r_{1,2}\sinh p(z-\epsilon)$ obey the following boundary conditions (11):

$$v'(L) = 0; \qquad u'_{r_1}(\epsilon) + r_1 u_{r_1}(\epsilon) = 0;$$
$$u'_{r_2}(\epsilon) + r_2 u_{r_2}(\epsilon) = 0. \tag{14}$$

Thus, for the ratio of determinants $Q_0(p)$ (12) it is obtained from (13) and from the explicit expressions for $v(z)$ and $u_{r_{1,2}}(z)$:

$$Q_0(p) = \frac{u_{r_1}(\epsilon)}{u_{r_2}(\epsilon)} \cdot \frac{u'_{r_2}v - u_{r_2}v'}{u'_{r_1}v - u_{r_1}v'} = \frac{p\sinh p(L-\epsilon) - r_2\cosh p(L-\epsilon)}{p\sinh p(L-\epsilon) - r_1\cosh p(L-\epsilon)}. \tag{15}$$

Let us demonstrate now the identity of BN expressions (12) and (15) with the GY formula for the ratio of determinants:

$$Q_0(p) = \frac{\mathrm{Det}_{r_2-N}\hat{D}_0(p)}{\mathrm{Det}_{r_1-N}\hat{D}_0(p)} = \frac{v'(\epsilon) + r_2 v(\epsilon)}{v'(\epsilon) + r_1 v(\epsilon)}, \tag{16}$$

where $v(z) = \cosh p(z-L)$ is the introduced above solution of homogeneous equation $\hat{D}_o v(z) = 0$ obeying the Neumann boundary condition at $z = L$.

The identity of expressions (15) and (16) is immediately seen from the explicit expression for $v(z)$ and also, in general, if we substitute in (15) $u'_{r_{1,2}}(\epsilon) = -r_{1,2}u_{r_{1,2}}(\epsilon)$ from boundary conditions in (14). This simple observation proves to be quite useful in subsequent analysis.

The difference of one-loop energies corresponding to the ratio of determinants (12)

$$V^{(d)}_{r_2} - V^d_{r_1} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\ln Q_0(p) \tag{17}$$

is UV divergent if Robin coefficients r_1, r_2 are fixed constants, as is seen from the explicit dependence $Q_0(p)$ given in (15). We shall show, however, that application of this logic to asymptotic boundary condition (3) makes r_1, r_2 in (15) dependent on f_1, f_2 and on momentum p in a way that makes $Q_0(p) \to 1$ at $p \to \infty$; hence, the integral in (17) is UV finite in this case.

An analogy of asymptotic, at $z \to 0$, expression (2) for elementary differential operator (10) is $\phi = \alpha z + \beta$ [this formally corresponds to $d = 1$, $\nu = 1/2$ in (2)]. And an analogy of Gubser-Mitra Euclidean Green function (7) (although here we take $L < \infty$) obeying Neumann boundary condition at $z = L$ and double-trace asymptotic boundary condition $\alpha = f\beta$ at $z \to 0$ is the following Green function:

$$G^{(0)}_{f-N}(p;z,z') = \frac{u_f(z)v(z')\theta(z'-z) + (z \leftrightarrow z')}{u'_f v - u_f v'} = \frac{[\cosh pz + \bar{f}\sinh pz]\cosh p(z'-L)\theta(z'-z) + (z \leftrightarrow z')}{p(\sinh pL + \bar{f}\cosh pL)};$$
$$\bar{f} = \frac{f}{p}. \tag{18}$$

Now, according to the BN prescription, we take the ratio of two Green functions (18) for two double-trace parameters f_1, f_2 at $z = z' = \epsilon$ and, following (12), *define* with this ratio the ratio of corresponding determinants:

$$\frac{G^{(0)}_{f_1-N}(p;z,z')}{G^{(0)}_{f_2-N}(p;z,z')}\bigg|_{z=z'=\epsilon} = \frac{\cosh p\epsilon + \bar{f}_1\sinh p\epsilon}{\sinh pL + \bar{f}_1\cosh pL} \cdot \frac{\sinh pL + \bar{f}_2\cosh pL}{\cosh p\epsilon + \bar{f}_2\sinh p\epsilon} = \frac{\mathrm{Det}_{f_2-N}\hat{D}_0(p)}{\mathrm{Det}_{f_1-N}\hat{D}_0(p)} = \frac{\prod_n \tilde{\lambda}_n^{(2)}}{\prod_n \tilde{\lambda}_n^{(1)}} \equiv \tilde{Q}_0(p). \tag{19}$$

Ratio (19) depends on ϵ which is not present in the definition of Green function (18), as well as it is not present in (7). As it was noted, formula (19) is actually a definition of the ratio of determinants, that is, a definition of corresponding eigenvalues $\tilde{\lambda}_n^{(1),(2)}$—just like authors of paper [2] *defined* the UV-finite one-loop energy $V_+^{(d)} - V_-^{(d)}$ with integral over z from ϵ to ∞ although the asymptotic boundary condition (3) is imposed at $z \to 0$ and the integrand (which is the difference of regular and irregular Green functions) does not know anything about $z = \epsilon$.

The difference of vacuum energies corresponding to the ratio of determinants (19) and given by $\int d^d p \ln \tilde{Q}_0(p)$ [cf. (9) or (17)] is UV finite since for $\bar{f}$ weakly depending on p [like in (18) (c.f. also (7)] $\tilde{Q}_0(p)$ in (19) $\to 1$ at $p \to \infty$.

There is the question: what Robin boundary condition at $z = \epsilon$ characterized by parameter r [like in (11)] corresponds to asymptotic condition (3) characterized by double-trace parameter f? Or in other words: what are the conditions of identity of $Q_0(p)$ and $\tilde{Q}_0(p)$ in the rhs of (12) and (19), correspondingly? The function $u_f(z) = \cosh pz + \bar{f} \sinh pz$ in the expression for the Green function (18) formally obeys at $z = \epsilon$ the Robin boundary condition $u'(\epsilon) + r_\epsilon u(\epsilon) = 0$ for the Robin parameter:

$$r_\epsilon = -\frac{u'(\epsilon)}{u(\epsilon)} = -\frac{p(\sinh p\epsilon + \bar{f}(p)\cosh p\epsilon)}{\cosh p\epsilon + \bar{f}(p)\sinh p\epsilon}, \qquad (20)$$

and it is easy to check that a substitution of (20) in BN ratio (15) gives BN ratio (19) identically.

Also, knowledge of $r_\epsilon = r_\epsilon(\bar{f})$ (20) permits us to put down the equations for spectra $\tilde{\lambda}_n^{(1),(2)}$ defined in (19):

$$\sqrt{\tilde{\lambda}_n - p^2}\tan\left[\sqrt{\tilde{\lambda}_n - p^2}(L-\epsilon)\right] = -r_\epsilon = \frac{p(\tanh p\epsilon + \bar{f}(p))}{1 + \bar{f}(p)\tanh p\epsilon}, \qquad (21)$$

$\bar{f}(p)$ see in (18). This equation is obtained from spectral equation $\hat{D}\phi_n = \lambda_n \phi_n$, and boundary conditions (11) where r_ϵ is taken from (20). At $\epsilon = 0$, (21) simplifies and also makes sense, as well as the ratio of determinants (19) makes sense in the limit $\epsilon \to 0$. However, in this case one-loop energy given by $\int d^d p \ln \tilde{Q}_0(p)$ is UV divergent. Thus, $\epsilon > 0$ really serves the UV regulator of quantum loops in d space; is not it curious to see this well-know, fact of AdS/CFT correspondence in the simplest example of this section.

Transcendental equation (21) for $\tilde{\lambda}_n$ is valid, in particular, for regular ($\bar{f} = \infty$) and irregular ($\bar{f} = 0$) asymptotics. It is also seen that (21) comes to spectral conditions for Neumann(ϵ)-Neumann(L) or Dirichlet(ϵ)-Neumann(L) boundary conditions for negative values of $\bar{f}$: $\bar{f} = -\tanh p\epsilon$ ($r_\epsilon = 0$) and $\bar{f} = -1/\tanh p\epsilon$ ($r_\epsilon = \infty$), correspondingly. Surely it is a sort of miracle that the BN approach gives a simple expression (19) for the ratio of infinite products of rather complex eigenvalues—solutions of Eq. (21).

III. ONE-LOOP QUANTUM ENERGY FOR ASYMPTOTIC BOUNDARY CONDITIONS IN ONE-BRANE ($L = \infty$) AND TWO-BRANE ($L < \infty$) RS MODELS

A. One-brane RS model

In parallel with the elementary example of Sec. II we apply the BN prescription [6–9] for the calculation of the ratio of determinants of operator $\hat{D}(p)$ (5) defined like in [2] for the zero boundary condition at IR infinity ($L = \infty$) and for two double-trace asymptotics (3). Like in (12) the ratio of determinants is equal to the ratio of Green functions (7) taken at $z = z' = \epsilon$:

$$\frac{\mathrm{Det}_{f_2}\hat{D}(p)}{\mathrm{Det}_{f_1}\hat{D}(p)} = \frac{G_{f_1}(p;\epsilon,\epsilon)}{G_{f_2}(p;\epsilon,\epsilon)} = \frac{I_{-\nu}(p\epsilon) + \bar{f}_1(p)I_\nu(p\epsilon)}{I_{-\nu}(p\epsilon) + \bar{f}_2(p)I_\nu(p\epsilon)} \cdot \frac{1+\bar{f}_2(p)}{1+\bar{f}_1(p)} \equiv Q(p,\epsilon). \qquad (22)$$

$\bar{f}_{1,2}$ are defined in (7). For regular ($f_2 = \infty$) and irregular ($f_1 = 0$) the asymptotics (3) ratio of corresponding determinants (22) is equal to $I_{-\nu}(p\epsilon)/I_\nu(p\epsilon)$. This was the result of "Remark B" in the Conclusion of [5].

For $\bar{f}_{1(2)}$ given in (7) $Q(p,\epsilon) \to 1$ at $p \to \infty$ [like $\tilde{Q}_0$ in (19)]. Then, the one-loop energy corresponding to the ratio of determinants (22) is UV finite:

$$V_{f_2}^{(d)} - V_{f_1}^{(d)} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln Q(p,\epsilon) < \infty. \qquad (23)$$

This conclusion is not valid for $\epsilon = 0$ in (22), which is in absence of the UV-brane screening AdS horizon. Thus here again—like in a simple example of Sec. II [cf. (19)]—ϵ plays a role of the UV regulator of UV divergencies of the one-loop vacuum energy (23).

In paper [2] $f_2 = f$ and $f_1 = 0$ (irregular asymptotic boundary condition denoted by index "-") were considered. And from (22) and (23) it follows that

$$\tilde{V}^{(d)}(f) \equiv V_f^{(d)} - V_-^{(d)} = -\frac{\Omega_{d-1}}{2(2\pi)^d \epsilon^d}\int_0^\infty y^{d-1} dy \ln\left[\frac{I_{-\nu}(y) + \bar{f}(y,\epsilon)I_\nu(y)}{I_{-\nu}(y)(1+\bar{f}(y,\epsilon))}\right], \qquad (24)$$

where $y = p\epsilon$, Ω_{d-1} is the volume of the $(d-1)$ sphere of the unit radius, and the function $\bar{f}(y,\epsilon)$ in (24) is easily seen from the definition of $\bar{f}(p)$ in (7): $\bar{f}(y,\epsilon) = f(2\epsilon)^{2\nu}\Gamma(1+\nu)/y^{2\nu}\Gamma(1-\nu)$. Thus, potential (24) is actually a function of dimensionless double-trace parameter $f\epsilon^{2\nu}$.

The formula for the difference of regular and irregular one-loop energies $V_+^{(d)} - V_-^{(d)}$ follows from (24) when $f = \infty$:

$$V_+^{(d)} - V_-^{(d)} = \frac{\Omega_{d-1}}{2(2\pi)^d \epsilon^d} \int_0^\infty y^{d-1} dy \ln\left[\frac{I_{-\nu}(y)}{I_\nu(y)}\right] = \frac{2\sin(\pi\nu)\Omega_{d-1}}{(2\pi)^{d+1} d\epsilon^d} \int_0^\infty \frac{y^{d-1} dy}{I_\nu(y) I_{-\nu}(y)}. \quad (25)$$

This expression differs from the ones, also different, received for $V_+ - V_-$ with the standard procedure (8) in [2–4] and in [5]. The visible drawback of formulas (23)–(25) is in their zero value for integer ν. However, this is the difficulty of the approach of papers [2–5] based upon different asymptotics at $z \to 0$ of I_ν and $I_{-\nu}$ coinciding at the ν integer.

Again, in parallel with the simple example of Sec. II, it is worthwhile to note that the nice GY formula (16) for the ratio of determinants now takes the following form:

$$\frac{\mathrm{Det}_{f_2} \hat{D}(p)}{\mathrm{Det}_{f_1} \hat{D}(p)} = \frac{\epsilon v'(p\epsilon) + r_2 v(p\epsilon)}{\epsilon v'(p\epsilon) + r_1 v(p\epsilon)}, \quad (26)$$

and it exactly coincides with ratio (22) if we use in (26) solutions of Eq. (5) determining Green functions (7), that is if it is taken $v(pz) = z^{d/2} K_\nu(pz)$ and $r_{1(2)}$ are built from $u_f = z^{d/2}[I_{-\nu} + \bar{f} I_\nu]$ in a way similar to (20):

$$r_{\epsilon 1(2)} = -\frac{\epsilon u'_{f_{1(2)}}(p\epsilon)}{u_{f_{1(2)}}(p\epsilon)} = -\frac{d}{2} - \frac{\epsilon I'_{-\nu}(p\epsilon) + \bar{f}_{1(2)} \epsilon I'_\nu(p\epsilon)}{I_{-\nu}(p\epsilon) + \bar{f}_{1(2)} I_\nu(p\epsilon)}. \quad (27)$$

B. Two-brane RS model

The introduction of the IR brane at finite $z = L < \infty$ does not make the task of the calculation of the one-loop quantum energy too much more complicated than in the case of the one-brane RS model considered above. The Green function $G_{f-r}^{(L)}(p; z, z')$ satisfying asymptotic boundary condition (3) at $z \to 0$ and a certain Robin boundary condition $zG' + rG = 0$ at $z = L$ is given by the expression similar to (7) where $z^{d/2} K_\nu(pz)$ must be changed to function $v_r(pz)$ obeying the Robin boundary condition $zv' + rv = 0$ at $z = L$:

$$v_r(pz) = \frac{\pi}{2\sin\pi\nu} z^{d/2} [I_{-\nu}(pz) - \gamma_r(pL) I_\nu(pz)],$$
$$\gamma_r(pL) = \frac{A_r[I_{-\nu}(pL)]}{A_r[I_\nu(pL)]},$$
$$A_r[\psi(pz)] = \left(\frac{d}{2} + r\right)\psi(pz) + z\psi'(pz). \quad (28)$$

Here for any value of the Robin parameter r: $\gamma_r(pL) \to 1$, $v_r(pz) \to z^{d/2} K_\nu(pz)$ at $L \to \infty$. Finally, the Green function $G_{f-r}^{(L)}$ is built from solutions of Eq. (5) $v_r(pz)$ (28) and $u_f(pz) = z^{d/2}[I_{-\nu}(pz) + \bar{f} I_\nu(pz)]$ [like in (7)]:

$$G_{f-r}^{(L)}(p; z, z') = -k^{d-1} \frac{u_f(z) v_r(z') \theta(z' - z) + (z \leftrightarrow z')}{u'_f v_r - u_f v'_r}$$
$$= -\frac{\pi k^{d-1} (zz')^{d/2}}{2\sin\pi\nu} \cdot \frac{[I_{-\nu}(pz) + \bar{f} I_\nu(pz)][I_{-\nu}(pz') - \gamma_r(pL) I_\nu(pz')]\theta(z' - z) + (z \leftrightarrow z')}{\gamma_r(pL) + \bar{f}(p)}, \quad (29)$$

where $\bar{f}(p)$ and $\gamma_r(pL)$ are defined in (7) and (28).

Thus, for $L < \infty$ the looked for ratio of one-loop determinants of differential operator (5) determined by two values of parameter f in the double-trace asymptotic condition (3) is given by a slightly modified BN formula (22):

$$\frac{\mathrm{Det}_{f_2-r} \hat{D}(p)}{\mathrm{Det}_{f_1-r} \hat{D}(p)} = \frac{G_{f_1-r}^{(L)}(p; \epsilon, \epsilon)}{G_{f_2-r}^{(L)}(p; \epsilon, \epsilon)} = \frac{I_{-\nu}(p\epsilon) + \bar{f}_1(p) I_\nu(p\epsilon)}{I_{-\nu}(p\epsilon) + \bar{f}_2(p) I_\nu(p\epsilon)} \cdot \frac{\gamma_r(pL) + \bar{f}_2(p)}{\gamma_r(pL) + \bar{f}_1(p)}. \quad (30)$$

Surely this expression for ratio of determinants is also given by the rhs of GY formula (26) if $v_r(p\epsilon)$ from (28) and $r_{\epsilon 1,2}$ from (27) are used in (26).

The visible feature of expression (30) is that its rhs includes two factors: one depending only on ϵ and the other one depending only on L. Therefore, one-loop vacuum energy $V_{f_2-r}^{(d)} - V_{f_1-r}^{(d)}$ corresponding to ratio (30) and given by standard expression (9) consists of two terms depending on ϵ and on L. In particular, taking in (30) $f_2 = \infty$ and $f_1 = 0$ the following formula for the difference of regular and irregular one-loop quantum energies is obtained in the two-brane RS model:

$$V_{+(L)}^{(d)} - V_{-(L)}^{(d)} = \frac{\Omega_{d-1}}{2(2\pi)^d} \frac{1}{\epsilon^d} \int_0^\infty y^{d-1} dy \ln\left[\frac{I_{-\nu}(y)}{I_\nu(y)}\right] - \frac{\Omega_{d-1}}{2(2\pi)^d} \frac{1}{L^d} \int_0^\infty y^{d-1} dy \ln\left[\frac{(\frac{d}{2}+r) I_{-\nu}(y) + y I_{-\nu}{}'(y)}{(\frac{d}{2}+r) I_\nu(y) + y I'_\nu(y)}\right]. \quad (31)$$

In receiving (31) from general formula (30) the definition of $\gamma_r(pL)$ given in (28) was used.

It is instructive to compare this result with the one-loop quantum energy in the RS model calculated in [14] and [15] where, not the asymptotic boundary condition (3) but rather the Robin boundary condition with a fixed Robin coefficient is imposed at $z = \epsilon$. Then, as it is shown in [14] and [15], a UV-finite nonlocal term of the one-loop quantum potential calculated for integer ν includes dependence on $\ln(L/\epsilon)$; hence, it gives hope for the dynamical explanation of the large mass hierarchy. There is nothing like this in expression (31). That is, the one-loop potential calculated for an asymptotic boundary condition cannot serve as a tool of stabilization of the IR brane.

IV. CONCLUSION: SOME TASKS FOR THE FUTURE

The main message of this paper perhaps may be expressed in one word "simplicity." The surprising simplicity of the BN and GY methods of calculation of quantum determinants hopefully opens new possibilities in studying quantum effects in higher dimensional models.

In particular, one-loop potential (24) as a function of double-trace parameter f may be of the Coleman-Weinberg-type in certain Schwinger-Dyson gap equations determining f self-consistently.

However, interesting results in this direction of thought may be expected for integer ν when formulas of the paper cannot be applied directly because $I_\nu = I_{-\nu}$ in this case. For the ν integer Green function of differential operator $\hat{D}(p)$, (5) may be easily constructed from solutions $z^{d/2}I_\nu(pz)$ and $z^{d/2}K_\nu(pz)$ in direct analogy with (7). Correspondingly the ratio of determinants of $\hat{D}(p)$ (5) may be put down for any boundary conditions at $z = L$ and at $z = \epsilon$ using the Barvinsky-Nesterov formula. Here the problem is in the lack of a physically motivated analogy of an asymptotic expression (2) when ν is the integer. Hence it is not clear in case of integer ν what may be the analogy of function $\bar{f}(p)$ in (7) introduced in [2] in the case of noninteger ν. Meanwhile, function $\bar{f}(p)$ essentially determines the form of the physically important potential (24) of the double-trace parameter f.

Another possible field of future study is the construction of the Schwinger-DeWitt expansion in the RS model on the basis of BN or GY methods applied in this paper. In [5] it was shown that in the one-brane RS model, the Schwinger-DeWitt expansion for curvature in d space is plagued by IR divergencies in higher terms of the expansion, and that these divergencies are regularized in the two-brane RS model, that is, when $L < \infty$. The same role of the term depending on L in expression (31) may be expected. This is the question for future research.

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APPENDIX: GY FORMULA AND CASIMIR EFFECT IN ONE LINE

A short explanation of the Gelfand-Yaglom approach was given in the Introduction. Here we demonstrate on some examples that the GY method immediately gives familiar results obtained conventionally in a rather lengthy way. The examples considered below refer to flat $(d+1)$-dimensional space and to an elementary differential operator $\hat{D}_0(p)$ (10).

(1) Classical Dirichlet-Dirichlet problem ($0 < z < L$): $\phi(0) = 0$, $\phi(L) = 0$. $v = C \cdot \sinh(pz)$ is a solution of the following equation $\hat{D}_0\phi = 0$, satisfying the boundary condition at $z = 0$. Then, according to the GY method, the "Dirichlet-Dirichlet" determinant $\mathrm{Det}_{D-D}\hat{D}_0 \sim \sinh(pL)$. This yields an expression for the quantum potential in d dimensions:

$$V^{(d)}_{D-D} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\ln[\sinh(pL)] = A + BL - \frac{1}{L^d}\frac{\Omega_{d-1}}{(2\pi)^d 2^{d+1} d}\int_0^\infty \frac{y^d dy}{e^y - 1}, \tag{A1}$$

where the volume of the sphere of the unit radius of dimension zero must be taken equal to 2 ($\Omega_{1-1} = 2$); A, B are irrelevant divergent constants. The last term in (A1) which is UV finite and tends to zero at $L \to \infty$ is the Casimir potential $V^{(d)}_{CasD-D}$. It is easy to check that (A1) gives its well-known [16] values in (1+1) and in (3+1) dimensions: $V^{(1)}_{CasD-D}L = -\pi/24$, $V^{(3)}_{CasD-D}L^3 = -\pi^2/1440$ (for the electromagnetic field this result must be multiplied by 2—the number of polarizations of the electromagnetic field).

(2) In the same way, the Casimir potential may be calculated in the Dirichlet-Neumann problem $[\mathrm{Det}_{D-N}\hat{D}_0(p) \sim \cosh pL]$ and in many other problems. One of the striking examples of the power of the GY method is the calculation of Casimir potential in $M^d \times S^1$ when z is a circle of length $L = 2\pi\rho$. In this case, spectra of periodic (untwisted) or antiperiodic (twisted) modes are found from the following equations: $\cos(\sqrt{\lambda_n - p^2}L) = \pm 1$. Then according to GY,

in the untwisted case, for example, $\mathrm{Det}_{untw}\hat{D}(p) \sim (\cosh pL - 1)$ [$\mathrm{Det}_{tw}\hat{D}(p) \sim (\cosh pL + 1)$ for twisted modes]. The UV-finite term of vacuum energy,

$$V^{(d)}_{untw} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln[\cosh(pL) - 1] = A + BL - \frac{1}{L^d}\frac{\Omega_{d-1}}{(2\pi)^d d}\int_0^\infty \frac{y^d dy}{e^y - 1}, \qquad (A2)$$

gives, in particular, well-known results for $d = 1$: $V_{Cas\,untw} = 4V_{CasD-D}$—cf. (A2) and (A1), and, for the Casimir effect on torus in 5 dimensions, i.e., for $d = 4$: $V^{(4)}_{Cas\,untw} \cdot \rho^4 = -3\zeta(5)/(2\pi)^6$, received in [16] and [17] with rather complex calculations. It is easy to get in the same way, well-known values of the Casimir potential for twisted modes.

(3) The GY method also gives at once, the final formula for the Casimir potential in the case of the general mixed Robin boundary conditions imposed on solutions of the following equation: $\hat{D}_0\phi = \lambda\phi$, on both borders $z = a$ and $z = b$ (prime means derivative over z):

$$\phi'(a) + r_a\phi(a) = 0; \quad \phi'(b) + r_b\phi(b) = 0, \qquad (A3)$$

where $r_{a,b}$ are Robin "masses." $v(z) = r_a \sinh p(z - a) - p \cosh p(z - a)$ is a solution of homogeneous equation $\hat{D}_0 v = 0$ obeying Robin boundary condition at $z = a$. Then, the GY method says that $\mathrm{Det}\hat{D}_0 \sim (v'(b) + r_b v(b))$. This gives straightaway for the Casimir potential (which is a UV-finite term of $V = 1/2\int d^d p \ln[v'(b) + r_b v(b)]$) expression identically coinciding with the massless version of formula (22) of paper [18] (after substitutions $p \to x,\ b - a \to a,\ r_a \to \beta_2^{-1},\ r_b \to \beta_1^{-1}$).

The generalization of the above formulas for the case of the massive scalar field is obvious.

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