PHYSICAL REVIEW D **95,** 085021 (2017)

$\mathbb{C}P^N$ sigma model on a finite interval revisited

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(Received 3 January 2017; published 27 April 2017)

In this paper we will revisit the large N solution of the $\mathbb{C}P^N$ sigma model on a finite interval of length L. We will find a family of boundary conditions for which the large N saddle point can be found analytically. For a certain choice of the boundary conditions the theory has only one phase for all values of L. Also, we will provide an example when there are two phases: for large L there is a standard phase with an unbroken $U(1)$ gauge symmetry and for small L there is a Higgs phase with a broken gauge symmetry.

DOI: 10.1103/PhysRevD.95.085021

I. INTRODUCTION

The two-dimensional $\mathbb{C}P^N$ sigma model in the large N limit was first solved in [1,2]. The theory exhibits a plethora of nontrivial properties: asymptotic freedom, confinement, and dynamical scale Λ generation via the dimensional transmutation:

$$\Lambda^2 = \Lambda_{\mathrm{uv}}^2 \exp\left(-\frac{4\pi}{g^2}\right) \tag{1}$$

where g is the coupling constant.

Physically, the 2D $\mathbb{C}P^N$ model naturally arises as a low-energy effective action of non-Abelian strings in QCD-like models; see [3] for a review. Therefore, a finite interval geometry corresponds to a string stretched between two branes or a monopole-antimonopole pair. Such configuration was studied in [4].

Recently the $\mathbb{C}P^N$ sigma model on a finite interval of length L with Dirichlet boundary conditions (BC) was investigated in [5] and [6] using large N expansion. In the earlier work [5] the large N saddle-point equations were solved only approximately and two distinct phases were found. In [6] saddle-point equations were solved numerically and it was argued that there is only one phase. In this paper we will find a set of boundary conditions for which the saddle-point equations can be solved analytically. Strictly speaking, we will study the $\mathbb{C}P^{2N}$ sigma model. We will consider two different boundary conditions:

(i) Mixed Dirichlet-Neumann(D-N) boundary conditions which will break global $SU(2N+1)$ to $SU(N)\times SU(N)$. We will show that the system has at least two phases: for $L > \pi/4\Lambda$ there is a standard "Coulomb" phase with an unbroken $U(1)$ gauge symmetry. This phase takes place for the $\mathbb{C}P^N$ model on usual $\mathbb{R}^2$. For $L < \pi/4\Lambda$ there is the "Higgs" phase with broken $U(1)$. Global $SU(N)\times SU(N)$ stays unbroken in both phases.

(ii) Dirichlet-Dirichlet and Neumann-Neumann (D-D and N-N) boundary conditions which will break $SU(2N+1)$ to $SU(N)\times SU(N+1)$. In this case, for all values of L there is a standard phase with an unbroken $U(1)$ gauge symmetry. The Higgs phase is prohibited in this case, because it will break global $SU(N)\times SU(N+1)$ to $SU(N)\times SU(N)$.

The main motivation of the present paper was to demonstrate that the phase structure is very sensitive to the amount of symmetries preserved by the boundary conditions. Since in two dimensions spontaneous symmetry breaking cannot occur, we can have more than one phase only if the corresponding symmetry which distinguishes these two phases is already broken at the boundary. Two exactly soluble examples considered in this paper support this claim.

As was mentioned before, simple Dirichlet-Dirichlet boundary conditions have been already studied in [5,6] and the conclusions obtained in these two papers were contradictory. In [6], the $\mathbb{C}P^N$ sigma model on a finite interval was studied numerically, and the result is quite surprising: authors claim that they see no difference between the Dirichlet-Dirichlet and Neumann-Neumann boundary conditions [7]. It seems strange, since the N-N boundary conditions do not break $SU(N+1)$ global symmetry, whereas D-D boundary conditions do break it. Although the solution found in the original paper [5] was only an approximate solution, we expect that the conclusion obtained there is correct, and the $\mathbb{C}P^N$ sigma model with D-D [8] boundary conditions does have a phase transition, contrary to the results of [6]. Finally, let us note that the large N $\mathbb{C}P^N$ model on a cylinder also possesses multiple phases [9].

II. GENERALIZED SADDLE-POINT EQUATIONS

Let us study the $\mathbb{C}P^{2N}$ model in the large N limit. The field content consists of $2N+1$ fields $n^i, i=0,\ldots,2N$, real vector field A_μ and real scalar λ. In the Euclidean space the Lagrangian reads as

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$$\mathcal{L} = (D_\mu n^i)^*(D^\mu n^i) + \lambda(n^{*i}n^i - r) \tag{2}$$

where $D_\mu = \partial_\mu - iA_\mu$, $\mu = t$, x, and $r = 2N/g^2$. The time coordinate t takes values from $-\infty$ to $+\infty$ and $x \in [0, L]$.

Nondynamical Lagrangian multipliers A_μ and λ force n^i to lie on $\mathbb{C}P^{2N}$ space: integration over λ yields $\sum_i n^{*i}n^i = r$ and A_μ is responsible for $U(1)$ invariance $n^i \sim e^{i\phi}n^i$.

We will proceed in a standard fashion: we will integrate out $2N$ fields n^i, $i = 1, \ldots, 2N$ fields and then find the large N saddle-point values of λ, A_μ and the remaining n^0 which we will denote by $\sigma = n^0$. After integrating out $2N$ n^i, fields we have

$$S_{\text{eff}} = \operatorname{tr}\log(-D_x^2 - D_t^2 + \lambda) + \int d^2x((D\sigma)^2 + \lambda(|\sigma|^2 - r)). \tag{3}$$

So far we do not have a factor of $2N$ in front of the determinant because we will impose different boundary conditions for these $2N$ fields.

We will study this model on a finite interval of length L with various boundary conditions. Note that the translational symmetry in the x direction is explicitly broken. However, we still have the time translations so we will consider only time translation invariant saddle points. By the choice of gauge we can always set $A_t = 0$. This allows us to rewrite Eq. (3) as

$$S_{\text{eff}} = \sum_n E_n + \int d^2x((D_x\sigma)^2 + \lambda(|\sigma|^2 - r)). \tag{4}$$

Note that we have already integrated out time frequencies, so we have energies E_n instead of of the usual log det. The sum over n is the sum over the eigenvalues E_n^2 of the following equation:

$$(-D_x^2 + \lambda(x))\psi_n = E_n^2\psi_n(x) \tag{5}$$

ψ_n are required to be normalized.

Varying effective action (4) with respect to λ we get the first saddle-point equation:

$$\frac{1}{2}\sum_n \frac{|\psi_n(x)|^2}{E_n} + |\sigma(x)|^2 - r = 0. \tag{6}$$

To obtain this equation we have used the standard quantum mechanical first order perturbation theory for (5).

The second saddle-point equation coincides with the σ equation of motion:

$$D_x^2\sigma - \lambda(x)\sigma = 0. \tag{7}$$

Finally, we have to vary with respect to A_x:

$$\frac{i}{2}\sum_n \frac{\psi_n(D_x\psi_n)^* - \psi_n^* D_x\psi_n}{E_n} = i\sigma(D_x\sigma)^* - i\sigma_n^* D_x\sigma_n \tag{8}$$

Below we will study the case $A_x = 0$ with real ψ_n and σ and so this equation will be trivially satisfied.

III. D-N BOUNDARY CONDITIONS: TWO PHASES

Now it is time to choose boundary conditions. Let us consider the following: For N fields n^i, $i = 1, \ldots, N$ we will use Dirichlet-Neumann:

$$n^i(0) = 0, \qquad D_x n^i(L) = 0. \tag{9}$$

And for N fields $n^i, i = N+1, \ldots, 2N$ we will use Neumann-Dirichlet:

$$D_x n^i(0) = 0, \qquad n^i(L) = 0. \tag{10}$$

And for σ we will impose Neumann-Neumann:

$$D_x\sigma(0) = D_x\sigma(L) = 0. \tag{11}$$

This choice breaks global $SU(2N+1)$ to $SU(N) \times SU(N)$.

Then in the D-N sector we have

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x(n-1/2)}{L}\right),$$
$$E_n^2 = \left(\frac{\pi(n-1/2)}{L}\right)^2 + \lambda, \quad n = 1, \ldots \tag{12}$$

In the N-D sector:

$$\tilde{\psi}_n(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x(n-1/2)}{L}\right),$$
$$E_n^2 = \left(\frac{\pi(n-1/2)}{L}\right)^2 + \lambda, \quad n = 1, \ldots \tag{13}$$

If we plug this into the first-saddle point equation (6) we will notice that $\sin^2$ and $\cos^2$ will sum up to 1 and the x dependence will disappear.. So we can consider σ to be constant. Let us first study the phase with nonzero λ. From the second saddle-point equation (7) we see that we have to put $\sigma = 0$. We will call this phase "Coulomb" phase because n^i has zero vacuum expectation value (vev) leaves the $U(1)$ unbroken.

The first saddle-point equation now reads as

$$\frac{N}{\pi}\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n-1/2)^2 + (\lambda L/\pi)^2}} - r = 0. \tag{14}$$

We need to separate the divergent part:

$$\frac{N}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(n-1/2)^2+(\lambda L/\pi)^2}}-\frac{1}{n}\right) + \frac{N}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}-r=0. \qquad (15)$$

Introducing the cutoff,

$$\sum_{n=1}^{\infty}\frac{\exp(-n\pi/L\Lambda_{\mathrm{uv}})}{n} = -\log(1-\exp(-\pi/L\Lambda_{\mathrm{uv}}))\approx -\log(\pi/L\Lambda_{\mathrm{uv}}) \qquad (16)$$

Renormalizing r using Eq. (1) we will have

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(n-1/2)^2+(\lambda L/\pi)^2}}-\frac{1}{n}\right)=\log(\pi/\Lambda L). \qquad (17)$$

Now it is easy to see the presence of two phases: the maximum of the left-hand side is reached when $\lambda = 0$, the corresponding value is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n-1/2}-\frac{1}{n}\right)=\log(4). \qquad (18)$$

It means that if $\log(\pi/\Lambda L)>\log(4)$ the saddle-point equations do not have a solution with nonzero λ.

Let us consider the limit $L\to 0$. We can expand the left-hand side in power series in λL:

$$\frac{1}{\sqrt{(n-1/2)^2+(\lambda L/\pi)^2}} = \frac{1}{n-1/2}-4\left(\frac{\lambda L}{\pi}\right)^2\frac{1}{(2n-1)^3}+\ldots \qquad (19)$$

Using the following identity,

$$\sum_{n=1}^{\infty}\frac{4}{(2n-1)^3}=\frac{7}{2}\zeta(3) \qquad (20)$$

we have

$$\frac{7\zeta(3)}{2}\left(\frac{\lambda L}{\pi}\right)^2=\log(4\Lambda L/\pi). \qquad (21)$$

We see that the Coulomb phase does not exist for $L<\pi/4\Lambda$.

Let us now show that the Higgs phase $\sigma = \mathrm{const}$, $\lambda = 0$ exists only for $L<\pi/4\Lambda$. We call this phase Higgsbecause nonzero σ breaks $U(1)$ gauge symmetry. In this case the second saddle-point equation is satisfied. The first one reads as

$$\frac{N}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n-1/2}-\frac{1}{n}\right)+\sigma^2=\frac{N}{\pi}\log(\pi/\Lambda L). \qquad (22)$$

Again using Eq. (20) we have

$$\sigma^2=\frac{N}{\pi}\log(\pi/4\Lambda L). \qquad (23)$$

IV. D-D AND N-N BOUNDARY CONDITIONS: ONE PHASE

Instead of the D-N and N-D boundary conditions let us investigate the case with Dirichlet-Dirichlet and Neumann-Neumann boundary conditions. As we will see shortly, the Coulomb phase is possible for all values of L. For the D-D case we have the following set of eigenfunctions:

$$\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x n}{L}\right), \quad E_n^2=\left(\frac{\pi n}{L}\right)^2+\lambda, \quad n=1,\ldots \qquad (24)$$

And for N-N,

$$\psi_n(x)=\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x n}{L}\right), \quad E_n^2=\left(\frac{\pi n}{L}\right)^2+\lambda, \quad n=0,\ldots \qquad (25)$$

Note that now we can have $n = 0$ which corresponds to a constant mode. Moreover if $\lambda = 0$ we have a genuine zero mode. It means that the Higgs phase with $\lambda = 0$ cannot exist for this choice of boundary conditions. In the saddle-point equations $\cos^2$ and $\sin^2$ again sum to 1, so we can have a saddle point with constant σ and λ. From now on, we will assume that $\lambda = \mathrm{const} \neq 0$. Then from the second saddle-point equation it follows that $\sigma = 0$. The first saddle-point equation now reads as

$$\frac{N}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^2+(\lambda L/\pi)^2}}-\frac{1}{n}\right)+\frac{N}{\lambda L}+\frac{N}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}-r=0. \qquad (26)$$

After r renormalization we have

$$\frac{N}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^2+(\lambda L/\pi)^2}}-\frac{1}{n}\right)+\frac{N}{\lambda L}=\frac{N}{\pi}\log(\pi/\Lambda L). \qquad (27)$$

Unlike the D-N and N-D case now the left-hand side is not bounded from above because of the $\frac{N}{\lambda L}$ term, which is essentially the contribution from the N-N constant mode.

It easy to show that for a fixed Λ and L we can always find the corresponding value of λ (for example one can plot the left-hand side as a function of λ and see that it takes values from $-\infty$ to $+\infty$).

V. CONCLUSION

In this paper we studied the large N $\mathbb{C}P^N$ model on a finite interval. We have shown that for a specific choice of boundary conditions the saddle-point equations admit a simple analytical solution. Under the Dirichlet-Dirichlet and Neumann-Neumann boundary condition the system possesses a Coulomb phase with the uniform λ vev, usual for the $\mathbb{C}P^N$ in the infinite space. This phase exists for all values of the interval length L. However, under the mixed Dirichlet-Neumann boundary conditions the system has two phases: the Coulomb phase which exists for $L > \pi/4\Lambda$ and the unusual Higgs phase for $L < \pi/4\Lambda$ with the uniform n^0 vev. Strictly speaking, it is possible to have additional phases with nonconstant vev's, similar to the FFLO [10,11] phase in superconductivity. It is even possible that the Coulomb and Higgs phases in the N-D case are not adjacent on the phase diagram because of the presence of additional phases. We will postpone this analysis for future work.

ACKNOWLEDGMENTS

We are thankful to A. Gorsky for numerous discussions and F. Popov for reading the manuscript. A. M. is grateful to RFBR for Grant No. 15-02-02092 for travel support.

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[7] To be more precise, the claim is that for the phase they have found, the eigenfunctions of $-\partial_x^2 + \lambda(x)$ satisfy both Dirichlet and Neumann boundary conditions at the same time. This is possible only for singular $\lambda(x)$.

[8] Also, we think that N-N boundary conditions do not lead to a phase transition; actually the published version of [5] contains the corresponding computation. In this aspect [5,6] agree.

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