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Low-energy effective worldsheet theory of a non-Abelian vortex in high-density QCD revisited: A regular gauge construction

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Color symmetry is spontaneously broken in quark matter at high density as a consequence of di-quark condensations with exhibiting color superconductivity. Non-Abelian vortices or color magnetic flux tubes stably exist in the color-flavor locked phase at asymptotically high density. The effective worldsheet theory of a single non-Abelian vortex was previously calculated in the singular gauge to obtain the $\mathbb{C}P^2$ model [1,2]. Here, we reconstruct the effective theory in a regular gauge without taking a singular gauge, confirming the previous results in the singular gauge. As a byproduct of our analysis, we find that non-Abelian vortices in high-density QCD do not suffer from any obstruction for the global definition of a symmetry breaking.

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I. INTRODUCTION

Quark matter at high temperature and/or high density is one of the important subjects in both the theoretical and experimental points of view. At high density, quark matter are expected to condensate by constituting Cooper pairs. Then, color symmetry is spontaneously broken, with exhibiting color superconductivity [3,4]; see Refs. [5,6] as a review. The two-flavor pairing may occur in the two-SC phase in which up and down quarks participate in condensations at intermediate density. At asymptotically high densities, if we can neglect the strange quark mass, the system possesses an $SU(3)$ global flavor symmetry. In that region, it may be possible to have a three-flavor pairing state, which is known as the "color-flavor locked (CFL)" phase, in which up, down, and strange quarks participate in condensations. The Ginzburg-Landau (GL) free energy [7–9] shows that in the CFL ground state the baryon number $U(1)_{\mathrm{B}}$, color $SU(3)_{\mathrm{C}}$, and flavor $SU(3)_{\mathrm{F}}$ symmetries are spontaneously broken down to the diagonal subgroup $SU(3)_{\mathrm{C+F}}$. In particular, $U(1)_{\mathrm{B}}$ and color symmetry breakings lead to superfluidity and color superconductivity, respectively. Therefore, when the CFL medium rotates, $U(1)_{\mathrm{B}}$ superfluid vortices with the quantized circulations are created along the rotation axis [7,8,10] as in the case of helium superfluids and ultracold atomic gases. Compared to the quantized unit circulation of $U(1)_{\mathrm{B}}$ superfluid vortices, vortices with smaller circulations ($1/3$ quantized circulations) exist, which also carry color magnetic fluxes. They are non-Abelian vortices or color magnetic flux tubes [11–14]; see Ref. [15] for a review. It was conjectured that one $U(1)_{\mathrm{B}}$ superfluid vortex is energetically split into a set of three color flux tubes with total color cancelled out [12], which has been recently confirmed numerically [16].

One non-Abelian vortex breaks the color-flavor symmetry $SU(3)_{\mathrm{C+F}}$ further into its subgroup in the vicinity of the core, generating Nambu-Goldstone modes (or collective coordinates) which parametrize a complex projective space $\mathbb{C}P^2 \simeq SU(3)_{\mathrm{C+F}}/[SU(2)\times U(1)]$. These $\mathbb{C}P^2$ modes are localized around the vortex core and propagate along the vortex line as gapless excitations [1,2]. A lot of rich physics were obtained from these $\mathbb{C}P^2$ modes. When the coupling of the $\mathbb{C}P^2$ target space to electromagnetic fields is introduced [17], it implies that a vortex lattice system behaves as a polarizer [18]. It also shows the Aharanov-Bohm scattering of charged particles such as electrons and muons [19]. The quantum mechanically induced gap shows the confinement of monopoles in the CFL phase; that is, quark condensations lead monopole confinement [20,21], which gives evidence of hadron-quark duality to the confinement phase in which quark confinement is expected to occur due to monopole condensations. Vortices interact with gluons by a topological interaction [22], implying that in the system of multiple vortices such as a vortex lattice, $\mathbb{C}P^2$ modes in individual vortices are aligned by the interaction, exhibiting color ferromagnetism [23]. Even with such discoveries of rich physics, there remains one technical problem in the derivation of the effective Lagrangian in Refs. [1,2]; in these references, a singular gauge was taken to construct the effective $\mathbb{C}P^2$ model. It is well known that, in general, one needs a careful treatment in the singular gauge. For instance, in the Abelian-Higgs model relevant for conventional metallic superconductors, a magnetic flux of a vortex is unphysically removed by taking the singular gauge. Therefore, one needs carefully to check whether the results in Refs. [1,2] for the effective action are correct and whether or not any additional term exists.

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In this paper, we construct the effective Lagrangian of a single non-Abelian vortex in a regular gauge without taking a singular gauge and confirm that the result of the effective Lagrangian calculated in the singular gauge in Refs. [1,2] is correct and no further term exists. To do this, we generalize the ansatz for the gauge field used in the singular gauge because it does not solve the Gauss-law constraint in the regular gauge. We introduce two different profile functions that depend on both the radial coordinate r and the azimuthal angle θ, in contrast to the singular gauge for which the profile function depends only on r. The profile functions of the zero modes are expanded in terms of partial waves, and we check the asymptotic behaviors of all the partial wave modes. By inserting the solutions of the partial wave modes into the original GL action, we derive the effective action of the vortex as the $\mathbb{C}P^2$ action living on the vortex worldsheet. The mode previously found in the singular gauge comes out as a normalizable mode among the partial wave modes discussed in this paper. By showing that the rest of the partial modes are all non-normalizable, we prove that the previous result on the effective theory on the vortex is correct.

This paper is organized as follows. In Sec. II, we review the GL effective theory, a non-Abelian vortex solution and its properties. In Sec. III, we construct the effective theory of a single vortex in a regular gauge. Section IV is devoted to a summary and discussion.

II. THE GINZBURG-LANDAU DESCRIPTION OF DENSE QCD AND A NON-ABELIAN VORTEX

A. Ginzburg-Landau effective theory

We start with the time-dependent GL Lagrangian for the CFL order parameters $\Phi_{\rm L}$ and $\Phi_{\rm R}$ which are defined as di-quark condensates,

$$\Phi_{{\rm L}a}^{\;A} \sim \epsilon_{abc}\epsilon^{ABC} q_{{\rm L}b}^{\;B}\mathcal{C} q_{{\rm L}c}^{\;C}, \qquad \Phi_{{\rm R}a}^{\;A} \sim \epsilon_{abc}\epsilon^{ABC} q_{{\rm R}b}^{\;B}\mathcal{C} q_{{\rm R}c}^{\;C}, \tag{1}$$

where $q_{\rm L/R}$ stand for left- and right-handed quarks with a, b, c as fundamental color ($SU(3)_{\rm C}$), A, B, C as fundamental flavor ($SU(3)_{\rm L/R}$) indices and $\mathcal{C}$ is the charge conjugation operator. Since at a high-density region a perturbative calculation shows mixing terms between $\Phi_{\rm L}$ and $\Phi_{\rm R}$ are negligible, we simply assume $\Phi_{\rm L} = -\Phi_{\rm R} = \Phi$ and fix their relative phase to unity. The transformation properties of the field Φ can be written as

$$\begin{aligned} &\Phi' = e^{i\theta_{\rm B}} U_{\rm C} \Phi U_{\rm F}^{-1}, \qquad e^{i\theta_{\rm B}} \in U(1)_{\rm B}, \\ &U_{\rm C} \in SU(3)_{\rm C}, \qquad U_{\rm F} \in SU(3)_{\rm F}. \end{aligned} \tag{2}$$

Here $SU(3)_{\rm F}$ is defined as the diagonal subgroup ($SU(3)_{\rm L+R}$) of the full flavor group $SU(3)_{\rm L} \times SU(3)_{\rm R}$. There is a redundancy of the discrete symmetries, and the actual symmetry group is given by

$$G = \frac{SU(3)_{\rm C} \times SU(3)_{\rm F} \times U(1)_{\rm B}}{\mathbb{Z}_3 \times \mathbb{Z}_3}. \tag{3}$$

The Lagrangian has been obtained as a low-energy effective theory of the high-density QCD in the CFL phase [7–9,24][1]

$$\begin{aligned} \mathcal{L}_{\rm GL} &= {\rm Tr}\Big[-\epsilon_3 F_{0i}F^{0i} - \frac{1}{2\lambda_3} F_{ij}F_{ij} + K_0 \nabla_0\Phi^\dagger \nabla_0 \Phi \\ &\quad - K_3 \nabla_i \Phi^\dagger \nabla_i \Phi - V(\Phi)\Big], \\ V(\phi) &= -m^2 \Phi^\dagger\Phi + \beta[({\rm Tr}[\Phi^\dagger\Phi])^2 + {\rm Tr}\{(\Phi^\dagger\Phi)^2\}] \\ &\quad + \frac{3m^4}{16\beta}, \end{aligned} \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_s [A_\mu, A_\nu]$, $\nabla_\mu = \partial_\mu - i g_s A_\mu$, $\mu = 0, 1, 2, 3$ is the space-time index, $\{i, j\} = \{1, 2, 3\}$ are spatial indices, λ_3 is a magnetic permeability and ϵ_3 is a dielectric constant for gluons. Here we ignore the strange quark mass. The static GL free energy functional can be defined as

$$\mathcal{E}_{\rm GL} = {\rm Tr}\left[\frac{1}{2\lambda_3} F_{ij}F^{ij} + K_3 \nabla_i\Phi^\dagger \nabla_i \Phi + V(\Phi)\right]. \tag{5}$$

The coefficients in the expression above may be calculated directly from the QCD Lagrangian using perturbative techniques. We quote here the standard results obtained in the literature [7–9] through perturbative calculations in QCD as $\beta = \frac{7\zeta(3)}{8(\pi T_c)^2} N(\mu)$, $K_3 = \frac{2}{3}\beta$, $K_0 = 3K_3$, $m^2 = -4N(\mu)\log\frac{T}{T_c}$, $N(\mu) = \frac{\mu^2}{2\pi^2}$, $g_s = \sqrt{\frac{24\pi^2\lambda}{27\log\mu/\Lambda}}$, $T_c \sim \mu\exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right)$, where μ is the chemical potential, Λ the QCD scale and T_c the critical temperature.

The vacuum expectation value of Φ can be computed by minimizing the potential defined at Eq. (4) as

$$\langle\Phi\rangle = \Delta_{\rm CFL}\mathbf{1}_3, \qquad \Delta_{\rm CFL}^2 \equiv \frac{m^2}{8\beta}. \tag{6}$$

In the ground state [Eq. (6)], the full symmetry group G is spontaneously broken down to $H = \frac{SU(3)_{\rm C+F}}{\mathbb{Z}_3}$ and the order parameter space becomes

$$G/H \simeq \frac{SU(3) \times U(1)}{\mathbb{Z}_3} = U(3). \tag{7}$$

[1] In this paper, we are ignoring the first-order time derivative term for simplicity since it makes the vortex dyonic.

Masses of gauge bosons and scalars are given by the following [14], $m_g^2 = g_s^2\Delta_{\rm CFL}^2 K_3\lambda_3$, $m_\zeta^2 = \frac{2m^2}{K_3}$, $m_\chi^2 = \frac{4\lambda_2\Delta_{\rm CFL}^2}{K_3}$, $m_\varphi^2 = 0$, where φ is the massless Nambu-Goldstone boson related to the breaking of $U(1)_{\rm B}$ symmetry, and ζ and χ are, respectively, the trace and traceless part of Φ.

The static equations of motion can also be directly found from the free energy in Eq. (5), and they read as

$$\nabla_i F_{ij} = ig_s K_3\lambda_3\left[\nabla_j\Phi\Phi^\dagger - \Phi(\nabla_j\Phi)^\dagger - \frac{1}{3}{\rm Tr}(\nabla_j\Phi\Phi^\dagger - \Phi(\nabla_j\Phi)^\dagger)\right],$$
$$\nabla_j^2\Phi = \frac{1}{K_3}[-m^2 + 2\beta\{\Phi\Phi^\dagger + {\rm Tr}(\Phi^\dagger\Phi)\}]\Phi. \tag{8}$$

B. Non-Abelian vortex or color magnetic flux tube

Let us first briefly review a few primary features of the non-Abelian vortices in the CFL phase in the absence of the electromagnetic interaction. It can be easily noticed from Eq. (7) that $\pi_1(G/H) = \mathbb{Z}$. This nonzero fundamental group implies the existing vortices. Since the broken $U(1)_{\rm B}$ is a global symmetry, the vortices are global vortices or superfluid vortices [11]. The structure of these vortices can be understood by the orientation and winding of the configuration of the condensed scalar field Φ in the far away from vortex core. We place a vortex along the z direction and use the cylindrical coordinates in this paper. One can write down the ansatz as [11–14]

$$\Phi(r,\theta) = \Delta_{\rm CFL}\begin{pmatrix} e^{i\theta}f_1(r) & 0 & 0 \\ 0 & f_2(r) & 0 \\ 0 & 0 & f_2(r)\end{pmatrix},$$
$$A_i(r) = -\frac{1}{3g_s}\frac{\epsilon_{ij}x_j}{r^2}A(r)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}, \quad i = \{1,2\}, \tag{9}$$

where f_1, f_2, and $A(r)$ are the profile functions. The GL free energy can be written by inserting the ansatz into Eq. (4) as

$$\mathcal{E}_{\rm GL} = 2\pi\int rdr\left[\frac{2}{3g_s^2\lambda_3 r^2}(\partial_r A)^2 + K_3\Delta_{\rm CFL}^2\left\{(\partial_r f_1)^2 + 2(\partial_r f_2)^2 + \frac{f_1^2}{9r^2}(3-2A)^2 + 2\frac{f_2^2}{9r^2}A^2 + \Delta_{\rm CFL}^2[f_1^2 - f_2^2]^2 + 2\Delta_{\rm CFL}^2[f_1^2 + 2f_2^2 - 3]^2\right\}\right]. \tag{10}$$

The form of the profiles f_1, f_2, and $A(r)$ can be calculated numerically with the boundary condition,

$$f_1(0) = 0, \quad \partial_r f_2(r)|_0 = 0, \quad A(0) = 0,$$
$$f_1(\infty) = f_2(\infty) = 1, \quad A(\infty) = 1. \tag{11}$$

The vortex configuration in Eq. (9) breaks the unbroken color-flavor diagonal $SU(3)_{\rm C+F}$ symmetry as

$$SU(3)_{\rm C+F} \to SU(2)\times U(1), \tag{12}$$

showing the existence of degenerate solutions. This degeneracy is due to the existence of Nambu-Goldstone modes parametrizing a coset space $\frac{SU(3)}{SU(2)\times U(1)} \simeq \mathbb{C}P^2$ [12]. The low-energy excitation and interaction of these zero modes can be calculated by the effective $\mathbb{C}P^2$ sigma model action [1]. Generic solutions on the $\mathbb{C}P^2$ space can be found by just applying a global transformation by a reducing matrix [25],

$$U(\phi) = \frac{1}{\sqrt{X}}\begin{pmatrix} 1 & -\phi^\dagger \\ \phi & X^{\frac{1}{2}}Y^{-\frac{1}{2}}\end{pmatrix},$$
$$X = 1 + \phi^\dagger\phi, \quad Y = \mathbf{1}_3 + \phi\phi^\dagger, \tag{13}$$

where $\phi = \{\phi_1, \phi_2\}$ are inhomogeneous coordinates of the $\mathbb{C}P^2$. The vortex solution with a generic orientation takes the form

$$\Phi(r,\theta) = \Delta_{\rm CFL}U(\phi)\begin{pmatrix} e^{i\theta}f_1(r) & 0 & 0 \\ 0 & f_2(r) & 0 \\ 0 & 0 & f_2(r)\end{pmatrix}U^\dagger(\phi),$$
$$A_i(r) = -\frac{\epsilon_{ij}x_j}{3g_s r^2}A(r)U(\phi)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}U^\dagger(\phi). \tag{14}$$

III. THE CONSTRUCTION OF THE EFFECTIVE ACTION OF A NON-ABELIAN VORTEX

The effective action can be computed by prompting the moduli parameter $\boldsymbol{\phi}$ to fields fluctuating on the vortex worldsheet in the t-z plane, which is known as the moduli approximation [26] (see also Refs. [27,28]). So when one inserts the rotated solution in Eq. (14) into the free energy, the static energy part is separated out from the rest. The other terms which would be relevant for small fluctuations can be written as

$$\mathcal{L}_{\rm eff} = \sum_\alpha c_\alpha {\rm Tr}[F_{i\alpha}F_i^\alpha + \kappa_\alpha|\mathcal{D}^\alpha\Phi|^2], \tag{15}$$

where $\alpha = \{0,3\}$ is the worldsheet index, $i = \{1,2\}$, $c_0 = \epsilon_3$, $c_3 = \frac{1}{\lambda_3}$ and $\kappa_\alpha = \frac{K_\alpha}{c_\alpha}$. The raising and lowering of the index α are done by the Minkowski signature $(+,-)$ for $\{0,3\}$. So the equations of motion for zero (the Gauss's law) and the third component can be expressed as

$$\mathcal{D}_i F^{i\alpha} = -ig_s\kappa_\alpha T^a \mathrm{Tr}[\Phi^\dagger T^a \mathcal{D}^\alpha\Phi - (\mathcal{D}^\alpha\Phi)^\dagger T^a\Phi]. \quad (16)$$

These equations are generated due to the fluctuation of the zero mode along the vortex. The ansatz for the generated gauge fields which solve the above Eq. (16) can be expressed as [29]

$$A_\alpha = \rho_\alpha(r,\theta)W_\alpha + \eta_\alpha(r,\theta)V_\alpha, \qquad \alpha = 0,3, \quad (17)$$

where ρ_α and η_α are profile functions which are to be determined by minimizing the action or by solving Eq. (16) and

$$W_\alpha = i\partial_\alpha\tilde{T}\,\tilde{T}, \qquad V_\alpha = \partial_\alpha\tilde{T},$$
$$\tilde{T} = UTU^\dagger, \qquad T = \mathrm{diag}(1,-1,-1), \quad (18)$$

where $U(\phi)$ is defined in the last section Eq. (13). These satisfy the commutation relations: $[W_\alpha,\tilde{T}]=2iV_\alpha$, $[V_\alpha,\tilde{T}]=-2iW_\alpha$, $\mathrm{Tr}W_\alpha W^\alpha = \mathrm{Tr}V_\alpha V^\alpha$, $\mathrm{Tr}W_\alpha V^\alpha = 0$. Here, we can see that W_α and V_α are orthogonal to each other. W_α and V_α are also orthogonal to the direction of A_i defined in Eq. (14). We can see this if we expand all three matrices as

$$U(\phi)\frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} U^\dagger(\phi) = \frac{1}{6}\mathbf{1} + \frac{1}{2}\tilde{T},$$
$$W_\alpha = i[\partial_\alpha UU^\dagger - \tilde{T}\partial_\alpha UU^\dagger\tilde{T}],$$
$$V_\alpha = -i[i\partial_\alpha UU^\dagger, \tilde{T}]. \quad (19)$$

W_α is the Delduc-Valent projection on $\mathbb{C}P^2$ [25] and was used in singular gauge computation. Here we introduce V_α as a new component in the ansatz (17) to solve Gauss's law, getting A_α orthogonal to the A_i direction indicating the fluctuation of the Nambu-Goldstone mode in the orthogonal direction of the background field, which is true because the Nambu-Goldstone bosons are generated by broken generators.

To compute the effective action, we have to insert the ansatz in Eq. (17) into the action in Eq. (15). Before we do so, let us compute the field strength of the gauge field and matter coupling separately. The first term of the field strength($F_{i\alpha} = \partial_i A_\alpha - \mathcal{D}_\alpha A_i$) can be written as

$$\partial_i A_\alpha = \left(\frac{x_i}{r}\partial_r\rho_\alpha(r,\theta) - \frac{\epsilon_{ij}x^j}{r^2}\partial_\theta\rho_\alpha(r,\theta)\right)W_\alpha + \left(\frac{x_i}{r}\partial_r\eta_\alpha(r,\theta) - \frac{\epsilon_{ij}x^j}{r^2}\partial_\theta\eta_\alpha(r,\theta)\right)V_\alpha. \quad (20)$$

The second term becomes

$$\mathcal{D}_\alpha A_i = \partial_\alpha A_i - ig_s[A_\alpha, A_i] = \frac{\epsilon_{ijx^j}}{r^2}A(r)\{\eta_\alpha(r,\theta)W_\alpha - \sigma_\alpha(r,\theta)V_\alpha\}, \quad (21)$$

where we have defined $2g_s\sigma_\alpha = 1 + 2g_s\rho_\alpha$. So, we can insert the field strength,

$$F_{i\alpha} = \partial_i A_\alpha - \mathcal{D}_\alpha A_i = \left(\frac{x_i}{r}\partial_r\rho_\alpha - \frac{\epsilon_{ij}x^j}{r^2}(\partial_\theta\rho_\alpha + A\eta_\alpha)\right)W_\alpha + \left(\frac{x_i}{r}\partial_r\eta_\alpha - \frac{\epsilon_{ij}x^j}{r^2}(\partial_\theta\eta_\alpha - A\sigma)\right)V_\alpha, \quad (22)$$

into the kinetic term of the gauge field in Eq. (15) to yield

$$\mathrm{Tr}F_{i\alpha}F_i^\alpha = \left[(\partial_r\rho_\alpha)^2 + (\partial_r\eta_\alpha)^2 + \frac{1}{r^2}(\partial_\theta\rho_\alpha + A\eta_\alpha)^2 + \frac{1}{r^2}(\partial_\theta\eta_\alpha - A\sigma_\alpha)^2\right]\mathrm{Tr}V_\alpha V^\alpha. \quad (23)$$

From the covariant derivative of the matter field Φ,

$$\mathcal{D}_\alpha\Phi = \partial_\alpha\Phi - ig_sA_\alpha\Phi = \Delta_{\mathrm{CFL}}\left[\frac{f_1e^{i\theta} - f_2}{2}(1 + g_s\rho_\alpha) - ig_s\eta_\alpha\frac{f_1e^{i\theta} + f_2}{2}\right]V_\alpha - g_s\Delta_{\mathrm{CFL}}\left(\eta_\alpha\frac{f_1e^{i\theta} - f_2}{2} + i\rho_\alpha\frac{f_1e^{i\theta} + f_2}{2}\right)W_\alpha, \quad (24)$$

we compute the $|\mathcal{D}_\alpha\Phi|^2$ as

$$\frac{|\mathcal{D}_\alpha\Phi|^2}{\Delta_{\mathrm{CFL}}^2} = \frac{1}{4}[(1 + 2g_s\rho_\alpha + 2g_s^2\rho_\alpha^2)(f_1^2 + f_2^2) + 2g_s^2\eta_\alpha^2(f_1^2 + f_2^2) - (1 + 2g_s\rho_\alpha)2f_1f_2\cos\theta - 4g_sf_1f_2\eta_\alpha\sin\theta]\mathrm{Tr}V_\alpha V_\alpha. \quad (25)$$

By changing the variables from ρ_α to $\sigma_\alpha = \frac{1+2g_s\rho_\alpha}{2g_s}$, Eq. (25) can be simplified as

$$\frac{4|\mathcal{D}_\alpha\Phi|^2}{\Delta_{\mathrm{CFL}}^2} = \left[\frac{1}{2}(f_1^2 + f_2^2) + 2g_s^2(\sigma_\alpha^2 + \eta_\alpha^2)(f_1^2 + f_2^2) - 4g_sf_1f_2(\sigma_\alpha\cos\theta + \eta_\alpha\sin\theta)\right]\mathrm{Tr}V_\alpha V_\alpha. \quad (26)$$

Let us define here a complex scalar field as

$$\Psi_\alpha(r,\theta) = \sigma_\alpha(r,\theta) + i\eta_\alpha(r,\theta), \quad (27)$$

and we can rewrite the action in terms of these fields. Equation (23) becomes

$$\mathrm{Tr}F_{i\alpha}F_i^\alpha = \sum_\alpha\left[|\partial_r\Psi_\alpha|^2 + \frac{1}{r^2}|\mathcal{D}_\theta\Psi_\alpha|^2\right]\mathrm{Tr}W_\alpha W^\alpha, \quad (28)$$

where $\mathcal{D}_\theta = \partial_\theta - iA(r)$. Equation (26) can also be rewritten as

$$\frac{|\mathcal{D}_\alpha\Phi|^2}{\Delta_{\mathrm{CFL}}^2} = \frac{1}{4}\left[\frac{1}{2}(f_1^2+f_2^2) + 2g_s^2|\Psi_\alpha|^2(f_1^2+f_2^2) - 2g_s f_1 f_2(\Psi_\alpha e^{-i\theta} + \Psi_\alpha^* e^{i\theta})\right]\mathrm{Tr}W_\alpha W^\alpha. \quad (29)$$

The effective Lagrangian in Eq. (15) can be written as

$$\mathcal{L}_{\mathrm{eff}} = \sum_\alpha c_\alpha\int d^2x\mathrm{Tr}[F_{i\alpha}F_i^\alpha + \kappa_\alpha|\mathcal{D}^\alpha\Phi|^2] = \sum_\alpha \mathcal{I}_\alpha \mathrm{Tr}W_\alpha W^\alpha, \quad (30)$$

where $\mathcal{I}_\alpha$ are the coefficients of the $\mathbb{C}P^2$ action, defined by

$$\mathcal{I}_\alpha = c_\alpha\int rdrd\theta\left[|\partial_r\Psi_\alpha|^2 + \frac{1}{r^2}|\mathcal{D}_\theta\Psi_\alpha|^2 + \frac{\Delta_{\mathrm{CFL}}^2\kappa_\alpha}{4}\left\{\frac{1}{2}(f_1^2+f_2^2) + 2g_s^2|\Psi_\alpha|^2(f_1^2+f_2^2) - 2g_s f_1 f_2(\Psi_\alpha e^{-i\theta} + \Psi_\alpha^* e^{i\theta})\right\}\right]. \quad (31)$$

The effective Lagrangian can be written explicitly as the form of the $\mathbb{C}P^2$ model,

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{I}_0\{\dot{\boldsymbol{n}}^\dagger\dot{\boldsymbol{n}} + (\boldsymbol{n}^\dagger\dot{\boldsymbol{n}})(\boldsymbol{n}^\dagger\dot{\boldsymbol{n}})\} - \mathcal{I}_3\{\partial_z\boldsymbol{n}^\dagger\partial_z\boldsymbol{n} + (\boldsymbol{n}^\dagger\partial_z\boldsymbol{n})(\boldsymbol{n}^\dagger\partial_z\boldsymbol{n})\}, \quad (32)$$

where $\boldsymbol{n}$ are the homogeneous coordinates of the $\mathbb{C}P^2$ space, which can be written in terms of the inhomogeneous coordinates as

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\phi^\dagger\phi}}\begin{pmatrix}1\\ \phi\end{pmatrix}. \quad (33)$$

If we rescale the z coordinate as

$$z' = \sqrt{\frac{\mathcal{I}_0}{\mathcal{I}_3}}z, \quad (34)$$

then the effective Lagrangian becomes

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{I}_0[\partial_\alpha\boldsymbol{n}^\dagger\partial^\alpha\boldsymbol{n} + (\boldsymbol{n}^\dagger\partial_\alpha\boldsymbol{n})(\boldsymbol{n}^\dagger\partial^\alpha\boldsymbol{n})]. \quad (35)$$

By expanding the field Ψ_0 in terms of partial waves as

$$\Psi_0 = \sum_m \Psi_m(r)e^{im\theta}, \quad (36)$$

$\mathcal{I}_0$ can be written in terms of the partial wave modes as

$$\mathcal{I}_0 = c_0\sum_m\int rdrd\theta\left[(\partial_r\Psi_m)^2 + \frac{(m-A(r))^2}{r^2}\Psi_m^2 + \frac{\Delta_{\mathrm{CFL}}^2\kappa_0}{4}\left\{\frac{1}{2}(f_1^2+f_2^2) + 2g_s^2\Psi_m^2(f_1^2+f_2^2) - 4g_s f_1 f_2\Psi_m\cos(m-1)\theta\right\}\right]. \quad (37)$$

It is easy to check that the last term of Eq. (37) vanishes after the theta integration unless $m = 1$. So we write $\mathcal{I}_0$ as

$$\mathcal{I}_0 = \sum_m 2\pi c_0\int rdr\left[(\partial_r\Psi_m)^2 + \frac{(m-A(r))^2}{r^2}\Psi_m^2 + \frac{\Delta_{\mathrm{CFL}}^2\kappa_0}{4}\left\{\frac{1}{2}(f_1^2+f_2^2) + 2g_s^2\Psi_m^2(f_1^2+f_2^2) - 4g_s f_1 f_2\Psi_m\delta_{m1}\right\}\right]. \quad (38)$$

We minimize the above integral by solving the equations for the modes,

$$\frac{1}{r}\partial_r r\partial_r\Psi_1 - \frac{(1-A(r))^2}{r^2}\Psi_1 = \frac{\Delta_{\mathrm{CFL}}^2\kappa_0 g_s}{2}[(f_1^2+f_2^2)g_s\Psi_1 - f_1f_2], \quad (39)$$

$$\frac{1}{r}\partial_r r\partial_r\Psi_m - \frac{(m-A(r))^2}{r^2}\Psi_m = \frac{\Delta_{\mathrm{CFL}}^2\kappa_0 g_s^2}{2}(f_1^2+f_2^2)\Psi_m, \qquad m \neq 1. \quad (40)$$

These equations can also be derived directly from the equations of motion in Eq. (16).

One should notice that Eq. (39) for $m = 1$ was derived in the singular gauge in Ref. [1] (for $K_0 = K_3$) but the rest, where $m \neq 1$, were absent in the singular gauge. The solution of Eq. (39) is normalizable ($m = 1$) and can be solved [1] numerically with the boundary condition $\Psi_1(0) = 0$ and $\Psi_1(\infty) = \frac{1}{2g_s}$. Large distance behavior of the solution can be expressed as

$$\Psi_1 \to_{r\to\infty} \frac{1}{2g_s} - \frac{1}{\sqrt{\xi}}e^{-\xi}, \quad (41)$$

and it is easy to show that the large distance value of $\Psi_1 = \frac{1}{2g_s}$ transforms the ansatz A_α in Eq. (17) into a pure gauge form as

$$A_\alpha = \frac{i}{g_s} \boldsymbol{g}^\dagger \partial_\alpha \boldsymbol{g}, \quad \text{where } \boldsymbol{g} = e^{-i\theta T_8^*},$$
$$T_8^* = \frac{1}{3} U(\phi) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} U^\dagger(\phi). \tag{42}$$

The coefficient $\mathcal{I}_0(m=1)$ can be written in terms of the solution of the equations of motion derived in the above,

$$\mathcal{I}(m=1) = \frac{2\pi m_0^2 c_0}{8 g_s^2} \int r dr [f_1^2 + f_2^2 - 4 f_1 f_2 g_s \Psi_1], \tag{43}$$

where $m_0^2 = g_s^2 \Delta_{\rm CFL}^2 \kappa_0$. The integrand vanishes at large distances, so the integral is finite.

All other modes ($m \neq 1$) are non-normalizable divergent modes. In this work, we do not solve these equations numerically. However, one can understand the behavior of the solutions once we analyze the asymptotic forms of the solutions. For the case $m \neq 1$, Eq. (40) can be expressed as

$$\frac{1}{r} \partial_r r \partial_r \Psi_m = \left[\frac{(m - A(r))^2}{r^2} + \frac{\Delta_{\rm CFL}^2 \kappa_0 g_s^2}{2} (f_1^2 + f_2^2) \right] \Psi_m. \tag{44}$$

The right-hand side of the above equation is positive definite (by assuming Ψ_m is positive). So the solution cannot have a local maximum. At large distances, the equation becomes the modified Bessel equation,

$$\xi^2 \Psi_m'' + \xi \Psi_m' - [(m-1)^2 + \xi^2] \Psi_m = 0 \tag{45}$$

where $\xi = m_0 r$. The solution is $\Psi_m \sim \frac{1}{\sqrt{\xi}} e^{\pm \xi}$.

At small distances near the origin ($r = 0$), where $f_1(r) = 0$, Eq. (44) reduces to

$$\left(\partial_\xi^2 + \frac{1}{\xi} \partial_\xi \right) \Psi_m = \frac{m^2}{\xi^2} \Psi_m. \tag{46}$$

The solution is $\Psi_m \sim \xi^{\pm m}$ for $m \neq 0, 1$. So $\Psi_m(0) = 0$ for $m \neq 0, 1$ and we may conclude that the solution diverges as $\frac{1}{\sqrt{\xi}} e^{\xi}$ at large distances since it does not have any local maximum. For $m = 0$, near the origin the solution could be written as either $\log \xi$ or $C_0 + \xi^2 C_2$, where C_0 and C_2 are constants and C_2 is found to be positive. So the solution diverges at large distances even if we set the value as constant at the origin.

IV. SUMMARY AND DISCUSSION

In this paper, we have analyzed the orientational $\mathbb{C}P^2$ zero modes of a single non-Abelian vortex in high-density QCD. To do so, we have followed the standard procedure of zero-mode analysis and have written the effective action, as was done in the singular gauge case. In order to solve the Gauss's law constraint in the regular gauge, we have generalized the ansatz of the gauge field (used in the singular gauge) to $A_\alpha = \rho(r,\theta) W_\alpha + \eta(r,\theta) V_\alpha$ by introducing a profile function η together with a matrix V_α orthogonal to W_α, neither of which exists in the singular gauge. In the regular gauge, the two profile functions (ρ, η) do not only depend on the radial coordinate r but also on the azimuthal angle θ. These two profile functions can be combined to the real and imaginary components of a single complex profile function $\Psi(r,\theta)$. The insertion of A_α together with vortex profile functions into the action gives the $\mathbb{C}P^2$ effective action on the t-z plane with a front coefficient depending on the complex profile function $\Psi(r,\theta)$. The front coefficient has been determined by inserting $\Psi(r,\theta)$ after solving the equations of motion. We have expanded the complex profile function Ψ in the partial wave basis as $\Psi = \sum_m \Psi_m(r) e^{im\theta}$ and have analyzed the asymptotic behaviors of all the modes. We have found that only one mode Ψ_1 is normalizable, which is identical to what was found in the singular gauge analysis. We have shown that all the other modes diverge exponentially at large distances. Finally, we have concluded that our regular gauge analysis established the previously known result of the existence of normalizable zero mode derived in singular gauge and that the previously constructed effective $\mathbb{C}P^2$ Lagrangian of the single vortex is correct.

Here let us discuss some points which may shed light on the interesting features of the regular gauge. In this analysis, we have introduced a complex profile function which does not depend only on r but also on the azimuthal angle θ. This azimuthal angle dependence of a zero mode makes the system complicated. The $\mathbb{C}P^2$ Nambu-Goldstone zero modes arise as a consequence of the unbroken color-flavor group $SU(3)_{\rm C+F}$ in the bulk, further broken as $SU(3)_{\rm C+F} \to U(1) \times SU(2)$ inside the vortex core. The system restores $SU(3)_{\rm C+F}$ symmetry at large distances from the vortex core, where the unbroken $SU(3)_{\rm C+F}$ group elements commute with the order parameter. However, the presence of vortex makes the order-parameter position dependent at large distances. So the embedding of color-flavor diagonal group $SU(3)_{\rm C+F}$ inside the original symmetry group becomes space dependent. The generators of the $SU(3)_{\rm C+F}$ changes along a path around the vortex by the action of holonomy. The space dependence is true only for the few generators which belong to the $\mathbb{C}P^2$ subspace of $SU(3)$ and others remain unaffected. The azimuthal angle dependence of $\mathbb{C}P^2$ generators actually makes the zero-mode analysis complicated. So when we fluctuate the zero modes, the generated gauge field A_α depends on the angle in a complicated way. In general, the generators may not return back to their own after a complete rotation along an encircled path around the vortex. In this case, it is said that there is the so-called obstruction, for which the profile functions in general diverges as r^c at large distances with a constant c [29–32]. In our case, after expanding our complex profile function in the partial wave basis, we have found that the profile functions corresponding to different partial wave modes diverge exponentially except for one normalizable mode ($m = 1$). Therefore, as a byproduct of our analysis, we have shown that non-Abelian vortices in high-density QCD do not suffer from any obstruction.

There is an alternative way to show the absence of normalizable modes other than the $\mathbb{C}P^2$ modes and translational modes, that is, the index theorem. Fermionic zero modes around a single non-Abelian vortex [33,34] were studied by the index theorem applied to the Bogoliubov–de Gennes equation [35]. The index theorem applied to bosonic modes should be studied in the framework of the GL theory.

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