

# Higher order scheme-independent calculations of physical quantities in the conformal phase of a gauge theory

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We consider an asymptotically free vectorial  $SU(N_c)$  gauge theory with  $N_f$  massless fermions in a representation  $R$ , having an infrared fixed point (IRFP) of the renormalization group at  $\alpha_{\text{IR}}$  in the conformal non-Abelian Coulomb phase. The cases with  $R$  equal to the fundamental, adjoint, and symmetric rank-2 tensor representation are considered. We present scheme-independent calculations of the anomalous dimension  $\gamma_{\bar{\psi}\psi, \text{IR}}$  to  $O(\Delta_f^4)$  and  $\beta'_{\text{IR}}$  to  $O(\Delta_f^5)$  at this IRFP, where  $\Delta_f$  is an  $N_f$ -dependent expansion parameter. Comparisons are made with conventional  $n$ -loop calculations and lattice measurements. As a test of the accuracy of the  $\Delta_f$  expansion, we calculate  $\gamma_{\bar{\psi}\psi, \text{IR}}$  to  $O(\Delta_f^3)$  in  $\mathcal{N} = 1$   $SU(N_c)$  supersymmetric quantum chromodynamics and find complete agreement, to this order, with the exactly known expression. The  $\Delta_f$  expansion also avoids a problem in which an IRFP may not be manifest as an IR zero of a higher  $n$ -loop beta function.

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A fundamental problem in quantum field theory concerns the properties at a conformal fixed point of the renormalization group. A specific question under intensive current investigation concerns the properties of an asymptotically free (AF) non-Abelian Yang-Mills vectorial gauge theory (in  $d = 4$  spacetime dimensions) with a set of massless fermions at an IRFP of the renormalization group in the Coulomb phase, where it exhibits scale and conformal invariance [1,2]. Here we consider a theory of this type, with gauge group  $G = SU(N_c)$  and  $N_f$  massless fermions  $\psi_j$ ,  $1 \leq j \leq N_f$ , in a representation  $R$ , where  $R$  is the fundamental ( $F$ ), adjoint (adj), or symmetric rank-2 tensor ( $S$ ). The dependence of the gauge coupling  $g = g(\mu)$  on the Euclidean momentum scale  $\mu$  is described by the beta function,  $\beta = d\alpha/dt$ , where  $\alpha(\mu) = g(\mu)^2/(4\pi)$  and  $dt = d \ln \mu$ . The IRFP occurs at an IR zero of  $\beta$  at  $\alpha_{\text{IR}}$ . At this fixed point, an operator  $\mathcal{O}$  for a physical quantity exhibits scaling behavior with a dimension  $D_{\mathcal{O}} = D_{\mathcal{O}, \text{free}} - \gamma_{\mathcal{O}}$ , where  $D_{\mathcal{O}, \text{free}}$  is the free-field dimension and  $\gamma_{\mathcal{O}}$  is the anomalous dimension.

Two important quantities that characterize the properties at the IRFP  $\alpha_{\text{IR}}$  are  $\gamma_{\bar{\psi}\psi}$  [3] and  $\beta' \equiv d\beta/d\alpha$ , denoted  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$ . Here,  $\beta'_{\text{IR}}$  is equivalent to the anomalous dimension of  $F_{a, \mu\nu} F_a^{\mu\nu}$ , where  $F_a^{\mu\nu}$  is the (rescaled) field-strength tensor [4]. As physical quantities,  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$  are scheme-independent (SI) [5]. However, conventional series expansions of these quantities in powers of  $\alpha$ , calculated to a finite order, do not maintain this scheme independence beyond the lowest orders. Clearly, it is very valuable to calculate and analyze series expansions for  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$  that are scheme-independent at each order. Some early work was in [6,7]. A natural expansion variable is

$$\Delta_f = N_u - N_f, \quad (1)$$

where, for a given  $N_c$  and  $R$ ,  $N_u$  is the upper ( $u$ ) limit to  $N_f$  allowed by asymptotic freedom. Scheme-independent series expansions of  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$  are [8]

$$\gamma_{\bar{\psi}\psi, \text{IR}} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j \quad (2)$$

and [9]

$$\beta'_{\text{IR}} = \sum_{j=1}^{\infty} d_j \Delta_f^j, \quad (3)$$

where  $d_1 = 0$  for all  $G$  and  $R$ . For general  $G$  and  $R$ , the  $\kappa_j$  were calculated to order  $j = 3$  in [8] and the  $d_j$  to order  $j = 4$  in [9], and for  $G = SU(3)$  and  $R = F$ ,  $\kappa_4$  was computed in [10] and  $d_5$  in [9].

Here we report our calculations of these scheme-independent expansions of  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$  to the highest orders yet achieved, presenting  $\kappa_4$  and  $d_5$  for an asymptotically free  $SU(N_c)$  gauge theory with a conformal IR fixed point, for  $R = F, \text{adj}, S$ . We also report our calculation of  $\kappa_3$  for supersymmetric quantum chromodynamics (SQCD). We believe that our new results are a substantial advance in the knowledge of conformal field theory. Our results have the advantage of scheme independence at each order in  $\Delta_f$ , in contrast to scheme-dependent (SD) series expansions of  $\gamma_{\bar{\psi}\psi, \text{IR}}$  and  $\beta'_{\text{IR}}$  in powers of  $\alpha$  [11–16], and they complement other approaches to understanding conformal and

superconformal field theory, such as the bootstrap [17] and lattice simulations [18].

The conventional power-series expansions of  $\beta$  and  $\gamma_{\bar{\psi}\psi}$  are

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_{\ell} \left( \frac{\alpha}{4\pi} \right)^{\ell} \quad (4)$$

and

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_{\ell} \left( \frac{\alpha}{4\pi} \right)^{\ell}, \quad (5)$$

where  $b_{\ell}$  and  $c_{\ell}$  are the  $\ell$ -loop coefficients;  $b_1$  [19],  $b_2$  [20], and  $c_1 = 6C_f$  are scheme-independent, while the  $b_{\ell}$  with  $\ell \geq 3$  and the  $c_{\ell}$  with  $\ell \geq 2$  are scheme-dependent, i.e. they depend on the scheme used for regularization and renormalization [5]. We denote the  $n$ -loop ( $n\ell$ )  $\beta$  and  $\gamma_{\bar{\psi}\psi}$  as  $\beta_{n\ell}$  and  $\gamma_{\bar{\psi}\psi, n\ell}$  and the IR zero of  $\beta_{n\ell}$  as  $\alpha_{\text{IR}, n\ell}$ .

The calculation of  $\kappa_j$  requires, as inputs, the values of the  $b_{\ell}$  for  $1 \leq \ell \leq j+1$  and the  $c_{\ell}$  for  $1 \leq \ell \leq j$ . The calculation of  $d_j$  requires, as inputs, the values of the  $b_{\ell}$  for  $1 \leq \ell \leq j$ . Thus, importantly,  $\kappa_j$  does not receive any corrections from  $b_{\ell}$  with  $\ell > j+1$  or  $c_{\ell}$  with  $\ell > j$ , and similarly,  $d_j$  does not receive any corrections from any  $b_{\ell}$  with  $\ell > j$ .

The coefficients  $\kappa_j$  were calculated in [8] for an (AF vectorial) supersymmetric gauge theory (SGT) with gauge group  $G$  and  $N_f$  pairs of chiral superfields in the  $R$  and  $\bar{R}$  representation, for  $j = 1, 2$ . Complete agreement was found, to the order calculated, with the exactly known result in the conformal non-Abelian Coulomb phase (NACP) [21–23]

$$\gamma_{\text{IR}, \text{SGT}} = \frac{\frac{2T_f}{3C_A} \Delta_f}{1 - \frac{2T_f}{3C_A} \Delta_f}. \quad (6)$$

In this theory,  $N_u = 3C_A/(2T_f)$ , and the conformal NACP is the interval  $N_{\ell} < N_f < N_u$ , where  $N_{\ell} = N_u/2$ , so that  $\Delta_f$  varies from 0 to a maximum of  $(\Delta_f)_{\text{max}} = 3C_A/(4T_f)$  in the NACP [24]. Hence,  $\gamma_{\text{IR}, \text{SGT}}$  increases monotonically from 0 to 1 as  $N_f$  decreases from  $N_u$  to  $N_{\ell}$ , saturating the upper bound  $\gamma_{\bar{\psi}\psi, \text{IR}, \text{SGT}} < 1$  from conformal invariance in this SGT [25].

As a test of the accuracy of the  $\Delta_f$  expansion, we have now calculated  $\kappa_3$  for SQCD with  $R = F$ , using inputs from [26]. We find  $\kappa_3 = 1/(3N_c)^3$ , in perfect agreement, to this order, with the exact result, Eq. (6). This agreement explicitly illustrates the scheme independence of the  $\kappa_j$ , since our calculations in [8] and here used inputs computed in the  $\overline{\text{DR}}$  scheme, while (6) was derived in the NSVZ scheme [21]. Our new result has a far-reaching implication:

it strongly suggests that  $\kappa_j = [2T_f/(3C_A)]^j$  for all  $j$ , so that the expansion (2) for this supersymmetric gauge theory, calculated to order  $O(\Delta_f^p)$ , agrees with the exact result to the given order for all  $p$ .

Because of electric-magnetic duality [22], as  $N_f \rightarrow N_{\ell}$  in the NACP, the physics is described by a magnetic theory with coupling strength going to zero, or equivalently, by an electric theory with divergent  $\alpha_{\text{IR}}$ . Hence, another important finding here is that the complete agreement that we obtain in SQCD to  $O(\Delta_f^3)$  between Eq. (2) and the exact Eq. (6) holds for arbitrarily strong  $\alpha_{\text{IR}}$ . Even apart from the issue of scheme dependence in Eq. (5), this agreement could not be achieved with the conventional expansion (5) of  $\gamma_{\bar{\psi}\psi, \text{IR}}$  in powers of  $\alpha$ .

The  $\Delta_f$  expansion also avoids a problem in which an IRFP may not be manifest as a physical IR zero of the  $n$ -loop beta function for some  $n$ . Indeed, although  $\beta_{n\ell}$  has a physical  $\alpha_{\text{IR}, n\ell}$  in SQCD for  $n = 2, 3$  loops [27], we have analyzed  $\beta_{4\ell}$  (in the  $\overline{\text{DR}}$  scheme), and we find that for a range of  $N_f$  in the NACP, it does not exhibit a physical  $\alpha_{\text{IR}, 4\ell}$ . This is analogous to the situation that we found for  $\alpha_{\text{IR}, 5\ell}$  in the nonsupersymmetric gauge theory [16]. In both cases, the  $\Delta_f$  expansions (2) and (3) circumvent this problem of a possible unphysical  $\alpha_{\text{IR}, n\ell}$  that one may encounter in using the convention expansions (4) and (5).

We next present our results for  $\kappa_4$  and  $d_5$  for a (non-supersymmetric)  $SU(N_c)$  gauge theory, making use of the impressive recent computation of  $b_5$  in [28]. (We have actually calculated  $\kappa_4$  and  $d_5$  for general  $G$  and  $R$  [29], but only present results here for  $R = F, \text{adj}, S$ .) The two-loop beta function has an IR zero (IRZ) in the interval  $I_{\text{IRZ}}: N_{\ell} < N_f < N_u$ , with upper and lower ( $\ell$ ) ends at  $N_u = 11N_c/(4T_f)$  and  $N_{\ell} = 17C_A^2/[2T_f(5C_A + 3C_f)]$  [24]. The non-Abelian Coulomb phase extends downward in  $I_{\text{IRZ}}$  from  $N_u$  to a lower value denoted  $N_{f, \text{cr}}$  [30]. Since chiral symmetry is exact in the NACP, one can classify the bilinear fermion operators according to their flavor transformation properties. These operators include the flavor-singlet  $\bar{\psi}\psi$  and the flavor-adjoint  $\bar{\psi}T_a\psi$ , where  $T_a$  is a generator of  $SU(N_f)$ . These have the same anomalous dimension [31], which we write simply as  $\gamma_{\bar{\psi}\psi}$ . For general  $G$  and  $R$ , the coefficients  $b_{\ell}$  were computed up to loop order  $\ell = 4$  [32] (checked in [33]) and the  $c_{\ell}$  also up to loop order  $\ell = 4$  [34], in the widely used  $\overline{\text{MS}}$  scheme [35]. These results were used in [8] to calculate the  $\kappa_j$  to order  $j = 3$  and in [9] to calculate  $d_j$  to order  $j = 4$ . For  $N_c = 3$  and  $R = F$ ,  $b_5$  was computed in [36], and this was used to calculate  $\kappa_4$  in [10] and  $d_5$  in [9] for this case (see also [16]).

We first report our results for  $\kappa_4$  and  $d_5$  for  $R = F$ , using  $b_5$  from [28]. We denote the Riemann zeta function as  $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ . We obtain

$$\begin{aligned}
 \kappa_{4,F} = & \frac{4(N_c^2 - 1)}{3^4 N_c^4 (25N_c^2 - 11)^7} [(263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 \\
 & - 290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352) \\
 & + 384(25N_c^2 - 11)(4400N_c^{10} - 123201N_c^8 + 480349N_c^6 - 486126N_c^4 + 84051N_c^2 + 1089)\zeta_3 \\
 & + 211200N_c^2(25N_c^2 - 11)^2(N_c^6 + 3N_c^4 - 16N_c^2 + 22)\zeta_5] \quad (7)
 \end{aligned}$$

and

$$\begin{aligned}
 d_{5,F} = & \frac{2^5}{3^6 N_c^3 (25N_c^2 - 11)^7} [N_c^{12}(-298194551 - 423300000\zeta_3 + 528000000\zeta_5) \\
 & + N_c^{10}(414681770 + 1541114400\zeta_3 - 821040000\zeta_5) + N_c^8(80227411 - 4170620256\zeta_3 + 2052652800\zeta_5) \\
 & + N_c^6(210598856 + 5101712352\zeta_3 - 4268183040\zeta_5) + N_c^4(-442678324 - 2250221952\zeta_3 + 2744628480\zeta_5) \\
 & + N_c^2(129261880 + 304571520\zeta_3 - 534103680\zeta_5) + 3716152 + 1022208\zeta_3], \quad (8)
 \end{aligned}$$

where the simple factorizations of the denominators have been indicated. For this  $R = F$  case, we find that  $\kappa_4 > 0$ , as was also true of  $\kappa_j$  with  $1 \leq j \leq 3$  (indeed,  $\kappa_1$  and  $\kappa_2$  are manifestly positive for any  $G$  and  $R$ ). We also find the same positivity results for  $R = \text{adj}$  and  $R = S$ . The property that for all of these representations  $R$ ,  $\kappa_j > 0$  for  $1 \leq j \leq 4$  and for all  $N_c$  implies two important monotonicity results. First, for these  $R$ , and with a fixed  $p$  in the interval  $1 \leq p \leq 4$ ,  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^p}$  is a monotonically increasing function of  $\Delta_f$  for  $N_f \in I_{\text{IRZ}}$ . Second, for these  $R$ , and with a fixed  $N_f \in I_{\text{IRZ}}$ ,  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^p}$  is a monotonically increasing function of  $p$  in the range  $1 \leq p \leq 4$ . In addition to the manifestly positive  $\kappa_1$  and  $\kappa_2$ , a plausible conjecture is that, for these  $R$ ,  $\kappa_j > 0$  for all  $j \geq 3$ . Note that the exact result (6) for the supersymmetric gauge theory shows that in that theory,  $\kappa_j > 0$  for all  $j$  and for any  $G$  and  $R$ .

In Figs. 1 and 2 we plot  $\gamma_{\text{IR}, \Delta_f^p}$  for  $R = F$ ,  $N_c = 2, 3$  and  $1 \leq p \leq 4$ . In Table I we list values of these  $\gamma_{\text{IR}, \Delta_f^p}$  [37]. These all satisfy the upper bound  $\gamma_{\text{IR}} < 2$  from conformal invariance [25]. Below, we will often omit the  $\bar{\psi}\psi$  subscript, writing  $\gamma_{\bar{\psi}\psi, \text{IR}} \equiv \gamma_{\text{IR}}$  and  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^p} \equiv \gamma_{\text{IR}, \Delta_f^p}$ .

For this  $R = F$  case we first remark on the comparison of  $\gamma_{\text{IR}, \Delta_f^4}$  with calculations of  $\gamma_{\text{IR}, n\ell}$  from analyses of power series in  $\alpha$ , which were performed to  $n = 4$  loop level in [11–14] using  $b_\ell$  and  $c_\ell$  in the  $\overline{\text{MS}}$  scheme (with studies of scheme dependence in [15]) and extended to  $n = 5$  loop level for  $N_c = 3$  in [16]. We have noted that  $\beta_{5\ell}$  does not have a physical  $\alpha_{\text{IR}, 5\ell}$  for  $N_f$  in the lower part of the interval  $I_{\text{IRZ}}$  [16]. Although we were able to surmount this problem via Padé approximants in [16], these are still scheme-dependent, while the  $\Delta_f$  expansion has the advantage of being scheme-independent. In general, we find that for a given  $N_c$  and  $N_f$ , the value of  $\gamma_{\text{IR}, \Delta_f^p}$  that we calculate to highest order, namely  $p = 4$ , is somewhat larger than  $\gamma_{\text{IR}, n\ell}$

calculated to its highest order [10,13]. For example, for  $N_c = 3$ ,  $N_f = 12$ ,  $\gamma_{\text{IR}, 4\ell} = 0.253$ ,  $\gamma_{\text{IR}, 5\ell} \approx 0.255$  (using a value of  $\alpha_{\text{IR}, 5\ell}$  from a Padé approximant [10,16]), while  $\gamma_{\text{IR}, \Delta_f^4} = 0.338$  and an extrapolation yields the estimate 0.400(5) for  $\gamma_{\text{IR}} = \lim_{p \rightarrow \infty} \gamma_{\text{IR}, \Delta_f^p}$  [10]. Similarly, for  $N_c = 2$  and  $N_f = 8$ ,  $\gamma_{\text{IR}, 4\ell} = 0.204$ , while  $\gamma_{\text{IR}, \Delta_f^4} = 0.298$ ; and for  $N_c = 4$ ,  $N_f = 16$ ,  $\gamma_{\text{IR}, 4\ell} = 0.269$ , while  $\gamma_{\text{IR}, \Delta_f^4} = 0.352$ .

We next compare our new results with lattice measurements, restricting to cases where the lattice studies are consistent with the theories being IR-conformal [18,30]. For  $N_c = 3$ , we compared our calculations of  $\gamma_{\text{IR}, \Delta_f^4}$  with lattice measurements for  $N_f = 12$  in [10], finding general consistency with the range of lattice results, although our  $\gamma_{\text{IR}, \Delta_f^4}$  and extrapolation to the exact  $\gamma_{\text{IR}}$  were higher than

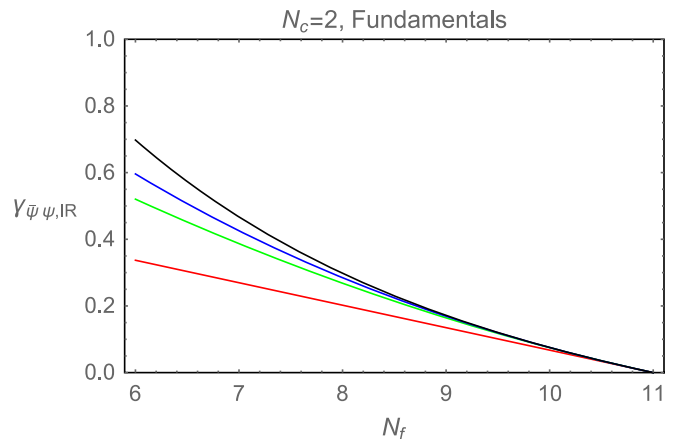


FIG. 1. Plot of  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^p}$  for  $R = F$ ,  $N_c = 2$ , and  $1 \leq p \leq 4$  as a function of  $N_f \in I_{\text{IRZ}}$ . From bottom to top, the curves (with colors online) refer to  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f}$  (red),  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^2}$  (green),  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^3}$  (blue), and  $\gamma_{\bar{\psi}\psi, \text{IR}, \Delta_f^4}$  (black).

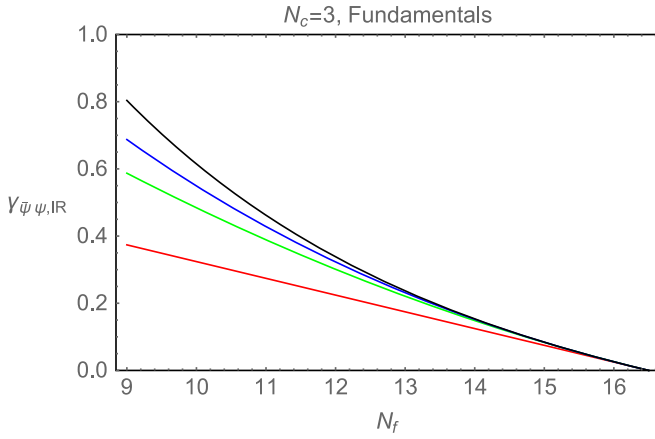


FIG. 2. Plot of  $\gamma_{\psi\psi,IR,\Delta_f^p}$  for  $R = F$ ,  $N_c = 3$ , and  $1 \leq p \leq 4$  as a function of  $N_f \in I_{IRZ}$ . From bottom to top, the curves (with colors online) refer to  $\gamma_{\psi\psi,IR,\Delta_f}$  (red),  $\gamma_{\psi\psi,IR,\Delta_f^2}$  (green),  $\gamma_{\psi\psi,IR,\Delta_f^3}$  (blue), and  $\gamma_{\psi\psi,IR,\Delta_f^4}$  (black).

some of the lattice values. We also found consistency for the cases  $N_f = 10$  and  $N_f = 8$  [10]. Here, we compare with lattice results for  $\gamma_{IR}$  in the case  $N_c = 2$ ,  $N_f = 8$ . [It is not clear from lattice studies if the  $SU(2)$ ,  $R = F$ ,  $N_f = 6$  theory has a conformal IRFP or not [18,30,38].] Following lattice studies of the  $SU(2)$ ,  $R = F$ ,  $N_f = 8$  theory by several groups [18,39], a recent measurement is  $\gamma_{IR} = 0.15 \pm 0.02 \equiv 0.15(2)$  [40]. Our value  $\gamma_{IR,\Delta_f^4} = 0.298$  is somewhat higher than this lattice result.

We proceed to discuss  $d_5$  for  $R = F$ . In Fig. 3 we plot  $\beta'_{IR,\Delta_f^p}$  for  $R = F$ ,  $N_c = 3$ , and  $2 \leq p \leq 5$ . In Table II we list values of  $\beta'_{IR,\Delta_f^p}$  for  $R = F$ ,  $N_c = 2, 3$  and  $2 \leq p \leq 5$ . For  $R = F$  and general  $N_c$ ,  $d_2$  and  $d_3$  are positive, while  $d_4$  and  $d_5$  are negative. For the case  $SU(3)$ ,  $N_f = 12$ , we get  $\beta'_{IR,\Delta_f^5} = 0.228$ . The conventional  $n$ -loop calculation

TABLE I. Values of the scheme-independent anomalous dimension  $\gamma_{IR,\Delta_f^p}$  with  $1 \leq p \leq 4$  for  $R = F$  and  $N_c = 2, 3$ .

$N_c$	$N_f$	$\gamma_{IR,\Delta_f}$	$\gamma_{IR,\Delta_f^2}$	$\gamma_{IR,\Delta_f^3}$	$\gamma_{IR,\Delta_f^4}$
2	6	0.337	0.520	0.596	0.698
2	7	0.270	0.387	0.426	0.467
2	8	0.202	0.268	0.285	0.298
2	9	0.135	0.164	0.169	0.172
2	10	0.0674	0.07475	0.07535	0.0755
3	9	0.374	0.587	0.687	0.804
3	10	0.324	0.484	0.549	0.615
3	11	0.274	0.389	0.428	0.462
3	12	0.224	0.301	0.323	0.338
3	13	0.174	0.221	0.231	0.237
3	14	0.125	0.148	0.152	0.153
3	15	0.0748	0.0833	0.0841	0.0843
3	16	0.0249	0.0259	0.0259	0.0259

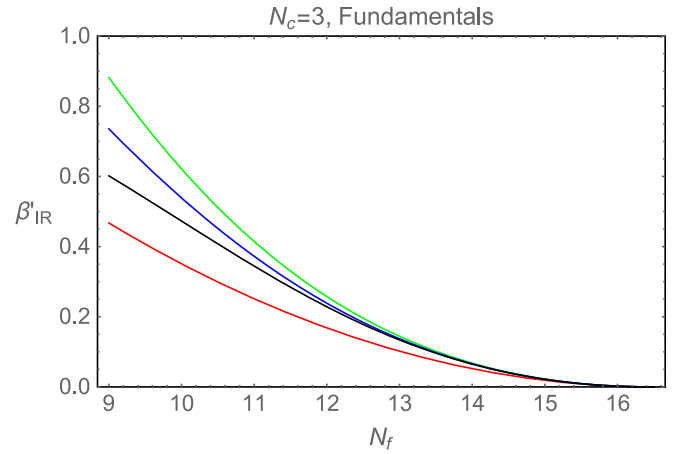


FIG. 3. Plot of  $\beta'_{IR,\Delta_f^p}$  for  $R = F$ ,  $N_c = 3$ , and  $2 \leq p \leq 4$  as a function of  $N_f \in I_{IRZ}$ . From bottom to top, the curves (with colors online) refer to  $\beta'_{IR,\Delta_f^2}$  (red),  $\beta'_{IR,\Delta_f^3}$  (green),  $\beta'_{IR,\Delta_f^4}$  (blue), and  $\beta'_{IR,\Delta_f^5}$  (black).

yielded  $\beta'_{IR,3\ell} = 0.2955$  and  $\beta'_{IR,4\ell} = 0.282$  [41], so  $\beta'_{IR,\Delta_f^5}$  is slightly smaller than  $\beta'_{IR,4\ell}$ . A recent lattice measurement yields  $\beta'_{IR} = 0.26(2)$  [42], consistent with both our  $\beta'_{IR,\Delta_f^5}$  and  $\beta'_{IR,4\ell}$ .

We next discuss the case  $R = \text{adj}$ , for which  $N_u = 11/4$  and  $N_\ell = 17/16$ , so  $I_{IRZ}$  includes the single integer value  $N_f = 2$  (whence  $\Delta_f = N_u - 2 = 3/4$ ). Results for this case were given for  $\kappa_p$  with  $1 \leq p \leq 3$  in [8] and for  $d_p$  with  $1 \leq p \leq 4$  in [9]. Here we find

$$\begin{aligned} \kappa_{4,\text{adj}} &= \frac{53389393}{2^7 \cdot 3^{14}} + \frac{368}{3^{10}} \zeta_3 + \left( -\frac{2170}{3^{10}} + \frac{33952}{3^{11}} \zeta_3 \right) N_c^{-2} \\ &= 0.0946976 + 0.193637 N_c^{-2} \end{aligned} \quad (9)$$

TABLE II. Scheme-independent values of  $\beta'_{IR,\Delta_f^p}$  with  $2 \leq p \leq 4$  for  $R = F$ ,  $N_c = 2, 3$  as functions of  $N_f$  in the respective intervals  $I_{IRZ}$ . The notation  $ae-n$  means  $a \times 10^{-n}$ .

$N_c$	$N_f$	$\beta'_{IR,\Delta_f^2}$	$\beta'_{IR,\Delta_f^3}$	$\beta'_{IR,\Delta_f^4}$	$\beta'_{IR,\Delta_f^5}$
2	6	0.499	0.957	0.734	0.6515
2	7	0.320	0.554	0.463	0.436
2	8	0.180	0.279	0.250	0.243
2	9	0.0799	0.109	0.1035	0.103
2	10	0.0200	0.0236	0.0233	0.0233
3	9	0.467	0.882	0.7355	0.602
3	10	0.351	0.621	0.538	0.473
3	11	0.251	0.415	0.3725	0.344
3	12	0.168	0.258	0.239	0.228
3	13	0.102	0.144	0.137	0.134
3	14	0.0519	0.0673	0.0655	0.0649
3	15	0.0187	0.0220	0.0218	0.0217
3	16	2.08e-3	2.20e-3	2.20e-3	2.20e-3

and

$$d_{5,\text{adj}} = -\frac{7141205}{2^3 \cdot 3^{16}} + \frac{5504}{3^{12}} \zeta_3 - \left( \frac{30928}{3^{14}} + \frac{465152}{3^{13}} \zeta_3 \right) N_c^{-2} \\ = -(0.828739 \times 10^{-2}) - 0.357173 N_c^{-2}. \quad (10)$$

We remark on the  $SU(2)$ ,  $N_f = 2$ ,  $R = \text{adj}$  theory, which has been of interest [43]. Extensive lattice studies of this theory have been performed and are consistent with IR conformality [18]. We get  $\beta'_{\text{IR},\Delta_f^5} = 0.147$ ; and  $\gamma_{\text{IR},\Delta_f^2} = 0.465$ ,  $\gamma_{\text{IR},\Delta_f^3} = 0.511$ , and  $\gamma_{\text{IR},\Delta_f^4} = 0.556$ . These  $\gamma_{\text{IR},\Delta_f^p}$  values are close to our  $n$ -loop calculations in [13] for this theory, namely  $\gamma_{\text{IR},3\ell} = 0.543$ ,  $\gamma_{\text{IR},4\ell} = 0.500$ . Lattice measurements of this theory have yielded a wide range of values of  $\gamma_{\text{IR}}$  including 0.49(13) [44], 0.22(6) [45], 0.31(6) [46], 0.17(5) [47], 0.20(3) [48], 0.50(26) [49], and 0.15(2) [40] (see references for details of uncertainty estimates).

Finally, we discuss the case  $R = S$ . For  $SU(2)$ ,  $S = \text{adj}$ , already discussed above. For  $SU(3)$ , we focus on the

$N_f = 2$  theory, for which we find  $\beta'_{\text{IR},\Delta_f^5} = 0.333$ ; and  $\gamma_{\text{IR},\Delta_f^2} = 0.789$ ,  $\gamma_{\text{IR},\Delta_f^3} = 0.960$ , and  $\gamma_{\text{IR},\Delta_f^4} = 1.132$  [37]. For comparison, our  $n$ -loop results from [13] for this case are  $\gamma_{\text{IR},3\ell} = 0.500$  and  $\gamma_{\text{IR},4\ell} = 0.470$ . Lattice studies of this theory include one that concludes that it is IR-conformal and gets  $\gamma_{\text{IR}} < 0.45$  [50] and another that concludes that it is not IR-conformal and gets an effective  $\gamma_{\text{IR}} \simeq 1$  [51].

In summary, we have presented calculations of  $\gamma_{\bar{\psi}\psi,\text{IR}}$  and  $\beta'_{\text{IR}}$  at a conformal IR fixed point of an asymptotically free gauge theory with fermions, to the highest orders yet achieved. We believe that these results are of fundamental value for the understanding of conformal field theory, especially because they are scheme-independent.

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