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Area term of the entanglement entropy of a supersymmetric $O(N)$ vector model in three dimensions

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We studied the leading area term of the entanglement entropy of the $\mathcal{N} = 1$ supersymmetric $O(N)$ vector model in $2 + 1$ dimensions close to the line of the second order phase transition in the large N limit. We found that the area term is independent of the varying interaction coupling along the critical line, unlike what is expected in a perturbative theory. Along the way, we studied noncommuting limits $n - 1 \to 0$ vs UV cutoff $r \to 0$ when evaluating the gap equation and found a match only when the appropriate counterterm is introduced and the coupling of which is chosen to take its fixed point value. As a bonus, we also studied fermionic Green functions in the conical background. We made the observation of a map between the problem and the relativistic hydrogen atom.

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I. INTRODUCTION

Entanglement entropy has emerged as a very powerful tool in characterizing important properties of many body systems. It has led to new insights for example in the discovery and classification of new phases of matter, such as, to name a few, these exotic symmetry protected topological phases and topological orders [1–3]. Since the beginning, it has been observed that the entanglement entropy of ground states of local field theories [4,5], or more generally ground states of local Hamiltonians even in discrete systems, satisfy a so-called area law. The area law is the observation that for some choice of region A in configuration space, of which the entanglement entropy with its complement one calculates, the leading term in the large region size limit is proportional to the area of the codimension-1 surface bounding region A. Apart from one-dimensional systems where there is an exact proof [6], the area law remains a conjecture in other dimensions—although new insights that edge toward a complete proof of the statement are emerging more recently [7,8]. The emergence of area laws is believed to be profoundly connected to quantum gravity theories, given the similarities between entanglement entropies and the Bekenstein-Hawking black hole entropy.

There is a series of works that explores how the entanglement entropy and, in particular, the area law change in the presence of perturbations to some given theories, such as free theories or conformal theories. (It is impossible to exhaust the literature on these topics. See for example Refs. [9–12], which are some of the early papers on the subject). It is known that the area law term is not universal in the sense that it can have dependence on the precise regularization scheme, and there are some recent efforts that extract universal contributions to the area term from relevant perturbations [13,14].

On the other hand, precisely for reasons of generic dependence of regularization schemes in field theories, attention has often been focused on subleading terms in the entanglement entropy, such as the logarithmic terms in even dimensions and the constant terms in odd dimensions, which are known to be scheme independent and are connected to important characteristics of the underlying theory, such as central charges, or the “F charge” in odd dimensions, at conformal fixed points. (See for example the seminal papers [15–18] that elaborated these connections.) These works are generally independent of the details of individual theories and are based on very general symmetries, such as Lorentz symmetries and conformal invariance.

It is a curiosity therefore to ask how the entanglement entropy depends on the strength of interaction coupling. For strongly coupled theories, there are very restricted tools at our disposal. We have a plethora of holographic results (for a very recent comprehensive review, see for example Ref. [19]). In some supersymmetric theories, a deformed supersymmetry preserving “entanglement entropy” can be computed exactly even in strongly interacting theories, first considered in Ref. [20], although attention is not usually paid to the coupling dependence of the area term, if there is a continuous coupling to be tuned in these calculations at all.

On the other hand, we have large N theories, where entanglement entropies can be computed in the large N saddle point limit for generic interacting couplings. This has been considered near the fixed point in Ref. [21], and more recently the flow of the entanglement under Renormalization (RG) flow of the renormalized mass was obtained, making use of the entanglement first law [22].

In this paper, we would like to consider another example in which large N techniques would be useful. We study the $\mathcal{N}=1$ supersymmetric $O(N)$ vector model in $2+1$ dimensions. The theory has a critical line that controls phase transitions between the $O(N)$ symmetry preserving phase and the $O(N)$ spontaneously broken phase [23]. The virtue of this is that there is a one parameter family of theories sitting on the critical line such that we can study the dependence of the entanglement entropy on this coupling. For simplicity, we will work in the leading large N limit and compute the entanglement entropy of half-space i.e. $y>0$. We will, in particular, focus on the area term. As we will see, one issue of interest is that there are various new counterterms that are required as soon as we employ the replica trick and obtain the gap equation there. Not all values of the counterterms can be easily fixed based on physical requirements. The minimal choice would suggest that the bosonic renormalized mass has no dependence on the coupling and takes exactly the same value as in the bosonic $O(N)$ vector model [21]. The fermionic renormalized mass depends on the inverse of the coupling for any nonzero coupling and thus does not admit a smooth limit back to the free theory. Surprisingly, however, the final form of the area term has no dependence on the coupling. That the area term is rigid in the leading large N limit comes as a surprise and is possibly an artifact of the large N expansion.

Before we end the Introduction, let us reiterate here why the study of the area term is a well-defined question in the current context, even though it is considered in many circumstances as being nonuniversal, with cutoff dependence. As discussed in Ref. [24], the entanglement entropy is an expansion in L/ϵ, $L\mu$, etc., where L is the region size, ϵ is the UV cutoff, and μ is any other mass scales in the theory. The change of the UV cutoff would for example have an interplay with the RG flow. Here, we focus on the entanglement of half-space near the critical line, such that $L\to\infty$. Since we stay on the critical line as g is tuned, there is no further complication of changing the cutoff scheme once it is fixed once at a given g. This should render the physics question we are posing sufficiently well defined.

We will begin with a brief review of the supersymmetric $O(N)$ vector model in Sec. II. Then, we will present the details of the computation of the entanglement entropy in Sec. III.

We will conclude in Sec. IV and relegate some excessive details to the Appendix.

II. SUPERSYMMETRIC O(N) VECTOR MODEL

The action is given by

$$S(\phi,\psi)=1/2\int d^3x\Big[\partial_\mu\phi\partial^\mu\phi-\mu_\phi^2\phi^2+\bar\psi(i\gamma^\mu\partial_\mu-\mu_\psi)\psi-2\frac{g\mu}{N}(\phi^2)^2-\frac{g^2}{N^2}(\phi^2)^3-\frac{g}{N}\phi^2(\bar\psi.\psi)-2\frac{g}{N}(\phi.\bar\psi)(\phi.\psi)\Big],\tag{1}$$

where the bosons ϕ_i and fermions ψ_i are in the fundamental representation of $O(N)$ and the Lorentz signature is chosen as $(1,-1,-1,-1)$ here.

After doing the Wick rotation, introducing the auxiliary fields, and integrating out the fermions and bosons fields, the effective action can be written as [23,25]

$$S_{\mathrm{eff}}=\int d^3x\left[-\frac{\lambda\rho}{2}+\frac{g^2\rho^3}{2}+g\mu\rho^2\right]+\frac{1}{2}Tr\ln(-\Box+\mu_\phi^2+\lambda)-\frac{1}{2}\mathrm{Tr}\ln(\partial\!\!\!/+\mu_\psi+g_0\rho).\tag{2}$$

Note that in the above action, μ_ϕ, μ_ψ, and μ are bare parameters of the theory. When they are equal, the theory preserves supersymmetry, which is the case we will focus on i.e.

$$\mu_\phi=\mu_\psi=\mu=\mu_0.\tag{3}$$

In the leading large N limit, the gap equation is given by

$$m_\psi=\mu_0+g\rho,\qquad m_\phi^2=\mu_0^2+4g\mu\rho+3g^2\rho^2-g\chi,\tag{4}$$

where

$$\rho=G_\phi(x,x),\qquad \chi=\mathrm{tr}G_\psi(x,x),\tag{5}$$

and the trace above refers to the trace with respect to spinor indices. In flat space, these means

$$G_\phi(x,x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+m_\phi^2},\qquad \mathrm{tr}G_\psi(x,x)=\mathrm{tr}\int\frac{d^3p}{(2\pi)^3}\frac{p\!\!\!/+m_\psi}{p^2+m_\psi^2}.\tag{6}$$

We note that m_ψ and m_ϕ are physical masses and therefore take finite values. However, at $d=3$, both propagators are linearly divergent in the UV. In fact, the linear divergence is given by

$$\text{divergence}(G_\phi) = \frac{\Lambda}{(2\pi)^2} = \frac{1}{2m_\psi}\text{divergence}(G_\psi), \quad (7)$$

where Λ is a UV cutoff. Therefore, this means that the bare couplings μ_0 must in fact be divergent so as to cancel the divergence of G to recover a finite physical mass.

The detailed phase structure in the leading large N limit can be found in Ref. [23].

At criticality, we arrange that

$$m_\psi = 0, \qquad \mu_0 = -g\rho. \quad (8)$$

Recall that the other gap equation is automatically satisfied after picking the above value for μ_0 for any value of g. This means that no extra divergent parameters are needed to remove any further singularities. In fact, it is convenient to compute

$$m_\phi^2 - m_\psi^2 = 2gm_\psi\rho - g\chi, \quad (9)$$

which clearly shows that the divergences of ρ and χ cancels each other, leading to a finite value of m_ϕ as soon as m_ψ is made finite. Supersymmetry ensures that this in fact vanishes along the supersymmetric preserving saddle points at all masses all the way to $m_\psi = m_\phi = 0$.

III. AREA TERM IN THE ENTANGLEMENT ENTROPY OF HALF-SPACE

Having briefly reviewed the theory, we would like to explore its entanglement entropy in this section. For simplicity, we will consider entanglement of half-space i.e. $y \geq 0$. We will employ the replica trick to extract the entanglement entropy. At replica index n, it is equivalent to putting the Euclidean path integral in a conical space, in which an angle deficit located in the $y - t$ plane is given by $2\pi(n-1)$. Translation invariance remains intact in the orthogonal direction that we call x. The boundary of half-space is thus the real line x, which has infinite length. We will regulate it only at the end when we extract the area term.

A. Green function in conical space

We will collect all the ingredients necessary to recover the entanglement entropy. First, we need to recover the Green function of both the bosons and fermions in the n-replicated space.

As it was observed already in the scalar $O(N)$ model, it is expected that the masses m_ψ and m_ϕ would generically acquire r dependence. If we were working with a critical theory at $n = 1$, it would then be expected purely from dimensional grounds that

$$m_\phi^2 = \frac{a_n}{r^2}, \qquad m_\psi = \frac{b_n}{r}, \quad (10)$$

where a_n $b_n \to 0$ as $n \to 1$. Currently, we adopt the strategy of computing the gap equation by obtaining ρ_n, χ_n perturbatively in $n - 1$.

1. Bosonic Green function

In three dimensions, the bosonic Green function of which the mass is dependent on the conical place satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{a_n}{r^2} + \frac{\partial^2}{\partial x^2}\right)G_B\left(m = \frac{a_n}{r^2}, \boldsymbol{r};\boldsymbol{r}_1\right) = -\delta(\boldsymbol{r} - \boldsymbol{r}_1). \quad (11)$$

The Green function could be solved by mode expansion, which gives

$$G_B(\boldsymbol{r};\boldsymbol{r_1}) = \sum_{l=-\infty}^{\infty}\frac{e^{i\nu(\theta-\theta_1)}}{2\pi n}\int_{-\infty}^{\infty}\frac{dk_\perp}{2\pi}e^{ik_\perp(x-x_1)} \times \int_0^\infty kdk\frac{J_{\nu_l}(kr)J_{\nu_l}(kr_1)}{k^2+k_\perp^2} \quad (12)$$

in which $\nu_l = \sqrt{\frac{l^2}{n^2} + a_n}$ for $|l| > 0$ and $\nu_l = \alpha_n$ for $l = 0$. We define $a_n \equiv {\alpha_n}^2$ with α_n taking either positive or negative values. For $l \neq 0$, we take the positive branch of the solution, whereas at precisely $l = 0$, the gap equation appears to force upon us the negative branch of the solution. This issue has been discussed in Ref. [21], which is related to the threshold of bound state formation.

Now, we would like to calculate the leading $n - 1$ correction to the Green function in the conical space. There are two contributions. First, because of the altered periodicity in the presence of the cone, the Green function at vanishing mass carries $n - 1$ dependence. To linear order in $n - 1$, we have

$$G_n(\boldsymbol{r};\boldsymbol{r}) - G_1(\boldsymbol{r};\boldsymbol{r}) = -\frac{(n-1)}{32r} \quad (13)$$

which is a special case of $D = 3$ of Eq. (4.65) in Ref. [21].

Now, however, there is an extra mass term depending on a_n that carries $n - 1$ dependence. To compute the leading correction coming from a_n, we can treat a_n as a perturbation of the conical space Laplacian. The correction to the Green function as a power series expansion in a_n is then obtained using

$$\left(\Box_n + \frac{a_n}{r^2}\right)(G_n + \delta G_n)(\boldsymbol{r};\boldsymbol{r_0}) = -\delta^3(\boldsymbol{r} - \boldsymbol{r_0}). \quad (14)$$

Naively, therefore, we have

$$
\begin{aligned}
\delta G_n(\boldsymbol{r},\boldsymbol{r}) &= \lim_{\boldsymbol{r}''\to\boldsymbol{r}} - a_n \int d^3x' \frac{1}{r'^2} G_n(\boldsymbol{r}'',\boldsymbol{r}') G_n(\boldsymbol{r}',\boldsymbol{r}') \\
&= \lim_{x''\to x} - a_n \int \frac{dr'}{r'} \sum_l \frac{1}{2\pi n} \int \frac{dk_\perp}{2\pi} \frac{e^{ik_\perp(x-x'')}}{(k^2+k_\perp^2)(k'^2+k_\perp^2)} \int_0^\infty kdkJ_{\nu_l}(kr)J_{\nu_l}(kr') \int_0^\infty k'dk'J_{\nu_l}(k'r')J_{\nu_l}(k'r) \\
&= \lim_{x''\to x} - a_n \sum_l \frac{1}{2\pi n} \int \frac{dk_\perp}{2\pi} \frac{e^{ik_\perp(x-x'')}}{(k^2+k_\perp^2)(k'^2+k_\perp^2)} \int_0^\infty kdkJ_{\nu_l}(kr) \int_0^k k'dk'J_{\nu_l}(k'r) \left(\frac{k'}{k}\right)^\nu \frac{1}{\nu_l} \\
&= \lim_{x''\to x} - a_n \sum_l \frac{1}{4\pi n} \int_0^\infty dk \int_0^k \frac{k'e^{-k(x-x'')} - ke^{-k'(x-x'')}}{k'^2-k^2} dk' J_{\nu_l}(kr) J_{\nu_l}(k'r) \left(\frac{k'}{k}\right)^\nu \frac{1}{\nu_l} \\
&= -a_n \frac{1}{4\pi^{3/2} nr} \sum_l \frac{1}{\nu_l} \int_0^1 dt \frac{t^{2\nu_l}}{1+t} {}_2F_1\left[\frac{1}{2},\frac{1}{2}+\nu;1+\nu,t^2\right] \frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu)},
\end{aligned} \tag{15}
$$

where we take $x = x''$ and perform the integral over k' only in the last step and made a change of variables, defining $k' = kt$.

In the above calculation, strictly speaking, we should have taken G_n to be evaluated at $a_n = 0$. However, supposedly, if the expression is regular in a_n, then to leading order in a_n it would not have made a difference had we set $a_n \to 0$ in G_n only in the last step. We now investigate this limit $\alpha_n \to 0$. The important surprise is that the $l = 0$ term contains a pole in $1/\alpha_n$ inherited from $1/\nu_l$, and therefore that term alone is of $O(\alpha_n)$. The leading a_n contribution to the Green function is therefore not linear in a_n, but depending on $\sqrt{a_n}$. Focusing on the $l = 0$ term, we finally get

$$
\begin{aligned}
&- a_n \frac{1}{4\pi^{3/2} nr} \frac{1}{\alpha_n} \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \int_0^1 dt \frac{1}{1+t} {}_2F_1\left[\frac{1}{2},\frac{1}{2};1,t^2\right] \\
&\quad = -a_n \frac{1}{4\pi^{3/2} nr} \frac{1}{\alpha_n} \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \frac{2}{\pi} \int_0^1 dt \frac{\mathrm{EllipticK}(t^2)}{1+t} \\
&\quad = -\frac{\alpha_n}{16nr}.
\end{aligned} \tag{16}
$$

We note immediately that this result is half of that obtained in Ref. [21]. The reason is that our expansion assumed that this is a power series expansion in a_n which, however, is in fact a function of $\sqrt{a_n}$. When we computed the linear order term in a_n, it was effectively a first derivative of the function subsequently evaluated near $a_n = 0$. Now, noting that for $z = x^2$,

$$
\frac{d}{dz} f(x) = \frac{d}{dx} f(x) \times \frac{1}{2x}, \tag{17}
$$

we reckon the factor of 2 we obtained can be attributed to treating the expansion as a function of a_n when it is in fact a function of α_n. Correcting this subtlety, we arrive at

$$
\boxed{\delta G_n(\boldsymbol{r},\boldsymbol{r}) = -\frac{\alpha_n}{8nr},} \tag{18}
$$

recovering correctly the result in Ref. [21].

The $|l| > 0$ terms can in fact be summed, and they are evaluated to $-a_n/(16\pi nr)$.

There is an alternative way to think about the pole in α obtained above. If we focus on the $r' \to 0$ limit of the integral and compute the $k_\perp$ integral first, one can see that the ν_0 term would contribute to a logarithmic divergence in the r' integral precisely if we first take the limit $a_n \to 0$. Therefore, the α pole observed above can also be alternatively taken as a logarithmic divergence localized at the conical singularity, $r \to 0$. To confirm such an expectation, let us extract the logarithmic divergence explicitly. One very convenient way is to recall that Eq. (13) implies that if we use the Euclidian Green function G_0 to calculate the correction above, the difference would be of order $O(n-1)^3$, assuming that $\alpha_n \sim O(n-1)$. Let us note that this assumption is supported by evidence in the solution of the gap equation of the $O(N)$ scalar model in Ref. [21] and also the expectation that the free energy should remain analytic in $n-1$ in the limit $n \to 1$. As we will see, this assumption is confirmed when we solve the gap equation in our case. So, up to $O(n-1)^2$, the result would be the same if we replace G_n by G_1. The massless bosonic Green function in 3D Euclidean space is

$$
G_1(\boldsymbol{r_0},\boldsymbol{r_1}) = \frac{1}{4\pi r_{(3)}} \tag{19}
$$

in which $r_{(3)}$ is the three-dimensional distance which is given by

$$
r_{(3)} = \sqrt{r_0^2 + r_1^2 - 2r_0 r_1 \cos(\theta_0 - \theta_1) + (x_0 - x_1)^2}. \tag{20}
$$

Therefore, to extract the leading divergent terms proportional to a_n, we can return to (15) and replace $G_n \to G_1$, which gives

$$\begin{aligned}&\frac{\alpha_n^2}{16\pi^2}\int_0^\infty \frac{dr_1}{r_1}\int_0^{2\pi n} d\theta_1 \int_{-\infty}^{\infty} dx_1 (r_0^2+r_1^2-2r_0r_1\cos(\theta_0-\theta_1)\\ &\quad +(x_0-x_1)^2)^{-1}\\ &=\frac{\alpha_n^2}{16\pi}\int_0^{r_0}\frac{dr_1}{r_0r_1}\int_0^{2\pi n} d\theta_1 (1+a^2-2a\cos(\theta_0-\theta_1))^{-1/2}\\ &\quad+\frac{\alpha_n^2}{16\pi}\int_{r_0}^{\infty}\frac{dr_1}{r_1^2}\int_0^{2\pi n} d\theta_1(1+a^2-2a\cos(\theta_0-\theta_1))^{-1/2}\\ &\approx\frac{\alpha_n^2}{4\pi r_0}\left[\int_{\frac{\epsilon}{r_0}}^1 \frac{da}{a}\mathrm{EllipticK}(a^2)+\int_0^1 da \mathrm{EllipticK}(a^2)\right]\\ &=\frac{\alpha_n^2}{4r_0}\left(\log 2-\frac{1}{2}\log\frac{\epsilon}{r_0}\right), \qquad (21)\end{aligned}$$

where we have defined $a \equiv r_1/r_0 < 1$.

The first term in the second-to-last line is divergent as $a \to 0$. Thus, we introduce a short range cutoff $\frac{\epsilon}{r_0}$, which shows a logarithmic divergence located at the conical singularity. The coefficient of the logarithmic divergence is given by $-1/(8r)$, precisely that anticipated in (18) in the $n \to 1$ limit. We note the similarity of this divergence to that observed in Ref. [21] that requires the counterterm of the form $\int d^3x\delta^2(r)\phi^2$ localized at the conical singularity.

2. Localized counterterms and conformal fixed points

Now let us introduce the counterterm $\frac{c}{2}\int d^3x\delta^2(r)\phi^2$, so that the Green function would be modified and take the form

$$G_B^c(\boldsymbol{r};\boldsymbol{r}') = \frac{1}{-\square_n + \frac{\alpha_n^2}{r^2} + c\frac{\delta(r)}{r}}. \qquad (22)$$

The calculation of the correction induced by the counterterm is straightforward, and it turns out that to the second order in c, the correction is (a detailed calculation is displayed in Appendix A)

$$\delta_c G_B^c(\boldsymbol{r},\boldsymbol{r}) = -\frac{c}{16nr_0} + \frac{c^2}{32r_0}(-\log k\epsilon + \log 2 - \gamma). \qquad (23)$$

We find that one proper form of c that could subtract the divergence can be chosen as

$$\begin{aligned}c &= c_r + \frac{c_r^2}{2}(-\log k\epsilon + \log 2 - \gamma)\\ &\quad - 4\alpha_n^2\left(\log 2 - \frac{1}{2}\log\epsilon/r_0\right) + \mathcal{O}(n-1)^3 \qquad (24)\end{aligned}$$

in which c_r is the coefficient c after renormalization.

So, we can immediately see that the beta function for c_r now takes the form

$$\beta(c_r) = \frac{c_r^2}{2} - 2\alpha_n^2. \qquad (25)$$

The fixed points for the theory are

$$c_{r\pm} = \pm 2\alpha_n. \qquad (26)$$

We note that only c_{r+} is a stable fixed point.

There are now two different expressions describing the same corrections to the Green's function in the replicated space. They are given by

$$G_B^c(\boldsymbol{r};\boldsymbol{r}) - G_1(\boldsymbol{r};\boldsymbol{r}) = \begin{cases} -\frac{\alpha_n}{8nr_0} - \frac{n-1}{32r_0} + \mathcal{O}(n-1)^2, \\ -\frac{c_{r+}}{16nr_0} - \frac{n-1}{32r_0} + \mathcal{O}(n-1)^2, \end{cases} \qquad (27)$$

where the first expression is obtained by keeping finite a_n, obtaining a finite expression that carries a pole in α, whereas the second expression requires regulating the log-divergence when the $a_n \to 0$ limit is first taken, and then counterterms, the coupling of which is taken to be 1 at the stable fixed point, are introduced. Reassuringly, these two answers match.

The importance of the introduction of the counterterm was already noticed in Ref. [21] and revisited in Ref. [22], where it is shown that the boundary term can be understood as following from the conformal coupling of the scalar to the Ricci scalar. Here, it is of note to see that the effect of this term can in fact be replaced by taking the $n \to 1$ limit last.

3. Fermionic Green function

Like bosons, the fermionic Green function could acquire a mass term in the conical space as well, such that

$$\begin{aligned}S_F &= \left(\partial\!\!\!/ + \frac{b_n}{r}\right)^{-1}\\ &= \begin{pmatrix} \partial_x + \frac{b_n}{r} & -e^{-i\hat{\theta}/n}\left(\frac{n}{r}\partial_{\hat{\theta}} + i\partial_r\right) \\ -e^{i\hat{\theta}/n}\left(\frac{n}{r}\partial_{\hat{\theta}} - i\partial_r\right) & -\partial_x + \frac{b_n}{r}\end{pmatrix}^{-1}. \qquad (28)\end{aligned}$$

And we define G_F as

$$\square G_F(r,r_1) = \partial\!\!\!/^2 G_F(r,r_1) = -\delta(\boldsymbol{r}-\boldsymbol{r_1}) \qquad (29)$$

which is related to the free fermionic Green function by

$$\partial\!\!\!/^{-1} = \partial\!\!\!/ G_F. \qquad (30)$$

Let us clarify here that the coordinates we are using is such that the metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + dx^2 = dx^2 + dy^2 + dz^2,$$
$$y = r\cos\theta; \qquad z = r\sin\theta, \qquad \theta = n\hat{\theta}, \tag{31}$$

where θ has periodicity $2\pi n$. Of course, the above coordinates are not single valued. However, if we are only interested in evaluating the Green function at the same point far away from the conical singularity at $r = 0$, space is essentially flat, and this coordinate can be used in a patch by patch fashion, which it is single valued. Then, G_F is solved in one patch and then transformed to the $r, \hat{\theta}$ coordinates, corresponding to patching all the patches together to recover one single valued Green function.

In the n-replicated space, therefore, the fermionic Green function has a mode expansion as follows:

$$\begin{aligned} G_F(\boldsymbol{r}, \boldsymbol{r_1}) &= \sum_{l=-\infty}^{\infty} \frac{e^{i\nu_l(\theta-\theta_1)}}{2\pi n} \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} e^{ik_\perp(x-x_1)} \int_0^\infty kdk \frac{J_{\nu_l}(kr) J_{\nu_l}(kr_1)}{k^2 + k_\perp^2} \\ &= \sum_{l=-\infty}^{\infty} \frac{e^{i\nu_l(\theta-\theta_1)}}{2\pi n} \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} e^{ik_\perp(x-x_1)} I_{\nu_l}(k_\perp r) K_{\nu_l}(k_\perp r_1) (\text{for } r < r_1). \end{aligned} \tag{32}$$

Here, $\nu_l = |\frac{2l+1}{2n}|$, so that the antiperiodic boundary condition is satisfied around the θ circle.

Just like the bosonic Green function, we tried to get the whole spectrum of the fermion (in order to calculate the Renyi entropy), and we found that it is closely related to the hydrogen atom, which is shown in Appendix C. However, the significant difference from the hydrogen atom is that the potential is an imaginary one, so the current problem does not have bound state solutions, unlike the hydrogen atom. However, the scattering problem for the hydrogen atom does not have a rigorous analytic expression up to our limited knowledge. But if we only calculate the entanglement entropy, only the order $(n-1)$ terms are important, which leads to the strategy similar to the case of bosons here, and we use the $(n-1)$ expansion to obtain the leading order corrections of the propagator from b_n,

$$\begin{aligned} \delta S_F(\boldsymbol{r_0}, \boldsymbol{r_1}) &= -\frac{1}{\partial\!\!\!/} \frac{b_n}{r} \frac{1}{\partial\!\!\!/} = -\partial\!\!\!/ G_F \frac{b_n}{r} \partial\!\!\!/ G_F \\ &= -\int r_2 dr_2 \int dx_2 \int d\theta_2 \partial\!\!\!/_0 G(r_0, r_2) \frac{b_n}{r_2} \partial\!\!\!/_2 G(r_2, r_1) \\ &= b_n \int dr_2 \int dx_2 \int d\theta_2 \partial\!\!\!/_2 G(r_0, r_2) \partial\!\!\!/_2 G(r_2, r_1). \end{aligned} \tag{33}$$

Taking the trace of the above expression, we get

$$\begin{aligned} \mathrm{tr}\delta S_F(\boldsymbol{r_0}, \boldsymbol{r_1}) &= 2b_n \int dr_2 \int dx_2 \int d\theta_2 \partial_{x_2} G(r_0, r_2) \partial_{x_2} G(r_2, r_1) + \partial_{r_2} G(r_0, r_2) \partial_{r_2} G(r_2, r_1) + \frac{1}{r_2^2} \partial_{\theta_2} G(r_0, r_2) \partial_{\theta_2} G(r_2, r_1) \\ &= -2b_n \int dr_2 \int dx_2 \int d\theta_2 \left(G(r_0, r_2) \Box G(r_2, r_1) - G(r_0, r_2) \frac{\partial_{r_2}}{r_2} G(r_2, r_1) \right) \\ &= 2\frac{b_n}{r_1} G(r_0, r_1) + 2b_n \int dr_2 \int dx_2 \int d\theta_2 G(r_0, r_2) \frac{\partial_{r_2}}{r_2} G(r_2, r_1). \end{aligned} \tag{34}$$

Now, we focus on the second term in Eq. (34) and take $\boldsymbol{r_0} \to \boldsymbol{r_1}$ hereafter,

$$\begin{aligned} 2b_n \int dr_2 \int dx_2 \int d\theta_2 G(r_0, r_2) \frac{\partial_{r_2}}{r_2} G(r_2, r_0) &= \frac{b_n}{2\pi^2 n} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_\perp \left(\int_0^{r_0} \frac{dr_2}{r_2} I_\nu(k_\perp r_2) K_\nu^2(k_\perp r_0) \partial_{r_2} I_\nu(k_\perp r_2) \right. \\ &\quad \left. + \int_{r_0}^{\infty} \frac{dr_2}{r_2} K_\nu(k_\perp r_2) I_\nu^2(k_\perp r_0) \partial_{r_2} K_\nu(k_\perp r_2) \right) \\ &= \frac{2b_n}{\pi^2 n r_0^2} \sum_{l=1}^{\infty} \frac{1}{2(4\nu^2 - 1)} + \frac{b_n}{4\pi^2 r_0^2} (1 - 2\log(\epsilon/r_0)) - \frac{b_n}{2\pi^2 r_0^2} \\ &= -\frac{b_n}{2\pi^2 r_0^2} \log(\epsilon/r_0) + b_n \times \mathcal{O}(n-1). \end{aligned} \tag{35}$$

The ϵ is a cutoff in the distance to the conical singularity. Much like what happens in the case of bosons, the calculation is strongly suggestive of the need to include a counterterm that is localized at $r \to 0$. However, the term of the form $\gamma \int d^3x \delta(r)\bar{\psi}\psi$ is such that γ is dimensionful and thus would not produce a divergence term that behaves in the same way as the one observed above. The above calculation is suggestive of a nonlocal counterterm perhaps of the form $\int \bar{\psi}\partial\!\!\!/^{-1}\partial_r\partial\!\!\!/^{-1}\psi$. Without knowing the precise form of the counterterm, we resort to a different strategy. Much like what happens for bosons, we can compute the above correction keeping n general and taking the $n \to 1$ limit only at the end. The details of this calculation are relegated to Appendix C. In that case, the integral is finite, and we obtain

$$-\frac{b_n}{2\pi^2 r_0^2(n-1)} + \frac{b_n g(n-1)}{24r^2} + \mathcal{O}(n-1)^2, \quad (36)$$

where we find no $\mathcal{O}(n-1)^0$ term exactly as in (35) above when a cutoff was introduced.

B. Gap equation in replicated space in $n = 1$ expansion

We now turn to the gap equations (4). We may expect that the interaction would break the supersymmetry, so we restore the notation to show possible deviation from the supersymmetric critical line,

$$m_\psi = \mu_\psi + g_\psi \rho, \qquad m_\phi^2 = \mu_\phi^2 + 4g_{\phi_1}\mu\rho + 3g_{\phi_2}^2\rho^2 - g_\psi \chi, \quad (37)$$

in which μ_ψ and μ_ϕ^2 are the bare masses of the fermion and boson. Like in the flat $n = 1$ case, these bare masses need to be renormalized, absorbing divergences in ρ and χ. To make these precise, we consider computing ρ and χ using Pauli-Villars regularization so that the divergences can be isolated clearly.

1. Pauli-Villars regularization of the bosonic and fermionic propagator

We would like to use Pauli-Villars[1] regularization to modify the Euclidian Green function a little. The idea is to consider a modified propagator which is the original propagator subtracted by one corresponding to a boson/fermion with a mass M; i.e. replace

$$G_{F/B} \to G_{F/B}^R = G_{F/B}(M = 0) - G_{F/B}(M). \quad (38)$$

In the limit $M \to \infty$, this extra term $G_{F/B}(M)$ approaches zero. We will keep the mass M finite and take $M \to \infty$ only at the end.

Under mode expansion, the regulator for the fermion now takes the form of

$$\begin{aligned} G_F(M) &= \sum_{l=-\infty}^{\infty} \frac{e^{i\nu(\theta-\theta_1)}}{2\pi n} \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} e^{ik_\perp(x-x_1)} \int_0^\infty kdk \frac{J_\nu(\sqrt{k^2+M^2}r)J_\nu(\sqrt{k^2+M^2}r_1)}{k^2+M^2+k_\perp^2} \\ &= \sum_{l=-\infty}^{\infty} \frac{e^{i\nu(\theta-\theta_1)}}{2\pi n} \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} e^{ik_\perp(x-x_1)} I_\nu\left(\sqrt{k_\perp^2+M^2}r\right) K_\nu\left(\sqrt{k_\perp^2+M^2}r_1\right)(r < r_1). \end{aligned} \quad (39)$$

Using the uniform expansion for both order and argument of the Bessel functions [26],

$$I_\nu(kr)K_\nu(kr) \approx \frac{1}{2\sqrt{k^2r^2+\nu^2}}. \quad (40)$$

The regulated fermionic Green function at replica index n then takes the form

$$\begin{aligned} G_{Fn}^R(r;r) &= \frac{M_\psi}{4\pi} - \frac{1}{4\pi^2 nr}\left(\log(2\cosh\pi) + \sum_l \left(\psi\left(\left|\frac{l+1/2}{n}\right| + \frac{1}{2}\right) - \frac{1}{2}\log\left(\left(\frac{l+1/2}{n}\right)^2 + 1\right)\right)\right) \\ &= \frac{M_\psi}{4\pi} - \frac{c_F(n)}{4\pi^2 nr} \end{aligned} \quad (41)$$

in which $\psi(x)$ is the polygamma function and $c_F(n)$ is a constant that depends on n and in the first order of $n-1$ takes the form of $c_F(n) = 1 + \frac{1}{2}(n-1)$. It is noteworthy that this term is nonvanishing in the limit $n \to 1$.

Taking the same strategy, the regulated bosonic Green function is given by

$$G_{Bn}^R(r;r) = \frac{M_\phi}{4\pi} - \frac{1}{4\pi^2 nr}\left(\log(2\sinh\pi) + \sum_l \left(\psi\left(\left|\frac{l}{n}\right| + \frac{1}{2}\right) - \frac{1}{2}\log\left(\left(\frac{l}{n}\right)^2 + 1\right)\right)\right) = \frac{M_\phi}{4\pi} - \frac{c_B(n)}{4\pi^2 nr} \quad (42)$$

[1]We thank S. Sachdev for sharing his notes explaining how this is done in replicated space.

in which $c_B(1)$ is exactly 0 and $c_B(n) = (n-1)\pi^2/8$ to the leading order of $(n-1)$. We note that this result is consistent with Eq. (13).

2. Renormalization of the gap equation

In Euclidian space, as pointed out in Eq. (3), all the mass scales coincide with each other when supersymmetry is exact, and all the couplings are related to each other. So we may expect that in the replicated space the masses would deviate from each other. The deviation should be proportional to powers of $(n-1)$ which vanishes at $n = 1$. So, in the renormalization process, we put in the ansatz

$$\mu_\phi^2 = \mu_\psi^2 + A, \qquad \mu = \mu_\psi + B, \tag{43}$$

while μ_ψ is chosen such that the first gap equation is properly renormalized.

We find that (taking $n \to 1$ last in the fermion integral to avoid the logarithmic divergence)

$$\mu_\psi = \frac{b_n}{r} - g\rho = -\frac{M}{4\pi} \tag{44}$$

$$A = 0 \tag{45}$$

$$B = -\frac{b_n}{2r}. \tag{46}$$

Indeed, in the limit $n \to 1$, A and B vanish as b_n vanishes, so that flat space supersymmetry is recovered.

So, after the renormailzation, the gap equations now take the form of

$$\frac{b_n}{r} = g\left(-\frac{\alpha_n}{8nr} - \frac{(n-1)}{32r}\right) \tag{47}$$

$$\frac{a_n}{r^2} = \frac{b_n^2}{r^2} - \frac{gb_n}{2\pi^2 r^2(n-1)} + \frac{b_n g(n-1)}{24r^2} + b_n\mathcal{O}(n-1)^2. \tag{48}$$

3. Solution of the gap equations

A set of self-consistent solutions perturbatively in $n-1$ is

$$\alpha_n = -\frac{(n-1)}{4}, \tag{49}$$

$$b_n = -\frac{\pi^2(n-1)^3}{8g_\psi}. \tag{50}$$

We find that a_n is independent of all the coupling constants, which is exactly the same number as (6.27) in Ref. [21]. This is not surprising since the fermionic part is just an order $(n-1)^3$ one. On the other hand, to the lowest order, b_n is inversely proportional to $g_\psi \equiv g$, which is a manifestation of the nonperturbative nature in the large N calculation. This set of solutions does not admit a smooth limit back to $g \to 0$. At precisely $g = 0$, the only solution is $a_n = b_n = 0$, as expected of a noninteracting theory.

C. Area term

The entropy takes the form of

$$S_{EE} = \partial_n S_{\mathrm{eff}}(n)|_{n=1} - S_{\mathrm{eff}}(1). \tag{51}$$

With a little bit of rearrangement, Eq. (2) now takes the form of

$$\int d^3x\sqrt{\det g_n}\rho(m_\phi^2 - m_\psi^2) + \frac{1}{2}\left[\mathrm{Tr}\log\left(-\Box + \frac{a_n}{r^2}\right) - \mathrm{Tr}\log\left(\partial\!\!\!/ + \frac{b_n}{r}\right)\right]. \tag{52}$$

If evaluated $\partial_n S_{\mathrm{eff}}(n)|_{n=1}$, we would find all the other terms vanishing, since they are $\mathcal{O}(n-1)^2$ or higher, while the last term vanishes for the trace over γ matrices to leading order in $(n-1)$. The remaining terms are

$$\frac{1}{2}\left[\mathrm{Tr}[G_n(a_n)\frac{\partial_n a_n}{r^2}]|_{n=1} - \mathrm{Tr}\log(-\Box)\right]. \tag{53}$$

Naively thinking it is of order $\mathcal{O}(n-1)^2$ exactly like a_n. However, as we previously showed, there is a nontrivial pole existing in the infrared limit which makes this term the only one contributing to the entropy. And it indeed gives the area law as we will show in the following:

$$\partial_n a_n \frac{1}{2}\mathrm{Tr}\left[G_n(a_n)\frac{1}{r^2}\right] = \frac{1}{2}\partial_n a_n \int \sqrt{\det g_n}\frac{d^3x}{r^2}\int\frac{dk_\perp}{2\pi}e^{ik_\perp(x_\perp - x'_\perp)}\sum_l \frac{e^{il(\theta-\theta')}}{2\pi n}\int_0^\infty kdk\frac{J_{\nu_l}(kr)J_{\nu_l}(kr')}{k^2+k_\perp^2}. \tag{54}$$

Performing the $k_\perp$ integral, we get

$$\begin{aligned}\frac{1}{2}\partial_n a_n \mathrm{Tr}\left[G_n(a_n)\frac{1}{r^2}\right] &= \frac{1}{4}\partial_n a_n \int \sqrt{\det g_n}\frac{d^3x}{r^2}\sum_l \frac{e^{il(\theta-\theta')}}{2\pi n}\int_0^\infty e^{-k\epsilon}dkJ_{\nu_l}(kr)J_{\nu_l}(kr)\\ &= \frac{1}{4}\partial_n a_n \sum_l \frac{1}{2\pi n}\int_0^{2\pi} nd\theta \int_0^\infty \frac{dr}{r}J_{\nu_l}(kr)J_{\nu_l}(kr)\int_\infty^\infty dx_\perp \int_0^\infty e^{-k\epsilon}dk\\ &= \frac{1}{4}\partial_n a_n \sum_l \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{2\nu_l}\int_{-\infty}^\infty dx_\perp \frac{1}{\epsilon}. \end{aligned} \tag{55}$$

We find that in the limit $n \to 1$, the $|l| > 1$ terms vanish since $\partial_n a_n$ is of order $O(n-1)$. However, the $l = 0$ term remains. Therefore, the leading term in the entanglement entropy is given by

$$S_{EE} = \frac{\hat{\alpha}L}{2\epsilon} + \cdots; \tag{56}$$

here, $\cdots$ could include a subleading term in the large area expansion. Also, $\epsilon = \Delta x_\perp$ is the short range cutoff in the $x_\perp$ direction, and L is the scale of the box in the dimension $x_\perp$, while $\hat{\alpha} = \lim_{n\to 1}\alpha_n/(n-1)$. If we take the solution of the gap equation in (49), we find that the area term in the entanglement entropy to leading order in the large N limit admits no dependence on the interaction coupling g.

IV. CONCLUSION

Motivated by a lack of computable examples of entanglement entropy of interacting field theories, we studied the $\mathcal{N} = 1$ supersymmetric $O(N)$ vector model in $d = 3$ near the second order phase transition line and computed the entanglement entropy of the half-space, extracting the leading area term. By considering the entanglement of the half-space, the volume and also boundary area of which diverge, we *a priori* made it almost impossible to extract the subleading universal constant term in the entanglement. Yet, since the area term itself encapsulates in reality most of the quantum entanglement of the ground state, and the physical significance of the emergence of an area term in a local field theory in the first place, we would like to understand whether the variation of the interaction coupling makes any qualitative difference to this leading term.

It turns out the supersymmetric theory has lots of similarities with the scalar $O(N)$ model at the critical point. The correction of the massless fermionic propagator in conical space computed perturbatively in $n-1$ has some new divergences, the counterterms of which we have not been able to pin down uniquely. Nevertheless, this term remains finite at any finite $n-1$ and acquires a $1/(n-1)$ pole enhancement. We solved the gap equations of the system perturbatively in $n-1$ and found surprisingly that the bosonic mass acquires exactly the same value as in the critical scalar theory found in Ref. [21], independently of the coupling constant g that can be varied freely along the critical line. An interesting note here is that we found two distinct ways of computing this quantity, by changing the order of limits—one in which $n-1$ is taken as the smallest scale and expanded first, such that a logarithmic divergence near the conical singularity would arise and call for a localized counterterm, and one in which the r integral is done first before the $n \to 1$ limit is taken, as in Ref. [21]. It turns out that the two match, if the couplings of the counterterm are chosen to take its fixed point value, suggesting that the value is robust and unique. Nonetheless, combining with the fermionic results, we arrive at the leading area term of the entanglement entropy that is only sensitive to the bosonic mass, and thus independent of the coupling g. We suspect this is a large N artifact and that a $1/N$ correction should reveal more intricate dependence of the coupling.

We made other interesting observations along the way. Particularly, we noticed the connection between, on the one hand, the equation of the fermionic Green function in conical space in the presence of a mass term b_n/r and the Dirac equation describing an electron in a relativistic hydrogen atom on the other. The bound states of the relativistic hydrogen atom have been carefully studied, and it is a subject discussed in textbooks. A good review can be found for example in Ref. [27]. These bound state solutions diverge when substituting in the parameters relevant in our problem. However, we believe that scattering states should have a sensible interpretation. The connection is to be studied in more detail in future work.

As mentioned above, the computation of the correction to the fermionic propagator also allowed the choice of two orders of limits, much like the bosonic ones. In this case, however, the logarithmic divergence is not obviously associated to a local counterterm. Our computation has assumed that the final result should agree with that following from the other order of the limit. But in the case of the unpalatable scenario in which counterterms with differing subtraction schemes could alter the result, we studied some typical possibilities and found that unsurprisingly the coupling constant could in fact enter into the area term if such counterterm ambiguities do exist. The physics question we asked is unambiguous, and we do not believe such ambiguity could exist, as we demonstrated in the bosonic case. Nonetheless, this is an important question whether the replica trick does recover uniquely the entanglement entropy of a given wave function.

The subleading universal term also holds key information, along with $1/N$ corrections. We would like to leave these important questions for future investigations.

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APPENDIX A: COUNTERTERM CALCULATION OF BOSONIC PROPAGATOR

In this part, we show explicitly the calculation of the counterterm of the boson mass in Eq. (23).

1. $-G(r;r_1)c\frac{\delta(r_1)}{r_2}G(r_1;r)$

The first order correction in c could be written as

$$\begin{aligned}
&-c\int_0^\infty \delta(r_1)dr_1\int_0^{2\pi n}d\theta_1\int_{-\infty}^\infty dx_1\frac{1}{(4\pi n)^2}(r_0^2+r_1^2-2r_0r_1\cos(\theta_0-\theta_1)\\
&\quad+(x_0-x_1)^2)^{-1/2}(r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)(x_1-x_2)^2)^{-1/2}\\
&=-\frac{c}{16\pi n}\int_{-\infty}^\infty dx_1(r_0^2+(x_0-x_1)^2)^{-1/2}(r_2^2+(x_1-x_2)^2)^{-1/2}\\
&=-\frac{c}{16\pi n}\int_{-\infty}^\infty dx_1(r_0^2+(x_0-x_1)^2)^{-1}(\text{take}r_0=r_2,x_0=x_2)\\
&=-\frac{c}{16nr_0}.
\end{aligned}\tag{A1}$$

2. $G(r;r_1)c\frac{\delta(r_1)}{r_1}G(r_1;r_2)c\frac{\delta(r_2)}{r_2}G(r_2;r)$

The second order correction in c takes the form

$$\begin{aligned}
&\frac{c^2}{(4\pi n)^3}\int_0^\infty\delta(r_1)dr_1\int_0^{2\pi n}d\theta_1\int_{-\infty}^\infty dx_1\int_0^\infty\delta(r_2)dr_2\int_0^{2\pi n}d\theta_2\int_{-\infty}^\infty dx_2(r_0^2+r_1^2-2r_0r_1\cos(\theta_0-\theta_1)\\
&\quad+(x_0-x_1)^2)^{-1/2}(r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)+(x_1-x_2)^2)^{-1/2}(r_2^2+r_3^2-2r_2r_3\cos(\theta_2-\theta_3)+(x_2-x_3)^2)^{-1/2}\\
&=\frac{c^2}{(2\pi n)^3}\int_0^\infty\delta(r_1)dr_1\int_0^{2\pi n}d\theta_1\int_{-\infty}^\infty dx_1\int_0^\infty\delta(r_2)dr_2\int_0^{2\pi n}d\theta_2\int_{-\infty}^\infty dx_2\int_{-\infty}^\infty\frac{dk_1}{2\pi}\int_{-\infty}^\infty\frac{dk_s}{2\pi}\int_{-\infty}^\infty\frac{dk_3}{2\pi}\\
&\quad\times e^{ik_1(x_0-x_1)}e^{ik_2(x_1-x_2)}e^{ik_3(x_2-x_3)}K_0[k_1(r_0^2+r_1^2-2r_0r_1\cos(\theta_0-\theta_1))^{1/2}]\\
&\quad\times K_0[k_2(r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2))^{1/2}]K_0[k_3(r_2^2+r_3^2-2r_2r_3\cos(\theta_2-\theta_3))^{1/2}]\\
&=\frac{c^2}{(2\pi n)^3}\int_0^\infty\delta(r_1)dr_1\int_0^{2\pi n}d\theta_1\int_0^\infty\delta(r_2)dr_2\int_0^{2\pi n}d\theta_2\int_{-\infty}^\infty\frac{dk_1}{2\pi}e^{ik_1(x_1-x_3)}K_0[k_1(r_0^2+r_1^2-2r_0r_1\cos(\theta_0-\theta_1))^{1/2}]\\
&\quad\times K_0[k_1(r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2))^{1/2}]K_0[k_1(r_2^2+r_3^2-2r_2r_3\cos(\theta_2-\theta_3))^{1/2}]\\
&=\frac{c^2}{8\pi n}\int_{-\infty}^\infty\frac{dk_1}{2\pi}e^{ik_1(x_1-x_3)}K_0[k_1r_0]K_0[k_10]K_0[k_1r_3](\text{take }r_0=r_3,x_0=x_3)\\
&=\frac{c^2}{32nr_0}K_0[k_10]\\
&=\frac{c^2}{32nr_0}(-\log k\epsilon+\log 2-\gamma).
\end{aligned}\tag{A2}$$

Since $K_0[k0]$ is divergent, we hereby introduce ϵ as a point spliting cutoff. The last line is valid as long as $k\epsilon$ is small. The divergence arises as $r_1 \to 0,\ r_2 \to 0$.

APPENDIX B: FERMIONIC GREEN FUNCTION AND THE HYDROGEN ATOM

Besides the perturbative calculation for the fermionic Green function in the main text, we can actually use the method of mode expansion to express the Green function as a sum of eigenfunctions, and this method might have possible further use since it will allow us to calculate the Renyi entropy. And we will show that the eigenfunctions have close connection with the $3+1$-dimensional hydrogen atom.

The eigenfunction equation for eigenvalue E is

$$\left(\partial\!\!\!/ + \frac{b_n}{r}\right)\psi = E\psi. \tag{B1}$$

After changing the coordinate into a polar coordinate, we get

$$\left(\partial\!\!\!/ + \frac{b_n}{r}\right) = \begin{pmatrix} \partial_x + \frac{b_n}{r} & -e^{-\frac{i\theta}{n}}\left(n\frac{\partial_\theta}{r} + i\partial_r\right) \\ -e^{\frac{i\theta}{n}}\left(n\frac{\partial_\theta}{r} - i\partial_r\right) & -\partial_x + \frac{b_n}{r} \end{pmatrix}. \tag{B2}$$

Then, we expand using the eigenfunctions for θ and x, and we have

$$\psi = \int \frac{dk_x}{2\pi} e^{ik_x} \sum_l e^{i(l+\frac{1}{2})\frac{\theta}{n}} \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix} \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix}; \tag{B3}$$

then, we get the eigenfunctions for r:

$$\left(ik_x + \frac{b_n}{r} - E\right)\psi_1 + \left(-i\frac{(2l+1+n)}{2nr} - i\partial_r\right)\psi_2 = 0 \tag{B4}$$

$$\left(-i\frac{(2l+1-n)}{2nr} + i\partial_r\right)\psi_1 + \left(ik_x + \frac{b_n}{r} - E\right)\psi_2 = 0. \tag{B5}$$

Now, we introduce dimensionless functions $F(r) = \sqrt{r}\psi_1$ $G(r) = \sqrt{r}\psi_2$, and the equations become

$$\left(k_x - i\frac{b_n}{r} + iE\right)F - \partial_r G - \frac{2l+1}{2nr}G = 0 \tag{B6}$$

$$\left(-k_x - i\frac{b_n}{r} + iE\right)G + \partial_r F - \frac{2l+1}{2nr}F = 0. \tag{B7}$$

Compared with the equations of the $3+1$-dimensional hydrogen atom, the eigenfunctions are

$$\left(m_c - \frac{\alpha}{r} - \epsilon m_c\right)F - \partial_r G - kG = 0 \tag{B8}$$

$$\left(-m_c - \frac{\alpha}{r} - \epsilon m_c\right)G + \partial_r F - kF = 0 \tag{B9}$$

Using the well-known result for bound states of the hydrogen atom, we have the eigenvalues for the bound states,

$$E = \frac{ik_x}{\left(1 - {b_n}^2\left(n - |\frac{2l+1}{2nr}| + \sqrt{|\frac{2l+1}{2nr}|^2 + {b_n}^2}\right)^{-2}\right)^{1/2}}. \tag{B10}$$

However, this just means that there are no bound states for the current problem, since now $|E| > k_x$, which is inconsistent with the assumption used to solve the bound state problem. This example gives the reason why in these kinds of problems we never use bound states to construct the set of complete bases.

On the other hand, calculation and summation for the scattering problem of the hydrogen atom has not been fully considered in the literature, at least within the limited knowledge of the writers. So, we end our discussion here before we can go further for the problem of getting the whole spectrum and Renyi entropy. So we took a step back and settle with obtaining an $(n-1)$ expansion as described in the main text.

APPENDIX C: CORRECTION OF THE FERMIONIC GREEN FUNCTION

In this section, we may show that the logarithmic divergence in Eq. (35) can be subtracted in the form of a pole in $1/(n-1)$.

We start with

$$\begin{aligned}\chi_n(\mathrm{b}_n) &= \mathrm{tr}[\partial\!\!\!/^{-1} - \partial\!\!\!/^{-1}(\delta)\partial\!\!\!/^{-1}] = \mathrm{tr}\left[\partial\!\!\!/_\mathrm{r} G(r,r'')\left(\frac{b_n}{r''}\right)\partial\!\!\!/^{-1}(r'',r')\right] = \mathrm{tr}\left[G(r,r'')\overleftarrow{\partial\!\!\!/}_\mathrm{r}\left(\frac{b_n}{r''}\right)\partial\!\!\!/^{-1}(r'',r')\right] \\ &= \mathrm{tr}\left[-G(r,r'')\overleftarrow{\partial\!\!\!/}_{\mathrm{r}''}\left(\frac{b_n}{r''}\right)\partial\!\!\!/^{-1}(r'',r')\right] = \mathrm{tr}\left[G(r,r'')\left(\left(\frac{b_n}{r''}\right)\partial\!\!\!/^{-1}(r'',r')\right)\right] = \mathrm{tr}[G(\partial\!\!\!/\delta\partial\!\!\!/)\partial\!\!\!/^{-1} + G(\delta\partial\!\!\!/)\partial\!\!\!/\partial\!\!\!/^{-1}] \\ &= 2\frac{b_n}{r}(G_n(m_\psi = 0, \gamma = 1/2)) - b_n\int\frac{dxd\theta dr}{r^2}G_n^2(b_n, \gamma = 1/2) + b_n\int d\theta dx\left[\frac{G(m_\psi = 0, \gamma = 1/2)^2}{r}\right]\Bigg|_{r\to 0}^{r\to\infty},\end{aligned} \tag{C1}$$

where the last surface term can be shown to be zero.

Now, we focus on the second term above:

$$\begin{aligned}\int \frac{G^2}{r'^2} d^3x' &= \int_0^{2n\pi} d\theta' \int dx' \int_0^\infty \frac{dr'}{r'^2} \sum_{l=-\infty}^{l=\infty} \frac{e^{i\nu_l(\theta-\theta')}}{2\pi n} \sum_{l'=-\infty}^{l'=\infty} \frac{e^{i\nu_{l'}(\theta'-\theta'')}}{2\pi n} \\ &\quad\times \int \frac{dk_\perp}{2\pi} e^{ik_\perp(x-x')} \int \frac{dk'_\perp}{2\pi} e^{ik'_\perp(x'-x'')} \int_0^\infty \frac{J_{\nu_l}(kr)J_{\nu_l}(kr')}{k^2+k_\perp^2} kdk \int_0^\infty \frac{J_{\nu_l}(k'r'')J_{\nu_l}(k'r')}{k'^2+k'^2_\perp} k'dk' \\ &= \sum_{l=-\infty}^{l=\infty} \frac{1}{4\pi n} \int_0^\infty \frac{dr'}{r'^2} \int_0^\infty dk \int_0^\infty dk' \frac{J_{\nu_l}(kr)J_{\nu_l}(k'r'')J_{\nu_l}(kr')J_{\nu_l}(k'r')}{k+k'} \\ &= \sum_{l=-\infty}^{l=\infty} \frac{1}{4\pi^{3/2}n} \int_0^\infty dk \int_0^1 \frac{t^{\nu_l}}{1+t} \frac{\Gamma(-1/2+\nu_l)}{\Gamma(1+\nu_l)} {}_2F_1(-1/2,-1/2+\nu_l;1+\nu_l,t^2) J_{\nu_l}(kr)J_{\nu_l}(ktr) \\ &= \sum_{l=-\infty}^{l=\infty} \frac{1}{8\pi^2 n r^2} \frac{1}{\nu_l^2-1/4} = \frac{\tan(\frac{\pi}{2}n)}{4\pi r^2}. \end{aligned} \tag{C2}$$

When we investigate the limit $n \to 1$, we expand the expression to get

$$\frac{b_n \tan(\frac{\pi}{2}n)}{4\pi r^2} = -\frac{b_n}{2\pi^2 r^2} \frac{1}{(n-1)} + \frac{b_n(n-1)}{24 r^2} + O(n-1)^2. \tag{C3}$$

We find that the result is in correspondence to Eq. (35) in that the coefficient of the divergence as well as the O(1) term are exactly the same.

APPENDIX D: OTHER SOLUTIONS OF THE GAP EQUATION?

In the case of bosons, we were able to pin down a fixed point value of these couplings of counterterms, leading to an answer that appears to be robust against the choice of different normalization schemes. For fermions, this issue is not well understood, and so here we explore the consequence should any scheme dependence in the gap equation actually survive. Suppose we subtract the leading term $\frac{b_n}{2\pi^2 r^2}$ in the gap equation by hand without referring to any fixed point value of a counterterm. In that case, Eqs. (47) and (48) become a set of homogeneous equations, with both α_n and b_n proportional to $(n-1)$. So, generically, we write

$$\alpha_n = (n-1)\alpha(g_{\phi_2}, g_\psi), \quad b_n = (n-1)b(g_{\phi_2}, g_\psi) \tag{D1}$$

in which α and b are functions that depend only on the coupling constants.

Two sets of the solutions to Eqs. (D1) are

$$\begin{aligned} \alpha &= \frac{9g_{\phi_2}^2 + 2g_\psi + 2\sqrt{432g_{\phi_2}^2 + g_\psi(g_\psi+192)}}{768-36g_{\phi_2}^2}, \\ b &= \frac{g_2\left(g_\psi + \sqrt{432g_{\phi_2}^2 + g_\psi(g_\psi+192)} + 96\right)}{48(3g_{\phi_2}^2-64)} \\ \alpha &= \frac{9g_{\phi_2}^2 + 2g_\psi - 2\sqrt{432g_{\phi_2}^2 + g_\psi(g_2+192)}}{768-36g_{\phi_2}^2}, \\ b &= -\frac{3g_\psi}{g_\psi + \sqrt{432g_{\phi_2}^2 + g_\psi(g_\psi+192)} + 96}. \end{aligned} \tag{D2}$$

A noteworthy feature of the solution is that the solution would be divergent at given value $g_{\phi_2} = 8/\sqrt{3}$, regardless of other parameters. This indicates a possible phase transition at this specific value. However, there exists another set of solution which remains regular for all values of the coefficients. Whether such a scenario would ever arise will be explored in future work.

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