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Two-dimensional $f(\tilde{R})$ Hořava-Lifshitz gravity

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We study two-dimensional $f(\tilde{R})$ Hořava-Lifshitz gravity from the Hamiltonian point of view. We determine constraints structure with emphasis on the careful separation of the second class constraints and global first class constraints. We determine number of physical degrees of freedom and also discuss gauge fixing of the global first class constraints.

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I. INTRODUCTION AND SUMMARY

Study of two-dimensional quantum gravity is very useful when we can understand principles and puzzles of quantum gravity. Two-dimensional models are much simpler than four-dimensional gravity but share some interesting features with four-dimensional gravity. Further, two-dimensional gravity plays a fundamental role in the modern formulation of string theory [1] where a propagating string in d-dimensional flat target space-time can be described as a theory of d- free scalar fields coupled to two dimensional gravity.

It is well known that there is no nontrivial gravitational dynamics in space-time dimension lower than four. In three dimensions, the Riemann tensor is proportional to the Ricci tensor and the source-free theory is trivial. In two dimensions the Einstein tensor is zero and the Einstein-Hilbert action is topological invariant. As a result there are no equations of motion and hence we cannot formulate meaningful theory. In order to resolve this issue it was proposed in [2] that the appropriate model for two-dimensional gravity is the constant curvature equation ${}^{(2)}R - 2\Lambda = 0$, where ${}^{(2)}R$ denotes the two dimensional Ricci scalar. In order to study quantum properties of this theory we need an action principle from which this equation can be derived. It turned out that the only invariant action is the nongeometric action that involves scalar field Φ as a Lagrange multiplier

$$S = \int d^2 x \Phi({}^{(2)}R - 2\Lambda), \tag{1}$$

that leads to desired equations of motion when we perform variation with respect to Λ. The exact solution of this model was found in [3].

A few years ago P. Hořava formulated its famous model of power counting renormalizable theory of gravity known as Hořava-Lifshitz gravity (HL) [4] which is the theory of gravity that is not invariant under full four-dimensional diffeomorphism but under reduced group of diffeomorphism known as a foliation preserving diffeomorphism in order to have theory with anisotropic scale invariance. In fact, the requirement of the anisotropic scale invariance is central for the power counting renormalizability of this theory. On the other hand the reduced group of diffemorphism has a very strong impact on the structure of the theory since there are additional modes with important phenomenological and theoretical consequences on the consistency of the theory.

This theory has an improved behavior at high energies due to the presence of the higher order spatial derivatives in the action which implies that the theory is not invariant under full diffeomorphism but it is invariant under so called foliation preserving diffeomorphism ($\mathrm{Diff}_{\mathcal{F}}$)

$$t' = f(t), \qquad x'^i = x^i(\mathbf{x}, t). \tag{2}$$

This property offers the possibility that the space and time coordinates have different scaling at high energies

$$t' = k^{-z} t, \qquad x'^i = k^{-1} x^i, \tag{3}$$

where k is a constant. A consequence of this fact is that in $3+1$ dimensions the theory contains terms with 2 time derivatives and at least $2z$ spatial derivatives since the minimal amount of the scaling anisotropy that is needed for the power-counting renormalizability of this theory is $z = 3$. Then collecting all terms that are invariant under $\mathrm{Diff}_{\mathcal{F}}$ symmetry leads to the general action [5,6]

$$S = \frac{M_p^2}{2} \int dt d^3\mathbf{x} N \sqrt{g} K_{ij} \mathcal{G}^{ijkl} K_{kl} - S_V, \tag{4}$$

where

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - D_i N_j - D_j N_i), \tag{5}$$

and where we introduced generalized De Witt metric $\mathcal{G}^{ijkl}$ defined as [7]

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$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}, \tag{6}$$

where λ is an arbitrary real constant. Finally note that D_i is the covariant derivative defined with the help of the metric g_{ij}.

The action S_V is the potential term action in the form

$$S_V = \frac{M_p^2}{2}\int dtd^3\mathbf{x}N\sqrt{g}\mathcal{V} = \frac{M_p^2}{2}\int dtd^3\mathbf{x}N\sqrt{g}\left(\mathcal{L}_1 + \frac{1}{M_*^2}\mathcal{L}_2 + \frac{1}{M_*^4}\mathcal{L}_3\right), \tag{7}$$

where $\mathcal{L}_n$ contain all terms that are invariant under foliation preserving diffeomorphism and where $\mathcal{L}_n$ contain $2n$ derivatives of the ADM variables (N, g_{ij}). In the UV when $k \gg M_*$ the dominant contributions come from the higher derivative terms that lead to the modified dispersion relation $\omega^2 \propto k^6$ that implies that this theory is power counting renormalizable. In the opposite regime $k \ll M_*$ the dispersion relation is relativistic and it can be shown that the theory has regions in the parameter space where it is in agreement with observation.

This theory has a very interesting property which is the presence of the vector a_i that contains spatial derivative of lapse N. These terms are forbidden in the theory invariant under full diffeomorphism which implies an existence of the local first class Hamiltonian constraint. In case of HL gravity the canonical structure is much more complicated as was shown previously in [8–11]. More precisely, two second class constraints were identified which should be solved for lapse N and conjugate momentum. However generally this constraint is a second order partial differential equation for lapse whose explicit solution was very difficult to find. For that reason it is instructive to perform an analysis of much simpler models as is for example two dimensional HL gravity. This was done previously in [12]. Our goal is to generalize this analysis to the case of two dimensional $f(\tilde{R}) -$ HL gravity which is more complex and allows local degrees of freedom on the reduced phase space. We also discuss the subtle point of the global first class constraints [11]. We argue that in order to solve the second class constraints we have to fix these global constraints. This is very important observation for the structure of the reduced phase space when we determine equations of motion for variables that define reduced phase space and we show that it takes a rather complicated form. As a result we are not able to derive Hamiltonian on the reduced phase space that is apparently nonlocal due to the necessity to fix global first class constraints with global gauge fixing functions.

As the check of the validity of our procedure we discuss two special cases of the choice of the parameters in this theory. The first one corresponds to the diffeomorphism invariant two dimensional $f(R)$ theory. We determine the canonical structure of this theory and we argue that it has the same form as in seminal papers [2,3]. Then we proceed to the analysis of the reduced phase space theory when we fix all first class constraints. We show that there are no physical degrees of freedom on the reduced phase space and we show that with suitable chosen gauge fixing function we derive equations for lapse and for scalar field that are in agreement with the equations derived in [13] which is also a nice consistency check of our analysis. Finally we consider the case when the function that defines $f(\tilde{R})$ theory is identically equal to one. This situation corresponds to the nonprojectable HL gravity in two dimensions that was analyzed previously in [12]. We perform the canonical analysis of this theory from a different point of view with emphasis on the existence of two global first class constraints and their gauge fixing. Solving all constraints we show that there are no physical degrees of freedom left and that these constraints lead to the solution that is in agreement with the analysis performed in [12].

Let us outline our results. We performed canonical analysis of two dimensional $f(\tilde{R})$ HL gravity and we show that the equations on the reduced phase space are rather complicated and contain integration over the whole space interval as a consequence of the gauge fixing of the global constraints. We mean that this is a very important result that should be valid in higher dimensional nonprojectable theory as well and which certainly makes the canonical analysis even more complicated than it is.

This paper is organized as follows. In the next section (II) we introduce two dimensional $f(\tilde{R})$ HL gravity and define basics notations. Then in Sec. III we perform Hamiltonian analysis of this theory and determine all constraints. In Sec. IV we consider special values of parameters that correspond to $f(R)$–gravity in two dimensions and we perform its Hamiltonian analysis. Finally in Sec. V we analyze pure nonprojectable HL gravity in two dimensions from a Hamiltonian point of view.

II. TWO-DIMENSIONAL $f(\tilde{R})$-HORAVA-LIFSHITZ GRAVITY

In this section we formulate two-dimensional HL $f(\tilde{R})$ gravity. Clearly the action for this system is the special case of higher dimensional $f(\tilde{R})$ HL gravities that were studied before, see for example [14–17]. Let us consider the following model of two-dimensional nonprojectable HL $f(\tilde{R})$ gravity

$$S = \frac{1}{\kappa}\int dtdxN\sqrt{g}f(\tilde{R}), \tag{8}$$

where $\kappa = 8\pi G_N$ and where $\tilde{R}$ is defined as

$$\tilde{R} = \mathcal{L}_K - \mathcal{L}_V, \tag{9}$$

where

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2 + \frac{2\mu}{\sqrt{g}N}\partial_\mu(\sqrt{g}Nn^\mu K) - \frac{2\mu}{\sqrt{g}N}\partial_i(\sqrt{g}g^{ij}\partial_j N), \tag{10}$$

with $K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - D_i N_j - D_j N_i)$ where D_i denotes the covariant derivative of the metric g_{ij} and N^i is the shift vector $N^i = g^{ij}N_j$. Finally n^μ is a future pointing normal vector to the surface Σ_t that in ADM variables is equal to $n^0 = \frac{1}{N}$, $n^i = -\frac{N^i}{N}$. Finally μ is a free parameter that approaches 1 in the low energy limit.

Let us now discuss the potential term $\mathcal{L}_V$ that is made of R, D_i, and $a_i = \frac{\partial_i N}{N}$ where R is Ricci scalar of the leaves $t = \text{const}$ that identically vanishes at one dimension $R = 0$. It can be shown [12] that in $d = 1$ dimensions $\mathcal{L}_V$ has the form

$$\mathcal{L}_V = 2\Lambda - \beta a_i a^i, \tag{11}$$

where Λ is cosmological constant and β is another dimensionless coupling constant.

To deal with $f(\tilde{R})$ gravity in two dimensions we introduce two scalar fields and write the action as

$$\begin{aligned} S &= \frac{1}{\kappa}\int dtdxN\sqrt{g}(f(A) + B(\tilde{R} - A)) \\ &= \frac{1}{\kappa}\int dtdxN\sqrt{g}(f(A) - BA + B(K_{ij}K^{ij} - \lambda K^2) \\ &\quad - 2\mu\partial_\mu Bn^\mu K + 2\mu\partial_i Bg^{ij}a_j - 2\Lambda B + \beta Ba_i a^i). \end{aligned} \tag{12}$$

In $1 + 1$ dimensions g_{ij} has only one components that we denote, following [12] as

$$\gamma \equiv \sqrt{g_{11}}, \qquad g_{11} = \gamma^2, \qquad g^{11} = \frac{1}{\gamma^2} \tag{13}$$

so that we have the following nonzero component of Γ^1_{11}

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{\gamma}\gamma', \qquad \gamma' \equiv \frac{\partial}{\partial x}\gamma. \tag{14}$$

Then we easily find that the action has the form

$$S = \frac{1}{\kappa}\int dtdxN\gamma\left(f(A) - BA + B(1-\lambda)K^2 - 2\mu\nabla_n BK + 2\mu\frac{1}{\gamma^2}B'a - 2\Lambda B + \beta Ba^2\frac{1}{\gamma^2}\right), \tag{15}$$

where

$$K = g^{11}K_{11} = \frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma} - \frac{N_1'}{\gamma^2} + \frac{\gamma'}{\gamma^3}N_1\right),$$
$$\nabla_n B = \frac{1}{N}(\dot{B} - N^1 B'), \qquad a \equiv a_1, \tag{16}$$

where $\dot{B} = \partial_t B$, $B' = \partial_1 B$. The action (15) will be the starting point of our canonical analysis that will be performed in the next section.

III. HAMILTONIAN ANALYSIS

Now we proceed to the Hamiltonian analysis of the theory specified by the action (15). Before we do, it is useful to simplify this action with the help of the fact that the variable A has no dynamics and can be eliminated by solving its equation of motion. In more details, the equation of motion for A has the form

$$\frac{df}{dA} - B = 0. \tag{17}$$

If we presume that there is a function Ψ that is inverse to $\frac{df}{dA}$ we find that the Eq. (17) has the solution

$$A = \Psi(B). \tag{18}$$

Inserting this solution into the action (15) we obtain the final form of the action

$$S = \frac{1}{\kappa}\int dtdxN\gamma\left(B(1-\lambda)K^2 - 2\mu\nabla_n BK + 2\mu\frac{1}{\gamma^2}B'a - U(B) + \beta Ba^2\frac{1}{\gamma^2}\right), \tag{19}$$

where

$$U(B) = f(\Psi(B)) - B\Psi(B). \tag{20}$$

Starting with the action (19) we find following conjugate momenta

$$\begin{aligned} \pi_N &= \frac{\delta L}{\delta\dot{N}} \approx 0, \qquad \pi^1 = \frac{\delta L}{\delta\dot{N}_1} \approx 0, \\ \pi &= \frac{\delta L}{\delta\dot{\gamma}} = \frac{2B}{\kappa}(1-\lambda)K - \frac{2\mu}{\kappa}\nabla_n B, \\ P &= \frac{\delta L}{\delta\dot{B}} = -\frac{2\mu}{\kappa}\gamma K. \end{aligned} \tag{21}$$

Then it is easy to perform Legendre transformation in order to find a corresponding Hamiltonian

$$H = \int dx(\pi\dot{\gamma} + P\dot{B} - \mathcal{L}) = \int dx\left(N\mathcal{H}_T + N_1\frac{1}{\gamma^2}\mathcal{H}_1\right), \tag{22}$$

where

$$\begin{aligned}\mathcal{H}_T &= -\frac{\kappa}{4\mu^2\gamma}B(1-\lambda)P^2 - \frac{\kappa}{2\mu}P\pi - \frac{2\mu}{\kappa\gamma}B'a \\ &\quad + \frac{\gamma}{\kappa}U(B) - \frac{\beta}{\kappa}\frac{Ba^2}{\gamma}, \\ \mathcal{H}_1 &= -\gamma\pi' + PB'\end{aligned} \tag{23}$$

using

$$K = -\frac{\kappa}{2\mu\gamma}P, \qquad -\frac{2\mu}{\kappa}\nabla_n B = \pi + \frac{2B}{2\mu\gamma}(1-\lambda)P. \tag{24}$$

Now we have to analyze the requirement of the preservation of the primary constraints $\pi_N \approx 0$, $\pi^1 \approx 0$

$$\begin{aligned}\partial_t\pi_N &= \{\pi_N, H\} \\ &= -\mathcal{H}_T - \frac{2\mu}{\kappa\gamma}\frac{N'}{N}B' - \left(\frac{2\mu}{\kappa}\frac{B'}{\gamma}\right)' - \frac{2\beta}{\kappa}\frac{B}{\gamma}a^2 - \left(\frac{2\beta}{\kappa}\frac{B}{\gamma}a\right)' \\ &\equiv -\mathcal{C} \approx 0, \\ \partial_t\pi^i &= \{\pi^1, H\} = -\mathcal{H}_1 \approx 0.\end{aligned} \tag{25}$$

Note that $\mathcal{C}$ obeys an important relation

$$\int dxN\mathcal{C} = \int dxN\mathcal{H}_T \tag{26}$$

using integration by parts and also the fact that we presume suitable asymptotic behavior of all fields so that the contributions from spatial infinities can be ignored. As in higher dimensional nonprojectable HL gravity we introduce the global primary constraint

$$\Pi_N = \int dx\pi_N N \tag{27}$$

and split the original constraint π_N into $\infty - 1$ local ones

$$\tilde{\pi}_N = \pi_N - \frac{\gamma}{\int dx\gamma N}\Pi_N \tag{28}$$

that obeys the relation

$$\int dxN\tilde{\pi}_N = 0. \tag{29}$$

Then the requirement of the preservation of the primary constraint Π_N implies

$$\partial_t\Pi_N = \{\Pi_N, H\} = -\int dxN\mathcal{H}_T \equiv -\Pi_T \approx 0 \tag{30}$$

using

$$\{\Pi_N, N\} = -N, \qquad \{\Pi_N, \pi_N\} = \pi_N, \qquad \{\Pi_N, a\} = 0 \tag{31}$$

and hence $\{\Pi_N, \mathcal{H}_T\} = 0$. In other words we have second global constraint $\Pi_T \approx 0$. We again split $\mathcal{C}$ into $\infty - 1$ local constraints $\tilde{\mathcal{C}}$ and one global constraint $\Pi_T \approx 0$ where we define $\tilde{\mathcal{C}} \approx 0$ as

$$\tilde{\mathcal{C}} = \mathcal{C} - \frac{\gamma}{\int dx\gamma N}\Pi_T \tag{32}$$

that obeys $\int dxN\tilde{\mathcal{C}} = 0$. To proceed further we introduce united notation for the second class constraints as $\Psi_A = (\tilde{\pi}_N, \tilde{\mathcal{C}})$. Since clearly $\{\tilde{\mathcal{C}}(x), \tilde{\mathcal{C}}(y)\} \neq 0$ we find that the matrix of Poisson brackets has the schematic form

$$\{\Psi_A(x), \Psi_B(y)\} = \triangle_{AB} \equiv \begin{pmatrix} 0 & X \\ Y & M \end{pmatrix} \tag{33}$$

so that the inverse matrix $\triangle^{AB}$ has the form

$$\triangle^{AB} = \begin{pmatrix} -Y^{-1}MX^{-1} & Y^{-1} \\ X^{-1} & 0 \end{pmatrix}. \tag{34}$$

As the final step we have to ensure that Π_T and Π_N are the first class constraints. Π_N clearly is since it has vanishing Poisson brackets with all constraints on the constraints surface. In case of Π_T this is not true but we can introduce the following combination of the constraints

$$\tilde{\Pi}_T = \Pi_T - \{\Pi_T, \Psi_A\}\triangle^{AB}\Psi_B \tag{35}$$

that obeys the equation

$$\begin{aligned}&\{\tilde{\Pi}_T, \Psi_A\} = \{\Pi_T, \Psi_A\} - \{\Pi_T, \Psi_C\}\triangle^{CB}\{\Psi_B, \Psi_A\} = 0, \\ &\{\tilde{\Pi}_T, \tilde{\Pi}_T\} = 0.\end{aligned} \tag{36}$$

For further purposes it is useful to determine the explicit form of $\tilde{\Pi}_T$. First of all we calculate the Poisson bracket between Π_T and $\tilde{\pi}_N$

$$\begin{aligned}\{\Pi_T, \tilde{\pi}_N(x)\} &= \{\Pi_T, \pi_N(x)\} \\ &= -\left\{\int dyN\tilde{\mathcal{H}}_T, \pi_N(x)\right\} = \mathcal{C}(x) \approx 0.\end{aligned} \tag{37}$$

In case of the constraint $\mathcal{C}$ we only need to know that this Poisson bracket is nonzero. Schematically we have $\{\Pi_T, \vec{\Psi}\} = (0, *)$, where $*$ is the nonzero expression. Then we obtain

$$\tilde{\Pi}_T = \Pi_T - (0,*)\begin{pmatrix} -Y^{-1}MX^{-1} & Y^{-1} \\ X^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ * \end{pmatrix} = \Pi_T - *X^{-1}\tilde{\pi}_N \quad (38)$$

which is a very important result that shows that $\tilde{\Pi}_T$ does not depend on the constraint $\tilde{\mathcal{C}} \approx 0$.

Now we proceed to the analysis of the constraint $\mathcal{H}_1$. We add to it the following expression proportional to the primary constraint $\pi_N \approx 0$

$$\tilde{\mathcal{H}}_1 = \mathcal{H}_1 + \pi_N\partial_1 N = -\gamma\partial_1\pi + P\partial_1 B + \pi_N\partial_1 N \quad (39)$$

and introduce its smeared form

$$\mathbf{H}_S(M^1) = \int dx M^1\tilde{\mathcal{H}}_1 \quad (40)$$

that has following Poisson brackets with canonical variables

$$\begin{aligned} \{\mathbf{H}_S(M^1),\gamma\} &= -(M^1\gamma)', \\ \{\mathbf{H}_S(M^1),\pi\} &= -M^1\pi', \\ \{\mathbf{H}_S(M^1),B\} &= -M^1B', \\ \{\mathbf{H}_S(M^1),P\} &= -(M^1P)', \\ \{\mathbf{H}_S(M^1),N\} &= -M^1N', \\ \{\mathbf{H}_S(M^1),\pi_N\} &= -(M^1\pi_N)' \end{aligned} \quad (41)$$

and also

$$\{\mathbf{H}_S(M^1),a\} = -(M^1)'a - M^1a'. \quad (42)$$

From these Poisson brackets we see that all constraints have vanishing Poisson brackets with $\mathbf{H}_S$ on constraint surface and hence $\tilde{\mathcal{H}}_1 \approx 0$ is the local first class constraint.

Let us now return to the second class constraints $\mathbf{\Psi}_A$ and try to find their solutions. The problem is that these second class constraints contain the global first class constraints in their definition. For that reason it is natural to fix the global first class constraints by appropriate global gauge fixing functions. Note that Π_N generates pure time dependent rescaling of N and π_N. For that reason it is natural to introduce the following gauge fixing function

$$\mathbf{G}_N = \int dx\gamma N - C \approx 0, \quad (43)$$

where C is a constant.[1] Now this gauge fixing function has a nonzero Poisson bracket

$$\{\Pi_N,\mathbf{G}_N\} = -\int dx\gamma N \approx -C \neq 0. \quad (44)$$

However this is not the end of the story due to the presence of the second global constraint $\tilde{\Pi}_T$. We have to fix this first class constraint in order to be able to solve $\mathcal{C} \approx 0$ for N. Let us propose following gauge fixing function

$$\mathbf{G}_T = \int dx\gamma\pi - C_\pi(t) \approx 0, \quad (45)$$

where we have to presume nontrivial dependence of C_π on time in order to find nontrivial dynamics. It is also easy to see that

$$\{\mathbf{H}_S(N^1),\mathbf{G}_T\} = 0 \quad (46)$$

and also

$$\begin{aligned} \{\mathbf{G}_T,\tilde{\Pi}_T\} &\approx \{\mathbf{G}_T,\Pi_T\} \\ &= \Pi_T - \int dxN\left(\frac{\kappa}{2\mu\gamma}\pi P - \frac{2\gamma}{\kappa}U(B)\right) \\ &\approx -\int dxN\left(\frac{\kappa}{2\mu\gamma}\pi P - \frac{2\gamma}{\kappa}U(B)\right). \end{aligned} \quad (47)$$

Finally we fix the diffeomorphism constraint. There is a number of possibilities how to fix it. For example, we could use the gauge fixing condition $\gamma = 1$. However this condition does not fix the gauge completely and there remains global diffeomorphism. For that reason we consider another possibility when we impose the gauge fixing function

$$\mathcal{G}_C = B - f(x), \quad (48)$$

where $f(x)$ is a prescribed function that obeys the regularity condition at infinity. Then we have

$$\{\tilde{\mathcal{H}}_1(x),\mathcal{G}_C(y)\} = B'(x)\delta(x-y) \approx f'(x)\delta(x-y). \quad (49)$$

Now we are ready to analyze the time evolution of all constraints and gauge fixing functions in order to show that all Lagrange multipliers are fixed. Recall that the total Hamiltonian with gauge fixing functions included has the form

$$\begin{aligned} H_T = (1+\lambda_T)\tilde{\Pi}_T + \lambda_N\Pi_N + V_T\mathbf{G}_T + V_N\mathbf{G}_N \\ + \int dx(\omega^A\mathbf{\Psi}_A + N^1\tilde{\mathcal{H}}_1 + M_1\mathcal{G}_C), \end{aligned} \quad (50)$$

where we extended the original Hamiltonian Π_T in order to coincide with $\tilde{\Pi}_T$ by appropriate linear combinations of constraints.

[1] In principle this could be time dependent function but we consider it to be constant for simplicity.

First of all we start with the constraint $\tilde{\mathcal{H}}_1 \approx 0$. Since the Hamiltonian was diffemorphism invariant we find

$$\partial_t \tilde{\mathcal{H}}_1(x) = \{\tilde{\mathcal{H}}_1(x), H_T\} \approx \int dy M_1(y)\{\tilde{\mathcal{H}}_1(x), \mathcal{G}_C(y)\} = M_1 f'(x). \quad (51)$$

Since by presumption $f'(x) \neq 0$ for all x we see that the only possibility how to obey this equation is to demand that $M_1 = 0$. Then the time evolution of the constraint $\tilde{\Pi}_T$ implies

$$\partial_t \tilde{\Pi}_T = \{\tilde{\Pi}_T, H_T\} = V_T\{\tilde{\Pi}_T, \mathbf{G}_T\} = 0 \quad (52)$$

which implies that $V_T = 0$. In the same way time evolution of Π_N implies

$$\partial_t \Pi_N = \{\Pi_N, H_T\} = V_N\{\Pi_N, \mathbf{G}_N\} = 0 \quad (53)$$

and we find $V_N = 0$. However these results also imply that the time evolution of the constraints Ψ_A simplify considerably since

$$\partial_t \Psi_A(x) = \{\Psi_A(x), H_T\} = \int dy \omega^B(y)\{\Psi_A(x), \Psi_B(y)\} = 0 \quad (54)$$

due to the fact that $V_T = V_N = M_1(x) = 0$. Since the matrix of Poisson brackets of the second class constraints is nonsingular we find that the equation above has the solution $\omega^B = 0$.

Finally we proceed to the requirement of the preservation of the constraints $\mathbf{G}_N$, $\mathbf{G}_T$, and $\mathcal{G}_C$. In the case of $\mathbf{G}_T$ we obtain

$$\frac{d}{dt}\mathbf{G}_T = \frac{\partial \mathbf{G}_T}{\partial t} + \{\mathbf{G}_T, H_T\} = \partial_t \mathbf{G}_T + (1+\lambda_T)\{\mathbf{G}_T, \tilde{\Pi}_T\} = 0 \quad (55)$$

using the fact that $\{\mathbf{G}_T, \Pi_N\} = 0$. Then we obtain

$$\lambda_T = -1 - \frac{\dot{C}_\pi}{\{\mathbf{G}_T, \tilde{\Pi}_T\}}. \quad (56)$$

In the case of $\mathbf{G}_N$ we find

$$\frac{d\mathbf{G}_N}{dt} = \{\mathbf{G}_N, H_T\} = \lambda_N\{\mathbf{G}_N, \Pi_N\} + (1+\lambda_T)\{\mathbf{G}_N, \tilde{\Pi}_T\} = 0 \quad (57)$$

that can be solved for λ_N. Finally the time evolution of the constraint $\mathcal{G}_C$ has the form

$$\partial_t \mathcal{G}_C(x) = \{\mathcal{G}_C(x), H_T\} = (1+\lambda_T)\{\mathcal{G}_C(x), \tilde{\Pi}_T\} + \int dy N^1(y)\{\mathcal{G}_C(x), \tilde{\mathcal{H}}_1(y)\} = -(1+\lambda_T)N\left(\frac{\kappa}{2\mu^2\gamma}B(1-\lambda)P + \frac{\kappa}{2\mu\gamma}\pi\right) + N^1(x)f'(x) = 0, \quad (58)$$

where we used the fact that $\tilde{\Pi}_T$ does not depend on $\tilde{\mathcal{C}}$. The previous equation can be solved for N^1 as

$$N^1 = \frac{\kappa(1+\lambda_T)}{2\mu^2\gamma f'}N(B(1-\lambda)P + \pi). \quad (59)$$

We see that we completely fixed all Lagrange multipliers.

Now we proceed to the analysis of the dynamics of the variables B, P, π, γ and π_N and N. In case of π_N we find that it is zero thanks to the constraint $\pi_N = 0$. B is determined by the constraint $\mathcal{G}_C = 0$ that implies

$$B = f(x). \quad (60)$$

Further, the conjugate momentum P can be expressed using the constraint $\mathcal{H}_1$ and we find

$$P = \frac{\gamma \pi'}{f'(x)}. \quad (61)$$

Finally we have to find N as a function of dynamical variables π, γ. To do this we use the fact that the constraint $\mathcal{C}$ has the form

$$\mathcal{C} = \frac{\kappa}{4\mu^2}B(1-\lambda)\left(\frac{\pi'}{B'}\right)^2 + \frac{\kappa}{2\mu}\frac{\pi'}{B'} - \frac{1}{\kappa}U(B) - \frac{\beta}{\kappa}\frac{BN'^2}{N^2} - \left(\frac{2\mu}{\kappa}B'\right)' - \left(\frac{2\beta}{\kappa}B\frac{N'}{N}\right)' = 0. \quad (62)$$

Introducing variable $y = \frac{N'}{N}$ we can rewrite the equation above to the form of the Riccati equation

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2, \quad (63)$$

where

$$q_0(x) = \frac{\kappa^2}{8\beta\mu^2}(1-\lambda)\left(\frac{\pi'}{B'}\right)^2 + \frac{\kappa^2}{4\beta\mu}\frac{\pi'}{BB'} - \frac{1}{2\beta B}U(B) - \frac{\mu}{\beta}\left(\frac{B'}{B}\right)',$$
$$q_1(x) = -\frac{B'}{B}, \qquad q_2 = -\frac{1}{2}. \quad (64)$$

This equation can be explicitly solved as $N = N(\pi, \gamma)$ however the explicit form of this solution is not important for us. We see that the remaining dynamical variables are π, γ whose equations of motion have the form

$$\begin{aligned}\partial_t\pi(x) &= \{\pi(x), H_T\} \\ &= -(1+\lambda_T)\frac{N}{\gamma}\left(\frac{\kappa}{4\mu^2\gamma}B(1-\lambda)P^2 + \frac{\kappa}{2\mu\gamma}\pi P \right.\\ &\quad \left. + \gamma U(B) + \frac{2\mu}{\kappa}\frac{B'}{\gamma}a + \frac{\beta}{\kappa}\frac{Ba^2}{\gamma}\right) \\ &\quad + \frac{\kappa\lambda_T}{2\mu^2\gamma f'}N(B(1-\lambda)P+\pi)\pi', \\ \partial_t\gamma(x) &= \{\gamma(x), H_T\} = -N\frac{\kappa}{2\mu}P + \partial_1(N^1\gamma) \end{aligned} \quad (65)$$

using again the fact that $\tilde{\Pi}_T$ does not depend on $\tilde{\mathcal{C}}$. It is important to stress that N, N^1, λ_T all depend on γ and π as follows from (59) and (64). Further, $1+\lambda_T$ is determined in (56) and we see that it is given as an integral over spatial section. In summary, the equation of motion for γ, π are very complicated and it is not possible to determine Hamiltonian on the reduced phase space. In other words, even $1+1f(\tilde{R})-\mathrm{HL}$ gravity has rather complicated structure so that it is hard to see whether it can be explicitly solved.

IV. THE CASE $\lambda = 1$, $\beta = 0$

It is instructive to perform Hamiltonian analysis of the $f(\tilde{R})-\mathrm{HL}$ gravity with special values of parameters. In this section we consider the case when $\lambda = 1$, $\beta = 0$ when the action has the form

$$S = \frac{1}{\kappa}\int dtdxN\gamma\left(-2\mu\nabla_n BK + 2\frac{\mu}{\gamma^2}B'a - U(B)\right). \quad (66)$$

From (66) we obtain conjugate momenta:

$$\pi = -\frac{2\mu}{\kappa}\nabla_n B, \qquad \pi_N \approx 0, \qquad \pi^1 \approx 0, \qquad P = -\frac{2\mu}{\kappa}N\gamma K \quad (67)$$

and hence the Hamiltonian has the form

$$H = \int dx\left(N\mathcal{H}_T + N_1\frac{1}{\gamma^2}\mathcal{H}_1\right), \quad (68)$$

where

$$\mathcal{H}_T = -\frac{\kappa}{2\mu}\pi P + \left(\frac{2\mu}{\kappa\gamma}B'\right)' + \frac{\gamma}{\kappa}U(B), \qquad \mathcal{H}_1 = -\gamma\pi' + PB', \quad (69)$$

where we used integration by parts in order to have a theory linear in N. As usually the preservation of the primary constraints $\pi_N \approx 0$, $\pi^1 \approx 0$ implies the secondary constraints $\mathcal{H}_T \approx 0$, $\mathcal{H}_1 \approx 0$. Now we have to analyze their preservation again. In order to do this we have to calculate corresponding Poisson brackets of the smeared form of these constraints $\mathbf{H}_T(X) = \int dxX\mathcal{H}_T$

$$\{\mathbf{H}_T(X), \mathbf{H}_T(Y)\} = \int dx(XY' - X'Y)\frac{1}{\gamma^2}(PB' - \gamma\pi') = \mathbf{H}_S\left((XY' - YX')\frac{1}{\gamma^2}\right) \quad (70)$$

and also

$$\{\mathbf{H}_S(X^1), \mathbf{H}_T(Y)\} = \mathbf{H}_T(-X^1Y'). \quad (71)$$

We see that there is a crucial difference with the analysis performed in previous sections since now there is local first class constraint $\mathcal{H}_T \approx 0$ together with spatial diffeomorphism constraint $\mathcal{H}_1 \approx 0$ and the first class constraints $\pi_N \approx 0$, $\pi^1 \approx 0$.

Let us now proceed to the gauge fixing of all constraints. At this place however we should be very careful with the variables N and N_1. To see this in more detail remember that we are free to add secondary constraints $\mathcal{H}_T$, $\mathcal{H}_1$ with arbitrary Lagrange multipliers to the total Hamiltonian H_T. Let us also presume that we couple the gravity with matter in the form of free scalar field

$$S_{\mathrm{mat}} = \frac{1}{2}\int dtdxN\gamma\left(\nabla_n\phi\nabla_n\phi - \frac{1}{\gamma^2}\phi'^2\right) \quad (72)$$

with corresponding matter contribution to the Hamiltonian in the form

$$H_{\mathrm{matter}} = \int dx\left[N\left(\frac{1}{2\gamma}P_\phi^2 + \frac{1}{2\gamma}(\phi')^2\right) + N_1\frac{1}{\gamma^2}P_\phi\phi'\right]. \quad (73)$$

Now when we include the secondary constraints to the total Hamiltonian we find that it has the form

$$H_{T,\mathrm{matter}} = \int dx\left[(N+\lambda_T)\left(\frac{1}{2\gamma}P_\phi^2 + \frac{1}{2\gamma}(\phi')^2\right) + (N_1+\lambda_1)\frac{1}{\gamma^2}P_\phi\phi'\right]. \quad (74)$$

In order to return to the Lagrange formalism we have to calculate the equation of motion for ϕ

$$\dot{\phi} = \{\phi, H\} = (N+\lambda_T)P_\phi + (N_1+\lambda_1)\frac{1}{\gamma^2}\phi' \quad (75)$$

that allows us to express P_ϕ as

$$P_\phi = \frac{1}{N+\lambda_T}\left(\dot{\phi} - (N_1+\lambda_1)\frac{1}{\gamma^2}\phi'\right). \qquad (76)$$

From this expression we immediately see that the components of the metric as it is seen by scalar field are $N+\lambda_T$, $N_1+\lambda_1$ instead of the original ones. For that reason it is convenient to consider N, N_1 as Lagrange multipliers and hence it does not make sense to speak about their conjugate momenta and fix them. Rather we should fix N, N_1 by the requirement of the preservation of the gauge fixing functions during the time evolution.[2] In other words the total Hamiltonian with gauge fixing constraints included has the form

$$H_T = \int dx (N\mathcal{H}_T + N^1\mathcal{H}_1 + \lambda^{\mathcal{H}_T}\mathcal{G}_{\mathcal{H}_T} + \lambda^{\mathcal{H}_1}\mathcal{G}_{\mathcal{H}_1}). \qquad (77)$$

Of course, there is a freedom in the choice of the gauge fixing functions $\mathcal{G}_{\mathcal{H}_T}$, $\mathcal{G}_{\mathcal{H}_1}$ when we only demand that they have nonzero Poisson brackets with $\mathcal{H}_T$, $\mathcal{H}_1$. On the other hand when we impose the condition that the solutions of the constraints correspond to the static solution we choose the following form of these constraints

$$\mathcal{G}_{\mathcal{H}_1} = \gamma^2 - N \approx 0, \qquad \mathcal{G}_{\mathcal{H}_T} = P \approx 0, \qquad (78)$$

where now we have the following nonzero Poisson brackets

$$\begin{aligned}
\{\mathcal{G}_{\mathcal{H}_1}(x), \mathcal{H}_T(y)\} &= -\frac{\kappa}{\mu}\gamma P \approx 0,\\
\{\mathcal{G}_{\mathcal{H}_1}(x), \mathcal{H}_1(y)\} &= -\gamma(x)\gamma(y)\frac{\partial}{\partial y}\delta(x-y),\\
\{\mathcal{G}_{\mathcal{H}_T}(x), \mathcal{H}_T(y)\} &= -\frac{2\mu}{\kappa}\frac{\partial}{\partial y}\left(\frac{\partial_y\delta(x-y)}{\gamma}\right)\\
&\quad -\frac{\gamma}{\kappa}\frac{\delta U(B)}{\delta B}\delta(x-y),\\
\{\mathcal{G}_{\mathcal{H}_T}(x), \mathcal{H}_1(y)\} &= -P\partial_y\delta(x-y) \approx 0. \qquad (79)
\end{aligned}$$

Then the requirement of the preservation of the constraint $\mathcal{H}_T \approx 0$ implies

$$\begin{aligned}
\partial_t\mathcal{H}_T(x) = \{\mathcal{H}_T(x), H_T\} &\approx \int dy \lambda^{\mathcal{H}_T}(y)\{\mathcal{H}_T(x), \mathcal{G}_{\mathcal{H}_T}(y)\}\\
&= \frac{\gamma}{\kappa}\lambda^{\mathcal{H}_T}\frac{\delta U}{\delta B} + \frac{2\mu}{\kappa}\frac{\partial}{\partial x}\left(\frac{\partial_x\lambda^{\mathcal{H}_T}}{\gamma}\right) \qquad (80)
\end{aligned}$$

that has clearly the solution $\lambda^{\mathcal{H}_T} = 0$. In the same way we find

$$\partial_t\mathcal{H}_1(x) = \{\mathcal{H}_1(x), H_T\} = 2\gamma\partial_x(\lambda^{\mathcal{H}_1}\gamma) = 0 \qquad (81)$$

that has again solution $\lambda^{\mathcal{H}_1} = 0$. Let us proceed to the analysis of the evolution of the constraints $\mathcal{G}_{\mathcal{H}_1} \approx 0$, $\mathcal{G}_{\mathcal{H}_T} \approx 0$

$$\partial_t\mathcal{G}_{\mathcal{H}_1}(x) = \{\mathcal{G}_{\mathcal{H}_1}(x), H_T\} = 2\gamma(x)\partial_x(\gamma N^1) = 0 \qquad (82)$$

which is equal to zero for $N^1 = \frac{D(t)}{\gamma}$ where $D(t)$ is an arbitrary time-dependent function. However, in order to have a solution with the asymptotic behavior $N^1 \to 0$ for $x \to \infty$, we choose $D = 0$. Then the requirement of the preservation of the constraint $\mathcal{G}_{\mathcal{H}_T}$ has the form

$$\begin{aligned}
\partial_t\mathcal{G}_{\mathcal{H}_T}(x) &= \{\mathcal{G}_{\mathcal{H}_T}(x), H_T\}\\
&= -\frac{2\mu}{\kappa}\partial\left(\frac{\partial N}{N}\right) - \frac{N^2}{\kappa}\frac{\delta U}{\delta B} = 0. \qquad (83)
\end{aligned}$$

It is convenient to parametrize N as $N = e^\omega$ so that the equation above has the form

$$2\mu\omega'' = -e^{2\omega}\frac{\delta U}{\delta B} \qquad (84)$$

that is generalization of the equation found in [13] to the case of $\mu \neq 1$. Further, the Hamiltonian constraint on the constraint surface implies

$$2\mu\left(\frac{B'}{N}\right)' + NU(B) = 0 \qquad (85)$$

that can be written as

$$2\mu B'' - 2\mu B'\omega' + e^{2\omega}U = 0. \qquad (86)$$

This equation is again in agreement with the combinations of Eqs. (2.14) and (2.15) presented in [13].

V. NONPROJECTABLE HL GRAVITY WITH $f(x) = 1$

Finally we perform the Hamiltonian analysis of the special case when $f(x) = 1$.

To begin with note that in case $f(x) = 1$ the equation of motion for A implies that $B = 1$ identically and hence the action has the form

$$S = \frac{1}{\kappa}\int dt dx N\gamma\left((1-\lambda)K^2 - 2\Lambda + \beta a^2\frac{1}{\gamma^2}\right)$$

which is the action studied in [12]. However our goal is to carefully identified global first class constraints so that we again proceed to the Hamiltonian formulation of this theory.

[2]Alternatively, we can still keep N and N_1 as dynamical fields and then it is possible to fix their values by fixing primary constraints $\pi_N \approx 0, \pi^1 \approx 0$. Then however gauge fixing of $\mathcal{H}_T$, $\mathcal{H}_1$ determine Lagrange multipliers λ_T, λ_1 that have to be included in the resulting metric as it is clear from the discussion presented above.

Starting with the action (19) we find following conjugate momenta

$$\pi_N = \frac{\delta L}{\delta \dot{N}} \approx 0, \qquad \pi^1 = \frac{\delta L}{\delta \dot{N}_1} \approx 0,$$
$$\pi = \frac{\delta L}{\delta \dot{\gamma}} = \frac{2}{\kappa}(1-\lambda)K. \tag{87}$$

Then it is easy to perform the Legendre transformation in order to find the corresponding Hamiltonian

$$H = \int dx(\pi\dot{\gamma} - \mathcal{L}) = \int dx\left(N\mathcal{H}_T + N_1\frac{1}{\gamma^2}\mathcal{H}_1\right), \tag{88}$$

where

$$\mathcal{H}_T = \gamma\left(\frac{\kappa}{4(1-\lambda)}\pi^2 - \frac{\beta}{\kappa}\frac{a^2}{\gamma^2} + \frac{2}{\kappa}\gamma\Lambda\right),$$
$$\mathcal{H}_1 = -\gamma\partial_1\pi. \tag{89}$$

Again the requirement of the preservation of the primary constraints $\pi_N \approx 0$, $\pi^1 \approx 0$ implies two secondary constraints

$$\partial_t\pi_N = \{\pi_N, H\} = -\mathcal{H}_T - \frac{2\beta}{\kappa}\frac{1}{\gamma}a^2 - \left(\frac{2\beta}{\kappa}\frac{1}{\gamma}a\right)'$$
$$= -\frac{\kappa}{4(1-\lambda)}\gamma\pi^2 - \frac{2}{\kappa}\gamma\Lambda - \frac{\beta}{\kappa}\frac{a^2}{\gamma} - \left(\frac{2\beta}{\kappa}\frac{1}{\gamma}a\right)' \equiv -\mathcal{C} \approx 0,$$
$$\partial_t\pi^i = \{\pi^1, H\} = -\mathcal{H}_1 \approx 0, \tag{90}$$

where $\mathcal{C}$ obeys the property

$$\int dxN\mathcal{C} = \int dxN\mathcal{H}_T. \tag{91}$$

Now we should proceed completely as in Sec. III and we will find that the theory has identical structure of constraints. For that reason we immediately skip to the analysis of the gauge fixed theory. We again fix the constraint Π_N with the gauge fixing function

$$\mathbf{G}_N = \int dx\gamma N - C \approx 0, \tag{92}$$

where C is a constant. Now this gauge fixing function has nonzero Poisson bracket

$$\{\Pi_N, \mathbf{G}_N\} = -\int dx\gamma N = -C \neq 0, \tag{93}$$

while we have

$$\{\mathbf{G}_N, \tilde{\pi}_N(x)\} = 0, \qquad \{\mathbf{H}_S(N^1), \mathbf{G}_N\} = 0. \tag{94}$$

The global constraint $\tilde{\Pi}_T$ is fixed by gauge fixing function

$$\mathbf{G}_T = \int dx\gamma\pi - C_\pi(t) \approx 0 \tag{95}$$

so that

$$\{\mathbf{G}_T, \Pi_T\} = \int dxN\mathcal{H}_T - \frac{4\beta}{\kappa^2}\int dxN\gamma\Lambda \approx -\frac{4\beta}{\kappa}\Lambda C \tag{96}$$

and we see that $\mathbf{G}_T$ cannot be gauge fixing function in case when $\Lambda = 0$ since in this case the theory possesses global scale gauge symmetry with $\mathbf{G}_T$ corresponding generator. We return to this problem below. It is also easy to see that

$$\{\mathbf{H}_S(N^1), \mathbf{G}_T\} = 0. \tag{97}$$

Finally we fix the diffemorphism constraint using the gauge fixing condition

$$\mathcal{G}_S\colon\ \gamma - g(t) \approx 0, \tag{98}$$

where $g(t)$ is an arbitrary time dependent function. Note that $\mathcal{G}_S$ has the following nonzero Poisson bracket with $\mathbf{H}_S(N^1)$

$$\{\mathcal{G}_S(x), \mathbf{H}_S(N^1)\} = (N^1)' \tag{99}$$

that is zero for $N^1 = N^1(t)$. Now from $\mathcal{H}_1 = 0$ we find that $\pi = \pi(t)$ and then the gauge fixing condition $\mathbf{G}_T$ implies

$$\int dx\gamma\pi(t) = g(t)\pi(t)\int dx = C_\pi(t) \tag{100}$$

and hence

$$\pi(t) = \frac{C_\pi(t)}{g(t)L}, \tag{101}$$

where L is the regularized length of the system.

Now we are ready to determine Lagrange multipliers for all constraints and gauge fixing functions. Recall that the total Hamiltonian with gauge fixing functions included has the form

$$H_T = (1+\lambda_T)\tilde{\Pi}_T + \lambda_N\Pi_N + V_T\mathbf{G}_T + V_N\mathbf{G}_N$$
$$+ \int dx(\omega^A\Psi_A + N^1\tilde{\mathcal{H}}_1 + M_1\mathcal{G}_S). \tag{102}$$

From the previous expression we see that the effective lapse is $(1+\lambda_T)N$ instead of N. However the value of λ_T is fixed by the requirement of the preservation of all constraints. First of all we start with the constraint $\tilde{\mathcal{H}}_1 \approx 0$. Since the Hamiltonian is diffemorphism invariant we find

$$\partial_t \tilde{\mathcal{H}}_1(x) = \{\tilde{\mathcal{H}}_1(x), H_T\} = \int dy M_1(y) \{\tilde{\mathcal{H}}_1(x), \mathcal{G}_C(y)\} = M_1'(x) = 0 \tag{103}$$

and this is equal to zero for $M^1 = M^1(t)$. On the other hand, we have to demand that the Lagrange multipliers have correct asymptotic behavior at infinity so that the only possible solution is $M^1 = 0$.

Then the time evolution of the constraint $\tilde{\Pi}_T$ implies

$$\partial_t \tilde{\Pi}_T = \{\tilde{\Pi}_T, H_T\} = V_T \{\tilde{\Pi}_T, \mathbf{G}_T\} = 0 \tag{104}$$

which implies $V_T = 0$. In the same way time evolution of Π_N implies

$$\partial_t \Pi_N = \{\Pi_N, H_T\} = V_N \{\Pi_N, \mathbf{G}_N\} = 0 \tag{105}$$

and we find $V_N = 0$. Then, exactly as in Sec. III, we find that $\omega^A = 0$.

Finally we proceed to the requirement of the preservation of the constraint $\mathbf{G}_T$, $\mathbf{G}_N$, and $\mathcal{G}_S$. In case of $\mathbf{G}_T$ we obtain

$$\frac{d}{dt}\mathbf{G}_T = \frac{\partial \mathbf{G}_T}{\partial t} + \{\mathcal{G}_T, H_T\} = \partial_t \mathcal{G}_T + (1+\lambda_T)\{\mathbf{G}_T, \tilde{\Pi}_T\} = 0 \tag{106}$$

and hence we find

$$\lambda_T = -1 - \frac{\dot{C}_\pi}{\{\mathbf{G}_T, \tilde{\Pi}_T\}} = -1 - \frac{\dot{C}_\pi}{\frac{4\beta}{\kappa}\Lambda C}. \tag{107}$$

In case of $\mathbf{G}_N$ we find

$$\partial_t \mathbf{G}_N = \{\mathbf{G}_N, H_T\} = \lambda_N \{\mathbf{G}_N, \Pi_N\} + (1+\lambda_T)\{\mathbf{G}_N, \Pi_T\} = 0 \tag{108}$$

using the fact that $\omega^A = 0$. The previous equation can be solved for λ_N but the explicit solution is not important for us. Finally the time evolution of the constraint $\mathcal{G}_S$ has the form

$$\partial_t \mathcal{G}_S(x) = \frac{\partial \mathcal{G}_S}{\partial t} + \{\mathcal{G}_S(x), H_T\} - \dot{g} + (1+\lambda_T)\{\mathcal{G}_S(x), \tilde{\Pi}_T\} + \int dy N^1(y) \{\mathcal{G}_S(x), \tilde{\mathcal{H}}_1(y)\} = -\dot{g} + (1+\lambda_T)\frac{\kappa}{2(1-\lambda)} N\gamma\pi + N_1' g(t) = 0. \tag{109}$$

The previous equation can be solved for N_1 at least in principle. However it is important to stress that N_1 is off diagonal component of the metric so that if we demand that the metric is diagonal we have to impose the condition $N_1 = 0$. Then the previous equation implies

$$\dot{g} = (1+\lambda_T)\frac{\kappa}{2(1-\lambda)} N g(t)\pi = -\frac{\kappa}{8\beta} C_\pi \dot{C}_\pi N, \tag{110}$$

where in the final step we used (101). Let us now return to the condition $\mathcal{C} = 0$ that can be solved for N. However we simplify the calculation considerably when we demand that the (00)-component of the effective metric is equal to -1. This requirement implies that we have to demand that $N = \frac{1}{1+\lambda_T}$ that with the help of (107) implies

$$N = -\frac{\frac{4\beta}{\kappa}\Lambda C}{\dot{C}_\pi} \tag{111}$$

and hence

$$\dot{g} = -\frac{\kappa}{2(1-\lambda)} C_\pi. \tag{112}$$

Note that (111) implies that $N = N(t)$ and then the constraint $\mathcal{C}$ simplifies considerably and leads to the result

$$C_\pi^2 = -\frac{8(1-\lambda)L^2}{\kappa^2} g^2 \Lambda. \tag{113}$$

Inserting this expression into (110) we obtain a differential equation for g

$$\dot{g} = \pm\sqrt{\frac{2\Lambda L^2}{\lambda - 1}}\, g \tag{114}$$

that can be easily integrated with the result

$$g = C e^{\pm\sqrt{\frac{2\Lambda L^2}{\lambda-1}} t}. \tag{115}$$

In other words we found in the process of the gauge fixing that all dynamical fields are fixed and that the line element has the form

$$ds^2 = -dt^2 + C e^{\pm\sqrt{\frac{2\Lambda L^2}{\lambda-1}} t} d^2 x \tag{116}$$

which is in complete agreement with the result derived in [12].

Finally we briefly mention the case of zero cosmological constant $\Lambda = 0$. In this case we cannot use the gauge fixing function $\mathbf{G}_T = \int dx\gamma\pi$ since it commutes with Π_T. Let us propose another gauge fixing function

$$\mathbf{G}_T(f) = \int dx \gamma f(\pi) - C_\pi(t), \tag{117}$$

where $\{\mathbf{H}_S(N^1), \mathbf{G}_T(f)\} = 0$ which follows from the fact that π is scalar. Using this gauge fixing function we find

$$\{\mathbf{G}_T(f),\Pi_T\} = \int dx\left(\frac{\kappa^2\gamma\pi}{4(1-\lambda)}\left(2f-\frac{df}{d\pi}\pi\right)-\frac{\beta^2}{\kappa^2}\frac{a}{\gamma}\frac{df}{d\pi}\right) \tag{118}$$

that is clearly nonzero and which also does not vanish on the constraint surface. Then we can proceed as in previous case. First of all the gauge fixing of the diffeomorphism constraint implies $\pi=\pi(t)$. Further, if we demand that (00)-component of the effective metric is equal to -1 we immediately obtain that $N=N(t)$ and hence $\pi=0$ as follows from $\mathcal{C}=0$. If we again require that the metric is diagonal we obtain Eq. (110) that implies $g=\text{const}$ for $\pi=0$ and we can choose this constant to be equal to one. In other words, the flat line element

$$ds^2=-dt^2+d^2x \tag{119}$$

is the solution of the gauge fixed $\Lambda=0$ nonprojectable HL gravity

To conclude, we found that in case of two-dimensional nonprojectable HL gravity all dynamical fields are fixed and there are no physical degrees of freedom left which is in agreement with the analysis performed in [12].

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