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Premetric equivalent of general relativity: Teleparallelism

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In general relativity (GR), the metric tensor of spacetime is essential since it represents the gravitational potential. In other gauge theories (such as electromagnetism), the so-called premetric approach succeeds in separating the purely topological field equation from the metric-dependent constitutive law. We show here that GR allows for a premetric formulation, too. For this purpose, we apply the teleparallel approach of gravity, which represents GR as a gauge theory based on the translation group. We formulate the metric-free topological field equation and a general linear constitutive law between the basic field variables. The requirement of local Lorentz invariance turns the model into a full equivalent of GR. Our approach opens a way for a natural extension of GR to diverse geometrical structures of spacetime.

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I. INTRODUCTION

The premetric formalism is an alternative representation of a classical field theory in which the field equations are formulated without the spacetime metric. Only the constitutive relations between the basic field variables, excitation H and field strength F, can involve the metric of the underlying manifold. This idea can be traced back to the early 1920s where it appears in the publications of Kottler [1,2]. Various applications of this construction to the formal structure of electrodynamics were worked out by Post [3]. The premetric formalism was studied intensively in the book [4]. For an account of the recent developments in this area, see our review [5].

One advantage of the premetric formalism is that the validity of a certain premetric model can be extended to a more general spacetime geometry. The premetric construction works pretty well in Maxwell's classical electrodynamics. In this case, all basic ingredients, such as the field equations, the conserved quantities of electric charge and of magnetic flux, and the Lorentz force expression are presented in a metric-free form. Only the constitutive relation between the excitation and the field strength are formulated with the use of the metric tensor. And this relation can be straightforwardly extended to a local and linear relation thereby getting rid of the metric altogether. Let us briefly recall the various outputs of this approach:

(i) natural extension of standard electrodynamics by axion, skewon, and dilaton fields;
(ii) metric-free dispersion relation for electromagnetic waves in a medium with general linear response behavior;
(iii) metric-free Green tensor (photon propagator);
(iv) metric-free jump conditions that include boundary conditions between two media, initial Cauchy and wave-front conditions;
(v) derivation of the metric from the local and linear constitutive relation by prohibiting birefringence in electromagnetic wave propagation;
(vi) natural account of Lorentz violation models.

Although Kottler's premetric program works well in Newtonian gravity [1] and even in a flat gravitomagnetism model [5], it seems to be completely unacceptable in general relativity (GR). This is due to the well-known fact that Einstein's theory is essentially based on a pseudo-Riemann geometry with the metric tensor as its primary variable. Nevertheless, in this paper, we will show that a premetric construction of GR is possible if one turns to its teleparallel reformulation.

The organization of the paper is as follows: In Sec. II, we construct a teleparallel model for the coframe field. It is a vector-valued analog of electromagnetic theory with a well-defined gravitational energy-momentum current and a Lorentz-type force density. The general local linear constitutive law between the coframe excitation and the

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coframe field strength is defined by the use of a constitutive tensor density of 6th rank. In Sec. III, we consider the coframe model on a pseudo-Riemannian manifold. This restriction naturally requires the localization of the group of coframe transformations. Moreover, when the constitutive tensor density is assumed to be constructed from the metric, the model turns out to be fully equivalent to GR. Section IV is devoted to the Lorentz force density as an interaction term in the equation of motion for a particle. We construct a metric-free equation for a congruence of trajectories with a constitutive law between the momentum covector and the velocity vector. Its restriction to the metric manifold yields a geodesic curve in the gravitational case and a trajectory of a charge in an exterior field in the electromagnetic case. In the concluding section, we discuss the main properties of our construction and propose some possible extensions of standard GR. In the Appendix, we provide some technical calculations.

II. PREMETRIC ELECTRODYNAMICS AND ITS COFRAME ANALOG IN GRAVITY

As was shown in [4], classical electrodynamics can be expressed in a premetric way. In this section, we briefly recall the basic electromagnetic quantities and construct their coframe analogs.

Our key assumption is that a gauge field model of gravity must be based on a conserved current, here on the macroscopic ("bosonic") energy-momentum current of matter; see Blagojević *et al.* [6]. This is in analogy to the electric current that serves as a basis of electromagnetic theory. We use a covector-valued 3-form as a representation of the material energy-momentum current and construct a vector-valued field-theoretical model. It represents a *vector-valued* analog of the electromagnetic theory. Recall that the latter is expressed in terms of ordinary, scalar-valued differential forms. Although at this stage, our construction appears to be only formal, its justification is based on its relation to the energy-momentum conservation law. Incidentally, the existence of an additional independent conserved 2-form, which is untwisted, is naturally related to the definition of a special coframe field on the manifold.

A. Geometric structure

Let us consider a differential manifold $\mathcal{M}$ endowed with a coframe field ϑ^α. The 1-forms ϑ^α, with $\alpha = 0, 1, 2, 3$, are assumed to be linearly independent at each point of $\mathcal{M}$. At this stage, we postulate that all equations are invariant under *rigid linear transformations* of the coframe ϑ^α. The transformed coframe $\vartheta^{\alpha'}$ then becomes

$$\vartheta^{\alpha'} = L_\alpha{}^{\alpha'}\vartheta^\alpha, \qquad L_\alpha{}^{\alpha'} = \text{const}, \tag{1}$$

with a constant invertible matrix $L_\alpha{}^{\alpha'} \in GL(4,\mathbb{R})$.

The coframe and its exterior products (taken in increasing order) generate the bases

$$\vartheta^\alpha, \qquad \vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta, \qquad \vartheta^{\alpha\beta\gamma} := \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma,$$
$$\vartheta^{\alpha\beta\gamma\delta} := \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta \tag{2}$$

of the spaces of untwisted differential forms of the order 1, 2, 3, and 4, respectively. Under the transformation (1), the basis forms (2) transform as tensors.

In order to express the *twisted forms*, we need the volume element (a non-negative measure) defined on $\mathcal{M}$. Relative to the basis ϑ^α, it is defined as a twisted scalar-valued 4-form

$$\text{vol} = \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta \otimes s, \tag{3}$$

Here $\varepsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita permutation symbol [7] that is normalized to $\varepsilon_{0123} = 1$, while s is a section of the orientation line bundle. In [8], Eq. (3) is represented symbolically as the absolute value of the untwisted 4-form. Under the transformation of the coframe (1), the volume element (3) transforms according to the law

$$\text{vol} \to |\det L|\text{vol}, \tag{4}$$

with $\det L$ as the Jacobian of the coframe transformation. Thus, the volume element (3) remains positive for all admissible coframes.

The frame field e_α is uniquely defined as the inverse of the coframe,

$$e_\alpha \rfloor \vartheta^\beta = \vartheta^\beta(e_\alpha) = \delta_\alpha^\beta. \tag{5}$$

Under the coframe transformation (1), the frame obeys the transformation law

$$e_\alpha \to e_{\alpha'} = (L^{-1})_{\alpha'}{}^\alpha e_\alpha, \tag{6}$$

with $(L^{-1})_{\alpha'}{}^\alpha L_\alpha{}^{\beta'} = \delta_{\alpha'}^{\beta'}$.

With these definitions at hand, the sets

$$\text{vol}, \qquad \epsilon_\alpha = e_\alpha \rfloor \text{vol}, \qquad \epsilon_{\alpha\beta} = e_\beta \rfloor \epsilon_\alpha,$$
$$\epsilon_{\alpha\beta\gamma} = e_\gamma \rfloor \epsilon_{\alpha\beta}, \qquad \epsilon_{\alpha\beta\gamma\delta} = e_\delta \rfloor \epsilon_{\alpha\beta\gamma}, \tag{7}$$

with the indices taken in increasing order, serve as basis forms for the spaces of twisted 4-forms, 3-forms, 2-forms, 1-forms, and 0-forms, respectively. These basis forms transform with an additional factor $|\det L|$. All forms in (7) are totally antisymmetric. It is worthwhile to note that the Levi-Civita permutation symbol $\varepsilon_{\alpha\beta\gamma\delta}$ is an untwisted tensor density, and one can check that the values of its components do not change under the frame transformation. In contrast, the 0-form $\epsilon_{\alpha\beta\gamma\delta}$ is a twisted density, which means that its components, so to say, are sensitive to the orientation: they either remain the same or change their sign when the frame transformation preserves orientation or changes orientation, respectively. In technical terms, the behavior of $\epsilon_{\alpha\beta\gamma\delta}$ depends on the sign of determinant $\det L$. This explains a different notation for $\epsilon_{\alpha\beta\gamma\delta}$ and $\varepsilon_{\alpha\beta\gamma\delta}$. From the definitions (7) we can straightforwardly check the identity

$$\vartheta^\alpha \wedge \epsilon_\beta = \delta^\alpha_\beta \mathrm{vol}. \tag{8}$$

Since at the first stage, we allow only *global* (rigid) transformations of the coframe, the exterior derivatives of the basis forms (2) and (7) transform as tensors. Hence, one does not need here an exterior *covariant* derivative of the forms. Subsequently, we will discuss how the symmetry transformation (1) can be generalized to the case of a point dependent $L_\alpha{}^{\alpha'}(x)$.

B. Excitation

1. Electromagnetism

In electromagnetism, the inhomogeneous field equation can be treated as a result of the electric charge conservation law. In order to describe, in a given spatial volume, the electric charge with a prescribed sign, we must use the twisted 3-form J of the electric current. Its expression in a twisted basis reads

$$J = J^\alpha \epsilon_\alpha. \tag{9}$$

Under the coframe transformations (1), we have $\epsilon_\alpha \to \epsilon_{\alpha'} = |\det L|(L^{-1})_{\alpha'}{}^\alpha \epsilon_\alpha$, and the components of the 3-form J transform as

$$J^\alpha \to J^{\alpha'} = (\det L)^{-1} L_\alpha{}^{\alpha'} J^\alpha, \tag{10}$$

or

$$J \to \frac{|\det L|}{\det L} J. \tag{11}$$

The 3-form J remains the same under orientation preserving transformations, while picking up an additional sign under transformations which reverse the orientation of the coframe. This additional sign compensates the change of the orientation of the integration domain. Consequently, the integral $\int_{\Omega_3} J$ (in particular, the total charge for a closed spatial domain Ω_3) is invariant under the coframe transformations.

The law of electric charge conservation in integral and differential forms is given by

$$\int_{\partial\Omega_4} J = 0 \quad \text{and} \quad dJ = 0, \tag{12}$$

respectively. Locally, the latter relation is equivalent to the inhomogeneous Maxwell equation

$$dH = J, \tag{13}$$

where H is the twisted 2-form of the *electromagnetic excitation*. In the $\vartheta^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ bases, it reads

$$H = \frac{1}{2} H_{\alpha\beta} \vartheta^{\alpha\beta} = \frac{1}{2} \check{H}^{\alpha\beta} \epsilon_{\alpha\beta}, \quad \text{with}$$
$$\check{H}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} H_{\gamma\delta}. \tag{14}$$

By construction, $H_{\alpha\beta}$ is a covariant twisted tensor, whereas $\check{H}^{\alpha\beta}$ is an untwisted contravariant tensor density.

2. Gravity

Similarly to this electrodynamics construction, we start our gravity model with a conservation law, now with energy-momentum conservation. In the canonical formalism, the standard energy-momentum tensor is replaced by the energy-momentum current Σ_α, a twisted covector-valued 3-form. We decompose it with respect to the 3-forms ϵ_β,

$$\Sigma_\alpha = \Sigma_\alpha{}^\beta \epsilon_\beta. \tag{15}$$

This is an object of 16 independent components; see [9]. Symmetry may only be imposed by the use of a metric tensor.

Taking into account (1), with constant $L_\beta{}^\alpha$, the conservation law for the energy-momentum current can be expressed as

$$\int_{\partial\Omega_4} \Sigma_\alpha = 0, \qquad d\Sigma_\alpha = 0. \tag{16}$$

Using the standard differential-geometric facts, we can solve Eq. (16) in a small topologically good region as

$$dH_\alpha = \Sigma_\alpha. \tag{17}$$

In this way, we define (up to a total derivative) the twisted covector-valued *gravitational excitation* 2-form

$$H_\alpha = \frac{1}{2} H_{\beta\gamma\alpha} \vartheta^{\beta\gamma} = \frac{1}{2} \check{H}^{\beta\gamma}{}_\alpha \epsilon_{\beta\gamma}. \tag{18}$$

It is of decisive importance to recognize that there is a fundamental difference to the electromagnetic case (13). The electromagnetic field does not carry electric charge (the "photon" is electrically neutral), the gravitational field, however, carries energy-momentum of its own. Hence the right-hand side of (17) reads $\Sigma_\alpha = {}^{(\mathrm{m})}\Sigma_\alpha + {}^{(\vartheta)}\Sigma_\alpha$. Here (m) denotes matter and (ϑ) the coframe field, and we assume additivity of the corresponding energy-momenta.

C. Field strength

1. Electromagnetism

In electrodynamics, the untwisted field strength 2-form

$$F = \frac{1}{2} F_{\alpha\beta} \vartheta^{\alpha\beta} \tag{19}$$

satisfies the equations

$$\int_{\partial\Omega_3} F = 0, \qquad dF = 0. \tag{20}$$

The homogeneous Maxwell equation $dF = 0$ is an expression of the conservation of the magnetic flux. The electromagnetic field strength F is determined operationally via the Lorentz force density, which acts on the electric current. We will discuss this below. The solution of Eq. (20) can be expressed in terms of the *electromagnetic potential* A,

$$dA = F. \tag{21}$$

In the coframe basis, this untwisted 1-form reads

$$A = A_\alpha \vartheta^\alpha. \tag{22}$$

It is defined up to the addition of a total derivative $A \to A + d\varphi$.

2. Gravity

In analogy to the field strength F of the electromagnetic theory, we introduce the *gravitational field strength* F^α. It is an untwisted vector-valued 2-form that satisfies the equation

$$dF^\alpha = 0. \tag{23}$$

The solution of this equation can be locally represented as

$$F^\alpha = d\theta^\alpha. \tag{24}$$

The set of four 1-forms θ^α is the analog of the electromagnetic potential A. We assume now that the potentials θ^α are linearly independent. It always can be reached due to the gauge invariance of Eq. (23). Indeed, we can redefine $\theta^\alpha \to \theta^\alpha + df^\alpha$, with four arbitrary scalar functions f^α.

We identify the reference coframe ϑ^α with the dynamical coframe θ^α and rewrite Eq. (24) as

$$F^\alpha = d\vartheta^\alpha. \tag{25}$$

Thus, we can consider the covector-valued forms Σ_α, H_α and the vector-valued F^α to be related to this special basis. In particular, we expand the untwisted form F^α relative to the untwisted basis $\vartheta^{\beta\gamma}$ as follows:

$$F^\alpha = \frac{1}{2} F_{\beta\gamma}{}^\alpha \vartheta^{\beta\gamma}. \tag{26}$$

D. Lorentz force

1. Electromagnetism

The force acting on electrically charged matter is described by a twisted covector-valued 4-form f_α. Being a top-order form, it can be represented as a vector-valued scalar $\mathfrak{f}_\alpha$ multiplied by the volume form $f_\alpha = \mathfrak{f}_\alpha \mathrm{vol}$. In electrodynamics (see [4] and also [10–12]), the Lorentz force is given by

$$f_\alpha = (e_\alpha \rfloor F) \wedge J. \tag{27}$$

Readers can refer to [4,10–12] for technical details explaining how one can compute the electric power and the total force of electromagnetic field acting on the matter by taking an appropriate integral of the Lorentz force density (27).

Expanding the current with respect to the 3-form basis,

$$J = J^\alpha \epsilon_\alpha, \tag{28}$$

and making use of (8), we recast the Lorentz force (27) into

$$f_\alpha = (J^\beta F_{\alpha\beta}) \mathrm{vol}. \tag{29}$$

The first factor represents the standard expression of the Lorentz force density

$$\mathfrak{f}_\alpha = J^\beta F_{\alpha\beta}. \tag{30}$$

By construction, J^α is an untwisted vector density, and accordingly $\mathfrak{f}_\alpha$ is an untwisted covector density. For a point particle, both the current density and the force density are proportional to a delta-function [13].

2. Gravity

Analogously to electromagnetism, we describe the Lorentz force for the coframe field by the 4-form

$$f_\alpha = (e_\alpha \rfloor F^\beta) \wedge {}^{(\mathrm{m})}\Sigma_\beta. \tag{31}$$

Expanding the energy-momentum current with respect to the 3-form basis,

$${}^{(\mathrm{m})}\Sigma_\alpha = \Sigma_\alpha{}^\beta \epsilon_\beta, \tag{32}$$

we introduce an untwisted energy-momentum tensor density $\Sigma_\alpha{}^\beta$ of massive matter. Substituting the representation (26) into (31) and using (32), we obtain the gravitational Lorentz force

$$f_\alpha = (\Sigma_\gamma{}^\beta F_{\alpha\beta}{}^\gamma)\mathrm{vol}. \tag{33}$$

The first factor on the right-hand side of (33) represents the covector of the *gravitational* Lorentz force density

$$\mathfrak{f}_\alpha = \Sigma_\gamma{}^\beta F_{\alpha\beta}{}^\gamma. \tag{34}$$

A comparison between (34) and (30) shows the deep analogy between gravity and electromagnetism.

E. Energy-momentum current of gravity

1. Electromagnetism

The energy-momentum current of the electromagnetic field (see [4]) is a covector-valued 3-form represented by

$${}^{(\mathrm{em})}\Sigma_\alpha = \frac{1}{2}[F\wedge(e_\alpha\rfloor H) - H\wedge(e_\alpha\rfloor F)]. \tag{35}$$

If the twisted electromagnetic Lagrangian 4-form

$${}^{(\mathrm{em})}\Lambda \coloneqq -\frac{1}{2}F\wedge H \tag{36}$$

can be specified, we can alternatively put it into the form

$$\begin{aligned}{}^{(\mathrm{em})}\Sigma_\alpha &= e_\alpha\rfloor{}^{(\mathrm{em})}\Lambda + F\wedge(e_\alpha\rfloor H)\\ &= -e_\alpha\rfloor{}^{(\mathrm{em})}\Lambda - H\wedge(e_\alpha\rfloor F).\end{aligned} \tag{37}$$

Using $F=dA$, we can rederive the field equation $dH=J$ and the current (35) from the Lagrangian (36).

One can straightforwardly verify the balance law [4]

$$d{}^{(\mathrm{em})}\Sigma_\alpha = f_\alpha + X_\alpha, \tag{38}$$

where f_α is the Lorentz force (27) and $X_\alpha = -\frac{1}{2}(F\wedge\mathcal{L}_\alpha H - H\wedge\mathcal{L}_\alpha F)$ describes an additional force determined by the constitutive law. Here $\mathcal{L}_\alpha$ denotes the Lie derivative along vector fields e_α.

2. Gravity

Similar to the electromagnetic case, we postulate the energy-momentum current of the coframe field as

$${}^{(\vartheta)}\Sigma_\alpha = \frac{1}{2}[F^\beta\wedge(e_\alpha\rfloor H_\beta) - H_\beta\wedge(e_\alpha\rfloor F^\beta)]. \tag{39}$$

We can also introduce the Lagrange 4-form for the coframe field,

$${}^{(\vartheta)}\Lambda = -\frac{1}{2}F^\alpha\wedge H_\alpha. \tag{40}$$

Then we can write its energy-momentum current in a form similar to (37),

$$\begin{aligned}{}^{(\vartheta)}\Sigma_\alpha &= e_\alpha\rfloor{}^{(\vartheta)}\Lambda + F^\beta\wedge(e_\alpha\rfloor H_\beta)\\ &= -e_\alpha\rfloor{}^{(\vartheta)}\Lambda - H_\beta\wedge(e_\alpha\rfloor F^\beta).\end{aligned} \tag{41}$$

Analogously to (38), one finds the balance law

$$d{}^{(\vartheta)}\Sigma_\alpha = f_\alpha + {}^{(\vartheta)}X_\alpha, \tag{42}$$

where f_α is gravitational Lorentz force (31) and ${}^{(\vartheta)}X_\alpha = -\frac{1}{2}(F^\beta\wedge\mathcal{L}_\alpha H_\beta - H_\beta\wedge\mathcal{L}_\alpha F^\beta)$ is an additional force to be determined by the corresponding constitutive law.

F. Constitutive relation

In order to complete the field-theoretical models of electromagnetism and gravity, a *constitutive relation* between the basic variables, namely between excitation H and field strength F should be introduced.

1. Electromagnetism

The system of the premetric field equations for electromagnetism (13) and (20) involves 8 equations for 12 independent variables, the components of the 2-forms H and F. This system is undetermined and has to be supplemented by an additional relation between the basic variables. In solid state electromagnetism, such relation can be of a rather complicated form. However, even the simplest case of a linear constitutive relation has a wide range of applications.

Using the expansions

$$H = \frac{1}{2}\check{H}^{\alpha\beta}\epsilon_{\alpha\beta}, \qquad F = \frac{1}{2}F_{\alpha\beta}\vartheta^{\alpha\beta}, \tag{43}$$

we postulate the most general local linear constitutive relation in the form of

$$\check{H}^{\alpha\beta} = \frac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}. \tag{44}$$

Due to this definition, the constitutive tensor density χ satisfies the symmetry relations

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} = -\chi^{\alpha\beta\delta\gamma}. \tag{45}$$

2. Gravity

Similarly, our coframe system must be endowed with the constitutive relation between F^α and H_α. We assume this relation to be linear and local. In analogy to electromagnetism, we use the expansions

$$H_\alpha = \frac{1}{2}\check{H}^{\beta\gamma}{}_\alpha\epsilon_{\beta\gamma}, \qquad F^\alpha = \frac{1}{2}F_{\beta\gamma}{}^\alpha\vartheta^{\beta\gamma}. \tag{46}$$

We postulate the most general local and linear constitutive relation in the form of

$$\check{H}^{\beta\gamma}{}_{\alpha} = \frac{1}{2}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}F_{\nu\rho}{}^{\mu}. \tag{47}$$

Here $\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}$ is the constitutive tensor density that obeys the symmetries

$$\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = -\chi^{\gamma\beta}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = -\chi^{\beta\gamma}{}_{\alpha}{}^{\rho\nu}{}_{\mu}. \tag{48}$$

G. Lagrange formalism

In this section, we apply the Lagrange formalism to derive the statements proposed above. In this way, we are able to justify the coframe model that was postulated in the previous section only by analogy.

1. Electromagnetism

Although the electromagnetic case is well-known, it is instructive to recall the variational procedure. This construction turns out to be completely metric-free. As only restriction, we will use an additional symmetry relation of the constitutive tensor density, namely

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta}. \tag{49}$$

In term of the irreducible decomposition [4], it means that the skewon part is assumed to be forbidden and the constitutive tensor density is left with only 21 independent components; then and only then a Lagrange formalism is possible.

We start with the Lagrange 4-form

$$\begin{aligned}\Lambda &= -\frac{1}{2}F\wedge H(F) + A\wedge J\\ &= \Big(-\frac{1}{2}F_{\alpha\beta}\check{H}^{\alpha\beta}(F_{\gamma\delta}) + A_{\alpha}J^{\alpha}\Big)\mathrm{vol}.\end{aligned} \tag{50}$$

The variation of this Lagrangian takes the form

$$\delta\Lambda = -\frac{1}{2}(\delta F\wedge H + F\wedge\delta H) + \delta A\wedge J. \tag{51}$$

In the case of the linear constitutive relation with the symmetry (49), the first two terms on the right-hand side of Eq. (51) are equal to one another. Indeed, using the component representation, we have

$$\begin{aligned}F\wedge\delta H &= -\frac{1}{2}(F_{\alpha\beta}\delta\check{H}^{\alpha\beta})\mathrm{vol}\\ &= -\frac{1}{4}(F_{\alpha\beta}\chi^{\alpha\beta\gamma\delta}\delta F_{\gamma\delta})\mathrm{vol}\\ &= -\frac{1}{2}(\check{H}^{\gamma\delta}\delta F_{\gamma\delta})\mathrm{vol} = \delta F\wedge H.\end{aligned} \tag{52}$$

Consequently, Eq. (51) takes the form

$$\begin{aligned}\delta\Lambda &= -d(\delta A)\wedge H + \delta A\wedge J\\ &= -d(\delta A\wedge H) - \delta A\wedge(dH - J).\end{aligned} \tag{53}$$

In order to derive the field equation from this expression, we remove, as usual, the total derivative term and require $\delta\Lambda$ to be zero for arbitrary variations of the potential. Then we obtain the inhomogeneous Maxwell equation and the electric charge conservation law as straightforward consequences,

$$dH = J, \qquad dJ = 0. \tag{54}$$

Let us now study relation (53) on shell, i.e, we assume that the inhomogeneous Maxwell equation (54) is already satisfied. Then we are left with

$$\delta\Lambda = -d(\delta A\wedge H). \tag{55}$$

For variations induced by frame transformations, we use $\delta_\alpha\Lambda$ instead of $\delta\Lambda$ and $\delta_\alpha A$ instead of δA. These variations are generated by the Lie derivatives relative to the frame vectors, $\delta_\alpha = \mathcal{L}_{e_\alpha}$. Thus, we have

$$\delta_\alpha\Lambda = \mathcal{L}_{e_\alpha}\Lambda = d(e_\alpha\rfloor\Lambda), \tag{56}$$

$$\delta_\alpha A = \mathcal{L}_{e_\alpha}A = d(e_\alpha\rfloor A) + e_\alpha\rfloor dA. \tag{57}$$

Substituting into (55), we obtain a conservation law

$$d\,{}^{(\mathrm{em})}\Sigma_\alpha = 0, \tag{58}$$

where

$${}^{(\mathrm{em})}\Sigma_\alpha = [e_\alpha\rfloor\Lambda + (e_\alpha\rfloor F)\wedge H] - (e_\alpha\rfloor A)\wedge J. \tag{59}$$

On the right-hand side of this equation, we recognize the energy-momentum of the electromagnetic field and the interaction term.

2. Gravity

Consider a Lagrangian of a system that includes the coframe field and a matter field

$$\Lambda = \frac{1}{2}F^\alpha\wedge H_\alpha + {}^{(\mathrm{m})}\Lambda. \tag{60}$$

Using (46), we rewrite it as

$$\Lambda = \frac{1}{2}(F_{\beta\gamma}{}^{\alpha}\check{H}^{\beta\gamma}{}_{\alpha})\mathrm{vol} + {}^{(\mathrm{m})}\Lambda. \tag{61}$$

Variation of this Lagrangian reads (see Appendix)

$$\delta\Lambda = -d(\delta\vartheta^\alpha\wedge H_\alpha) - \delta\vartheta^\alpha\wedge(dH_\alpha - {}^{(\vartheta)}\Sigma_\alpha - {}^{(\mathrm{m})}\Sigma_\alpha), \tag{62}$$

TABLE I. Premetric electromagnetism-gravity analogy.

Objects and Laws	Electromagnetism	Gravity
Source current	J	Σ_α
Conserved source current	$dJ = 0$	$d\Sigma_\alpha = 0$
Excitation	H	H_α
Inhomogeneous field equation	$dH = J$	$dH_\alpha = {}^{(\vartheta)}\Sigma_\alpha + {}^{(\mathrm{m})}\Sigma_\alpha$
Field strength	F	F^α
Homogeneous field equation	$dF = 0$	$dF^\alpha = 0$
Potential	A	ϑ^α
Potential equation	$dA = F$	$d\vartheta^\alpha = F^\alpha$
Lorentz force	$f_\alpha = (e_\alpha \rfloor F) \wedge J$	$f_\alpha = (e_\alpha \rfloor F^\beta) \wedge {}^{(\mathrm{m})}\Sigma_\beta$
Energy-momentum current	$\Sigma_\alpha = e_\alpha \rfloor \Lambda + F \wedge (e_\alpha \rfloor H)$	${}^{(\vartheta)}\Sigma_\alpha = e_\alpha \rfloor \Lambda + F^\beta \wedge (e_\alpha \rfloor H_\beta)$
Lagrangian	$\Lambda = -(1/2) F \wedge H$	$\Lambda = -(1/2) F^\alpha \wedge H_\alpha$
Constitutive tensor	$\chi^{\alpha\beta\gamma\delta}$	$\chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu$

where the energy-momentum current of the coframe field is specified by

$$ {}^{(\vartheta)}\Sigma_\alpha = e_\alpha \rfloor \Lambda + F^\beta \wedge (e_\alpha \rfloor H_\beta). \tag{63} $$

The matter energy-momentum current ${}^{(\mathrm{m})}\Sigma_\alpha$ is defined via the relation

$$ \delta^{(\mathrm{m})}\Lambda = \delta\vartheta^\alpha \wedge {}^{(\mathrm{m})}\Sigma_\alpha. \tag{64} $$

For variations of the coframe that vanish on the boundary, we are left with the field equation

$$ dH_\alpha = \Sigma_\alpha, \tag{65} $$

where the total energy-momentum current is given as a sum of the coframe current (63) and the matter current defined in (64)

$$ \Sigma_\alpha = {}^{(\vartheta)}\Sigma_\alpha + {}^{(\mathrm{m})}\Sigma_\alpha. \tag{66} $$

Note that the conservation law for this quantity, $d\Sigma_\alpha = 0$, follows straightforwardly from field equation (65).

H. Premetric electromagnetism-gravity correspondence

We can now summarize the analogy between the premetric coframe model of gravity and the standard electromagnetic theory in Table I.

III. FIELD-THEORETICAL MODELS ON METRIC MANIFOLDS

So far, all the ingredients in the electromagnetic as well as in the coframe model are premetric. Indeed, the metric is not involved in these formalisms at all. We will now consider these models on a manifold endowed with a pseudo-Riemannian metric. In the electromagnetic case, this structure allows us to describe vacuum electrodynamics. For the coframe field, we are able to reinstate standard GR in the context of a premetric formalism.

A. Coframe field and metric

We consider a manifold $\mathcal{M}$ endowed with a smooth metric g and restrict the coframe field ϑ^a to be orthonormal relative to this metric. Thus, the metric on our manifold can be expressed as

$$ g = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta, \tag{67} $$

where $g_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric. In other words, we restrict ourselves to the subgroup $O(1,3)$ of the orthogonal transformations of the coframe:

$$ \vartheta^\alpha \to \vartheta^{\alpha'} = L_\alpha{}^{\alpha'}\vartheta^\alpha. \tag{68} $$

Then the metric satisfies the relation

$$ g_{\alpha'\beta'} L_\alpha{}^{\alpha'} L_\beta{}^{\beta'} = g_{\alpha\beta}, \tag{69} $$

and hence $(\det L)^2 = 1$. We observe that the metric in (67) is invariant under a wider class of transformations that depend on a point $x \in \mathcal{M}$ with $L_\alpha{}^{\alpha'}(x)$, that is, we have *local coframe transformations*.

We can develop the coframe and the frame fields, respectively, in terms of local coordinates $\{x^i\}$ as follows:

$$ \vartheta^\alpha = \vartheta_i{}^\alpha dx^i, \qquad e_\alpha = e^i{}_\alpha \partial_i. \tag{70} $$

In these holonomic coordinates, the components of the metric tensor read

$$ g_{ij} = g_{\alpha\beta}\vartheta_i{}^\alpha \vartheta_j{}^\beta, \qquad g^{ij} = g^{\alpha\beta} e^i{}_\alpha e^j{}_\beta. \tag{71} $$

The volume element (3) takes now the form

$$ \mathrm{vol} = \sqrt{-g}\, d^4x = |\det \vartheta_i{}^\alpha| d^4x, \tag{72} $$

where $g = \det(g_{ij}) = -(\det \vartheta_i{}^\alpha)^2$. We recognize in this standard expression the twisted 4-form as defined in (3).

It is worthwhile to note that $\check{H}^{\alpha\beta}$ and $\check{H}^{\alpha\beta}{}_\gamma$ are true tensors under the restriction to the orthogonal group.

B. Vacuum electrodynamics

Standard Maxwell-Lorentz electrodynamics is recovered in the premetric framework provided the constitutive tensor is expressed in terms of the Minkowski metric as follows:

$$\chi^{\alpha\beta\gamma\delta} = \frac{1}{2}\lambda_0(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}). \qquad (73)$$

Here g is the determinant of the metric and $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$ denotes the vacuum impedance. In 'the International System of Units' (SI), its value is $\lambda_0 = 1/(377\Omega)$. If we only allow the metric $g^{\alpha\beta}$ to enter the constitutive tensor (73) as variable, then, due to the symmetries (45) of $\chi^{\alpha\beta\gamma\delta}$, the construction of (73) is well determined.

We expand the field strength 2-form in a coordinate basis

$$F = \frac{1}{2}F_{ij}dx^i \wedge dx^j \qquad (74)$$

and derive from $dF = 0$ the homogeneous Maxwell equation in its standard form in tensor calculus:

$$\epsilon^{ijkl}\partial_j F_{kl} = 0. \qquad (75)$$

If the constitutive tensor (73) is used, also the inhomogeneous field equation $dH = J$ can be rewritten in the standard tensor notation,

$$\partial_j(\sqrt{-g}F^{ij}) = \sqrt{-g}J^i. \qquad (76)$$

This results in the conservation law of the electric current,

$$\partial_i(\sqrt{-g}J^i) = 0. \qquad (77)$$

The Lorentz force in a coframe basis reads

$$f_\alpha = e_\alpha{}^i F_{ik}\sqrt{-g}J^k d^4x. \qquad (78)$$

The scalar factor of this 4-form presents the ordinary expression of the Lorentz force density covector

$$\mathfrak{f}_i = F_{ik}J^k. \qquad (79)$$

C. Constitutive tensor density of the coframe

We turn now to the gravitational model. We require the 6th rank constitutive tensor density to be expressed in terms of the metric tensor $g_{\alpha\beta}$ as variable alone. Due to the symmetries listed in Eq. (48), the most general expression of this type appears to be

$$\begin{aligned}\chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu = \frac{2}{\varkappa}\{&\beta_1 g_{\alpha\mu}(g^{\beta\nu}g^{\gamma\rho} - g^{\gamma\nu}g^{\beta\rho}) \\ &+ \beta_2[(g^{\gamma\rho}\delta^\beta_\alpha - g^{\beta\rho}\delta^\gamma_\alpha)\delta^\nu_\mu - (g^{\gamma\nu}\delta^\beta_\alpha - g^{\beta\nu}\delta^\gamma_\alpha)\delta^\rho_\mu] \\ &+ \beta_3[(g^{\gamma\rho}\delta^\beta_\mu - g^{\beta\rho}\delta^\gamma_\mu)\delta^\nu_\alpha - (g^{\gamma\nu}\delta^\beta_\mu + g^{\beta\nu}\delta^\gamma_\mu)\delta^\rho_\alpha]\},\end{aligned} \qquad (80)$$

provided we assume the additional "paircom" symmetry

$$\chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu = \chi^{\nu\rho}{}_\mu{}^{\beta\gamma}{}_\alpha. \qquad (81)$$

Here β_1, β_2, β_3 are dimensionless factors, $\varkappa$ is a dimensionful constant.

A remark is in order concerning the dimensions. The coframe and the gravitational field strength have the dimensions of a length, $[\vartheta^\alpha] = [d\vartheta^\alpha] = \ell$. Analogously, the gravitational current and the gravitational excitation have the same dimension as a momentum: $[\Sigma_\alpha] = [H_\alpha] = [\text{momentum}] = \frac{m\ell}{t} = ft$. As a result, $[F^\alpha \wedge H_\alpha] = ft\ell = [\text{action}]$. Hence the Lagrangian has, indeed, the correct dimension of an action. Consequently, the dimension of the constant $\varkappa$ is obtained as the ratio of the dimension of F^α divided by the dimension of H_α, that is, we have $[\varkappa] = \frac{t}{m}$. Thus, $[\kappa] = [\frac{\varkappa}{c}] = \frac{t^2}{m\ell} = \frac{1}{f}$ is Einstein's gravitational constant. This demonstrates a remarkable consistency of teleparallel gravity with Einstein's GR.

Observe that the symmetry (81) allows the coframe model to be derived from a Lagrangian. Using the constitutive tensor (80), we can write the coframe Lagrangian in (60) as

$$\begin{aligned}{}^{(\vartheta)}\Lambda &= \frac{1}{2}F^\alpha \wedge H_\alpha \\ &= \frac{1}{4\varkappa}F_{\beta\gamma\alpha}(\beta_1 F^{\beta\gamma\alpha} + \beta_2 g^{\alpha\beta}F_\nu{}^{\gamma\nu} + \beta_3 F^{\alpha\gamma\beta})\text{vol} \\ &= \frac{1}{2\varkappa}F_{\beta\gamma\alpha}(\alpha_1{}^{(1)}F^{\beta\gamma\alpha} + \alpha_2{}^{(2)}F^{\beta\gamma\alpha} + \alpha_3{}^{(3)}F^{\beta\gamma\alpha})\text{vol},\end{aligned} \qquad (82)$$

where ${}^{(I)}F^{\beta\gamma\alpha}$ are the three irreducible pieces of the field strength; see [9].

D. GR in terms of coframe variables

We constructed a set of coframe models parametrized by dimensionless numerical parameters α_1, α_2, and α_3 that turns out to be very similar to the electrodynamics system. The question is: How are these models connected to gravity, in particular to GR?

Recall that Einsteins theory is expressed by the field equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij}. \qquad (83)$$

Here $\kappa = 8\pi G/c^4$, with Newton's gravitational constant G. When the metric tensor (71) is substituted into the left-hand side of (83), we obtain an expression that includes second order derivatives of the coframe components plus the product of their first order derivatives. Exactly the same type of expressions we have in the coframe field equation $dH_\alpha = \Sigma_\alpha$. Thus, for some special values of the parameters α_1, α_2, and α_3, we can recover standard GR from the coframe field equation.

It seems technically simpler to deal with the Lagrangian. Recall that the left-hand side of (83) is derived from the action functional

$$\mathcal{W} = \frac{1}{2\kappa c}\int R\sqrt{-g}d^4x. \tag{84}$$

As it is well known, the scalar curvature R and, in turn, the Lagrangian in (84) can be expressed as a sum of two parts: a term that is quadratic in the first order derivatives of the metric plus a total divergence. In particular, up to a total derivative, Eq. (84) can be represented [see [14] Eq. (3.20)] as

$$\mathcal{W} = \frac{1}{2\kappa c}\int g^{ij}(\Gamma_{li}{}^{k}\Gamma_{kj}{}^{l} - \Gamma_{lk}{}^{k}\Gamma_{ij}{}^{l})\sqrt{-g}d^4x. \tag{85}$$

The expression of this Lagrangian in terms of the coframe is well-known. In a compact form (see [15]), this *teleparallel equivalent of GR* reads

$$\mathcal{W} = \frac{1}{2}\int F^\alpha \wedge H_\alpha, \tag{86}$$

where

$$H_\alpha = \frac{1}{\kappa c}{}^\star[g_{\alpha\beta}F^\beta - g_{\alpha\beta}\vartheta^\beta \wedge (e_\gamma \rfloor F^\gamma) - 2g_{\beta\gamma}e_\alpha\rfloor(\vartheta^\beta \wedge F^\gamma)]$$
$$= \frac{1}{\kappa c}g_{\alpha\beta}{}^\star\left(-{}^{(1)}F^\beta + 2{}^{(2)}F^\beta + \frac{1}{2}{}^{(3)}F^\beta\right). \tag{87}$$

In tensor form, (87) can be found in [16]; see Eq. (A.15).

There is a long development of this teleparallel theory of gravity. Relevant papers are, amongst others, Pellegrini and Plebanski [17], Kaempffer [18], Cho [19], Hehl, Nitsch and von der Heyde [16], Nitsch *et al.* [20], Muench *et al.* [21], Nester *et al.* [22–24], Obukhov & Pereira [25], Itin [26–30], Maluf [31], Aldrovandi & Pereira [32]. A review was given in [6].

We substitute (87) into (86) and compare the result with the coframe Lagrangian (82). The values of the free parameters turn out to be

$$(\beta_1 = 1, \beta_2 = -4, \beta_3 = 2) \quad \text{and}$$
$$\left(\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = \frac{1}{2}\right). \tag{88}$$

Since (40) includes all possible Lagrangians that are quadratic in the first order derivatives of the coframe components, we found that the Hilbert-Einstein Lagrangian is a special case of a coframe Lagrangian.

E. Local coframe transformations

In a premetric teleparallel formalism, GR turns out to be a special case of a general coframe model with the specific parameters of (88). This case, however, is very distinguished. Indeed, standard GR and its teleparallel equivalent are invariant under *local* Lorentz transformations of the coframe field,

$$\vartheta^{\alpha'} = L_\alpha{}^{\alpha'}(x)\vartheta^\alpha. \tag{89}$$

It can be checked (see [19]) that there exists, up to an arbitrary multiplicative constant, only one set of free parameters $(\alpha_1, \alpha_2, \alpha_3)$, which constitutes a locally Lorentz invariant coframe model with invariant Lagrangian and field equation. Other ingredients of the coframe model, such as field strength, excitation, energy-momentum current, and Lorentz-type force, are not locally invariant. This fact is very well known in GR, where the energy-momentum of gravity cannot be defined in a covariant way.

IV. LORENTZ FORCE AND GEODESICS

Equations of motion for test particles in an external gravitational field should not be postulated, they are rather the consequence of the conservation laws. Most conveniently, one can derive the equations of motion with the help of the multipole expansion methods by integrating conservation laws over an extended test body. Here we confine our attention to the lowest (monopole) order and consider the relativistic version of Newton's equation of motion with the gravitational Lorentz force on the right-hand side. We demonstrate that one can rewrite the latter as the standard geodesic equation of GR, provided we assume the metric dependent constitutive relation between the momentum p_α and the velocity u^α of a test particle.

A. Premetric equation of particle motion

In accordance with the expression (33) of the twisted covector-valued 4-form for the Lorentz force, the equation of motion of test particle reads, in the monopole approximation,

$$\frac{dp_\alpha}{ds} = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma. \tag{90}$$

Here p_α is the integrated momentum (1st moment) of an extended body. The body is characterized by an infinite set of multipole moments which are derived by integrating the energy-momentum current density $\Sigma_a{}^\beta$ over a cross-section of body's world tube. In the lowest approximation, we neglect effects of the dipole and higher order moments [33].

The equation (90) is invariant under arbitrary smooth reparametrization of the curve $s \to \lambda(s)$. Thus, even being expressed via the length parameter s, equation of motion (90) is premetric, provided we consider the momentum p_α and the 4-velocity u^α as independent variables. Moreover, Eq. (90) is invariant under a rescaling of the momentum $p_\alpha \to C p_a$. This symmetry manifests Einstein's principle of equivalence of inertial and gravitational mass, which is valid even in the premetric framework.

Let us rewrite Eq. (90) in a coordinate basis. Multiplying both sides of this equation by $\vartheta_i{}^\alpha$, we find

$$\begin{aligned}\vartheta_i{}^\alpha \frac{dp_\alpha}{ds} &= u^j (F_{\alpha\beta}{}^\gamma \vartheta_i{}^\alpha \vartheta_j{}^\beta) p_\gamma \\ &= u^j (\partial_i \vartheta_j{}^\gamma - \partial_j \vartheta_i{}^\gamma) p_\gamma. \end{aligned} \tag{91}$$

Consequently,

$$\vartheta_i{}^\alpha \frac{dp_\alpha}{ds} + \frac{d\vartheta_i{}^\alpha}{ds} p_\alpha = u^j p_\gamma \partial_i \vartheta_j{}^\gamma. \tag{92}$$

Thus, the equation of motion of a test particle takes the form

$$\frac{dp_i}{ds} = u^j p_\alpha \partial_i \vartheta_j{}^\alpha. \tag{93}$$

This equation is metric-free, and it is valid in a general geometric background.

B. Geodesic equation

Eventually, the metric g is introduced on the spacetime manifold. Recall the two equivalent representations of the metric tensor in terms of a coframe ϑ^α or of coordinates x^i, respectively:

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = g_{ij} dx^i \otimes dx^j. \tag{94}$$

We observe

$$p_\gamma \partial_i \vartheta_j{}^\gamma = g_{\beta\gamma} p^k \vartheta_k^\beta \partial_i \vartheta_j{}^\gamma = \frac{1}{2} p^k \partial_i g_{jk}. \tag{95}$$

As a result, (93) is recast into

$$\frac{dp_i}{ds} = u^j \partial_j p_i = \frac{1}{2} \partial_i g_{jk} p^j u^k. \tag{96}$$

So far, this equation contains two unknowns, the covector p_i, the momentum, and the vector u^i, the velocity. We now assume the *constitutive relation* between the momentum and the velocity of the particle to be local and linear,

$$p_i = m g_{ij} u^j, \tag{97}$$

where m is the mass of the particle. As a consequence, (96) reduces to

$$\frac{du_i}{ds} = u^j \partial_j u_i = \frac{1}{2} \partial_i g_{jk} u^k u^j. \tag{98}$$

This is equivalent to the standard geodesic equation; see [34]:

$$\frac{du^i}{ds} + \Gamma_{jk}{}^i u^j u^k = 0. \tag{99}$$

C. Particle motion in an electromagnetic field

The premetric framework above, which correctly produces a geodesic, can be extended to an electric point charge. The total force should be the sum of the gravitational and the electromagnetic Lorentz terms

$$\frac{dp_\alpha}{ds} = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma + q u^\beta F_{\alpha\beta}. \tag{100}$$

Here q is the lowest multipole moment arising from the integration of the electric current vector density J^α over a cross-section of body's world tube; it is interpreted as a total electric charge of a test body. Using the constitutive relation (97), we then end up with the standard equation of motion of a charge in a curved spacetime:

$$\frac{du^i}{ds} + \Gamma_{jk}{}^i u^j u^k = \frac{q}{m} F^{ij} u_j. \tag{101}$$

V. DISCUSSION

A. A gauge view at gravity

A gauge-theoretical understanding of gravitational theory was our tool for arriving at a premetric version of general relativity, namely teleparallelism, here specifically by considering a gauge theory of the *translation group*. However, it is the semidirect product of the translation group $T(4)$ with the *Lorentz group* $SO(1,3)$, the Poincaré group $T(4) \rtimes SO(1,3)$, which is the group of motion in Minkowski spacetime. The Poincaré group is connected with the energy-momentum and spin angular momentum of matter as Noether currents.

The gauging, that is, the localization of the Poincaré group, yields the Poincaré gauge theory of gravity (PG), see the review [6], Part B. If the spin of matter is suppressed, a (Inönü-Wigner type) group contraction of the PG leads to a translation gauge theory. This contraction is mathematically very delicate and is conventionally done in a heuristic manner. In this way, the teleparallelism theory is emerging. At the same time it becomes intelligible why teleparallelism has a number of unexpected and somewhat strange features. After all, the vanishing of the curvature, that is, the defining characteristics of teleparallelism theory, is hard to digest from a purely Einsteinian GR point of view (as already Pauli remarked to Einstein in the 1920s). However, from the point of view of PG, this is self-evident, since the curvature is the gauge field strength of the Lorentz group—and the suppression of the material spin, in turn, suppresses the Lorentz group as gauge group. And thus the Pauli objection can be invalidated. By the same token we recognize that teleparallelism can only be really understood in the context of PG. It is not comprehensible as a stand-alone theory.

B. Nonlocal extension of teleparallelism

A further success of the gauge-theoretical view at GR can be listed: When, in the early 2000s, Mashhoon recognized that Einstein's clock hypothesis is not sustainable as soon as high translational and rotational accelerations occur. Therefore, he looked for a classical nonlocal extension of GR and of the Einstein field equation. In spite of several attempts, he was not able to implement it on the basis of the Einstein equation and GR.

Again, as soon as one looked at gravity from a gauge-theoretical perspective, it evident of how one has to proceed: Switch from GR to the teleparallel approach to gravity. Its structure is closely related to electromagnetism. And in electromagnetism it is straightforward to generalize a local and linear constitutive law to a nonlocal and linear framework—already Volterra pointed this out in the 1910s.

Mashhoon and one of the present authors [35,36] took their "teleparallel" glasses and looked at the field equation of gravity. Following Volterra, they set up a nonlocal framework for a classical theory of gravity, extending thereby GR to the domain of high accelerations. This nonlocal theory of gravity was worked out in some detail by Mashhoon and collaborators and can be found in the forthcoming monograph of Mashhoon [37]. Quite unexpectedly, nonlocal gravity is able to describe the cosmos without taking recourse to dark matter; see the title of [35]. The nonlocal theory explains dark matter straightforwardly. Up to now, the astrophysical data seem to speak in favor of this new framework.

C. $U(1)$-axion field versus axial torsion vector field

Consider axion electrodynamics [38]: The $U(1)$-axion a is present in the third irreducible piece of the electromagnetic constitutive tensor in (44):

$$
\begin{aligned}
&{}^{(3)}\chi^{\alpha\beta\gamma\delta} = a\epsilon^{\alpha\beta\gamma\delta},\\
&[{}^{(3)}\chi^{\alpha\beta\gamma\delta}] = 1/(\text{electric resistance}).
\end{aligned} \tag{102}
$$

Similarly, the axial torsion piece $\mathcal{A} \coloneqq g_{\alpha\beta}{}^{\star}(\vartheta^\alpha \wedge F^\beta)$ is manifest in the third piece of the gravitational constitutive tensor in (47):

$$
\begin{aligned}
&{}^{(3)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu},\\
&[{}^{(3)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}] = \text{mass/time} = \text{force/velocity} = [1/\kappa c].
\end{aligned} \tag{103}
$$

The explicit form of ${}^{(3)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}$ can be read off most conveniently from the Lagrangian (82). Both quantities, the electrodynamical axion and the axial torsion, should contribute to the axial anomaly of quantum field theory; see Obukhov [39].

Moreover, Mielke *et al.* [40] tentatively assumed that the axial torsion $\mathcal{A}$, which is a geometric quantity characterizing spacetime, can be chosen as the gradient of a pseudoscalar field $\mathcal{P}$, that is, $\mathcal{A} = d\mathcal{P}$. Subsequently, without any physical argument to support it and without an appropriate dimensional analysis, $\mathcal{P}$ is identified with the axion field a of the internal $U(1)$ symmetry of Peccei-Quinn. This is what we call an *ad hoc* assumption. Moreover, our dimensional analysis in Eqs. (102) and (103) above shows how far-fetched such an assumption is.

Similar attempts were made by Castillo-Felisola *et al.* [41]. Corral argued that they don't consider torsion as a field strength related to translational gauging, but rather that they rely on "the geometrical interpretation of *torsion*." And this would make a difference. We cannot share this optimism: What else other than a geometric quantity is a translational gauge field strength, after all?

One could try the ansatz, with the superscript ${}^{(\vartheta)}$ denoting the constitutive tensor density for the coframe Lagrangian (82),

$$
\hat{\chi}^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = {}^{(\vartheta)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} + a'\varepsilon^{\beta\gamma\nu\rho}g_{\alpha\mu}, \tag{104}
$$

in order to link (102) with (103). However, the trace via $g^{\alpha\mu}$ of (104) can never yield the axion, unless one introduces in an *ad hoc* fashion a dimensionful factor in a'. In other words, in this way one cannot find an axion in a natural way.

The $U(1)$ axion is related to the *internal* group $U(1)$, whereas the axial torsion is related to the *external* translation group $T(4)$ via the Cartan circuit. One should not marry internal and external groups, unless one investigates supersymmetry, which allows such a mixing under certain circumstances.

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APPENDIX: VARIATION OF THE COFRAME LAGRANGIAN

We start with the premetric coframe Lagrangian,

$$\Lambda = \frac{1}{2} F^\alpha \wedge H_\alpha(F^\beta). \tag{A1}$$

Substituting the components of the forms (18), (26), we obtain

$$\Lambda = \frac{1}{8}(F_{\beta\gamma}{}^\alpha \check{H}^{\mu\nu}{}_\alpha)\vartheta^{\beta\gamma} \wedge \epsilon_{\mu\nu}. \tag{A2}$$

Applying the relation, which is a direct consequence of (8),

$$\vartheta^{\beta\gamma} \wedge \epsilon_{\mu\nu} = (\delta^\gamma_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\gamma_\nu)\mathrm{vol}, \tag{A3}$$

we derive the coframe Lagrangian in components,

$$\Lambda = \frac{1}{4}(F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha)\mathrm{vol}. \tag{A4}$$

Consequently, the variation of the Lagrangian takes the form

$$\delta\Lambda = \frac{1}{4}[\delta(F_{\beta\gamma}{}^\alpha)\check{H}^{\beta\gamma}{}_\alpha + F_{\beta\gamma}{}^\alpha \delta(\check{H}^{\beta\gamma}{}_\alpha)]\mathrm{vol} + \frac{1}{4}(F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha)\delta(\mathrm{vol}). \tag{A5}$$

Applying the local and linear constitutive relation (47) together with its symmetry property (81), we find

$$F_{\beta\gamma}{}^\alpha \delta(\check{H}^{\beta\gamma}{}_\alpha) = F_{\beta\gamma}{}^\alpha \chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu \delta(F_{\nu\rho}{}^\mu) = \delta(F_{\nu\rho}{}^\mu)\chi^{\nu\rho}{}_\mu{}^{\beta\gamma}{}_\alpha F_{\beta\gamma}{}^\alpha = \delta(F_{\nu\rho}{}^\mu)\check{H}^{\nu\rho}{}_\mu. \tag{A6}$$

Thus, Eq. (A5) takes the form

$$\delta\Lambda = \frac{1}{2}\delta(F_{\beta\gamma}{}^\alpha)\check{H}^{\beta\gamma}{}_\alpha \mathrm{vol} + \frac{1}{4}F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha \delta(\mathrm{vol}). \tag{A7}$$

In order to calculate the variation $\delta(F_{\beta\gamma}{}^\alpha)$, we use (26):

$$\delta(d\vartheta^\alpha) = \frac{1}{2}\delta(F_{\beta\gamma}{}^\alpha)\vartheta^{\beta\gamma} + F_{\beta\gamma}{}^\alpha \delta\vartheta^\beta \wedge \vartheta^\gamma. \tag{A8}$$

Hence,

$$\delta(F_{\beta\gamma}{}^\alpha) = e_\gamma \rfloor e_\beta \rfloor d(\delta\vartheta^\alpha) - F_{\beta\mu}{}^\alpha e_\gamma \rfloor(\delta\vartheta^\mu) + F_{\gamma\mu}{}^\alpha e_\beta \rfloor(\delta\vartheta^\mu). \tag{A9}$$

Thus, the first term of (A7) reads

$$\frac{1}{2}\delta(F_{\beta\gamma}{}^\alpha)\check{H}^{\beta\gamma}{}_\alpha \mathrm{vol} = -d(\delta\vartheta^\alpha) \wedge H_\alpha - F_{\beta\mu}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha \delta\vartheta^\mu \wedge (e_\gamma \rfloor \mathrm{vol}). \tag{A10}$$

In order to calculate the variation of the volume element, we apply the formula

$$\delta(\mathrm{vol}) = \delta\vartheta^\mu \wedge (e_\mu \rfloor \mathrm{vol}). \tag{A11}$$

Accordingly, the variation of the coframe Lagrangian (A7) takes the form

$$\delta\Lambda = -d(\delta\vartheta^\alpha) \wedge H_\alpha - \Sigma_\alpha \wedge \delta\vartheta^\alpha, \tag{A12}$$

where

$$\Sigma_\alpha = \left(F_{\beta\alpha}{}^\mu \check{H}^{\beta\rho}{}_\mu - \frac{1}{4}\delta^\rho_\alpha F_{\beta\gamma}{}^\mu \check{H}^{\beta\gamma}{}_\mu\right)(e_\rho \rfloor \mathrm{vol}). \tag{A13}$$

Using the components of the forms (18), (26), we obtain (39) and (41). We extract the total derivative as in (A12) and obtain finally

$$\delta\Lambda = -d(\delta\vartheta^\alpha \wedge H_\alpha) - \delta\vartheta^\alpha \wedge (dH_\alpha - \Sigma_\alpha). \tag{A14}$$

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