Casimir effect in extended theories of gravity

G. Lambiase,[1,2,*] A. Stabile,[3,4,†] and An. Stabile[1,2,‡]

[1] *Dipartimento di Fisica "E.R. Caianiello", Università degli Studi di Salerno, via G. Paolo II, Stecca 9, I—84084 Fisciano, Italy*

[2] *Dipartimento di Fisica "E.R. Caianiello", Istituto Nazionale di Fisica Nucleare (INFN) Sezione di Napoli, Gruppo collegato di Salerno, 84084 Fisciano, Italy*

[3] *Dipartimento di Ingegneria, Università degli Studi del Sannio, Palazzo Dell'Aquila Bosco Lucarelli, Corso Garibaldi, 107- 82100 Benevento, Italy*

[4] *Istituto Nazionale di Fisica Nucleare (INFN) Sezione di Napoli, Complesso Universitario di Monte SantAngelo, Edificio G, Via Cinthia, I-80126 Napoli, Italy*

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We study the Casimir vacuum energy density and the Casimir pressure for a massless scalar field confined between two nearby parallel plates in a slightly curved, static spacetime background, employing the weak-field approximation in the framework of extended theories of gravity (ETG). Following a perturbative approach, we find the gravity corrections to the Casimir vacuum energy density and pressure. The corrections to the vacuum energy density in the framework of general relativity (GR) are small, and today they are still undetected with current technology. However, future sensitivity improvements in gravitational interferometer experiments will give a useful tool to detect such effects induced by gravity. For these reasons, which are interesting from a theoretical point of view, we generalize the outcomes of GR in the context of ETG. Finally, we find the general relation to constrain the free parameters of the ETG.

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I. INTRODUCTION

Our Universe appears to be spatially flat and undergoing a period of accelerated expansion. Observational data has been used to probe this picture [1–6], but two unrevealed ingredients are needed in order to achieve this dynamical scenario, namely, *dark matter* on galactic and extragalactic scales, and *dark energy* on cosmological scales. The dynamical evolution of self-gravitating structures can be explained within Newtonian gravity, but a dark matter component is required in order to obtain agreement with observations [7].

Lately, *models of extended gravity* [8,9] have been considered as a viable theoretical mechanism to explain cosmic acceleration and galactic rotation curves. In such models one extends only the geometric sector, without introducing any exotic matter. Such models may result from an effective theory of a quantum gravity formulation (which may contain additional contributions with respect to general relativity) on galactic, extragalactic, and cosmological scales where otherwise large amounts of unknown dark components are required. In the context of models of extended gravity, one may consider that the gravitational interaction acts differently at different scales, while the robust results of general relativity (GR) on local and Solar System scales are preserved [10].

In the so-called weak-field approximation, any relativistic theory of gravitation in general yields corrections to the gravitational potentials (e.g., Ref. [11]) which—at the Newtonian and post-Newtonian levels—could constitute the test bed for these theories [12]. In fact, in extended theories of gravity (ETG) there are further gravitational degrees of freedom, and moreover the form of the gravitational interaction is no longer scale invariant. Hence, in a given situation, besides the Schwarzschild radius, other characteristic gravitational scales could arise from the dynamics. Such scales, in the weak-field approximation, should exhibit a form of gravitational confinement in this way [13].

Models of fourth-order gravity have been studied mainly in the Newtonian limit (weak-field and small velocity) [14,15], as well as in the Minkowskian limit [16]. In the former one finds modifications of the gravitational potential, while in the latter one obtains massive gravitational-wave modes [17,18]. The weak-field limit of such proposals has to be tested against realistic self-gravitating systems. Galactic rotation curves, stellar hydrodynamics, and gravitational lensing appear as natural candidates as test-bed experiments [19–24]. These corrections to the gravitational Lagrangian were already considered by several authors [25–39]. From a conceptual viewpoint, there is no reason *a priori* to restrict the gravitational Lagrangian to a linear function of the Ricci scalar minimally coupled to matter [40]. In particular, one may consider the generalization of $f(\mathcal{R})$ models

*lambiase@sa.infn.it
†arturo.stabile@gmail.com
‡anstabile@gmail.com

(where $\mathcal{R}$ is the Ricci scalar) through generic functions containing curvature invariants—such as the *Ricci squared* ($\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}$) or the *Riemann squared* ($\mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}^{\alpha\beta\gamma\delta}$)—that are not independent due to the Gauss-Bonnet invariant ($\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\sigma}\mathcal{R}^{\mu\nu\lambda\sigma}$). Note that the same remark applies to the Weyl invariant $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$. Hence, one may add a (massive propagating) scalar field coupled to geometry; this leads to the *scalar-tensor fourth-order gravity*.

A fundamental issue is the possibility that ETG can be plagued by pathologies, such as the appearance of ghosts (negative-norm states), which could allow for negative possibilities and consequently violations of unitarity [41–44]. In particular, while standard GR and the Gauss-Bonnet theory have the same field content, this is not the case for $f(\mathcal{R})$ gravity and Weyl gravity. The former is safe, even though it does not improve the ultraviolet behavior of the theory; $f(\mathcal{R})$ gravity has just an extra scalar and can be ghost free in its Newtonian limit [45]. The latter has an extra pathological spin-2 field, which however causes no problem in the low-energy regime since then the effects of higher-order terms give rise to small corrections to GR [45,46]. In our analysis, we will consider an action motivated from noncommutative geometry within the class of Weyl-type gravity. However, as we will discuss, this proposal will be considered in the spirit of an effective field theory, and hence it is free from any pathologies.

In this paper we consider the Casimir effect in curved space where the metric is given by a modified theory of gravity. Generally speaking, the Casimir effect [47,48] can be defined as the stress on the bounding surfaces when a quantum field is confined in a finite volume of space. The confinement of a physical field obviously restricts the modes of the corresponding quantum field, giving us measurable macroscopic effects. The Casimir effect has been widely discussed in flat space [49–59] and has been shown to have good agreement with the theory. Recently, some authors [60–69] have considered the influence of a gravitational field on the vacuum energy density of a quantum field inside a cavity. Indeed, possible modifications in the vacuum energy induced by gravity could play a relevant role in the dynamics of the Universe [62,70]. Yet, at a microscopic level, modifications of the Casimir energy induced by strong gravitational fields could become relevant in models of quark confinement based on string interquark potentials [71–74]. In connection with the equivalence principle, there is a further question, namely, whether vacuum fluctuations gravitate or not. Finally, the analysis of the possible gravitational effects on Casimir cavities faces the open issue concerning the limits of validity of GR at small distances [75]. In this paper we perform a perturbative evaluation up to second order of the gravitational corrections to the Casimir vacuum energy density for a massless scalar field confined in a cavity in a slightly curved static spacetime background. Our starting point is Ref. [63], where the spacetime was given by the approximated Schwarzschild solution, while we consider the Newtonian limit of the solution in the case of ETG.

The paper is organized as follows. In Sec. II we investigate the weak-field limit of a particular ETG, i.e., scalar-tensor-higher-order models, in view of constraining their parameters by the analysis of the Casimir effect. In Sec. III we analyze the dynamic of a massless scalar field in a weak gravitational field. Finally, in Sec. IV we discuss the theoretical constraints of the ETG considered. Our conclusions are drawn in Sec. V.

II. SCALAR-TENSOR FOURTH-ORDER GRAVITY

Let us consider the action

$$\mathcal{S} = \int d^4x\sqrt{-g}[f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m], \tag{1}$$

where f is an unspecified function of the Ricci scalar $\mathcal{R}$, the curvature invariant $\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} \doteq Y$ (where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor), and ϕ is a scalar field. We note that the Riemann tensor can be discarded since the Gauss-Bonnet invariant fixes it in the action (for details, see Ref. [76]). Here $\mathcal{L}_m$ is the Lagrangian density of the ordinary matter, $\omega(\phi)$ is a generic function of the scalar field, g is the determinant of the metric tensor $g_{\mu\nu}$, and[1] $\mathcal{X} = 8\pi G$.

In the metric approach, namely, when the gravitational field is fully described by only the metric tensor $g_{\mu\nu}$, the field equations are obtained by varying the action (1) with respect to $g_{\mu\nu}$, leading to

$$\begin{aligned} &f_{\mathcal{R}}\mathcal{R}_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2}g_{\mu\nu} \\ &\quad - f_{\mathcal{R};\mu\nu} + g_{\mu\nu}\Box f_{\mathcal{R}} + 2f_Y\mathcal{R}_\mu{}^\alpha\mathcal{R}_{\alpha\nu} \\ &\quad - 2[f_Y\mathcal{R}^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y\mathcal{R}_{\mu\nu}] + [f_Y\mathcal{R}_{\alpha\beta}]^{;\alpha\beta}g_{\mu\nu} \\ &\quad + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}T_{\mu\nu}, \end{aligned} \tag{2}$$

where $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter, $f_{\mathcal{R}} = \frac{\partial f}{\partial\mathcal{R}}$, $f_Y = \frac{\partial f}{\partial Y}$, and $\Box = {}_{;\sigma}{}^{;\sigma}$ is the D'Alembert operator. For the Ricci tensor we use the convention $\mathcal{R}_{\mu\nu} = \mathcal{R}^\sigma{}_{\mu\sigma\nu}$, while for the Riemann tensor we define $\mathcal{R}^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \cdots$. The affinity connections are the usual Christoffel symbols of the metric, namely, $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$, and we adopt the signature $(+,-,-,-)$. The trace of the field equation above reads

$$\begin{aligned} &f_{\mathcal{R}}\mathcal{R} + 2f_Y\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} - 2f + \Box[3f_{\mathcal{R}} + f_Y\mathcal{R}] + 2[f_Y\mathcal{R}^{\alpha\beta}]_{;\alpha\beta} \\ &\quad - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}T, \end{aligned} \tag{3}$$

[1]Here we use the convention $c = 1$.

where $T = T^{\sigma}{}_{\sigma}$ is the trace of energy-momentum tensor. Finally, by varying the action (1) with respect to the scalar field ϕ, we obtain the Klein-Gordon field equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi = 0, \tag{4}$$

where $\omega_\phi = \frac{d\omega}{d\phi}$ and $f_\phi = \frac{df}{d\phi}$.

A. Solutions for a point-like source in the weak-field limit

For many physical systems the study is carried out in the weak-field approximation; in particular, for our aim the Newtonian limit of the theory is adequate. In order to perform our approximation we have to perturb Eqs. (2), (3), and (4) in a Minkowski background $\eta_{\mu\nu}$ [14,77]. Neglecting the technical aspects, we can write the expression for the metric tensor $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} \cong \begin{pmatrix} 1 + g_{tt}^{(2)}(t,\mathbf{x}) + \cdots & 0 \\ 0 & -\delta_{ij} + g_{ij}^{(2)}(t,\mathbf{x}) + \cdots \end{pmatrix}$$
$$\doteq \begin{pmatrix} 1 + 2\Phi & 0 \\ & -\delta_{ij} + 2\Psi\delta_{ij} \end{pmatrix},$$
$$\phi \cong \phi^{(0)} + \phi^{(2)} + \cdots \doteq \phi^{(0)} + \varphi, \tag{5}$$

where Φ, Ψ, and φ are proportional to the power c^{-2} (Newtonian limit). The function f, up to the c^{-4} order, can be developed as

$$f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}, \phi) = f_{\mathcal{R}}(0,0,\phi^{(0)})\mathcal{R} + \frac{f_{\mathcal{R}\mathcal{R}}(0,0,\phi^{(0)})}{2}\mathcal{R}^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 + f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\mathcal{R}\phi + f_Y(0,0,\phi^{(0)})\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}, \tag{6}$$

while all other possible contributions in f are negligible [14,15,78]. By introducing the quantities

$$m_{\mathcal{R}}{}^2 \doteq -\frac{f_{\mathcal{R}}(0,0,\phi^{(0)})}{3f_{\mathcal{R}\mathcal{R}}(0,0,\phi^{(0)}) + 2f_Y(0,0,\phi^{(0)})},$$
$$m_Y{}^2 \doteq \frac{f_{\mathcal{R}}(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})},$$
$$m_\phi{}^2 \doteq -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}, \tag{7}$$

and setting $f_{\mathcal{R}}(0,0,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$ for simplicity, we get the complete set of differential equations

$$(\triangle - m_Y{}^2)\triangle\Phi + \left[\frac{m_Y{}^2}{2} - \frac{m_{\mathcal{R}}{}^2 + 2m_Y{}^2}{6m_{\mathcal{R}}{}^2}\triangle\right]\mathcal{R} + m_Y{}^2 f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\triangle\varphi = -m_Y{}^2\mathcal{X}\rho,$$
$$\left\{(\triangle - m_Y{}^2)\triangle\Psi - \left[\frac{m_Y{}^2}{2} - \frac{m_{\mathcal{R}}{}^2 + 2m_Y{}^2}{6m_{\mathcal{R}}{}^2}\triangle\right]\mathcal{R} - m_Y{}^2 f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\triangle\varphi\right\}\delta_{ij}$$
$$+ \left\{(\triangle - m_Y{}^2)(\Psi - \Phi) + \frac{m_{\mathcal{R}}{}^2 - m_Y{}^2}{3m_{\mathcal{R}}{}^2}\mathcal{R} + m_Y{}^2 f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\varphi\right\}_{,ij} = 0,$$
$$(\triangle - m_{\mathcal{R}}{}^2)\mathcal{R} - 3m_{\mathcal{R}}{}^2 f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\triangle\varphi = m_{\mathcal{R}}{}^2\mathcal{X}\rho,$$
$$(\triangle - m_\phi{}^2)\varphi + f_{\mathcal{R}\phi}(0,0,\phi^{(0)})\mathcal{R} = 0, \tag{8}$$

where the energy-momentum tensor $T_{\mu\nu}$ has also been expanded in the case of a perfect fluid when the pressure is negligible with respect to the mass density ρ.

The last two equations in Eq. (8) are a coupled system and—for a point-like source $\rho(\mathbf{x}) = M\delta(\mathbf{x})$—admit the solutions [78]

$$\varphi(\mathbf{x}) = \sqrt{\frac{\xi}{3}}\frac{2GM}{|\mathbf{x}|}\frac{1}{\omega_+ - \omega_-}[e^{-m_+|\mathbf{x}|} - e^{-m_-|\mathbf{x}|}],$$
$$\mathcal{R}(\mathbf{x}) = -2m_{\mathcal{R}}{}^2\frac{GM}{|\mathbf{x}|}\frac{1}{\omega_+ - \omega_-}[(\omega_+ - \eta^2)e^{-m_+|\mathbf{x}|} - (\omega_- - \eta^2)e^{-m_-|\mathbf{x}|}], \tag{9}$$

where

$$\omega_\pm = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2},$$
$$m_\pm^2 = m_{\mathcal{R}}^2\omega_\pm,$$
$$\xi = 3f_{\mathcal{R}\phi}(0,0,\phi^{(0)})^2,$$
$$\eta = \frac{m_\phi}{m_{\mathcal{R}}}.$$

The solutions of the gravitational potential Φ and Ψ, derived from the first two equations of Eq. (8), are

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)e^{-m_+|\mathbf{x}|} + \left[\frac{1}{3} - g(\xi,\eta)\right]e^{-m_-|\mathbf{x}|} - \frac{4}{3}e^{-m_Y|\mathbf{x}|}\right], \quad (10)$$

$$\Psi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 - g(\xi,\eta)e^{-m_+|\mathbf{x}|} - [1/3 - g(\xi,\eta)]e^{-m_-|\mathbf{x}|} - \frac{2}{3}e^{-m_Y|\mathbf{x}|}\right], \quad (11)$$

where

$$g(\xi,\eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi-1)^2 - 2\eta^2(\xi+1)}}{6\sqrt{\eta^4 + (\xi-1)^2 - 2\eta^2(\xi+1)}}.$$

Note that for $f_Y \to 0$, i.e., $m_Y \to \infty$, we obtain the same outcome for the gravitational potential as in Ref. [78] for a $f(\mathcal{R},\phi)$ theory. The absence of the coupling term between the curvature invariant Y and the scalar field ϕ as well as the linearity of the field equations (8) guarantees that the solution (10) is a linear combination of solutions obtained within a $f(\mathcal{R},\phi)$ theory and a $\mathcal{R} + Y/{m_Y}^2$ theory, generalizing the outcomes of the theory $f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta})$ [15].

III. DYNAMIC OF A MASSLESS SCALAR FIELD IN A WEAK GRAVITATIONAL FIELD

In a curved spacetime the massless scalar field $\psi(t,\mathbf{x})$ obeys the following field equation [79]:

$$(\Box + \xi\mathcal{R})\psi(t,\mathbf{x}) = \frac{1}{\sqrt{-g}}\partial_\alpha[\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi(t,\mathbf{x})] + \xi\mathcal{R}\psi(t,\mathbf{x}) = 0, \quad (12)$$

where ξ is a coupling parameter between geometry and matter. In our ETG framework and in the vacuum, the curvature scalar $\mathcal{R}$ is different from zero [see Eq. (9)]. For simplicity we consider the massless scalar field $\psi(t,\mathbf{x})$ confined between two parallel plates separated by a distance L and with extension S, placed at a distance R from the gravitational nonrotating source ($R \gg L, \sqrt{S}$) (see Fig. 1).

We choose a reference frame with the origin at one of the plates and the z axis along the radial direction. We can expand the metric tensor components—using the gravitational potentials $\Phi(\mathbf{x})$ [Eq. (10)], $\Psi(\mathbf{x})$ [Eq. (11)], and $\mathcal{R}(\mathbf{x})$ [Eq. (9)]—around the distance R along the z direction as follows:

$$\begin{aligned} g_{00}(\mathbf{x}) &\simeq 1 + 2\Phi_0(R) + 2\Lambda(R)z, \\ g_{ij}(\mathbf{x}) &\simeq -1 + 2\Psi_0(R) + 2\Sigma(R)z, \\ \mathcal{R}(\mathbf{x}) &\simeq \mathcal{R}_1(R) + \mathcal{R}_2(R)z, \end{aligned} \quad (13)$$

where

$$\begin{aligned}
\Phi_0(R) &= -\frac{GM}{R}\left[1 + g(\xi,\eta)e^{-m_+R} + \left(\frac{1}{3} - g(\xi,\eta)\right)e^{-m_-R} - \frac{4}{3}e^{-m_YR}\right], \\
\Lambda(R) &= \frac{GM}{R^2}\left[1 + g(\xi,\eta)(1+m_+R)e^{-m_+R} + \left(\frac{1}{3} - g(\xi,\eta)\right)(1+m_-R)e^{-m_-R} + -\frac{4}{3}(1+m_YR)e^{-m_YR}\right], \\
\Psi_0(R) &= -\frac{GM}{R}\left[1 - g(\xi,\eta)e^{-m_+R} - \left(\frac{1}{3} - g(\xi,\eta)\right)e^{-m_-R} - \frac{2}{3}e^{-m_YR}\right], \\
\Sigma(R) &= \frac{GM}{R^2}\left[1 - g(\xi,\eta)(1+m_+R)e^{-m_+R} - \left(\frac{1}{3} - g(\xi,\eta)\right)(1+m_-R)e^{-m_-R} + -\frac{2}{3}(1+m_YR)e^{-m_YR}\right], \\
\mathcal{R}_1(R) &= -2{m_\mathcal{R}}^2\frac{GM}{R}\frac{(\omega_+ - \eta^2)e^{-m_+R} - (\omega_- - \eta^2)e^{-m_-R}}{\omega_+ - \omega_-}, \\
\mathcal{R}_2(R) &= 2{m_\mathcal{R}}^2\frac{GM}{R^2}\frac{(\omega_+ - \eta^2)(1+m_+R)e^{-m_+R} - (\omega_- - \eta^2)(1+m_-R)e^{-m_-R}}{\omega_+ - \omega_-}.
\end{aligned} \quad (14)$$

Using the metric (13), the field equation for the scalar field $\psi(t,\mathbf{x})$ [Eq. (12)] becomes

$$\ddot{\psi}(t,\mathbf{x}) - [1 + 4\Phi + 4\gamma z]\Delta\psi(t,\mathbf{x}) + \xi[\mathcal{R}_1 + \mathcal{R}_2 z]\psi(t,\mathbf{x}) = 0, \quad (15)$$

where a dot represents a derivative with respect to t, and

$$\Phi \equiv \Phi_0(R) + \Psi_0(R) = -\frac{2GM}{R}[1 - e^{-m_YR}], \qquad \gamma \equiv \Lambda(R) + \Sigma(R) = \frac{2GM}{R^2}[1 - (1+m_YR)e^{-m_YR}]. \quad (16)$$

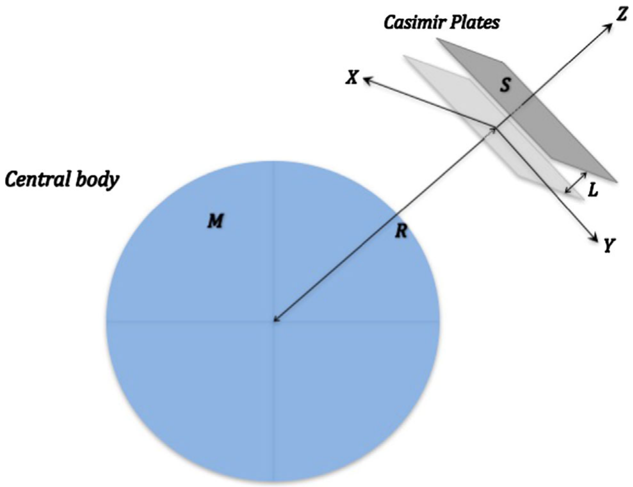

FIG. 1. The configuration of the Casimir-like system in a gravitational field.

To solve Eq. (15) we use the ansatz

$$\psi(t,\mathbf{x}) = N\Xi(z)e^{i(\omega t-\mathbf{k}_\perp\cdot\mathbf{x}_\perp)}, \tag{17}$$

where $\mathbf{k}_\perp \equiv (k_x,k_y)$, $\mathbf{x}_\perp \equiv (x,y)$, and N is a normalization constant. The field equation (15) becomes

$$(\partial_\lambda^2+\lambda)\Xi(\lambda)=0, \tag{18}$$

where

$$\begin{aligned}
&\lambda \equiv \lambda(z) = \alpha - \beta z \equiv \zeta\theta^{-2/3} - \theta^{1/3}z,\\
&\zeta = (1-4\Phi)\omega^2 - \mathbf{k}_\perp^2 - \xi\mathcal{R}_1,\\
&\theta = 4\omega^2\gamma + \xi\mathcal{R}_2.
\end{aligned} \tag{19}$$

The solution of Eq. (18) can be given in terms of Bessel functions:

$$\Xi(\lambda) = \sqrt{\lambda}\left[C_1 J_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right) + C_2 J_{-1/3}\left(\frac{2}{3}\lambda^{3/2}\right)\right], \tag{20}$$

where C_1 and C_2 are constants. We note that $\lambda \gg 1$ because $\theta \ll \zeta$. Then we can use the asymptotic form of Eq. (20):

$$\Xi(\lambda) \simeq \sqrt{\frac{3}{\pi\sqrt{\lambda}}}\sin\left[\frac{2}{3}\lambda^{3/2}+\tau\right]. \tag{21}$$

If we assume that the field ψ satisfies Dirichlet boundary conditions on the plates, that is,

$$\psi(z=0)=\psi(z=L)=0, \tag{22}$$

after some algebra, we find the relation

$$\frac{2}{3}[\lambda^{3/2}(0) - \lambda^{3/2}(L)] = n\pi, \tag{23}$$

where n is an integer. From Eq. (23) we find the energy spectrum:

$$\omega_n^2 = (1+4\Phi+2\gamma L)\left[\mathbf{k}_\perp^2 + \left(\frac{n\pi}{L}\right)^2\right] + \xi\left[\mathcal{R}_1 + \frac{1}{2}\mathcal{R}_2 L\right]. \tag{24}$$

This relation shows that the dependence on the parameters $m_\mathcal{R}$ and m_ϕ appears in the second term on the right-hand side of Eq. (24), i.e., the curvature terms. In other words, if the coupling parameter ξ is zero the energy spectrum in the case of $f(\mathcal{R},\phi)$ gravity reduces to that of GR.

Finally, using the scalar product defined in the theory of quantum fields in curved spacetimes,

$$(\psi_n,\psi_m) = -i\int_V [(\partial_\mu\psi_n)\psi_m^* - \psi_n(\partial_\mu\psi_m)^*] \times \sqrt{-g_3}\,n^\mu dxdydz, \tag{25}$$

we derive the normalization constant

$$N_n^2 = \frac{\alpha\beta}{3S\omega_n n(1-\Phi_0)}. \tag{26}$$

A. Casimir vacuum energy density

To calculate the mean vacuum energy density $\mathcal{E}$ between the plates, we use the general relation [79]

$$\mathcal{E} = \frac{1}{V_P}\sum_n \int d^2\mathbf{k}_\perp \int dxdydz\sqrt{-g_3}(g_{00})^{-1}T_{00}(\psi_n,\psi_n^*), \tag{27}$$

where

$$\begin{aligned}
&V_P = \int dxdydz\sqrt{-g_3} \simeq SL\left[1-3\Psi_0 - \frac{3}{2}\Sigma L\right],\\
&T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi.
\end{aligned} \tag{28}$$

V_P is the proper volume and $T_{00}(\psi_n,\psi_n^*)$ is a component of the energy-momentum tensor. Using the Schwinger proper-time representation and ζ-function regularization, we find the mean vacuum energy density:

$$\mathcal{E} = -[1+3(\Phi_0-\Psi_0) - (2\Sigma-\Lambda)L_P]\frac{\pi^2}{1440L_P^4} + \frac{\xi\mathcal{R}_2}{192L_P}, \tag{29}$$

where L_P is the proper length of the cavity,

$$L_P = \int dz\sqrt{-g_3} \simeq L\left[1 - \Psi_0 - \frac{1}{2}\Sigma L\right],$$

$$\Phi_0 - \Psi_0 = -\frac{2GM}{R}\left[g(\xi,\eta)e^{-m_+R} + \left(\frac{1}{3} - g(\xi,\eta)\right)e^{-m_-R} - \frac{1}{3}e^{-m_YR}\right],$$

$$2\Sigma - \Lambda = \frac{GM}{R^2}[1 - 3g(\xi,\eta)(1+m_+R)e^{-m_+R} - (1-3g(\xi,\eta))(1+m_-R)e^{-m_-R}]. \qquad (30)$$

The relation (29) gives the corrections to the Casimir vacuum energy density, up to the order $o(M/R^2)$.[2] Contrary to GR results, where corrections to the Casimir energy density occur to second order (M/R^2), in ETG one obtains corrections to first order, $o(M/R)$. This is a very interesting result, because it provides a physical quantity that directly tests ETG. Finally, we note that the first-order correction increases the Casimir vacuum energy density, while the second-order corrections decrease the Casimir vacuum energy density.

B. Casimir pressure

The attractive force observed between the cavity plates is obtained from the relation $\mathcal{F} = -\frac{\partial E}{\partial L_P}$, where $E = \mathcal{E}V_P$ is the Casimir vacuum energy. In the case of ETG, one gets

$$\mathcal{F} = -\left[1 + 3(\Phi_0 - 2\Psi_0) - \frac{7\Sigma - 2\Lambda}{3}L_P\right]\frac{\pi^2 S_P}{480 L_P^4}, \qquad (31)$$

where

$$\Phi_0 - 2\Psi_0 = \frac{GM}{R}[1 - 3g(\xi,\eta)e^{-m_+R} - (1-3g(\xi,\eta))e^{-m_-R}],$$

$$7\Sigma - 2\Lambda = \frac{GM}{R^2}[5 + 9g(\xi,\eta)(1+m_+R)e^{-m_+R} + 3(1-3g(\xi,\eta))(1+m_-R)e^{-m_-R} + -8(1+m_YR)e^{-m_YR}].$$

From Eq. (31) we note that the contributions of curvature are mathematically zero. Furthermore, the correction at first order does not depend on the parameter m_Y, while those at second order depend on all of the parameters. By introducing the Casimir pressure $\mathcal{P} = \mathcal{F}/S$, one finally gets

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_{\mathrm{ETG}},$$

$$\mathcal{P}_0 = -\frac{\pi^2}{480 L_P^4},$$

$$\mathcal{P}_{\mathrm{ETG}} = \left[3(\Phi_0 - 2\Psi_0) - \frac{7\Sigma - 2\Lambda}{3}L_P\right]\mathcal{P}_0, \qquad (32)$$

[2]In Eq. (29) we have neglected all products of order higher than c^{-2}.

where $\mathcal{P}_0$ is the Casimir pressure in the flat case, while $\mathcal{P}_{\mathrm{ETG}}$ is the correction to the pressure in the context of ETG.

IV. EXPERIMENTAL CONSTRAINTS

Our aim now is to test the compatibility of an ETG with the experimental data. This can be achieved by using the pressure as a measurable physical quantity. Imposing the constraint $|\mathcal{P}_{\mathrm{ETG}}| \lesssim \delta\mathcal{P}$, where $\delta\mathcal{P}$ is the experimental error, we obtain the relation

$$\left|3(\Phi_0 - 2\Psi_0) - \frac{7\Sigma - 2\Lambda}{3}L_P\right| \lesssim \frac{\delta\mathcal{P}}{\mathcal{P}_0}, \qquad (33)$$

which gives the constraint for the free parameters (7) of the ETG.

Now we analyze some models of ETG; see Table I. Let us first consider the case A in Table I; the only relevant quantity is $m_{\mathcal{R}}$. The relation (33) implies

$$|1 - 2e^{-m_{\mathcal{R}}R_\oplus}| \lesssim \frac{2}{3}\frac{R_\oplus}{R_\oplus^S}\frac{\delta\mathcal{P}}{\mathcal{P}_0}, \qquad (34)$$

where $R_\oplus$ and $R_\oplus^S$ are the radius and Schwarzschild radius of the Earth. As a special case of $f(\mathcal{R})$ theories one can consider the polynomial expression

$$f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2 + \sum_{n=3}^{N}\alpha_n\mathcal{R}^n. \qquad (35)$$

Note, however, that the characteristic scale $m_{\mathcal{R}}$ is only generated by the $\mathcal{R}^2$ term. In the literature an interesting model of $f(\mathcal{R})$ theories is that of Starobinsky, $f(\mathcal{R}) = \mathcal{R} - \mathcal{R}^2/\mathcal{R}_0$ [80], for which $m_{\mathcal{R}}^2 = \mathcal{R}_0/6$.

To generalize the previous results one has to include the curvature invariant $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$. For case B in Table I—corresponding to the general class of $f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta})$ theories and their characteristic scales $m_{\mathcal{R}}$ and m_Y—we use Eq. (33) to obtain

$$\left|1 - 2e^{-m_{\mathcal{R}}R_\oplus} - \frac{1}{6}\frac{L_P}{R_\oplus}[5 + 6(1+m_{\mathcal{R}}R_\oplus)e^{-m_{\mathcal{R}}R_\oplus} - 8(1+m_YR_\oplus)e^{-m_YR_\oplus}]\right| \lesssim \frac{2}{3}\frac{R_\oplus}{R_\oplus^S}\frac{\delta\mathcal{P}}{\mathcal{P}_0}.$$

Notice that to also constrain the parameter m_Y one needs corrections up to second order in (M/R^2). This class of theories includes the Weyl-square-type model, i.e., $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - \frac{2}{3}\mathcal{R}^2$, where only one characteristic scale appears ($m_{\mathcal{R}} \to +\infty$).

The same argument is also valid for the scalar-tensor case of the theory, for which in the Newtonian limit (see case D in Table I) the more general expression (6) reads

$$\left(1 - \phi^{(0)}\sqrt{\frac{\xi}{3}}\right)\mathcal{R} + \sqrt{\frac{\xi}{3}}\mathcal{R}\phi - \frac{{m_\phi}^2}{2}(\phi - \phi^{(0)})^2. \qquad (36)$$

TABLE I. Here $f_{\mathcal{R}}(0,0,\phi^{(0)}) = 1$ and $\omega(\phi^{(0)}) = 1/2$, and for case D we set also $c_0 + c_1\phi^{(0)} = 1$.

Case	ETG	Mass definition
A	$f(\mathcal{R})$	$m_{\mathcal{R}}{}^2 = -\frac{1}{3f_{\mathcal{R}\mathcal{R}}(0)}$ $m_Y{}^2 \to \infty, m_\phi{}^2 = 0$ $\xi = 0, \eta = 0, g(\xi,\eta) = 2/3$ $m_+ = m_{\mathcal{R}}, m_- \to \infty$
B	$f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta})$	$m_{\mathcal{R}}{}^2 = -\frac{1}{3f_{\mathcal{R}\mathcal{R}}(0,0)+2f_Y(0,0)}$ $m_Y{}^2 = \frac{1}{f_Y(0,0)}, m_\phi{}^2 = 0$ $\xi = 0, \eta = 0, g(\xi,\eta) = 2/3$ $m_+ = m_{\mathcal{R}}, m_- \to \infty$
C	$f(\mathcal{R}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_{\mathcal{R}}{}^2 = -\frac{1}{3f_{\mathcal{R}\mathcal{R}}(0,\phi^{(0)})}$ $m_Y{}^2 \to \infty, m_\phi{}^2 = -f_{\phi\phi}(0,\phi^{(0)})$ $\xi = 3f_{\mathcal{R}\phi}(0,\phi^{(0)})^2, \eta = \frac{m_\phi}{m_{\mathcal{R}}}$ $m_\pm = \sqrt{\frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}}m_{\mathcal{R}}$
D	$c_0\mathcal{R} + c_1\mathcal{R}\phi + f(\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_{\mathcal{R}}{}^2 \to \infty, m_Y{}^2 \to \infty, m_\phi{}^2 = -f_{\phi\phi}(\phi^{(0)})$ $\xi = 3c_1{}^2, \eta \to 0, g(\xi, 0) = \frac{2}{3(1-\xi)}$ $m_+ \to \infty, m_- \to \frac{m_\phi}{\sqrt{1-3c_1{}^2}}$
E	$f(\mathcal{R}, \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$m_{\mathcal{R}}{}^2 = -\frac{1}{3f_{\mathcal{R}\mathcal{R}}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2 = \frac{1}{f_Y(0,0,\phi^{(0)})}, m_\phi{}^2 = -f_{\phi\phi}(0,0,\phi^{(0)})$ $\xi = 3f_{\mathcal{R}\phi}(0,0,\phi^{(0)})^2, \eta = \frac{m_\phi}{m_{\mathcal{R}}}$ $m_\pm = \sqrt{\frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}}m_{\mathcal{R}}$

Thus, for the a general scalar-tensor (ST) theory in the Newtonian limit, one can consider the model[3]

$$f_{\mathrm{ST}}(\mathcal{R}, \phi) = c_0\mathcal{R} + c_1\mathcal{R}\phi - \frac{1}{2}m_\phi{}^2(\phi - \phi^{(0)})^2 + \frac{1}{2}\phi_{;\alpha}\phi^{;\alpha}. \tag{37}$$

Since for this case $m_{\mathcal{R}} \to \infty$, $m_Y \to \infty$, $\xi = 3c_1{}^2$, $\eta \to 0$, $m_+ \to \infty$, and $m_- = \frac{m_\phi}{\sqrt{1-3c_1{}^2}}$, from Eq. (33) it follows that

$$\left|1 - \frac{2e^{-m_- R_\oplus}}{1 - 3c_1^2}\right| \lesssim \frac{2}{3}\frac{R_\oplus}{R_\oplus^S}\frac{\delta\mathcal{P}}{\mathcal{P}_0}.$$

As a special case of a scalar-tensor fourth-order gravity model (case E), we consider noncommutative spectral geometry (NCSG) [81,82], for which at a cutoff scale (set as the grand unification scale) the purely gravitational part of the action coupled to the Higgs $\mathbf{H}$ reads [83]

$$\mathcal{S}_{\mathrm{NCSG}} = \int d^4x\sqrt{-g}\left[\frac{\mathcal{R}}{2\kappa_0{}^2} + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tau_0\mathcal{R}^\star\mathcal{R}^\star + \frac{\mathbf{H}_{;\alpha}\mathbf{H}^{;\alpha}}{2} - \mu_0{}^2\mathbf{H}^2 + -\frac{\mathcal{R}\mathbf{H}^2}{12} + \lambda_0\mathbf{H}^4\right], \tag{38}$$

where $\mathcal{R}^\star\mathcal{R}^\star$ is the topological term that integrates to the Euler characteristic, and hence is nondynamical. Since the square of the Weyl tensor can be expressed in terms of $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ as $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 2\mathcal{R}_{\mu\nu}R^{\mu\nu} - \frac{2}{3}\mathcal{R}^2$, the NCSG action is a particular case of the action (1). For this model, we have the following parameters:

$$m_R \to \infty, \qquad \xi = \frac{af_0\mathbf{H}_0^2}{12\pi^2}, \qquad m_Y = \sqrt{\frac{5\pi^2(k_0^2\mathbf{H}_0 - 6)}{36f_0k_0^2}},$$
$$\eta \to 0, \qquad m_\phi = \frac{af_0}{\pi^2}(2\mu_0 - 12\lambda_0\mathbf{H}_0^2),$$
$$m_+ \to \infty, m_- \to \frac{m_\phi}{\sqrt{1-\xi}}, \tag{39}$$

where we have set $f_R(0,0,\phi^{(0)}) = \frac{1}{2k_0{}^2} - \xi_0\frac{af_0}{\pi^2}\phi^{(0)2} = 1$. Using Eq. (33), one obtains

$$\left|1 - \frac{1}{6}\frac{L_P}{R_\oplus}[5 - 8(1 + m_Y R_\oplus)e^{-m_Y R_\oplus}]\right| \lesssim \frac{2}{3}\frac{R_\oplus}{R_\oplus^S}\frac{\delta\mathcal{P}}{\mathcal{P}_0}.$$

The total absolute experimental error (in experiments measuring the Casimir pressure between Au coated

[3] With the condition $\alpha_0 + \alpha_1\phi^{(0)} = 1$.

plates by means of a micromechanical torsional oscillator at the shortest separations) is as small as 0.2% ($\delta\mathcal{P}/\mathcal{P}_0 \simeq 0.002$) of the measured Casimir pressure [84]. Current technology is far from providing a direct experiment to check the influence of gravity on the Casimir pressure.[4]

For the sake of simplicity we can analyze the case of a $f(\mathcal{R})$ theory [Eq. (34)]. The term on the right-hand side of Eq. (34) is of the order[5] $o(10^6)$, while the one on the left-hand side is of the order of unity. Such a large difference implies that, at the moment, we cannot use the Casimir experiments [84] to measure the pressure in order to constrain the parameter $m_\mathcal{R}$ of the gravitational model: to be able to constrain the free parameters, we need to improve the experimental sensitivity on Earth by at least 6 orders of magnitude, $\frac{\delta\mathcal{P}}{\mathcal{P}_0} \lesssim 10^{-9}$. However, gravitational interferometers may provide a valid framework to test ETG. They have reached a high sensitivity, and therefore they could in principle be used as a tool to detect the small effects induced by gravity on the Casimir pressure, improving the required precision by 6 orders of magnitude. Moreover, future space missions with advanced technology might allow measurements around the planet Jupiter. In fact, in this case we get the best ratio of the radii,[6] $\frac{R_J}{R_J^S} \simeq 2.5 \times 10^7$, and therefore we only have to improve the sensitivity of the instruments by 4 orders of magnitude. This also suggests that in the future Jupiter might provide a different laboratory in the Solar System to test the influence of modified gravity on Casimir-like experiments.

V. CONCLUSIONS

Working in the weak-field approximation, we have solved the field equations in a curved spacetime for a massless scalar field, following a perturbative approach up to the $o(M/R^2)$ order. We have derived the corrections to the flat spacetime Casimir vacuum energy density and pressure in the context of ETG.

For the Casimir vacuum energy density (29), we have found that GR *only* gives a contribution to the second order, and not to the first order. This is a very interesting result because future experiments with higher sensitivity may directly test the models of ETG to the first order in $o(M/R^2)$.

Finally, we have also derived the Casimir pressure, given by Eq. (32). In this case GR gives a contribution to the first order. Requiring that the corrections for a given model of ETG decrease within the experimental errors, we have imposed the constraints (33) on the free parameters of the model. However, present technology [84] does not allow for a direct measurement of gravity effects on a Casimir-like apparatus. This is because the sensitivity of the experiments needs to be improved by at least 6 orders of magnitude in order to quantify the small effects induced by gravity. For these reasons, the gravitational interferometers—owing to the achieved high sensitivity—might give a direct check of the influence of gravity on the Casimir pressure and, as a consequence, they can be considered an alternative tool for testing the ETG. We have also discussed how the planet Jupiter, with appropriate space missions around it, might be a favorite laboratory in the Solar System to test some models of ETG by measuring the Casimir pressure.

[4]We note that for the Earth $\Phi_0 - 2\Psi_0 \simeq 10^{-9}$ and $(7\Sigma - 2\Lambda)L_P \simeq 10^{-22}$.

[5]We have $\frac{R_\oplus}{R_\oplus^S} \simeq 7.2 \times 10^8$ and $\frac{\delta\mathcal{P}}{\mathcal{P}_0} \simeq 10^{-3}$.

[6]Here we report some estimations of the parameter $\frac{R}{R^S}$ for different Solar System objects: Moon $\simeq 1.6 \times 10^{10}$, Mercury $\simeq 4.9 \times 10^9$, Venus $\simeq 8.4 \times 10^8$, Mars $\simeq 3.6 \times 10^9$, Saturn $\simeq 6.9 \times 10^7$, Uranus $\simeq 1.9 \times 10^8$, Neptune $\simeq 1.7 \times 10^8$, and Pluto $\simeq 6.2 \times 10^{10}$

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