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Renormalizability in *D*-dimensional higher-order gravity

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A simple expression for calculating the classical potential in D-dimensional gravitational models is obtained through a method based on the generating functional. The prescription is then used as a mathematical tool to probe the conjecture that renormalizable higher-order gravity models—which are, of course, nonunitary—are endowed with a classical potential that is nonsingular at the origin. It is also shown that the converse of this statement is not true, which implies that the finiteness of the classical potential at the origin is a necessary but not sufficient condition for the renormalizability of the model. The systems we have utilized to verify the conjecture were fourth- and sixth-order gravity models in D dimensions. A discussion about the polemic question related to the renormalizability of new massive gravity, which Oda claimed to be renormalizable in 2009 and which was shown to be nonrenormalizable by Muneyuki and Ohta three years later, is considered. We remark that the solution of this issue is straightforward if the aforementioned conjecture is employed. We point out that our analysis is restricted to local models in which the propagator has simple and real poles.

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I. INTRODUCTION

Higher-order gravity models are prime candidates as far as the construction of a renormalizable gravity theory is concerned. In fact, the higher-order terms of these systems are responsible, in general, for taming the wild ultraviolet divergences present in the Einstein-Hilbert action. In addition, as is well known, a pacific coexistence between renormalizability and unitarity is generally not attained in these models.

Recently, many authors [1–19] have addressed the problem of verifying a conjecture that—as far as we know—was hinted at for the first time by Stelle [20,21] in his analysis of the renormalizability of fourth-order gravity in four dimensions: renormalizable higher-order gravity models are endowed with a classical potential lacking a singularity at the origin. Nonetheless, neither Stelle nor the subsequent authors up to now seemed to perceive in their guesstimates that the converse of this premise is not true.

Our main goal here is to probe via some specific models whether the finiteness of the potential at the origin is a necessary but not sufficient condition for the renormalizability of the model.

A natural question must then be posed. What is the utility of this conjecture? The advantages that result from this surmise are very relevant. Indeed, by simply computing the classical potential at the origin, we can be absolutely certain that any higher-derivative gravity model with a divergent potential at the origin is nonrenormalizable. Additionally, if we are uncertain about the renormalizability of a given system—as is the case of new massive gravity (NMG) [22–25], which Oda [26] claimed to be renormalizable and, three years later, Muneyuki and Ohta [27] showed to be nonrenormalizable—using our conjecture, we would promptly conclude that this system is nonrenormalizable since its gravitational potential is singular at the origin. If we make a detailed comparison between the simplicity of our premise and the difficult computations required by the ordinary methods of quantum field theory, we come to the conclusion that our surmise is much easier to handle in the cases just mentioned. It is important to recall that the task of proving the renormalizability of a given higher-order gravity model is very hard work even for the experts on the subject, which can be easily seen by leafing through the aforementioned articles [26,27], as well as the ones by Stelle [20], Antoniadis and Tomboulis [28], and Johnston [29].

The models we shall use to probe the specified conjecture are fourth- and sixth-order gravity systems in D dimensions, and a particular sixth-order gravity system in four dimensions. They are defined by the following actions:

$$I^{(\mathrm{fourth-order})} = \int d^D x \sqrt{|g|}\left[\frac{2\sigma}{\kappa^2}R + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu} + \frac{\gamma}{2}R^2_{\mu\nu\alpha\beta} - \mathcal{L}_{\mathrm{M}}\right], \tag{1}$$

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$$I^{(\text{sixth-order})} = \int d^D x \sqrt{|g|} \frac{1}{\kappa^2} \left[2R + \frac{\alpha_0}{2} R^2 + \frac{\beta_0}{2} R^2_{\mu\nu} + \frac{\gamma_0}{2} R^2_{\mu\nu\alpha\beta} + \frac{\alpha_1}{2} R \Box R + \frac{\beta_1}{2} R_{\mu\nu} \Box R^{\mu\nu} + \frac{\gamma_1}{2} R_{\mu\nu\alpha\beta} \Box R^{\mu\nu\alpha\beta} - \mathcal{L}_{\text{M}} \right], \quad (2)$$

$$I = \int d^4 x \sqrt{|g|} \left(\frac{2}{\kappa^2} R + \alpha_0' R^2 + a_1' R \Box R + b_0' R^2_{\mu\nu} - \mathcal{L}_{\text{M}} \right),$$

where $\sigma = \pm 1, \alpha, \beta, \gamma, \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1, \alpha_0', a_1', b_0'$ are arbitrary constants, $\kappa^2 = 4\kappa_D$, and $\mathcal{L}_{\text{M}}$ is the Lagrangian for matter, with

$$\kappa_D = \left(\frac{D-2}{D-3} \right) G_D \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \quad (3)$$

being the D-dimensional Einstein constant for $D > 3$ (see Appendix A). Here, G_D is the Newton constant in D dimensions ($D > 3$), and Γ is the gamma function. Note that κ_D reduces to its usual value in four dimensions, namely, $\kappa_D = 8\pi G_4$. We remark also that the Einstein constant in $D = 3$ cannot be related to G_3 since general relativity in three dimensions is trivial and has no Newtonian limit. Nevertheless, for simplicity's sake, κ_3 will be used from now on as the symbol for the Einstein constant in $D = 3$, although it is unrelated to G_3.

Now since, in order to probe the conjecture at hand, we are required to compute the gravitational potential, the efficiency with which we will make the verification of this surmise will heavily depend on how skilled we are in building out a simple prescription for calculating this potential. Accordingly, in Sec. II we construct a straightforward method for calculating the D-dimensional gravity potential based on the generating functional. Using this prescription, the conjecture is verified for fourth- and sixth-order gravity models in D dimensions in Secs. III and IV, respectively. We point out that the analysis of the tree-level unitary of the aforementioned systems is made in the respective sections. The aim of this study is to confirm the general premise that renormalizable higher-order models are nonunitary. We present our conclusions in Sec. V. We remark also that, in this last section, special attention is devoted to NMG since it was the analysis of this model that inspired our conjecture.

It is worth mentioning that we will only deal with local models in which the poles of the propagator are simple and real. Technical details will be relegated to the appendixes. We use natural units throughout and our Minkowski metric is diag$(1, -1, -1, \ldots, -1)$.

II. SIMPLE PRESCRIPTION FOR CALCULATING THE D-DIMENSIONAL POTENTIAL FOR GRAVITATIONAL MODELS

From quantum field theory, we know that the generating functional for the connected Feynman diagrams $W_D(T)$ is related to the generating functional $Z_D(T)$ for linearized gravity theories by $Z_D(T) = e^{iW_D(T)}$ [30–32], where

$$W_D(T) = -\frac{\kappa_D}{2} \int d^D x d^D y T^{\mu\nu}(x) D_{\mu\nu,\alpha\beta}(x-y) T^{\alpha\beta}(y). \quad (4)$$

Here, $T^{\mu\nu}(x) (= T^{\nu\mu}(x))$ and $D_{\mu\nu,\alpha\beta}(x-y)$ are, respectively, the external conserved current and the propagator.

Now, keeping in mind that

$$D_{\mu\nu,\alpha\beta}(x-y) = \int \frac{d^D k}{(2\pi)^D} e^{ik(x-y)} D_{\mu\nu,\alpha\beta}(k),$$
$$T^{\mu\nu}(k) = \int d^D x e^{-ikx} T^{\mu\nu}(x),$$

we promptly obtain

$$W_D(T) = -\frac{\kappa_D}{2} \int \frac{d^D k}{(2\pi)^D} T^{\mu\nu}(k)^* \mathcal{P}_{\mu\nu,\alpha\beta}(k) T^{\alpha\beta}(k),$$

where $\mathcal{P}_{\mu\nu,\alpha\beta}(k)$ is the "modified propagator" in momentum space obtained by neglecting all terms of the usual Feynman propagator that are orthogonal to the external conserved currents.

Assuming then that the external conserved current is time independent, we get, from the preceding equation,

$$W_D(J) = -\frac{\kappa_D}{2} \int \frac{d^D k}{(2\pi)^{D-1}} \left[\delta(k^0) T \mathcal{P}_{\mu\nu,\alpha\beta}(k) \iint \times d^{D-1}\mathbf{x} d^{D-1}\mathbf{y} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} T^{\mu\nu}(\mathbf{x}) T^{\alpha\beta}(\mathbf{y}) \right], \quad (5)$$

where the time interval T is produced by the factor $\int dx^0$.

Simple algebraic manipulations, on the other hand, reduce (5) to the form

$$W_D(T) = -\kappa_D T \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \mathcal{P}_{\mu\nu,\alpha\beta}(\mathbf{k}) \Delta^{\mu\nu,\alpha\beta}(\mathbf{k}), \quad (6)$$

where $\mathcal{P}_{\mu\nu,\alpha\beta}(\mathbf{k}) \equiv \mathcal{P}_{\mu\nu,\alpha\beta}(k)|_{k^0=0}$, and

$$\Delta^{\mu\nu,\alpha\beta}(\mathbf{k}) \equiv \iint d^{D-1}\mathbf{x} d^{D-1}\mathbf{y} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} \frac{T^{\mu\nu}(\mathbf{x}) T^{\alpha\beta}(\mathbf{y})}{2}.$$

In the specific case of two masses M_1 and M_2 located, respectively, at $\mathbf{a_1}$ and $\mathbf{a_2}$, the current assumes the form

$$T^{\mu\nu}(\mathbf{x}) = \eta^{\mu 0}\eta^{\nu 0}[M_1\delta^{D-1}(\mathbf{x}-\mathbf{a_1}) + M_2\delta^{D-1}(\mathbf{x}-\mathbf{a_2})].$$

Therefore,

$$\Delta^{\mu\nu,\alpha\beta}(\mathbf{k}) = M_1 M_2 e^{i\mathbf{k}\cdot\mathbf{r}}\eta^{\mu 0}\eta^{\nu 0}\eta^{\alpha 0}\eta^{\beta 0}, \tag{7}$$

where $\mathbf{r} = \mathbf{a_2} - \mathbf{a_1}$.

As a consequence,

$$W_D(T) = -\kappa_D T \frac{M_1 M_2}{(2\pi)^{D-1}} \int d^{D-1}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \mathcal{P}_{00,00}(\mathbf{k}). \tag{8}$$

Bearing in mind that

$$Z_D(T) = \langle 0|e^{-iH_D T}|0\rangle = e^{-iE_D T}, \tag{9}$$

which implies that

$$E_D = -\frac{W_D(T)}{T}, \tag{10}$$

we find that the D-dimensional interparticle gravitational energy can be computed through the simple expression

$$E_D(r) = \kappa_D \frac{M_1 M_2}{(2\pi)^{D-1}} \int d^{D-1}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \mathcal{P}_{00,00}(\mathbf{k}). \tag{11}$$

Accordingly, the D-dimensional gravitational potential sourced by a mass M at rest is given by

$$V_D(r) = \kappa_D \frac{M}{(2\pi)^{D-1}} \int d^{D-1}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \mathcal{P}_{00,00}(\mathbf{k}). \tag{12}$$

Using the straightforward prescription above, it is possible to test the aforementioned conjecture easily, as will be shown in the next two sections.

III. VERIFYING THE CONJECTURE FOR FOURTH-ORDER GRAVITY SYSTEMS IN D DIMENSIONS

To find the gravitational potential, we first need to compute the propagator. Nonetheless, before obtaining this operator, it will be worthwhile to remember that this calculation demands knowledge only of the linearized quadratic part of the model. On the other hand, since linearized Gauss-Bonnet invariant is a total derivative in any spacetime dimension >3 (the restriction of $D = 4$ coming into play only when we take the full nonlinear structure into account) [33], and, in addition, both the curvature and Ricci tensors have the same number of components in $D = 3$ [34], we can drop the term of action (1) containing $R^2_{\mu\nu\alpha\beta}$ for $D > 2$ in the specified computation.

To compute the propagator, we recall that, for small fluctuations around the Minkowski metric $\eta_{\mu\nu}$, the full metric assumes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{13}$$

Linearizing the Lagrangian associated with the quadratic part of the action (1), namely,

$$\mathcal{L}^{(\text{fourth-order})} = \sqrt{|g|}\left[\frac{2\sigma}{\kappa^2}R + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right], \tag{14}$$

via the preceding equation and adding to the result the gauge-fixing Lagrangian, $\mathcal{L}_{\text{gf}} = \frac{1}{2\lambda}(\partial_\mu \gamma^{\mu\nu})^2$, where $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and λ is a gauge parameter (de Donder gauge), we find

$$\mathcal{L}^{(\text{fourth-order})} = \frac{1}{2} h_{\mu\nu}\mathcal{O}^{\mu\nu,\alpha\beta}h_{\alpha\beta}, \tag{15}$$

where, in momentum space,

$$\mathcal{O} = \left(\sigma + \frac{\beta\kappa^2 k^2}{4}\right)k^2 P^{(2)} + \frac{k^2}{2\lambda}P^{(1)} + \frac{k^2}{4\lambda}P^{(0-w)} - \frac{k^2}{4\lambda}\sqrt{D-1}[P^{(0-sw)} + P^{(0-ws)}] + \left[-(D-2)\sigma + (D-1)\alpha\kappa^2 k^2 + D\frac{\beta\kappa^2 k^2}{4} + \frac{D-1}{4\lambda}\right]k^2 P^{(0-s)}. \tag{16}$$

Inverting this operator, we obtain the propagator for fourth-order gravity in D dimensions, i.e.,

$$D^{(\text{fourth-order})} = \frac{1}{\sigma}\left[\frac{1}{k^2} - \frac{1}{k^2 - m_2^2}\right]P^{(2)} + \frac{2\lambda}{k^2}P^{(1)} + \frac{1}{\sigma(D-2)}\left[\frac{1}{k^2 - m_0^2} - \frac{1}{k^2}\right]P^{(0-s)} + \left[\frac{4\lambda}{k^2} + \frac{(D-1)m_0^2}{\sigma k^2(k^2 - m_0^2)(D-2)}\right]P^{(0-w)} + \frac{\sqrt{D-1}m_0^2}{(D-2)\sigma k^2(k^2 - m_0^2)}[P^{(0-sw)} + P^{(0-ws)}], \tag{17}$$

where $\{P^{(1)}, P^{(2)}, \ldots, P^{(0-ws)}\}$ is the usual set of D-dimensional Barnes-Rivers operators (see Appendix B), and

$$m_2^2 \equiv -\frac{4\sigma}{\beta\kappa^2}, \qquad m_0^2 \equiv \frac{4\sigma(D-2)}{\kappa^2[4\alpha(D-1) + D\beta]}. \tag{18}$$

Here, we are supposing that there are no tachyons in the model, which implies that $m_2^2 > 0$ and $m_0^2 > 0$.

The expression for the spatial part of the modified propagator can be trivially found by means of (17). Making the appropriate computations, we arrive at the following result:

$$\mathcal{P}_{\mu\nu,\alpha\beta}(\mathbf{k}) = \frac{1}{\sigma}\left\{\left[-\frac{1}{\mathbf{k}^2}+\frac{1}{\mathbf{k}^2+m_2^2}\right]\times\left[\frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\lambda}+\eta_{\mu\lambda}\eta_{\nu\kappa})-\frac{1}{D-1}\eta_{\mu\nu}\eta_{\kappa\lambda}\right]+\frac{1}{(D-1)(D-2)}\left[\frac{1}{\mathbf{k}^2}-\frac{1}{\mathbf{k}^2+m_0^2}\right]\eta_{\mu\nu}\eta_{\kappa\lambda}\right\}. \tag{19}$$

As a consequence,

$$\mathcal{P}_{00,00}(\mathbf{k}) = \frac{1}{\sigma}\left(-\frac{D-3}{D-2}\frac{1}{\mathbf{k}^2}+\frac{D-2}{D-1}\frac{1}{\mathbf{k}^2+m_2^2}-\frac{1}{(D-1)(D-2)(\mathbf{k}^2+m_0^2)}\right). \tag{20}$$

Therefore, the D-dimensional gravitational potential generated by a static mass M can be computed through the expression

$$V_D^{(\text{fourth-order})}(r) = -\frac{\kappa_D M}{\sigma(2\pi)^{D-1}}\left[\frac{D-3}{D-2}\int\frac{d^{D-1}\mathbf{k}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{r}}-\frac{D-2}{D-1}\int\frac{d^{D-1}\mathbf{k}}{\mathbf{k}^2+m_2^2}e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{1}{(D-2)(D-1)}\int\frac{d^{D-1}\mathbf{k}}{\mathbf{k}^2+m_0^2}e^{i\mathbf{k}\cdot\mathbf{r}}\right].$$

Performing the integrations, we find (see Appendix C)

$$V_D^{(\text{fourth-order})}(r) = -\frac{\kappa_D M}{\sigma(2\pi)^{\frac{D-1}{2}}}\left[\frac{D-3}{D-2}\frac{2^{\frac{D-5}{2}}}{r^{D-3}}\Gamma\left(\frac{D-3}{2}\right)-\frac{D-2}{D-1}\left(\frac{m_2}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_2 r)+\frac{1}{(D-1)(D-2)}\left(\frac{m_0}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_0 r)\right], \quad (D=4,5) \tag{21}$$

and

$$V_3^{(\text{fourth-order})}(r) = \frac{\kappa_3 M}{4\pi\sigma}[K_0(m_2 r)-K_0(m_0 r)], \tag{22}$$

where K_ν is the modified Bessel function of the second order of order ν.

Bearing in mind that

$$K_\nu(r) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-r}}{\sqrt{r}}\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right) \quad (r\to\infty), \tag{23}$$

it is trivial to see that (21) and the Newton gravitational potential agree asymptotically if and only if $\sigma=+1$. Accordingly, we assume from now on that $\sigma=+1$ for $D>3$.

Before proceeding, it is important to call attention to the fact that our discussion will be restricted to the systems in three, four, and five dimensions since these are the only models in which it is possible to compute the gravitational potential analytically.

We analyze now the small-distance behavior of the gravitational potential in the specified systems.

A. $D=3$

Remembering that, for $x \ll 1$,

$$K_0(x) \sim -\left(\gamma+\ln\frac{x}{2}\right)+\frac{x^2}{4}\left(1-\gamma-\ln\frac{x}{2}\right)+x^4\left(\frac{1}{128}(3-2\gamma)-\frac{1}{64}\ln\frac{x}{2}\right)+\ldots, \tag{24}$$

where γ is the Euler-Mascheroni constant, we may rewrite the expression for the gravitational potential (22) as

$$V_3(r) \sim \frac{\kappa_3 M}{4\pi\sigma}\left[\ln\frac{m_0}{m_2}+\frac{(m_2 r)^2}{4}\left(1-\gamma\ln\frac{m_2 r}{2}\right)-\frac{(m_0 r)^2}{4}\left(1-\gamma\ln\frac{m_0 r}{2}\right)+\ldots\right]. \tag{25}$$

Therefore, as $r\to 0$, we get

$$V_3(0) = \frac{\kappa_3 M}{4\pi\sigma}\ln\frac{m_0}{m_2}. \tag{26}$$

It follows then that full tridimensional fourth-order gravity theories, i.e., the models with no special relations between their parameters, have a gravitational potential that is finite at the origin. However, NMG [22], for instance, where their parameters are linked via the constraint $8\alpha+3\beta$, is singular at the origin. Note that σ for this system is equal to -1. We shall analyze the model alluded to in Sec. V.

B. $D=4$

Taking into account that $K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}$, we immediately obtain, from (21),

$$V_4(r) = -\frac{\kappa_4 M}{8\pi r}\left(1-\frac{4}{3}e^{-m_2 r}+\frac{1}{3}e^{-m_0 r}\right). \tag{27}$$

To check whether $V_4(r)$ is regular at the origin, we expand the exponentials at $r=0$ into power series. Doing so, it is easy to verify that the contributions of the higher-derivative terms cancel the Newtonian one, making the model free of singularity. In fact, the alluded potential can be written as

$$V_4(r) \sim MG_4 \frac{m_0 - 4m_2}{3} + \mathcal{O}(r). \qquad (28)$$

The singularity cancellation occurs because the zero-order terms containing higher derivatives produce a coefficient $+1$ responsible for canceling out the coefficient -1 from the original Newton term.

C. $D = 5$

Keeping in mind that, for $x \to 0$,

$$K_1(x) \sim \frac{1}{x} + \frac{x}{4}\left[2\gamma - 1 + \frac{1}{8}\left(2\gamma - \frac{5}{2}\right)x^2 + \frac{1}{192}\left(2\gamma - \frac{10}{3}\right)x^4 + \ldots\right] + \frac{x}{2}\ln\frac{x}{2}\left[1 + \frac{x^2}{8} + \frac{x^4}{192} + \ldots\right],$$

and we may write $V_5(r)$ as

$$V_5(r) \sim -\frac{\kappa_5 M}{48(2\pi)^2}\left[(m_0^2 - 9m_2^2)(2\gamma - 1 + 2\ln r) + m_0^2 \times \ln\frac{m_0^2}{4} - 9m_2^2 \ln\frac{m_2^2}{4} + \ldots\right]. \qquad (29)$$

As a consequence, the full fourth-order gravitational potential in five dimensions is divergent at the origin; nevertheless, if $m_0^2 = 9m_2^2$, this potential is finite at the cited point. Accordingly, we have found a nonsingular potential at the origin in five dimensions related to fourth-order gravity, with its value being given by

$$V_5(0)|_{m_0^2 = 9m_2^2} = -\frac{3\kappa_5 M m_2^2 \ln 3}{32\pi^2}. \qquad (30)$$

Let us then probe our conjecture for fourth-order gravity in D dimensions.

D. Testing the conjecture

According to our conjecture, the necessary condition for a D-dimensional higher-order model to be renormalizable is that it has a classical potential that is finite at the origin. As we have just shown, full fourth-order gravity systems in $D = 3, 4$ are finite at the origin, while in $D = 5$ the full model has a singularity at the aforementioned point. So, if the conjecture at hand is correct, both the three- and four-dimensional full models are expected to be renormalizable, whereas the five-dimensional one should be nonrenormalizable.

Now, since full fourth-order gravity models in $D = 3, 4$ are known to be renormalizable [20,27], they agree with our conjecture since, as we have just demonstrated, they lack a singularity at the origin.

As far as the five-dimensional system is concerned, it is trivial to show by power counting that the full model is nonrenormalizable. In fact, in this case the degree of superficial divergence is given by

$$\delta = 5 + \frac{1}{2}\left(\sum_{n=3}^{\infty}(n-2)(V_n - E)\right), \qquad (31)$$

which clearly shows that the system is nonrenormalizable since δ becomes greater as the vertex number increases. Remembering that this model is divergent at the origin, it is in agreement with our surmise because it asserts that renormalizable systems must always be finite at the origin.

On the other hand, the gravitational potential in NMG is divergent at the origin, as we shall prove in Sec. V, while the five-dimensional model with its parameters connected by the relation $m_0^2 = 9m_2^2$ has a potential that is free of singularity at the origin. Both systems are in accord with our conjecture. Indeed, new massive gravity is nonrenormalizable [27] and the five-dimensional model is nonrenormalizable by power counting. Note that our surmise says that the existence of a classical potential lacking a singularity at the origin is a necessary but not sufficient condition for the renormalizability of the theory.

For completeness' sake, we discuss now the tree-level unitarity of the fourth-order gravity models.

E. Unitarity of the fourth-order gravity systems

We show now that full fourth-order gravity models are nonunitary in $D = 3, 4, 5$. To do so, we make use of a method pioneered by Veltman [35] which has been extensively used since it was conceived. The prescription consists of saturating the propagator with conserved external currents and computing afterward the residues at the simple poles of the saturated propagator (SP) alluded to. If the residues at all poles are positive or null, the system is tree-level unitary, but if at least one of the residues is negative, the model is nonunitary at tree level.

For $D = 4$ and $D = 5$, we obtain from (19) the saturated propagator in momentum space (note that we have chosen $\sigma = +1$ for reasons already explained)

$$SP(k) = T_{\mu\nu}(k) D^{\mu\nu,\alpha\beta}(k) T_{\alpha\beta}(k) = \frac{A}{k^2} - \frac{B}{k^2 - m_2^2} + \frac{C}{k^2 - m_0^2}.$$

Here,

$$A \equiv T_{\mu\nu}^2 - \frac{T^2}{2}, \qquad B \equiv T_{\mu\nu}^2 - \frac{T^2}{3}, \qquad C \equiv \frac{T^2}{6},$$

where $T_{\mu\nu}$ is an external conserved current, with $T_{\mu\nu} = T_{\nu\mu}$.

Now, taking into account that

$$\left(T_{\mu\nu}^2 - \frac{T^2}{2}\right)\bigg|_{k^2=0} > 0,$$
$$\left(T_{\mu\nu}^2 - \frac{T^2}{3}\right)\bigg|_{k^2=m_2^2} > 0, \qquad (\text{see Ref.[33]}), \quad (32)$$

we come to the conclusion that

$$\mathrm{Res}(SP)|_{k^2=0} > 0,$$
$$\mathrm{Res}(SP)|_{k^2=m_0^2} > 0,$$
$$\mathrm{Res}(SP)|_{k^2=m_2^2} < 0,$$

implying that fourth-order gravity is nonunitary for $D = 4$ and $D = 5$.

If $D = 3$, the following results are found for the full theory (see Table I).

Thus, full tridimensional fourth-order gravity is nonunitaty for $\sigma = \pm 1$. In addition, it is also renormalizable [27].

NMG, in turn, is tree-level unitary and nonrenormalizable (see Sec. V), while fourth-order gravity in five dimensions—with their parameters constrained by the relation $m_0^2 = 9m_2^2$—is nonunitary and nonrenormalizable by power counting.

The preceding results confirm, as expected, that renormalizable higher-order gravity models will always be nonunitary.

IV. PROBING THE CONJECTURE FOR *D*-DIMENSIONAL SIXTH-ORDER GRAVITY MODELS

Since we are only interested in the linear part of the action (1), we did not take the γ_0 term into account. On the other hand, the quadratic part of the resulting action can be written as

$$I^{(\text{sixth-order})} = \int d^D x\sqrt{|g|}\frac{1}{\kappa^2}\left[2R + \frac{1}{2}RF_1(\Box)R + \frac{1}{2}R_{\mu\nu} \times F_2(\Box)R^{\mu\nu} + \frac{1}{2}R_{\mu\nu\alpha\beta}F_3(\Box)R^{\mu\nu\alpha\beta}\right], \qquad (33)$$

where

$$F_1(\Box) \equiv \alpha_0 + \alpha_1\Box,$$
$$F_2(\Box) \equiv \beta_0 + \beta_1\Box,$$
$$F_3(\Box) \equiv \gamma_1\Box.$$

Now, in the weak field approximation, we obtain

$$R_{\mu\nu\alpha\beta}F_3(\Box)R^{\mu\nu\alpha\beta} = 4R_{\mu\nu}F_3(\Box)R^{\mu\nu} - RF_3(\Box)R + \partial\Omega + \mathcal{O}(h^3). \qquad (34)$$

Substituting (34) into (33), we find

$$I^{(\text{sixth-order})} = \int d^D x\sqrt{|g|}\frac{1}{\kappa^2}\left[2R + \frac{1}{2}R(F_1(\Box) - F_2(\Box)) \times R + \frac{1}{2}R_{\mu\nu}(F_2(\Box) + 4F_3(\Box))R^{\mu\nu}\right]. \qquad (35)$$

Making the following redefinitions:

$$F_1(\Box) - F_3(\Box) \Rightarrow F_1(\Box),$$
$$F_2(\Box) + 4F_3(\Box) \Rightarrow F_2(\Box),$$

we come to the conclusion that the quadratic part of our original action reduces in this approximation to

$$I^{(\text{sixth-order})} = \int d^D x\sqrt{|g|}\frac{1}{\kappa^2}\left[2R + \frac{\alpha_0}{2}R^2 + \frac{\beta_0}{2}R_{\mu\nu}^2 + \frac{\alpha_1}{2}R\Box R + \frac{\beta_1}{2}R_{\mu\nu}\Box R^{\mu\nu}\right]. \qquad (36)$$

Taking the same series of actions which we have utilized for verifying our conjecture related to fourth-order gravity models in D dimensions, we find that the propagator in sixth-order gravity systems can be written in momentum space as

$$D(k) = \left[\frac{1}{k^2} + \frac{1}{m_{2_+}^2 - m_{2_-}^2}\left(\frac{m_{2_-}^2}{k^2 - m_{2_+}^2} - \frac{m_{2_+}^2}{k^2 - m_{2_-}^2}\right)\right]P^{(2)} - \frac{1}{D-2}\left[\frac{1}{k^2} + \frac{1}{m_{0_+}^2 - m_{0_-}^2}\left(\frac{m_{0_-}^2}{k^2 - m_{0_+}^2} - \frac{m_{0_+}^2}{k^2 - m_{0_-}^2}\right)\right]P^{(0-s)} + (\ldots). \qquad (37)$$

Here, $(\ldots)$ stands for the set of terms that are irrelevant to the spectrum of the theory, and

$$m_{2_\pm}^2 = \frac{\beta_0}{2\beta_1}\left(1 \pm \sqrt{1 + \frac{16\beta_1}{\beta_0^2}}\right),$$
$$m_{0_\pm}^2 = \frac{\xi_0}{2\xi_1}\left(1 \pm \sqrt{1 - \frac{4(D-2)\xi_1}{\xi_0^2}}\right),$$

where $\xi_l = (D-1)\alpha_l + \frac{D}{4}\beta_l (l = 0, 1)$.

As a consequence,

$$\mathcal{P}_{00,00}(\mathbf{k}) = -\frac{D-3}{D-2}\frac{1}{\mathbf{k}^2} + \frac{1}{(D-1)(D-2)}\frac{1}{m_{0_+}^2 - m_{0_-}^2}\left(\frac{m_{0_-}^2}{\mathbf{k}^2 + m_{0_+}^2} - \frac{m_{0_+}^2}{\mathbf{k}^2 + m_{0_-}^2}\right) - \frac{D-2}{D-1} \times \frac{1}{m_{2_+}^2 - m_{2_-}^2}\left(\frac{m_{2_-}^2}{\mathbf{k}^2 + m_{2_+}^2} - \frac{m_{2_+}^2}{\mathbf{k}^2 + m_{2_-}^2}\right).$$

It follows then that the D-dimensional gravitational potential for sixth-order models reads

$$V_3(r) = \frac{\kappa_3 M}{4\pi}\left\{\frac{m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2}K_0(m_{0_+}r) - \frac{m_{0_+}^2}{m_{0_+}^2 - m_{0_-}^2}K_0(m_{0_-}r) - \frac{m_{2_-}^2}{m_{2_+}^2 - m_{2_-}^2}K_0(m_{2_+}r) + \frac{m_{2_+}^2}{m_{2_+}^2 - m_{2_-}^2}K_0(m_{2_-}r)\right\}, \quad (38)$$

$$V_D(r) = -\frac{\kappa_D M}{(2\pi)^{\frac{D-1}{2}}}\left\{\frac{D-3}{D-2}\frac{2^{\frac{D-5}{2}}}{r^{D-3}}\Gamma\left(\frac{D-3}{2}\right) - \frac{1}{(D-1)(D-2)}\frac{m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2}\left(\frac{m_{0_+}}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_{0_+}r) + \frac{1}{(D-1)(D-2)}\frac{m_{0_+}^2}{m_{0_+}^2 - m_{0_-}^2}\left(\frac{m_{0_-}}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_{0_-}r) + \frac{D-2}{D-1}\frac{m_{2_-}^2}{m_{2_+}^2 - m_{2_-}^2}\left(\frac{m_{2_+}}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_{2_+}r) - \frac{D-2}{D-1}\frac{m_{2_+}^2}{m_{2_+}^2 - m_{2_-}^2} \times \left(\frac{m_{2_-}}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_{2_-}r)\right\} \quad (D = 4, 5). \quad (39)$$

It is trivial to see using (23) that (39) and the Newton gravitational potential coincide for $r \to \infty$. Our next step will be to make a thorough analysis of the behavior near the origin of the gravitational potential that we have just found.

A. $D = 3$

Taking (24) into account, we find that for $r \ll 1$, (38) assumes the form

$$V_3(r) \sim \frac{\kappa_3 M}{4\pi}\left(\frac{m_{2_-}^2 \ln m_{2_+} - m_{2_+}^2 \ln m_{2_-}}{m_{2_+}^2 - m_{2_-}^2} - \frac{m_{0_-}^2 \ln m_{0_+} - m_{0_+}^2 \ln m_{0_-}}{m_{0_+}^2 - m_{0_-}^2} + \ldots\right). \quad (40)$$

Consequently, the tridimensional sixth-order gravitational potential is finite at the origin and has the following value:

$$V_3(0) = \frac{\kappa_3 M}{4\pi}\left(\frac{m_{2_-}^2 \ln m_{2_+} - m_{2_+}^2 \ln m_{2_-}}{m_{2_+}^2 - m_{2_-}^2} - \frac{m_{0_-}^2 \ln m_{0_+} - m_{0_+}^2 \ln m_{0_-}}{m_{0_+}^2 - m_{0_-}^2}\right). \quad (41)$$

B. $D = 4$

In this case, the gravitational potential is given by

$$V_4(r) = \frac{\kappa_4 M}{4\pi r}\left(-\frac{1}{2} + \frac{1}{6}\frac{m_{0_-}^2 e^{-m_{0_+}r} - m_{0_+}^2 e^{-m_{0_-}r}}{m_{0_+}^2 - m_{0_-}^2} - \frac{2}{3}\frac{m_{2_-}^2 e^{-m_{2_+}r} - m_{2_+}^2 e^{-m_{2_-}r}}{m_{2_+}^2 - m_{2_-}^2}\right). \quad (42)$$

Expanding the exponentials at $r = 0$, we get

$$V_4(r) \sim \frac{\kappa_4 M}{4\pi}\left(\frac{2}{3}\frac{m_{2_-}^2 m_{2_+} - m_{2_+}^2 m_{2_-}}{m_{2_+}^2 - m_{2_-}^2} - \frac{1}{6}\frac{m_{0_-}^2 m_{0_+} - m_{0_+}^2 m_{0_-}}{m_{0_+}^2 - m_{0_-}^2}\right) + \mathcal{O}(r). \quad (43)$$

Thus, the gravitational potential for sixth-order gravity in four dimensions is finite at the origin, with its value at this point being equal to

$$V_4(0) = \frac{\kappa_4 M}{4\pi}\left(\frac{2}{3}\frac{m_{2_-}^2 m_{2_+} - m_{2_+}^2 m_{2_-}}{m_{2_+}^2 - m_{2_-}^2} - \frac{1}{6}\frac{m_{0_-}^2 m_{0_+} - m_{0_+}^2 m_{0_-}}{m_{0_+}^2 - m_{0_-}^2}\right). \quad (44)$$

C. $D = 5$

It is straightforward to show that, if $r \ll 1$, (39) reduces, for $D = 5$, to

$$V_5(r) \sim \frac{\kappa_5 M}{(2\pi)^2}\left\{-\frac{3}{8}\frac{m_{2_+}^2 m_{2_-}^2}{m_{2_+}^2 + m_{2_-}^2}\ln\frac{m_{2_+}}{m_{2_-}} + \frac{1}{24}\frac{m_{0_+}^2 m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2}\ln\frac{m_{0_+}}{m_{0_-}} + \ldots\right\}, \tag{45}$$

which converges to a finite value at the origin that is equal to

$$V_5(0) = -\frac{\kappa_5 M}{(2\pi)^2}\left\{\frac{3}{8}\frac{m_{2_+}^2 m_{2_-}^2}{m_{2_+}^2 - m_{2_-}^2}\ln\frac{m_{2_+}}{m_{2_-}} - \frac{1}{24}\frac{m_{0_+}^2 m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2}\ln\frac{m_{0_+}}{m_{0_-}}\right\}. \tag{46}$$

We probe now our conjecture for D-dimensional sixth-order gravity models.

D. Verifying the conjecture

It is not difficult to check by power counting that the superficial divergence related to the system at hand can be written as

$$\delta = D + \frac{6-D}{2}E - \frac{6-D}{2}\sum_{n=3}^{\infty}(n-2)V_n. \tag{47}$$

Therefore, we conclude that

(i) $3 \le D \le 5 \Rightarrow \delta$ decreases as the number of vertices increases $\Rightarrow$ super-renormalizable,

(ii) $D = 6 \Rightarrow \delta$ is independent of the number of vertices $\Rightarrow$ renormalizable, and

(iii) $D \ge 7 \Rightarrow \delta$ increases as the number of vertices increase $\Rightarrow$ nonrenormalizable.

Since the gravitational potential can only be computed analytically for $D = 3, 4, 5$, we restrict our analysis to these dimensions. On the other hand, we have proved that the gravitational potential for the full models is finite at $r = 0$ in the dimensions above. Accordingly, these models are in total accord with our surmise, which requires that they must be nonsingular at the origin. For completeness, we finally shall study the unitarity of the specified models.

E. Unitarity of the sixth-order gravity models

From (37) we find that the saturated propagator is given by the expression

$$\begin{aligned} SP(k) &= \frac{1}{k^2}\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{D-2}T^2\right) \\ &+ \left[\frac{1}{m_{2_+}^2 - m_{2_-}^2}\left(\frac{m_{2_-}^2}{k^2 - m_{2_+}^2} - \frac{m_{2_+}^2}{k^2 - m_{2_-}^2}\right)\right] \\ &\times \left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{D-1}T^2\right) - \frac{1}{(D-1)(D-2)} \\ &\times \left[\frac{1}{m_{0_+}^2 - m_{0_-}^2}\left(\frac{m_{0_-}^2}{k^2 - m_{0_+}^2} - \frac{m_{2_+}^2}{k^2 - m_{0_-}^2}\right)\right]T^2. \end{aligned} \tag{48}$$

Therefore,

$$\begin{aligned} \mathrm{Res}(SP(k))|_{k^2=0} &= \left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{D-2}T^2\right)\Big|_{k^2=0}, \\ \mathrm{Res}(SP(k))|_{k^2=m_{2_+}^2} &= \frac{m_{2_-}^2}{m_{2_+}^2 - m_{2_-}^2} \times \left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{D-1}T^2\right)\Big|_{k^2=m_{2_+}^2}, \\ \mathrm{Res}(SP(k))|_{k^2=m_{2_-}^2} &= -\frac{m_{2_+}^2}{m_{2_+}^2 - m_{2_-}^2} \times \left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{D-1}T^2\right)\Big|_{k^2=m_{2_-}^2}, \\ \mathrm{Res}(SP(k))|_{k^2=m_{0_+}^2} &= -\frac{1}{(D-1)(D-2)}\frac{m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2} \times T^2|_{k^2=m_{0_+}^2}, \end{aligned} \tag{49}$$

$$\mathrm{Res}(SP(k))|_{k^2=m_{0_-}^2} = \frac{1}{(D-1)(D-2)}\frac{m_{0_+}^2}{m_{0_+}^2 - m_{0_-}^2} \times T^2|_{k^2=m_{0_-}^2}. \tag{50}$$

Our next step is to obtain the signs related to the residues. To do that, however, we first must know how $m_{2_+}^2$ and $m_{2_-}^2$, as well as $m_{0_+}^2$ and $m_{0_-}^2$, are ordered. To facilitate this task, we redefine the following parameters:

$$\alpha_0 \mapsto \kappa^2\alpha_0, \quad \alpha_1 \mapsto \kappa^4\alpha_1, \quad \beta_0 \mapsto \kappa^2\beta_0, \quad \beta_1 \mapsto \kappa^4\beta_1,$$

which implies that, in terms of these redefined parameters, the masses $m_{\pm}^2$ and $m_{0_\pm}^2$ assume the form

$$m_{2_\pm}^2 = \frac{\beta_0}{2\kappa^2\beta_1}\left(1 \pm \sqrt{1 + \frac{16\beta_1}{\beta_0^2}}\right), \tag{51}$$

$$m_{0_\pm}^2 = \frac{\xi_0}{2\kappa^2\xi_1}\left(1 \pm \sqrt{1 - \frac{4(D-2)\xi_1}{\xi_0^2}}\right), \tag{52}$$

where $\xi_l = 3\alpha_l + \beta_l$ ($l = 0, 1$). Actually, we are interested in the following regions in the parametric spaces:

$$\Omega_\beta = \{(\beta_0,\beta_1) \in \mathbb{R}^2 | \kappa^2 m_{2_+}^2 > 0 \quad \text{and} \quad \kappa^2 m_{2_-}^2 > 0\},$$
$$\Omega_\xi = \{(\xi_0,\xi_1) \in \mathbb{R}^2 | \kappa^2 m_{0_+}^2 > 0 \quad \text{and} \quad \kappa^2 m_{0_-}^2 > 0\},$$
$$\Omega_\alpha = \left\{(\alpha_0,\alpha_1) = \left(\frac{4\xi_0 - D\beta_0}{4(D-1)}, \frac{4\xi_1 - D\beta_1}{4(D-1)}\right) \in \mathbb{R}^2 \times |(\beta_0,\beta_1) \in \Omega_\beta \quad \text{and} \quad (\xi_0,\xi_1) \in \Omega_\xi\right\}.$$

Taking (51) and (52) into account, we may write

$$\Omega_\beta = \{(\beta_0,\beta_1) \in \mathbb{R}^2 | \beta_0 < 0 \quad \text{and} \quad -\beta_0^2/16 < \beta_1 < 0\},$$
$$\Omega_\xi = \{(\xi_0,\xi_1) \in \mathbb{R}^2 | \xi_0 > 0 \quad \text{and} \quad 0 < \xi_1 < \xi_0^2/4(D-2)\}.$$

As a result, we find that, in these regions, the masses are ordered as

$$m_{2_+}^2 > m_{2_-}^2 \quad \text{and} \quad m_{0_+}^2 > m_{0_-}^2. \tag{53}$$

Now, from (32) and (53), we arrive at the conclusion that

$$\mathrm{Res}(SP(k))|_{k^2=0} > 0,$$
$$\mathrm{Res}(SP(k))|_{k^2=m_{2_+}^2} > 0, \qquad \mathrm{Res}(SP(k))|_{k^2=m_{2_-}^2} < 0,$$
$$\mathrm{Res}(SP(k))|_{k^2=m_{0_+}^2} < 0, \qquad \mathrm{Res}(SP(k))|_{k^2=m_{0_-}^2} > 0.$$

Consequently, the particle content of the model is made up of three healthy particles and two ghosts, which clearly shows that full sixth-order gravity is nonunitary. The results above confirm once more that renormalizable higher-order gravity models are nonunitarry.

V. FINAL COMMENTS

We have verified that renormalizable higher-order gravitational models, specifically fourth- and sixth-order gravity systems in D dimensions, possess a singularity-free classical potential at the origin. The converse is not necessarily true. Indeed, consider the gravity system in four dimensions defined by the Lagrangian [16]

$$\mathcal{L} = \sqrt{|g|}\left(\frac{2}{\kappa^2}R + \alpha_0 R^2 + a_1 R\Box R + b_0 R_{\mu\nu}^2\right),$$

where the masses of the modes related to higher-order terms are given by

$$m_{(0)\pm}^2 = \frac{3a_0 + b_0 \pm \sqrt{(3a_0 + b_0)^2 - 24a_1\kappa^{-2}}}{6a_1},$$
$$m_{(2)}^2 = \frac{4}{|b_0|\kappa^2}. \tag{54}$$

Here, $m_{(2)}$ and $m_{(0)+}$ are ghost excitations, while $m_{(0)-}$ is a healthy mode [12].

In this scenario, the potential is given by

$$V_4(r) = -\frac{G_4 M}{r}\left[1 - \frac{4}{3}e^{-m_{(2)}r} + \frac{1}{3}\left(\frac{m_{(0)-}^2}{m_{(0)-}^2 - m_{(0)+}^2} \times e^{-m_{(0)+}r} + \frac{m_{(0)+}^2}{m_{(0)+}^2 - m_{(0)-}^2}e^{-m_{(0)-}r}\right)\right], \tag{55}$$

and, as a consequence, in the region near the origin, it assumes the form

$$V_4(r) \sim G_4 M\left[-\frac{4}{3}m_{(2)} + \frac{1}{3}\frac{m_{(0)+}m_{(0)-} - (m_{(0)+} - m_{(0)-})}{m_{(0)+}^2 - m_{(0)-}^2}\right] + \mathcal{O}(r). \tag{56}$$

Therefore, the potential is finite at $r = 0$. Nonetheless, the model at hand is nonrenormalizable by power counting, which implies that the finiteness of the classical potential at the origin is a necessary—but certainly not sufficient—condition for the renormalizability of the model.

In summary, if a higher-derivative gravity model is renormalizable, it is necessarily nonunitary and, in addition, is endowed with a classical potential finite at the origin, but the opposite is not true in general. We have also confirmed the general premise that renormalizable higher-derivative gravity models are nonunitary.

Now we address the issue of NMG [22]. Our main interest in this system is in the fact that it was by analyzing its properties that the idea of the conjecture came to light. As is well known, this model aroused great interest in the physical community when it was conceived since it is a tree-level unitary higher-order gravity model; in fact, tree-level unitary higher-derivative gravity systems are extremely rare in physics. On the other hand, the aforementioned theory caused considerable controversy as far as its renormalizability is concerned. Actually, it was initially claimed to be renormalizable by Oda [26], only to be shown to be nonrenormalizable some years later by Muneyuki and Ohta [27]. It is exactly the disagreement between these results that we want to discuss in the framework of our conjecture. Nevertheless, for clarity's sake, we begin by presenting some important points related to the system at hand.

A. Tree-level unitarity

From (17), it is straightforward to obtain the saturated propagator, i.e.,

$$SP(k) = \frac{1}{\sigma}\left[\frac{1}{k^2} - \frac{1}{k^2 - m_2^2}\right]\left[T_{\mu\nu}^2 - \frac{1}{2}T^2\right] + \frac{1}{\sigma}\left[-\frac{1}{k^2} + \frac{1}{k^2 - m_0^2}\right]\frac{1}{2}T^2. \tag{57}$$

Equation (18), in trurn, furnishes the constraints

$$\frac{\sigma}{\beta} < 0, \qquad \frac{\sigma}{8\alpha + 3\beta} > 0. \tag{58}$$

Now, the residues of $SP(k)$ at the poles $k^2 = m_2^2$, $k^2 = 0$, and $k^2 = m_0^2$ are, respectively,

$$\mathrm{Res}(SP)|_{k^2=m_2^2} = -\frac{1}{\sigma}\left(T_{\mu\nu}^2 - \frac{1}{2}T^2\right)\Big|_{k^2=m_2^2}, \tag{59}$$

$$\mathrm{Res}(SP)|_{k^2=0} = -\frac{1}{\sigma}(T_{\mu\nu}^2 - T^2)|_{k^2=0}, \tag{60}$$

$$\mathrm{Res}(SP)|_{k^2=m_0^2} = -\frac{1}{2\sigma}(T^2)|_{k^2=m_0^2}. \tag{61}$$

Therefore, we arrive at the conclusion that $Res(SP)|_{k^2=m_2^2} > 0$ if $\sigma = -1$ (which we assume to be the case from now on), and $\mathrm{Res}(SP)|_{k^2=0}$. As a result, we need not worry about these poles; the troublesome one is $k^2 = m_0^2$ since $\mathrm{Res}(SP)|_{k^2=m_0^2} < 0$. A way out of this difficulty is to consider the $m_0 \to \infty$ limit of the model under discussion, which leads us to conclude that $\alpha = -\frac{3}{8}\beta$. Accordingly, the class of models defined by the Lagrangian

$$\mathcal{L} = \sqrt{|g|}\left[-\frac{2R}{\kappa^2} + \frac{\beta}{2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right)\right], \tag{62}$$

where $\kappa^2 = 4\kappa_3$, is ghost free at tree level. For convenience's sake, we replace β with $\frac{4}{\kappa^2 m_2^2}$. The resulting Lagrangian,

$$\mathcal{L}_{\mathrm{NMG}} = \sqrt{|g|}\left[-\frac{2R}{\kappa^2} + \frac{2}{\kappa^2 m_2^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right)\right], \tag{63}$$

defines the famous system referred to as new massive gravity [22–25].

At this point, it will be interesting to recall some comments that, in a sense, predicted the nonrenormalizability of NMG.

(i) It is not clear at all whether the specific ratio between α and β will survive renormalization at a given loop, even at one loop; in other words, unitarity beyond tree level has to be checked [36].

(ii) Most likely, NMG is nonrenormalizable since it only improves the spin-2 projections of the propagator but not the spin-0 projection [37].

Undoubtedly, these remarks have anticipated for a few years the definitive proof related to the nonrenormalizability of NMG.

B. Gravitational potential

From (22) we get, without any difficulty,

$$V_{\mathrm{NMG}}(r) = -\frac{\kappa_3 M}{4\pi}K_0(m_2 r). \tag{64}$$

Note that the potential in NMG has a logarithm singularity at the origin.

C. Discussing the renormalizability of NMG via our conjecture

According to Oda [26], NMG is renormalizable. However, the author made a mistake when he considered NMG a full three-dimensional gravity model (with $\sigma = -1$), with the latter being renormalizable. In other words, although the birth of NMG is the full gravity model just mentioned (see Fig. 1), the system under discussion has a constraint between its parameters ($\alpha = -\frac{3}{8}\beta$). It is precisely this special relation between the parameters that is responsible for breaking the renormalizability of the full model, as was demonstrated by Muneyuki and Ohta [27].

Examining the diagram depicted in Fig. 1, we clearly see that, as m_0 becomes greater and greater, the full potential $V_3(r)$ with $\sigma = -1$ and $m_2 < m_0$ [see (22)] rapidly approaches the potential in NMG, and eventually they coalesce. It is worth mentioning that, to arrive at the NMG potential from the full potential above, the latter must necessarily become singular at the origin which takes place in the $m_0 \to \infty$ limit. It is remarkable that this is precisely

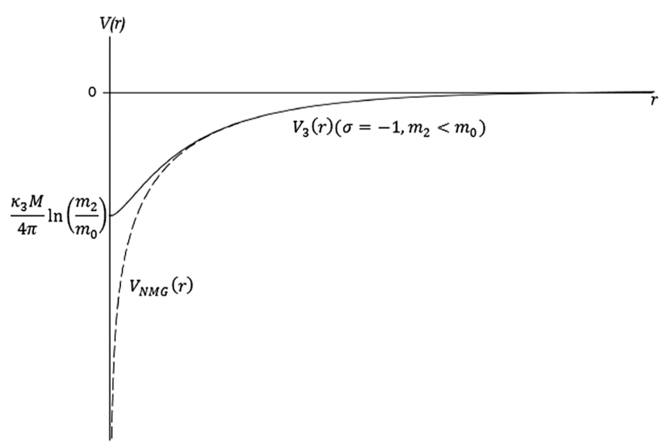

FIG. 1. Gravitational potential for both the full fourth-order gravity model in three dimensions with $\sigma = -1$ and $m_2 < m_0$ (the solid line) and NMG (the dashed line).

the condition for avoiding, at tree level, the massive spin-0 ghost that haunts full tridimensional fourth-order gravity. Accordingly, the presence of the singularity in NMG is correlated to the absence of the tree-level ghost, which means that the renormalizability of the model and its consequent nonunitary and the existence of a singularity in the potential are intertwined. In the diagram shown in Fig. 2, the behavior of full fourth-order gravity in three dimensions is depicted as far as its unitarity and renormalizability, and the existence of a finite gravitational potential at the origin, are concerned. A cursory glance at this diagram suggests that, in three dimensions, a unitary system is nonrenormalizable, with it being connected to a singular potential at the origin, while a renormalizable model is related to a potential finite at the origin, with it also being nonunitary. Interestingly enough, it was exactly the analysis of this model that led us to propose the conjecture analyzed in this paper.

We remark also that, although we have only tested our premise for some specific *D*-dimensional higher-derivative gravitational models, the surmise is completely general. In fact, our conjecture is valid for the most general *D*-dimensional gravitational action below,

$$I_D = \int d^D x\sqrt{|g|}\left(\frac{2\sigma}{\kappa^2}R + \frac{1}{2\kappa^2}RF_1(\Box)R + \frac{1}{2\kappa^2}R^{\mu\nu}F_2(\Box)R_{\mu\nu} + \frac{1}{2}R_{\mu\nu\alpha\beta}F_3(\Box)R^{\mu\nu\alpha\beta}\right).$$

Here,

$$F_1(\Box) = \sum_{n=0}^{p}\alpha_n(\Box)^n + f_1(\Box), \tag{65}$$

$$F_2(\Box) = \sum_{n=0}^{q}\beta_n(\Box)^n + f_2(\Box), \tag{66}$$

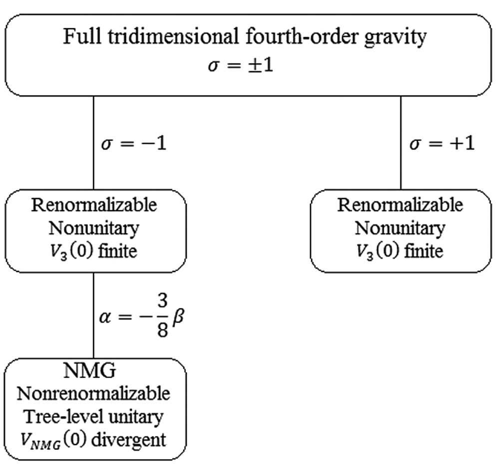

FIG. 2. The renormalizability, unitarity, and gravitational potential at the origin for full fourth-order gravity in three dimensions ($\sigma = \pm 1$).

$$F_3(\Box) = \sum_{n=0}^{r}\gamma_n(\Box)^n + f_3(\Box). \tag{67}$$

where $f_1(\Box)$, $f_2(\Box)$, and $f_3(\Box)$ are nonlocal functions, and $\alpha_n(n = 0, \ldots, p)$, $\beta_n(n = 0, \ldots, q)$, and $\gamma_n(n = 0, \ldots, r)$ are real coefficients. These results will be published elsewhere [38]. Last but not least we would like to draw the reader's attention to the article by [40], in which action (2) was built out for the first time.

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APPENDIX A: *D*-DIMENSIONAL EINSTEIN CONSTANT

As is well known, the *D*-dimensional Poisson equation can be written as

$$\nabla^2_{D-1}\varphi_D(\mathbf{x}) = G_D\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\rho, \tag{A1}$$

where ρ is the mass density.

On the other hand, the Schwarzschild metric in isotropic coordinates reads

$$ds^2 = \left[\frac{1+\frac{1}{2}\varphi_D(\mathbf{x})}{1-\frac{1}{2}\varphi_D(\mathbf{x})}\right]^2 dt^2 - \left[1-\frac{1}{2}\varphi_D(\mathbf{x})\right]^{\frac{4}{D-3}} \times [(dx^1)^2 + \cdots + (dx^{D-1})^2]. \tag{A2}$$

In the Newtonian limit, i.e., far form the mass distributions, the previous metric assumes the form

$$ds^2 = [1 + 2\varphi_D(\mathbf{x})]dt^2 - \left[1 - \frac{2}{D-3}\varphi_D(\mathbf{x})\right] \times [(dx^1)^2 + \cdots + (dx^{D-1})^2]. \tag{A3}$$

From the Einstein equations, namely, $G_{\mu\nu} = \kappa_D T_{\mu\nu}$, we then find

$$G_{00} = \kappa_D\rho = \frac{D-2}{D-3}\nabla^2_{D-1}\varphi_D(\mathbf{x}). \tag{A4}$$

Therefore, we come to the conclusion that

$$\kappa_D = \frac{D-2}{D-3}G_D\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}(D > 3). \tag{A5}$$

As we mentioned in the Introduction, in $D = 3$, κ_3 cannot be related to G_3; nonetheless, for simplicity's sake, κ_3 is used, in general, as the symbol for the tridimensional Einstein constant, although it is unrelated to G_3.

TABLE I. Signs of the residues of SP at the poles $k^2 = 0$, $k^2 = m_0^2$, $k^2 = m_2^2$ related to full fourth-order gravity in three dimensions.

$D = 3$	$\sigma = +1$	$\sigma = -1$
$\mathrm{Res}(SP(k))\|_{k^2=0}$	$= 0$	$= 0$
$\mathrm{Res}(SP(k))\|_{k^2=m_0^2}$	> 0	< 0
$\mathrm{Res}(SP(k))\|_{k^2=m_2^2}$	< 0	> 0

TABLE II. Multiplicative table for the Barnes-Rivers operators.

	$P^{(2)}$	$P^{(1)}$	$P^{(0-s)}$	$P^{(0-w)}$	$P^{(0-sw)}$	$P^{(0-ws)}$
$P^{(2)}$	$P^{(2)}$	0	0	0	0	0
$P^{(1)}$	0	$P^{(1)}$	0	0	0	0
$P^{(0-s)}$	0	0	$P^{(0-s)}$	0	$P^{(0-sw)}$	0
$P^{(0-w)}$	0	0	0	$P^{(0-w)}$	0	$P^{(0-ws)}$
$P^{(0-sw)}$	0	0	0	$P^{(0-sw)}$	0	$P^{(0-s)}$
$P^{(0-ws)}$	0	0	$P^{(0-ws)}$	0	$P^{(0-w)}$	0

APPENDIX B: D-DIMENSIONAL BARNES-RIVERS OPERATORS

The complete set of D-dimensional Barnes-Rivers operators in momentum space is given by

$$P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}(\theta_{\mu\kappa}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\kappa}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\kappa\lambda},$$
$$P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}(\theta_{\mu\kappa}\omega_{\nu\lambda} + \theta_{\mu\lambda}\omega_{\nu\kappa} + \theta_{\nu\lambda}\omega_{\mu\kappa} + \theta_{\nu\kappa}\omega_{\mu\lambda}),$$
$$P^{(0-s)}_{\mu\nu,\kappa\lambda} = \frac{1}{D-1}\theta_{\mu\nu}\theta_{\kappa\lambda}, \qquad P^{(0-w)}_{\mu\nu,\kappa\lambda} = \frac{1}{D-1}\omega_{\mu\nu}\omega_{\kappa\lambda},$$
$$P^{(0-sw)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{D-1}}\theta_{\mu\nu}\omega_{\kappa\lambda}, \qquad P^{(0-ws)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{D-1}}\omega_{\mu\nu}\theta_{\kappa\lambda},$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ and $\omega_{\mu\nu} \equiv \frac{k_\mu k_\nu}{k^2}$ are, respectively, the usual transverse and longitudinal vectorial projection operators. The multiplicative table for these operators is displayed in Table II.

APPENDIX C: SOME RELEVANT INTEGRALS

The integrals related to the models dealt with in this article can be generically written as

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} f(|\mathbf{k}|)e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{C1}$$

Now, keeping in mind that

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} f(|\mathbf{k}|)e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{1}{r^{\frac{D-3}{2}}}\int_0^\infty x^{\frac{D-1}{2}} f(x) J_{\frac{D-3}{2}}(xr)dx$$
$$(D > 2), \quad (\text{see Ref.[13]}),$$

where $x \equiv |\mathbf{k}|$, we promptly find the following results:

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{1}{r^{D-3}}\int_0^\infty y^{\frac{D-5}{2}} J_{\frac{D-3}{2}}(y)dy = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{1}{r^{D-3}} I_D,$$

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + m^2} = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{1}{r^{D-3}}\int_0^\infty \frac{y^{\frac{D-1}{2}}}{y^2 + m^2 r^2} J_{\frac{D-3}{2}}(y)dy = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{1}{r^{D-3}}\mathcal{I}_D(r).$$

Here,

$$I_D \equiv \int_0^\infty y^{\frac{D-5}{2}} J_{\frac{D-3}{2}}(y)dy, \tag{C2}$$

and

$$\mathcal{I}(r) \equiv \int_0^\infty \frac{y^{\frac{D-1}{2}}}{y^2 + m^2 r^2} J_{\frac{D-3}{2}}(y)dy.$$

From Gradshteyn and Ryzhik [39], we obtain

$$I_D = 2^{\frac{D-5}{2}}\Gamma\left(\frac{D-3}{2}\right), \qquad (D = 4, 5), \tag{C3}$$

$$\mathcal{I}_D(r) = (mr)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr), \qquad (D = 3, 4, 5). \tag{C4}$$

Accordingly,

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\frac{2^{\frac{D-5}{2}}}{r^{D-3}}\Gamma\left(\frac{D-3}{2}\right), \quad (D = 4, 5),$$

$$\int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + m^2} = \frac{1}{(2\pi)^{\frac{D-1}{2}}}\left(\frac{m}{r}\right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr). \quad (D = 3, 4, 5)$$

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