

Cyclic mixmaster universesJohn D. Barrow^{*} and Chandrima Ganguly[†]*DAMTP, Centre for Mathematical Sciences, University of Cambridge,
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(Received 20 March 2017; published 19 April 2017)

We investigate the behavior of bouncing Bianchi type IX “mixmaster” universes in general relativity. This generalizes all previous studies of the cyclic behavior of closed spatially homogeneous universes with and without an entropy increase. We determine the behavior of models containing radiation by analytic and numerical integration and show that an increase of radiation entropy leads to an increasing cycle size and duration. We introduce a null energy condition violating ghost field to create a smooth, nonsingular bounce of finite size at the end of each cycle and compute the evolution through many cycles with and without an entropy increase injected at the start of each cycle. In the presence of increasing entropy, we find that the cycles grow larger and longer and the dynamics approach flatness, as in the isotropic case. However, successive cycles become increasingly anisotropic at the expansion maxima which is dominated by the general-relativistic effects of anisotropic 3-curvature. When the dynamics are significantly anisotropic, the 3-curvature is negative. However, it becomes positive after continued expansion drives the dynamics close enough to isotropy for the curvature to become positive and for gravitational collapse to ensue. In the presence of a positive cosmological constant, radiation, and a ghost field, we show that, for a very wide range of cosmological constant values, the growing oscillations always cease and the dynamics subsequently approach those of the isotropic de Sitter universe at late times. This model is not included in the scope of earlier cosmic no-hair theorems because the 3-curvature can be positive. In the case of a negative cosmological constant, radiation, and an ultrastiff field (to create nonsingular bounces), we show that a sequence of chaotic oscillations also occurs, with sensitive dependence on initial conditions. In all cases, we follow the oscillatory evolution of the scale factors, the shear, and the 3-curvature from cycle to cycle.

DOI: [10.1103/PhysRevD.95.083515](https://doi.org/10.1103/PhysRevD.95.083515)**I. INTRODUCTION**

In 1922, Alexander Friedmann [1] first noted the existence of “periodic worlds” in his solutions of Einstein’s equations for isotropic and homogeneous universes with positive spatial curvature. But the physical study of cyclic universes in general-relativistic cosmology begins with the work of Tolman [2], who first considered the simple situation of a closed Friedmann universe with zero cosmological constant, Λ , and non-negative pressure. The evolution of Tolman’s cyclic universes can be continued periodically through the big crunch singularity if it is assumed that no new physics arises there and the evolution can be extended smoothly through it.¹ As a next step, Tolman incorporated the general consequences of an increase of entropy from cycle to cycle, in accordance with the second law of thermodynamics. This produced a

monotonic increase in the maximum size and length of successive cycles, which continues forever. If this entropy increase is modeled as an increase in the dimensionless entropy per baryon in a mixture of radiation and baryons (ignoring baryon nonconserving interactions), then the increase is a reflection of the asymmetry in the pressure, p , from cycle to cycle, as energy is transferred from the baryonic ($p = 0$) to the radiation ($p \neq 0$) gas. Tolman’s work attracted periodic interest by other astronomers like Bonnor [4] and Zanstra [5] in the 1950s, before Zeldovich and Novikov turned Tolman’s result into a theorem for rather restrictive equations of state of matter obeying Friedmann’s equations and the laws of thermodynamics. They also assumed that the cycles could not be continued indefinitely into the past because they would become smaller than the smallest finite-sized elementary particles (not assumed pointlike in those days). As the cycles continue to increase in size, an oscillating universe appears increasingly “flat,” although it is closed with positive spatial curvature [6]. This might even provide an explanation for the proximity of the expansion dynamics to flatness today that differs in detail from that of the standard one-cycle inflationary universe model (although the latter generates proximity to flatness by a large entropy increase in one cycle through “reheating” rather than by a

^{*}J.D.Barrow@damtp.cam.ac.uk[†]C.Ganguly@damtp.cam.ac.uk¹Tolman’s analysis, which assumed no equation of state linking the pressure, $p \geq 0$, to the density, ρ , required an additional assumption. To avoid a finite-time (“sudden”) singularity occurring where $p \rightarrow \infty$ with finite ρ before the expansion maximum is reached, one must stipulate some control over p , for example $p < C\rho$ for constant $C > 0$ [3].

progressive buildup over many cycles by all processes). However, in what follows, we will show that it does not share other features of an inflationary universe at late times.

Generalizations of this simple oscillatory Tolman universe produced some interesting new features. Barrow and Dabrowski [7] show that if there is a positive cosmological constant ($\Lambda > 0$) then the sequence of growing oscillations always comes to an end after a finite proper time and the dynamics evolve toward an ever-expanding de Sitter asymptote as $t \rightarrow \infty$ [7]. This end to the oscillations occurs no matter how small the positive value of the cosmological constant is; the cycles grow until they inevitably produce one that is large enough for the Λ term to eventually dominate the dynamics at large size, and its effect is to stop a further contraction from occurring. Notice that the final state is always one which is close to flatness and only just (depending on the exact size of the intercycle entropy jump) dominated by the Λ energy density—rather like our Universe, in fact.

Our paper is organized as follows. We begin in Sec. II by presenting some simple exact solutions with finite minima that illustrate the usefulness of introducing ghost fields into any model of bouncing universes. In Sec. III, we describe the thermodynamic aspects of the evolution. In Sec. IV, we give the diagonal Bianchi type IX metric and field equations and present some simple approximate parametric solutions for the case of the radiation-dominated axisymmetric type IX universe as well as a solution for the purely cosmological constant-dominated axisymmetric type IX universe (a case which is not encompassed by the cosmic no-hair theorems). These will serve to guide our interpretation of the numerical solutions of the full dynamical equations. In the next section, we present the results of numerically solving the full Bianchi type IX equations, first during a single cycle without the ghost field and then with the ghost field over many cycles. We study cases with and without the injection of radiation entropy. Finally, we study the same problems with the addition of positive or negative cosmological constants.

II. SIMPLE ISOTROPIC BOUNCES

The assumption of a “bounce” occurring at zero expansion scale and infinite density is computationally (and physically) awkward. However, it can be improved upon by introducing a simple “ghost” field, with negative density, $\rho < 0$, that will cause the expansion to go through a smooth minimum at the beginning and at the end of each cycle instead of through a singularity where $\rho = \infty$. Ghost fields have often been used in bouncing cosmology scenarios to effect a nonsingular bounce such as in Refs. [8,9] For illustrations, we can find two simple exact solutions which are of use in more complicated situations. Suppose that we have a closed Friedmann universe with scale factor, $a(t)$, containing two “fluids” having densities $\rho > 0$ and $\rho_g < 0$. The second ghost fluid with negative density, ρ_g , acts as a

model stress to dominate at small a and effect a bounce at $a = a_{\min}$, while the conventional fluid with positive density, ρ , dominates at larger a . This situation continues to exist until the spatial curvature creates an expansion maximum at $a = a_{\max}$. We give two solutions which are useful models of this type of behavior for more detailed analyses and illustrate the effects of the two fields:

- (i) Ghost fluid with $p_g = \rho_g \propto a^{-6} < 0$ and conventional radiation fluid with $p = \rho/3 \propto a^{-4} > 0$:

The Friedmann equation (setting $8\pi G = c = 1$) is

$$\frac{\dot{a}^2}{a^2} = -\frac{\Sigma}{a^6} + \frac{\Gamma}{a^4} - \frac{1}{a^2},$$

with $\Sigma \geq 0$ and $\Gamma \geq 0$ being constants. The exact solution for the scale factor, when $\Gamma^2 \geq 4\Sigma$, can be written simply in terms of the expansion maximum and minimum radii in conformal time, defined by $dt = ad\eta$, as [10]

$$a^2(\eta) = \frac{1}{2} [a_{\max}^2 + a_{\min}^2 + (a_{\max}^2 - a_{\min}^2) \sin 2(\eta + \eta_0)],$$

where the integration constant η_0 can be set to zero without loss of generality and

$$a_{\min}^2 \equiv \frac{\Gamma - \sqrt{\Gamma^2 - 4\Sigma}}{2},$$

$$a_{\max}^2 \equiv \frac{\Gamma + \sqrt{\Gamma^2 - 4\Sigma}}{2}.$$

Oscillatory solutions occur when $\Gamma^2 > 4\Sigma$, and $\Gamma^2 = 4\Sigma$ gives a static universe. In the high radiation entropy ($\propto \rho^{3/4} \propto \Gamma^{3/4}$) limit $\Gamma^2 \gg 4\Sigma$, we have $a_{\max} \rightarrow \Gamma$ and $a_{\min} \rightarrow \Sigma/\Gamma$, and we see that the maxima grow and the minima decrease in size if we let the radiation entropy grow from cycle to cycle.

- (ii) Ghost fluid with $p_g = \rho_g/3 \propto a^{-4} < 0$ and conventional dust fluid with $p = 0$, $\rho \propto a^{-3} > 0$:

The Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = -\frac{\Gamma}{a^4} + \frac{M}{a^3} - \frac{1}{a^2},$$

and, if $M^2 \geq 4\Gamma$, a new exact solution can be written simply in conformal time $dt = ad\eta$ in terms of the expansion maximum and minimum radii as [10]

$$a(\eta) = \frac{1}{4} [a_{\max} + a_{\min} + (a_{\max} - a_{\min}) \sin(\eta + \eta_0)],$$

where

$$a_{\min} = \frac{1}{2} [M - \sqrt{M^2 - 4\Gamma}],$$

$$a_{\max} = \frac{1}{2} [M + \sqrt{M^2 - 4\Gamma}].$$

In the limit $M^2 \gg 4\Gamma$, we have $a_{\min} \rightarrow \Gamma/M$ and $a_{\max} \rightarrow M$, so if we introduce an increase in matter entropy ($\propto M$) from cycle to cycle, then we will have successively increasing maxima and decreasing minima.

In what follows, we shall add a ghost field to an oscillating anisotropic, spatially homogeneous universe in order to produce a smooth bounce at finite values of the scale factor where the densities are nonsingular. It would be possible to effect a smooth bounce with a scalar field with quadratic potential which has been investigated in mixmaster universes [11]; however, the probability of this bounce occurring is small, $O(a_{\min}/a_{\max})$, in any universe with $a_{\min} \ll a_{\max}$.

All of the above discussion has focused upon simple isotropic closed universes with S^3 spatial topology. The situation in simple anisotropic universes of Kantowski-Sachs type was studied in detail by Barrow and Dąbrowski [7] and produces a more complicated scenario for cycle to cycle evolution. The Kantowski-Sachs universes have special $S^2 \times S^1$ spatial topology and are far from generic even among homogeneous anisotropic universes, although they have inhomogeneous generalizations with no symmetries found by Szekeres [12]. Only closed (compact space sections) universes with $S^2 \times S^1$ or S^3 topologies possess maximal hypersurfaces and so can recollapse and bounce when gravitationally attractive matter is present [13]. Whether or not they will do so depends on the matter content of the universe.

In this paper, we will study the dynamics of a cyclic Bianchi type IX “mixmaster” universe with S^3 spatial topology. This is the most general spatially homogeneous anisotropic closed universe, and it contains the closed Friedmann universe as an isotropic special case. Exact solutions are only known for the axisymmetric special cases in vacuum, containing stiff matter ($p = \rho$) or electromagnetic fields, or a combination of both [14]. We are particularly interested in the behavior of these anisotropic universes on approach to an expansion maximum of the volume and the behavior of the three expansion scale factors there. Do the cycles grow in maximum size, and do they become increasingly anisotropic from cycle to cycle? We will confine our attention to the case where the fluid in the universe is comoving, although in a subsequent study, we will generalize this to fluids with noncomoving velocities. The type IX universe behaves quite differently than the simple Bianchi type I anisotropic universe because it has both expansion anisotropy (shear) and 3-curvature anisotropy. The 3-curvature anisotropy has no Newtonian analog. The 3-curvature dynamics are complicated, and the sign of the 3-curvature varies in time and is only positive when the expansion dynamics are sufficiently

close to isotropy. An expansion maximum will only occur when the 3-curvature, ${}^{(3)}R$, becomes positive (as it is all the time in the closed Friedmann universes). We want to discover if increasing entropy increases the size of successive expansion maxima, as in the cyclic Friedmann models, but also determine what happens to the expansion anisotropy over successive cycles. We can also incorporate a positive or negative cosmological constant to see if it can terminate a sequence of oscillations in a cyclic type IX universe in the same way that it does in an isotropic closed universe.

In order to follow the Bianchi type IX evolution smoothly from cycle to cycle, we introduce a stiff ghost field to create an expansion minimum at nonzero volume in every cycle, as discussed above. This field has no significant effect on the expansion maximum or the behavior of the dynamics in its vicinity. It is well known that the Bianchi IX model displays formal chaotic behavior in all its degrees of freedom as the volume tends to zero [15]. However, there is only an unbounded number of chaotic oscillations of the scale factor on an open interval $0 < t < T$ around the time origin for finite T ; an infinite number of scale factor oscillations occur on any such interval no matter how small the value of T . In any finite interval $T_1 < t < T$, not including $t = 0$, the number of oscillations is finite and not technically chaotic. For realistic choices of $T_1 \approx 10^{-43}$ s for the start of classical cosmology, there will be fewer than about 12 mixmaster oscillations even if they continued all the way from T_1 up to the present day [16]. This is because the overall expansion scale changes rapidly with the number of scale factor oscillations, which occur in logarithmic time. Thus, if there is a bounce at *finite* volume, the issue of chaotic mixmaster oscillations [17] is irrelevant to a discussion of the long-term dynamics.

Some bouncing cosmologies deal with the situation at bounce by incorporating a so-called phase of ekpyrosis where there is effectively an ultrastiff isotropic fluid with a $p \gg \rho$ field added to the matter content of the universe [18]. If there are no anisotropic pressures, this drives the dynamics toward isotropy as a singularity is approached. However, we should expect anisotropic pressures to be larger than the energy density (as is assumed for the isotropic pressure), due to the dominance of collisionless particles when $T > 10^{15}$ GeV. The presence of these anisotropic pressures during the phase of ekpyrosis can reinstate the distorting effects of anisotropies, and they diverge on approach to the bounce [19,20].

In this work, we shall consider the effect of expansion and curvature anisotropies on a model of a bouncing type IX universe. This model incorporates several bounces, and we increase the entropy of the constituent matter content via an injection at each bounce. The increased entropy in each bounce leads to a higher maximum and longer cycles, as found in the original analysis of Tolman. The claim is that with increasing maxima simple isotropic bouncing

models, such as the isotropic Friedmann universe, approach flatness and begin to resemble the present-day universe in this respect. We shall investigate this claim in more general circumstances. We confirm the increasing volume maxima in successive bounces and the eventual cessation of bounces in the presence of a cosmological constant, but with a more complicated transitional evolution for the isotropization of the three scale factors. However, we find that in the absence of a cosmological constant successive cycles become increasingly anisotropic despite the increase in size and approach to flatness. This is quite different than the long-term evolution predicted by inflation.

We also investigate the effects of a negative cosmological constant. This always produces collapse to a future singularity [21]. Our aim in doing this is to construct an anisotropic version of the simple Friedmann universes which can all be transformed into simple harmonic oscillators in conformal time after rescaling the expansion scale factor [22]. These models are studied as a further simple example of a bouncing type IX universe. They include “domain-wall” matter ($p = -2/3\rho$) as well as a negative cosmological constant in a closed Friedmann universe. We can also include an ultrastiff matter field with isotropic pressures ($p = 5\rho$) to subdue the anisotropies on approach to the bounce. In the absence of pressure anisotropies, this should work and allow our model to propagate further without hitting a singularity. In a later study, we will include both noncomoving velocities and associated pressure anisotropies in the analysis of cyclic type IX universes.

A. Entropy

Our next task is to follow the consequences of a growth of entropy in the constituents of the system. The definition of a cosmological entropy is still debated. Here, we shall use only the thermodynamic entropy of the radiation or matter content and ignore any contribution from a “gravitational entropy” that might be associated with Weyl curvature, gravitational clustering, or the area of the particle horizon [23]. We will use a very simple toy model of entropy injection in our model of a bouncing universe. We are, in this analysis, not interested in the physical origin of the production of entropy by nonequilibrium processes like quantum particle production or viscous anisotropy damping, which are dramatic entropy producers as $t \rightarrow 0$ [24]. Rather, we will consider a sudden entropy increase at each expansion minimum and also use the dependence of the size of the expansion maximum on the entropy to determine the effect of increasing the entropy in a cycle. To circumvent issues regarding the nonconservation of baryon number, which would hence make the definition of entropy per baryon ambiguous, we shall consider the effects of an increase of entropy of radiation. To model this, we consider first the definition of the entropy of radiation, S , which is given by

$$S \propto T^3 V, \quad (1)$$

where T is the temperature at that instant and V is the volume of the universe. We assume that the entropy per unit volume, once injected at the minima, remains constant throughout the duration of the cycle until the next minimum (we ignore all particle-antiparticle annihilations and massive particles that become nonrelativistic). The radiation energy density varies as $\rho_r = C_r V^{-4/3}$. Hence, during each cycle, the quantity $T^3 V$ is a constant, implying that $\rho_r \propto T^4$. So, we can write the entropy as

$$S \propto T^3 \left(\frac{C_r}{\rho_r} \right)^{3/4} \propto C_r^{3/4}. \quad (2)$$

Thus, we can assess the effect of increasing the entropy of radiation by an increase in the constant C_r . It has been shown previously, in works such as Ref. [2], that an increase in the entropy of radiation leads to an increase in the expansion maxima in closed Friedmann universes. We expect this to occur also for a Bianchi IX universe. However, we also wish to study how the shape of the anisotropy behaves as the cycles get bigger with entropy injection. For this, we need to specify the form of the type IX metric and analyze the field equations.

III. MODEL OF THE BOUNCING BIANCHI TYPE IX UNIVERSE

A. Einstein equations for diagonal type IX universes

We aim to study the effects of entropy growth in anisotropic oscillating models of Bianchi type IX. We divide the problem into several subcases. We are interested in discovering whether the present-day universe would isotropize and approach flatness in a bouncing universe model after many cycles of entropy growth. In all cases, we take our cosmological model to be the spatially homogeneous Bianchi type IX spacetime containing radiation with pressure $p = \rho/3$ and a “dust” matter field with equation of state $p = 0$. We use the radiation field to assess the effects of increasing the entropy of radiation from cycle to cycle in a bouncing universe. If only these two fluids are present, the universe will evolve toward a strong curvature singularity and experience only one cycle unless we assume a periodic continuation through the singularity. If we add the ultrastiff ghost field with $\rho_g < 0$ and $p_g/\rho_g \gg 1$, then we create a smooth non-singular bounce and can follow the evolution through several cycles. We can then introduce a growth of entropy in the radiation field in the vicinity of the bounce in order to study the long-term effects of the shear anisotropy and the 3-curvature on the expansion maximum. In the last section of the paper, we introduce a positive cosmological constant and also a negative cosmological constant, to see how the evolution is changed by their presence. In the latter case, we

will add a domain-wall fluid ($p = -2\rho/3$) and an ultrastiff fluid ($p = 5\rho$) to facilitate smooth nonsingular bounces.

Note that when a ghost field is added to create a nonsingular bounce it means that the dynamics will be dominated by this isotropic matter field at the expansion minima and so it will have a small isotropizing effect over small time intervals around the minima. However, it is outweighed by the lengthening of the evolution time produced by the growing size of successive maxima. It would be possible to effect nonsingular bounces with an anisotropic ghost field, but this complication has been avoided here.

We study the solution of the Einstein equations for a diagonal spatially homogeneous Bianchi type IX universe with metric [25]

$$ds^2 = dt^2 - \gamma_{ab}(t)e_\mu^a e_\nu^b dx^\mu dx^\nu, \\ \gamma_{ab}(t) = \text{diag}[a^2(t), b^2(t), c^2(t)], \\ e_\mu^a = \begin{bmatrix} \cos z & \sin z \sin x & 0 \\ -\sin z & \cos z \sin x & 0 \\ 0 & \cos x & 1 \end{bmatrix}, \quad (3)$$

containing noninteracting perfect fluids, each with a perfect fluid equation of state $p = (\gamma - 1)\rho$. The equations of state parameters for the radiation, dust, and ghost fields are given by $\gamma_r = 4/3$, $\gamma_m = 1$, and $\gamma_g = 2$, respectively. The orthogonal expansion scale factors $a(t)$, $b(t)$, and $c(t)$ for the Bianchi type IX universe with the specified matter content satisfy the field equations

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{a^2}{4b^2c^2} + \frac{b^2}{4a^2c^2} - \frac{3c^2}{4a^2b^2} + \frac{1}{2a^2} + \frac{1}{2b^2} \\ - \frac{1}{2c^2} = - \sum_{i=r,m,g} (\gamma_i - 1)\rho, \quad (4)$$

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} + \frac{b^2}{4a^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3a^2}{4b^2c^2} + \frac{1}{2b^2} + \frac{1}{2c^2} \\ - \frac{1}{2a^2} = - \sum_{i=r,m,g} (\gamma_i - 1)\rho, \quad (5)$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}\dot{a}}{ca} + \frac{a^2}{4b^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3b^2}{4a^2c^2} + \frac{1}{2a^2} + \frac{1}{2c^2} \\ - \frac{1}{2b^2} = - \sum_{i=r,m,g} (\gamma_i - 1)\rho. \quad (6)$$

The constraint equation reduces to

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}\dot{a}}{ca} + \frac{1}{2a^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{a^2}{4b^2c^2} \\ - \frac{b^2}{4a^2c^2} - \frac{c^2}{4a^2b^2} = \sum_{i=r,m,g} \rho. \quad (7)$$

From the fluid continuity equations, we have

$$\rho_r(t) \propto (abc)^{-4/3} \\ \rho_g(t) \propto (abc)^{-2} \\ \rho_m(t) \propto (abc)^{-1}.$$

If we introduce a new time coordinate, τ , by defining

$$d\tau = dt/abc, \quad (8)$$

then the field equations become ($'$ denotes $d/d\tau$)

$$2(\ln a)'' + a^4 - (b^2 - c^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \quad (9)$$

$$2(\ln b)'' + b^4 - (c^2 - a^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \quad (10)$$

$$2(\ln c)'' + c^4 - (a^2 - b^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \quad (11)$$

and the constraint equation simplifies to

$$4[(\ln a)'(\ln b)' + (\ln b)'(\ln c)' + (\ln c)'(\ln a)'] \\ = a^4 + b^4 + c^4 - 2c^2(a^2 + b^2) - 2a^2b^2 \\ + 4a^2b^2c^2 \sum_{i=r,m,g} \rho_i. \quad (12)$$

Before we attempt to study the full numerical evolution of the type IX equations of motion with an increase in entropy of the radiation field in the presence of the ultrastiff ghost field and a dust field, we shall try to construct an approximate parametric solution for the type IX evolution containing only the radiation field. Detailed studies of the behavior of the most general Bianchi type universes at intermediate times have been conducted in Refs. [26,27]. What they reveal is that at a very early time there is a reduction in anisotropy by quantum effects which is significant. To the future of such a time, the evolution enters a long quasisymmetric phase. Two scale factors are larger than the third, and differences between the first two are insignificant compared to their size relative to the other. This situation is familiar from the evolutionary pattern during isotropization of Kasner metrics containing collisionless particles [28] where the anisotropic pressures created by the particles mimic the 3-curvature anisotropies in type IX. It is as if the dynamics has entered a time reverse of one of the long cycles that a Bianchi type IX universe

encounters on approach to small times. Following Doroshkevich *et al.* [26], we use the approximation $a = b \gg c$, which reduces Eqs. (9)–(11) to

$$(\ln a)'' + a^2 c^2 = \frac{1}{3} \rho_r a^4 c^2, \quad (13)$$

$$(\ln c)'' = \frac{1}{3} \rho_r a^4 c^2, \quad (14)$$

$$2(\ln a)'(\ln c)' + (\ln a)^2 = -a^2 c^2 + \rho_r a^4 c^2. \quad (15)$$

We find the parametric solution quoted in Ref. [26], as follows. Defining ω , the ratio of the two terms on the right-hand side of (14), by

$$\omega^2 \equiv \frac{a^2 c^2}{3 \rho_r a^4 c^2}, \quad (16)$$

we can express the radiation density in terms of the scale factors using $\rho_r = C_r (a^2 c)^{-4/3}$ and find ω in terms of the scale factors as $3\omega^2 = a^{2/3} c^{4/3} / C_r$. Hence, using (12)–(14), we can write

$$2(\ln \omega)'' = \frac{2}{3} (\rho_r a^4 c^2 - a^2 c^2) = \frac{2}{3} \rho_r a^4 c^2 (1 - 3\omega^2). \quad (17)$$

Inspecting the above equation, we see that the $\rho_r a^4 c^2$ term must be a function of the form $f(\omega, \omega', \omega'')$. Our next task is to determine the form of this function, so that once we substitute this form in, we can get a self-consistent solution for the evolution equation for the parameter ω , both by following the route of using Eq. (15) to obtain ω in terms of the scale factors and then using the equations of motion of the scale factors and also by using the functional form in terms of the parameter ω that we choose for $\rho_r a^4 c^2$ in its own evolution equation. After a few simple trials, it luckily turns out that the ansatz $\rho_r a^4 c^2 = \alpha \omega'$ gives us the self-consistent solution we desire, by following both of the routes we described. We shall demonstrate that this is in fact the case. We choose the ansatz $\rho_r a^4 c^2 = \alpha \omega'$ and absorb the constant of integration C_r into the constant α . Then, we have

$$(\ln \omega)' = \frac{1}{3} \alpha (\omega_0 + \omega - \omega^3), \quad (18)$$

where ω_0 is a constant of integration. We now examine the equation we get for $\rho_r a^4 c^2$ by substituting in the equations of motion. This is

$$(\ln \rho_r a^4 c^2)'' = \frac{2}{3} \rho_r a^4 c^2 (1 - 6\omega^2). \quad (19)$$

Using our ansatz, $\rho_r a^4 c^2 = \alpha \omega'$, we also find that

$$(\ln \alpha \omega')'' = \frac{2}{3} \alpha \omega' (1 - 6\omega^2). \quad (20)$$

We can write the left-hand side of this equation as $(\ln \alpha + \ln \omega')'' = (\ln \omega')''$ as $\ln \alpha$ is a constant. We can integrate the above equation once as

$$(\ln \omega')' = \frac{2}{3} (\omega_0 - 2\omega^3 + \omega). \quad (21)$$

We can write out the left-hand side of this equation as follows:

$$(\ln \omega')' = \omega' \frac{d}{d\omega} (\ln(\omega'/\omega) + \ln \omega). \quad (22)$$

Differentiating and canceling factors of ω' from the first term in the brackets in the above equation and then substituting in the right-hand side of Eq. (17) for $(\ln \omega)'$, we get

$$(\ln \omega')' = (\ln \omega)' + \frac{1}{3} \alpha \omega (1 - 3\omega^2). \quad (23)$$

Using the expression we found previously for $(\ln \omega)'$, in Eq. (17), we recover the same right-hand side as Eq. (21) up to some additive integration constants, confirming that our ansatz gives us a self-consistent solution. Hence, we can write the evolution equation for the parameter ω as

$$(\ln \omega)' = Q^{1/2} (\omega_0 + \omega - \omega^3), \quad (24)$$

where we have redefined our constant α to be the constant $3Q^{1/2}$ for notational consistency with Ref. [27]. In conclusion, we have the following radiation-era solution in terms of the parameter ω as in Ref. [27]:

$$a(\tau) = 3^{1/2} Q (\omega_0 + \omega - \omega^3), \quad (25)$$

and

$$c(\tau) = \frac{3^{1/2} \omega^{3/2}}{Q^{1/2} (\omega_0 + \omega - \omega^3)^{1/2}}. \quad (26)$$

By inspecting the solution, from the evolution equation of the parameter ω , that is (17), we find that, for $\omega = -1, 0, 1$, the equation yields a simple form,

$$(\ln \omega)' = Q^{1/2} \omega_0. \quad (27)$$

However, for the special choices of $\omega = -1, 0$, the scale factor $c(\tau)$ becomes imaginary and zero, respectively. On computing the volume maximum in terms of ω for all positive values of ω numerically, we find that the value of ω at which the volume maxima occur is very close to 1. Thus, we conclude that $\omega \sim 1$ is the point marking the volume

maximum as well as the end point of the validity of this parametric solution.²

We now use the axisymmetric solution for the type IX universe to see how the anisotropy behaves at the maximum of each cycle as we increase the radiation entropy, which is proportional to $C_r^{3/4}$. Keeping in mind that the limit $\omega \rightarrow 1$ corresponds to the instant of the maximum volume of the expansion, we can see how the ratios of the scale factors in the two directions behave in this limit. In the limit of $\omega \rightarrow 1$, the ratio a/c reduces to

$$\frac{a^2}{c^2} = Q^3 \omega_0^3. \quad (28)$$

We have seen that the quantity Q is related to the constant α , and hence to the constant C_r (which has been absorbed into the constant α as stated above). Therefore, an increase in the entropy of radiation [note that Eq. (8) ensures that $dS/dt > 0$ if and only if $dS/d\tau > 0$] causes a corresponding increase in the ratio of the scale factors, which indicates that the universe is becoming more anisotropic as the heights of the successive expansion maxima increase.

B. Adding a positive cosmological constant

We can use a similar approximation, but this time in comoving proper time, t , instead of the conformal time coordinate, τ . Using the approximation that $c \ll a = b$, we can reduce the Einstein equations to the following form,

$$\begin{aligned} 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} &= \Lambda, \\ \frac{\ddot{c}}{c} + \frac{\dot{a}}{a} + \frac{\dot{c}\dot{a}}{ca} &= \Lambda, \end{aligned} \quad (29)$$

where Λ is the cosmological constant. If we rewrite the first equation in the form

$$a \frac{d}{da} (\dot{a}^2) + \dot{a}^2 = \Lambda a^2 - 1, \quad (30)$$

it integrates to

$$\dot{a}^2 = \frac{\Lambda a^2}{3} - 1 + \frac{C_1}{a}, \quad (31)$$

²Well before the volume maximum is approached, the effects of the anisotropic 3-curvature are similar to the addition of a trace-free anisotropic pressure stress (long-wavelength homogeneous gravitational wave modes) on a background of simple Bianchi I form [29] or Friedmann form [30]. In the presence of isotropic blackbody radiation, the two scale factors evolve as $a(t) \propto t^{1/2} (\ln t)^{2n_1}$ and $c(t) \propto t^{1/2} (\ln t)^{n_2}$, with $2n_1 + n_2 = 0$, so the volume $a^2 b \propto t^{3/2}$ evolves as in Friedmann, to leading order [26,29], but the shear falls more slowly than when the 3-curvature is isotropic.

where C_1 is an integration constant. In order to determine the value of the constant C_1 , we can choose Kasner initial conditions where the scale factor $a(t) \sim t^{2/3}$ as $t \rightarrow 0$. Substituting this into Eq. (27), we get $C_1 = 4/9$ as $t \rightarrow 0$. The above equation resembles the equation for ordinary dust and a cosmological constant in an isotropic closed Friedmann universe. In the case of cosmological constant domination, the solution for $a(t)$ tends to the de Sitter solution, that is,

$$a(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right). \quad (32)$$

Under the approximation $a \sim b \gg c$, the Friedmann constraint becomes

$$\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}\dot{c}}{ac} + \frac{1}{a^2} = \Lambda. \quad (33)$$

Substituting in (27), we get

$$2\frac{\dot{a}\dot{c}}{ac} + \frac{C_1}{a^3} = \frac{2\Lambda}{3}. \quad (34)$$

Substituting in the solution for the cosmological constant dominated $a(t)$ and taking the limit of very late times (which is the limit in which we expect cosmological constant domination), we get

$$c(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right). \quad (35)$$

Thus, we see that for cosmological constant domination isotropization is achieved as $t \rightarrow \infty$, with the scale factors evolving toward

$$a(t) = b(t) \sim c(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}t\right). \quad (36)$$

It is worth noting that this result is not just a case of the standard cosmic no-hair theorem for spatially homogeneous universes due to Wald [31] because the 3-curvature can be positive in type IX universes and all cosmological no-hair theorems assume that ${}^3R \leq 0$ [32–34]. Ostensibly, this is to ensure the universe does not suffer collapse to a future singularity before the Λ term can dominate. However, the Kantowski-Sachs spatially homogeneous universe has ${}^3R > 0$ and need not approach the de Sitter metric at large t when $\Lambda > 0$ [35]. In fact, the conditions necessary (and sufficient) for type IX models with $\Lambda = 0$ to recollapse are extremely subtle [3,13], and examples have been found where type IX universes expand forever even though the sum of the density and the three principal pressures is positive [36]. Typically, the 3-curvature is *negative* as long as the dynamics are significantly

anisotropic (${}^3R = 2/c^2 - b^2/2c^4$ for axisymmetric (3) when $a = b \gg c$). This causes the expansion to continue until there is sufficient isotropization for the 3-curvature to become positive, and only then does an expansion maximum of the volume become possible. This occurs unless a positive cosmological constant comes to dominate before it is reached, as in Eq. (32).

Hence, we see that the results of Barrow and Dabrowski [7] showing the inevitable termination of oscillations in a closed oscillation universe with $\Lambda > 0$ continue to hold in Bianchi type IX universes. Eventually, cycles will grow large enough for the Λ term to dominate the dynamics. When it does so, it quickly stops the scale factors from reaching expansion maxima. They all continue to expand, and the dynamics are increasingly dominated by the Λ term and approach the de Sitter metric.

IV. NUMERICAL SOLUTIONS OF THE TYPE IX EQUATIONS

We move now to consider the numerical integration of the full (nonaxisymmetric) type IX equations, first, in the presence only of radiation, ρ_r , and then with radiation and a ghost field, ρ_g , that will create a smooth bounce at a finite-volume minimum. In each case, we will be interested in the behavior with ($\dot{S} > 0$) and without ($\dot{S} = 0$) an entropy increase and with and without a cosmological constant. These computations will enable us to confirm the general picture found from the analytic approximations of the previous section.

A. Radiation universe with no entropy increase: $\dot{S} = 0$

1. No ghost field: $\rho_g = 0$

In the case of the Bianchi IX universe containing just the radiation field, we have followed the initial assumption of Ref. [26] that two of the scale factors approach the same value in the expanding half of the cycle. Thus, on approach to the maximum of the scale factor, the universe resembles an axisymmetric type IX universe. The scale factors that are similar to each other approach their maxima first, and then, after reaching their maxima, they start contracting and then oscillate around an almost constant value. The third scale factor approaches its maximum at a later time and goes past the peaks of the other two scale factors before reversing into contraction. We can now follow the evolution of the quantities in this scenario, without needing to assume axial symmetry, by computing the behavior of the scale factors from various sets of initial conditions.

We start with initial conditions that are similar to the ones chosen in Ref. [26], that is,

$$[a(t), b(t), c(t)] \propto [t^{1/2}, t^{1/2}, t^{5/8}]. \quad (37)$$

In this case, in the absence of the ghost field, we find that the universe is unable to reexpand after collapsing. The addition of the radiation or dust field does not cause a qualitative change in the behavior of the scale factors but causes the system to reach stiffness faster. Thus, we choose values for the initial conditions for the density of radiation to be $\rho_r(t_i) = 1$ where t_i refers to the initial instant of time at which we start the integration. The scale factors in two directions oscillate about each other as they follow the evolution trend of the volume scale factor. The third scale factor is then smaller than the other two scale factors and does not display the oscillatory behavior undergone by the other two scale factors. The shear and the 3-curvature show oscillatory profiles before blowing up on approach to the strong curvature singularity at the ‘‘big crunch.’’ The presence of the singularity at the end of the collapse phase is inferred by the fact that the density of the matter and radiation components diverge there.

Running the simulation with arbitrarily selected initial conditions or even Kasner-like initial conditions makes the collapse occur closer to the starting instant, in comparison to the case done with the initial conditions in (33), and little information can be extracted from the results. For the Kasner initial conditions, however, before the collapse occurs, the individual scale factors show some oscillatory behavior.

We see a single oscillation of the volume scale factor in Fig. 1(a). The behaviors of the individual scale factors are seen in Fig. 1(b), where the blue dashed line corresponds to the scale factor $a(t)$ and the green dotted and the yellow solid lines correspond to the scale factors $b(t)$ and $c(t)$, respectively. As we have noted before, the scale factors $a(t)$ and $b(t)$ show small oscillations around each other, while the scale factor $c(t)$ has a much smaller amplitude and does not show such oscillations. This is reminiscent of a long era in the evolution of type IX on approach to an initial singularity seen in Ref. [17]. On looking at the shear and the spatial 3-curvature, we see that they display oscillatory behavior as well, before blowing up on approach to the singularity. The shear evolution is shown in Fig. 2(a), and the 3-curvature evolution is shown in Fig. 2(b).

2. Ghost field present: $\rho_g \neq 0$

We create a simple bouncing cosmological model by adding a ghost field to create a nonsingular bounce. The other fields in the system are the radiation fields and the dust fields. As before, we again choose initial values for the radiation, the dust, and the ghost fields to be of order 1, as $\rho_r(t_i) = 8$, $\rho_m(t_i) = 5$, and $\rho_g(t_i) = -5$, respectively. Again, as long as the initial conditions for the densities are of the same order, their exact numerical value does not much affect the results of the computation qualitatively. Changing these numbers significantly changes the number of bounces the system undergoes in the same time frame of integration, but qualitatively the features are not altered.

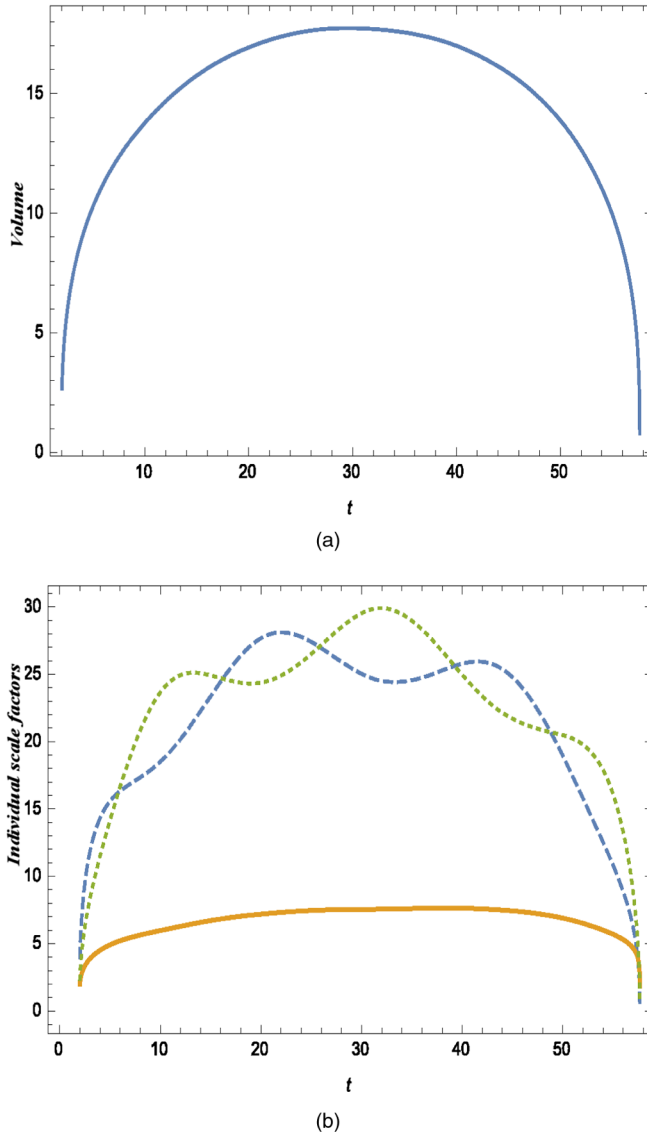


FIG. 1. Time evolution of the volume (top) and three orthogonal scale factors (bottom) of a type IX universe with only radiation and no ghost field in comoving proper time t during a single cycle. The blue dashed, green dotted, and yellow solid lines correspond to scale factors $a(t)$, $b(t)$, and $c(t)$, respectively.

Of course, as one might expect, changing the initial conditions for the density of the ghost field to be orders of magnitude smaller than the radiation and dust fields causes the system to collapse and not undergo the non-singular bounce. On evolving this system through several successions of bounces, we find that the scale factors oscillate rapidly from cycle to cycle as do the energy densities of the radiation, matter, and ghost field. The square of the shear tensor and the 3-curvature also show similar oscillatory behavior.

The evolution of the volume scale factor from cycle to cycle is oscillatory, as can be seen in Fig. 3(a), as are the behaviors of the scale factors in the three orthogonal

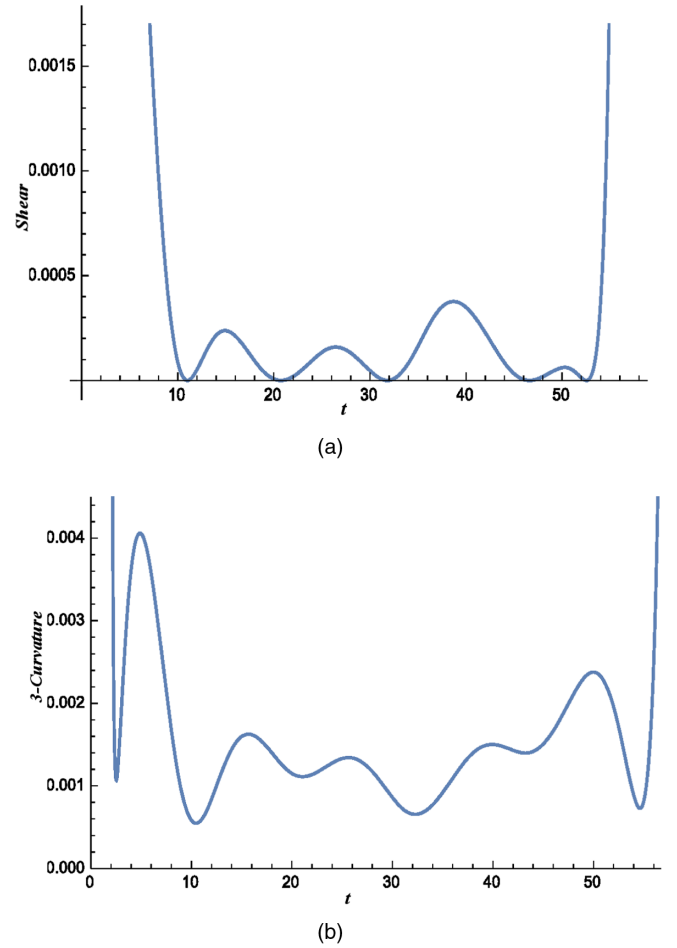


FIG. 2. From top to bottom: Evolution of shear and 3-curvature scalars during a single cycle of a type IX universe containing radiation and no ghost field, with comoving proper time, t .

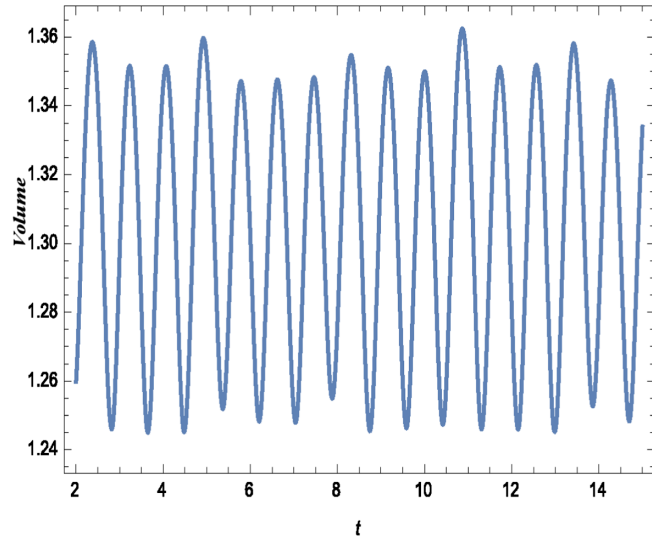
directions. As before, in Fig. 3(b), the blue dashed, solid yellow, and dotted green lines trace the $a(t)$, $b(t)$, and $c(t)$ scale factors. The shear and the 3-curvature also display oscillatory behavior and are shown in Figs. 4(a) and 4(b), respectively.

Therefore, we see that adding the ghost field is essential for avoiding collapse to a singularity after just one cycle and for allowing it to actually propagate smoothly through successive bounces. We will include the ghost field in the model in the rest of the paper when we examine the effects of entropy injection, or the effects of adding a cosmological constant, so that we can evolve the model through a series of cycles without encountering singularities.

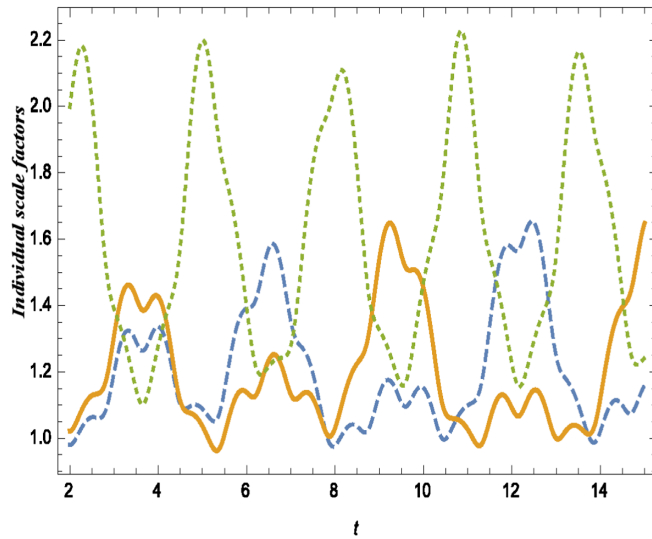
B. Radiation universe with entropy increase: $\dot{S} > 0$

1. Ghost field present: $\rho_g \neq 0$

We now consider our bouncing anisotropic cosmological model with dust, radiation, and a ghost field to prompt and to allow it to propagate through several cycles when there is an entropy increase from cycle to cycle.



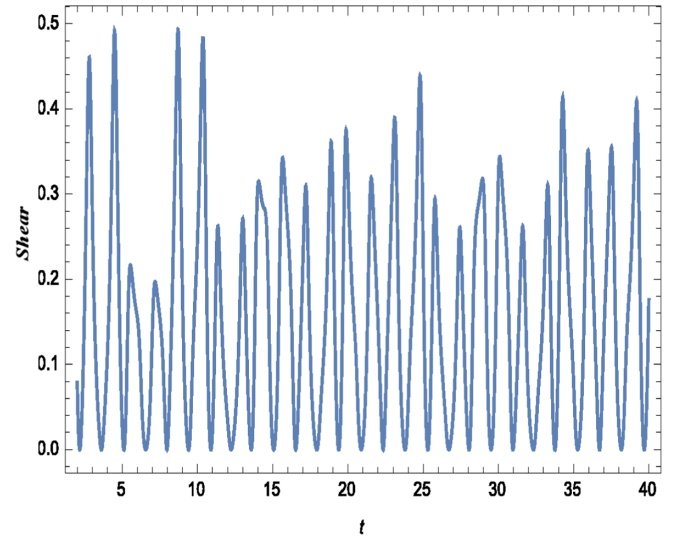
(a)



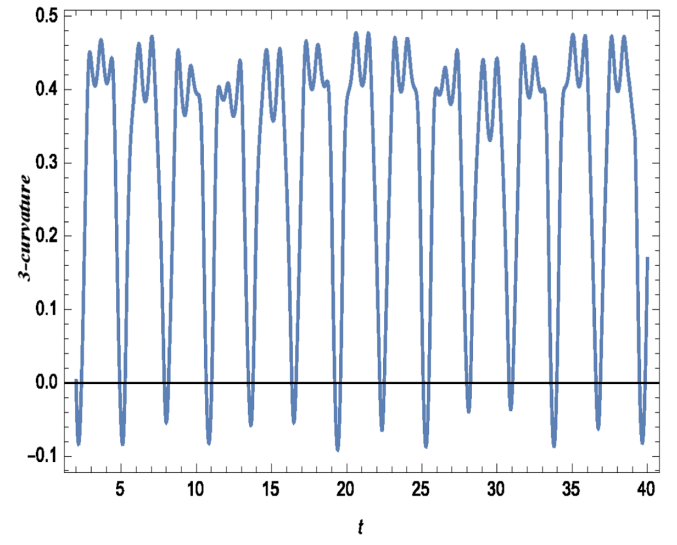
(b)

FIG. 3. Evolution of volume (top) and individual scale factors (bottom) with radiation, ordinary dust, and the ghost field included, with time. The blue dashed, green dotted, and yellow solid lines correspond to $a(t)$, $b(t)$, and $c(t)$, respectively.

We start with Kasner initial conditions and with the same initial scale-factor evolution equation (33) and the respective energy densities as before [$\rho_r(t_i) = 8$, $\rho_m(t_i) = 5$ and $\rho_g(t_i) = -5$]. This time, we increase the value of the constant of radiation (C_r) by a factor of 2, to model the effects of the entropy increase on the dynamics. We find that the volume scale factor shows an increase in cycle-to-cycle expansion maxima as expected. The individual scale factors proceed through several chaotic oscillations in each cycle, and the three directions seem to oscillate increasingly out of phase as the volume maxima get larger.



(a)

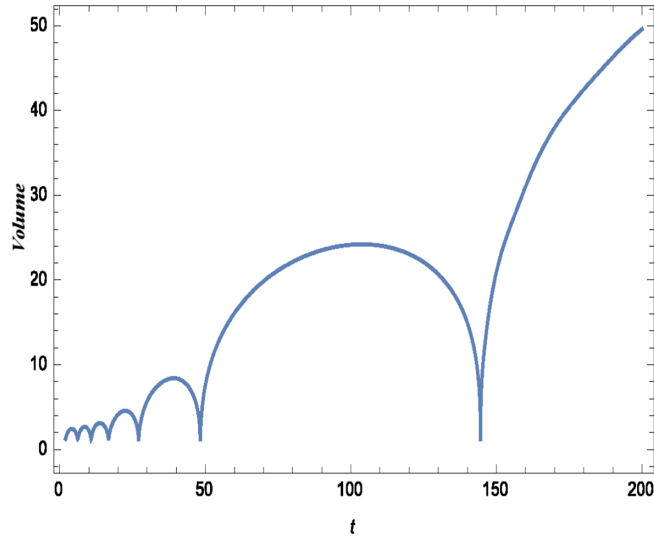


(b)

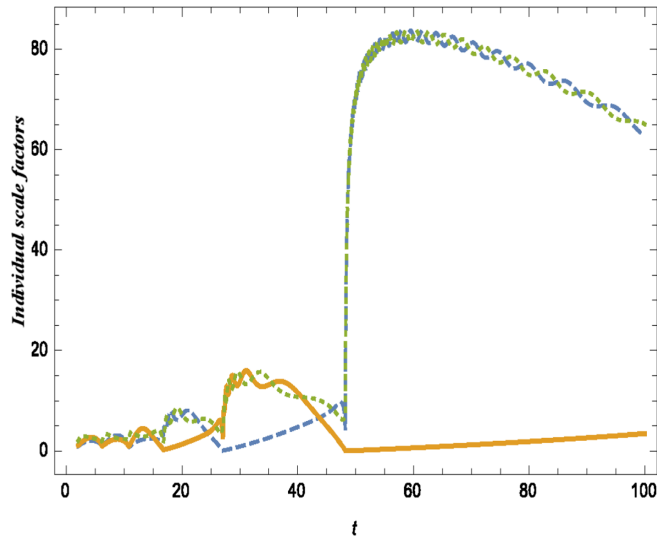
FIG. 4. From top to bottom: evolution of shear and curvature with radiation, ordinary dust, and the ghost field included, with time.

Figure 5(a) represents the evolution of the volume scale factor, and Fig. 5(b) represents the evolution of the individual scale factors.

To see if greater expansion volume maxima lead to an increase in the anisotropy, we plot the square of the shear tensor [Fig. 6(a)], denoted by σ^2 , and we see that the shear tensor indeed shoots up to larger and larger values, at each successive minimum, as the corresponding radiation maximum is increased. We can see that a similar increase occurs when we track the difference in the expansion rates of the scale factors in the three directions. A significant increase in the differences of the expansion rates in the a and the b directions and in the difference of the expansion rates in the b and c directions is seen as the expansion maxima get



(a)

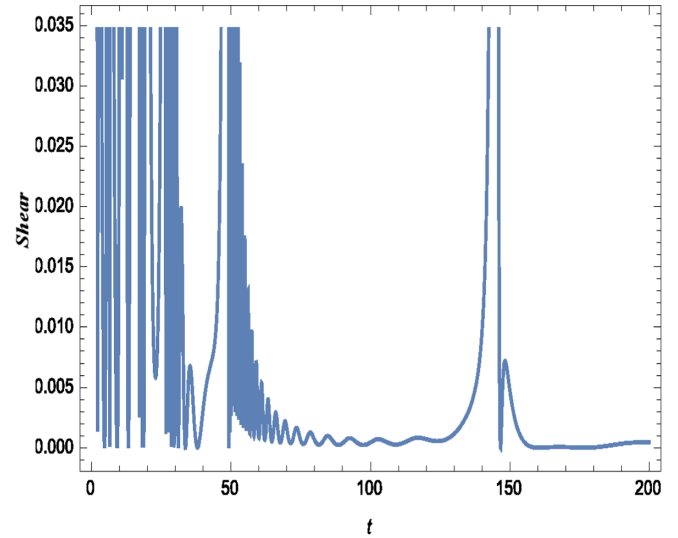


(b)

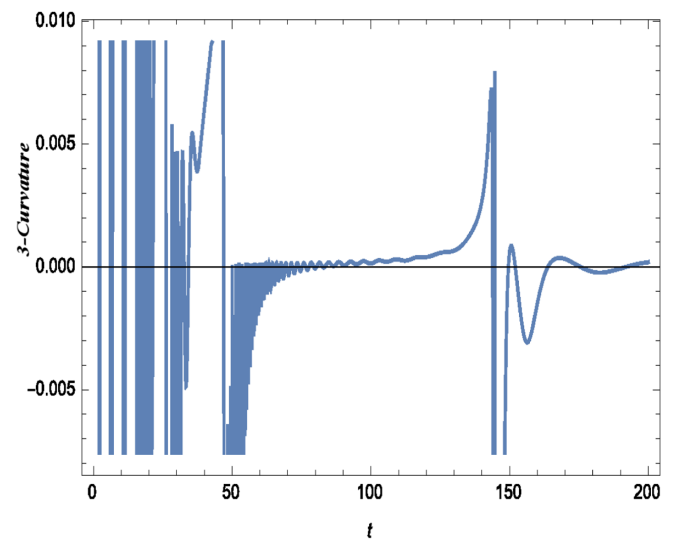
FIG. 5. Evolution of the volume (top) and individual scale factors (bottom) of a type IX universe with entropy increase from cycle to cycle ($dS/dt > 0$), with t time. The blue dashed, green dotted, and solid yellow lines correspond to the orthogonal scale factors $a(t)$, $b(t)$, and $c(t)$ respectively. The model includes radiation, dust matter, and the ghost field. The ghost field ensures that the cycles bounce smoothly at finite values of the volume.

bigger. There is not such a large increase in the difference in the expansion rates in the a and c directions. We can also look at the 3-curvature [Fig. 6(b)], and we find that it oscillates to a greater extent initially, but as time increases, the amplitude and frequency of these oscillations decrease, and the universe seems to approach flatness, albeit with strong expansion and 3-curvature anisotropy.

We can understand this intuitively in another way. The shear is at its highest near the minimum of each cycle. As the expanding phase of the cycle begins, the shear is diluted



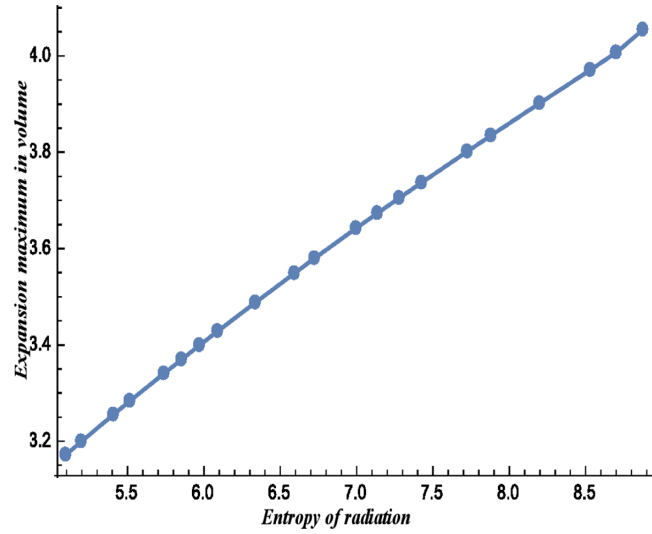
(a)



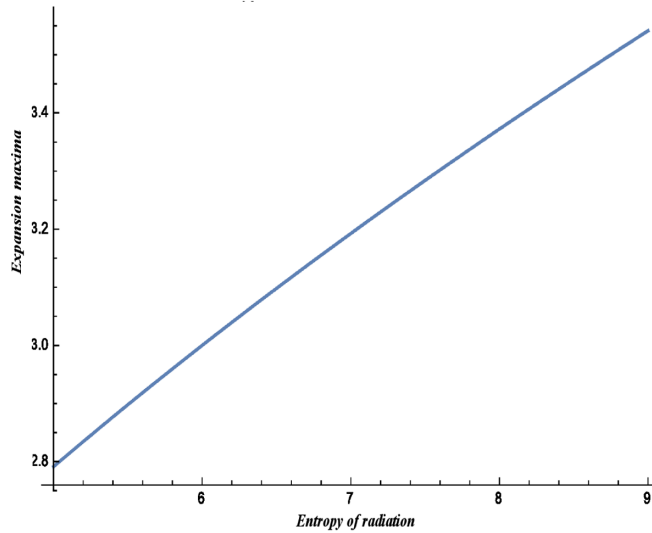
(b)

FIG. 6. Evolution of shear (top) and the 3-curvature (bottom) scalars with time in a type IX universe with entropy increase. The model includes radiation, dust, and the ghost field to create smooth nonsingular bounces.

rather slowly as $\sigma \sim (t \ln t)^{-1}$. Until the shear is diluted sufficiently, the universe cannot recollapse. When the shear enters the contracting phase, shear anisotropy accumulates. The longer the model evolves before the bounce, the greater the amount of shear anisotropy is accumulated. Thus, by injecting radiation entropy and effectively increasing the expansion maxima, we give the shear anisotropy more time to increase in the contracting phase, and thus more time to dilute in the expanding phase. Hence, despite the increase in the shear anisotropy, we still see the universe recollapse and bounce.



(a)



(b)

FIG. 7. (Top) Volume maxima plotted against the entropy of radiation in Bianchi IX universes and (bottom) in isotropic Friedmann universes.

We can also compare the pattern of entropy growth in the anisotropic Bianchi IX case with its isotropic subcase, the closed Friedmann universe.

We see in Fig. 7(a) for the case of the Bianchi IX universe and in Fig. 7(b) for the case of the isotropic closed Friedmann universe that the variation of successive volume scale factor maxima in the Friedmann and the type IX universes with increasing radiation entropy show very similar, almost linear behavior (this would be different if we injected entropy according to a different rule). We also show what happens to the range of the bounce with entropy injection in the Bianchi IX case.

It should come as no surprise that the range is also increasing fairly linearly with the injection of entropy as

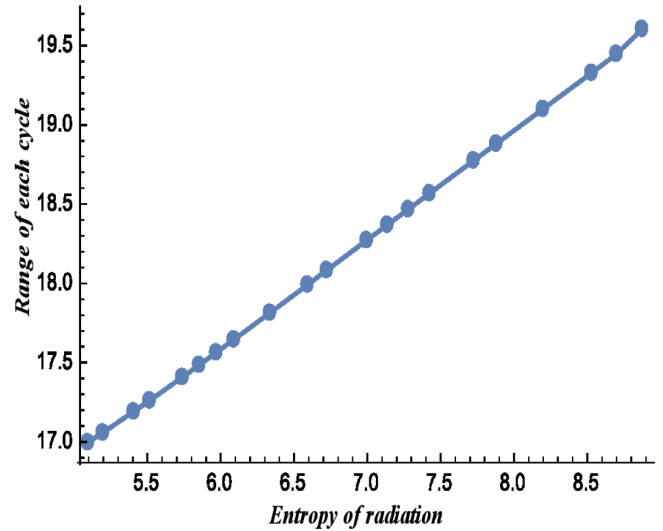


FIG. 8. Range of bounces vs entropy of radiation in the presence of ordinary dust and radiation in oscillating Bianchi IX universes.

can be seen in Fig. 8. The increase in volume maxima simply means that the model takes a longer time to recollapse.

V. ADDING A COSMOLOGICAL CONSTANT

We have seen in previous work that the addition of cosmological constant to the closed Friedmann model which incorporates increasing volume maxima with the injection of radiation entropy results in the model ceasing to oscillate before expanding exponentially toward the de Sitter metric [7]. Now, we study the effects of adding both a positive or a negative cosmological constant in type IX universes. The motivation for doing this, for the case of the positive cosmological constant, is to see if a similar exponential expansion to the isotropic case takes place. It is also interesting to investigate the effect the expansion prompted by the cosmological constant has on the anisotropy and the spatial 3-curvature. The negative cosmological constant models create an interesting anisotropically recollapsing counterpart to the closed Friedmann models. These models are among the simplest versions of a closed isotropic bouncing universe as they incorporate a negative cosmological constant as well as a curvature “field” which behaves as a fluid with equation of state $p = -1/3\rho$. They admit a simple periodic solution in the isotropic case. It is expected that when anisotropy is included, in the absence of an ultrastiff matter field, the universe will quickly approach an anisotropic singularity since the negative Λ term is only influential at large volumes to affect a collapse but has a negligible effect as the future singularity is reached.

A. Positive cosmological constant

We study the effect of adding a positive cosmological constant to a type IX model containing radiation with

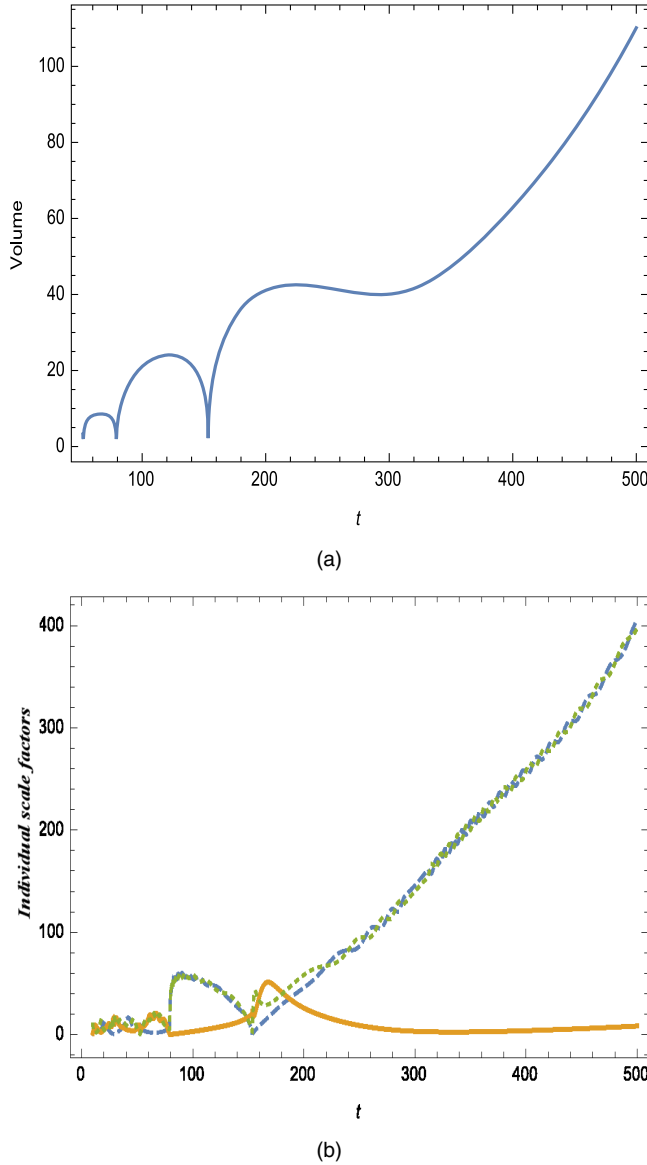


FIG. 9. Evolution of (top) the volume and (bottom) the individual scale factors of a type IX universe with positive Λ , with time, t . The blue dashed, green dotted, and solid yellow lines correspond to scale factors $a(t)$, $b(t)$, and $c(t)$, respectively. The model includes radiation, dust, and a ghost field to create nonsingular bounces. Note that the oscillations cease after a finite time when Λ term dominates the dynamics at large volume. All scale factors then asymptote to the de Sitter expansion after a few transitional oscillations.

entropy injection, a dust field, and the ultrastiff ($p = 5\rho$) ghost field to ensure a smooth nonsingular bounce. The results depend on the value of the cosmological constant relative to the initial energy densities of the other fields. If the cosmological constant is set to have a value of the same order of magnitude as the other components in the system, then the model displays a very sudden increase in all the scale factors in each of the three directions without any of the oscillating behavior that we have seen in the other

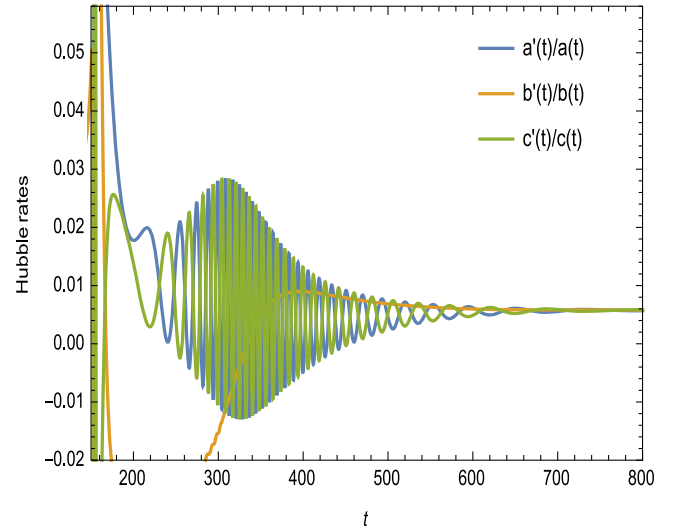
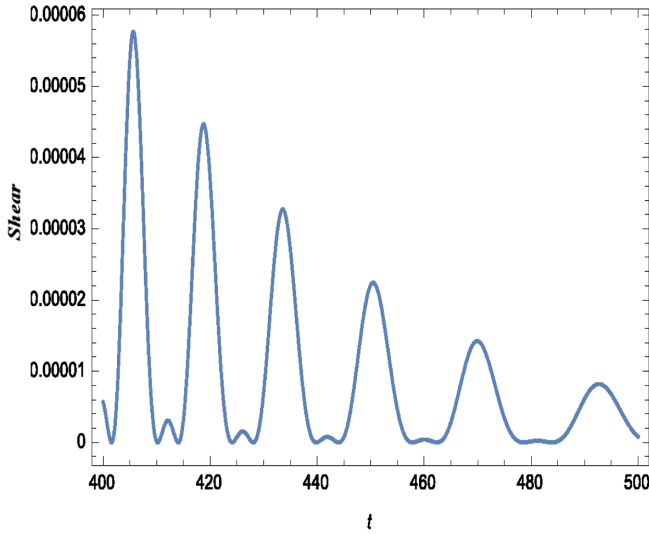


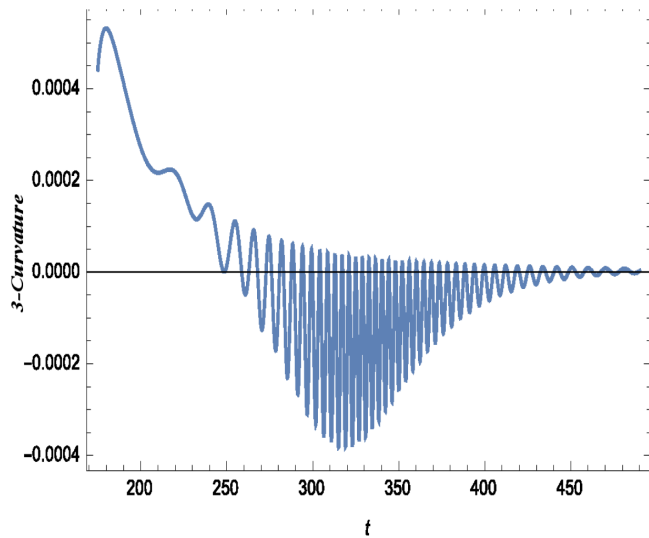
FIG. 10. Evolution of the Hubble rates in the presence of a positive cosmological constant with time. The blue dashed, green dotted, and solid yellow lines trace the Hubble rates $\dot{a}(t)/a(t)$, $\dot{b}(t)/b(t)$, and $\dot{c}(t)/c(t)$. Oscillations cease when Λ dominates. The Hubble rates then undergo an anisotropic transition phase before eventually approaching isotropic de Sitter-like expansion where the individual Hubble rates approach the same constant value.

cases; it quickly asymptotes to de Sitter behavior. On the other hand, if the cosmological constant is too small, then its effects cannot be seen in the time interval that we have set for integration. Thus, for an intermediate range of values of the cosmological constant, we find that as soon as the cosmological constant starts to dominate the volume scale factor, which was hitherto showing an oscillatory behavior with increasing maxima from cycle to cycle, does not recollapse beyond the point of maxima. Instead, it enters a final phase of exponential expansion. The individual scale factors all undergo exponential expansion, asymptoting to de Sitter behavior. However, they have different rates of expansion in this phase, with two of the scale factors having nearly the same rate, oscillating around each other in this phase (this is reminiscent of the axisymmetric behavior that motivated our analytic approximation in Sec. III above). The third scale factor expands much less than the other two. As soon as the cosmological constant starts to dominate, the shear and the 3-curvature, which were oscillating before Λ domination as in the previous case with $\Lambda = 0$, now start oscillating with smaller and smaller amplitudes as time progresses. For the purposes of our computation, we set the cosmological constant to be a value which is approximately $\Lambda \approx 3H^2$ which can be taken to mark the onset of cosmological constant domination.

In Fig. 9(a), we show the evolution of the volume scale factor, and in Fig. 9(b), we show the evolution of the individual scale factors, where the blue dashed, yellow solid, and green dotted lines represent the scale factors $a(t)$,



(a)



(b)

FIG. 11. (Top) Evolution with time, t , of the shear and (bottom) 3-curvature scalars in type IX universes with positive Λ . The model includes radiation, dust, and a ghost field to create nonsingular bounces.

$b(t)$, and $c(t)$, respectively. We can see the domination of the Λ term leading to de Sitter-like expansion in all of the scale factor directions by looking at the Hubble rates.

We see in Fig. 10 that the Hubble rates undergo several oscillations and enter into a transition phase when the cosmological constant starts to dominate. However, with positive cosmological constant domination, the oscillations cease, and as the scale factors undergo exponential expansion, their Hubble rates tend to a constant value.

Figures 11(a) and 11(b) show the behavior of the shear tensor squared σ^2 and of the 3-curvature, respectively. They show a decrease with time, with the shear showing oscillations, as before, but with decreasing amplitude.

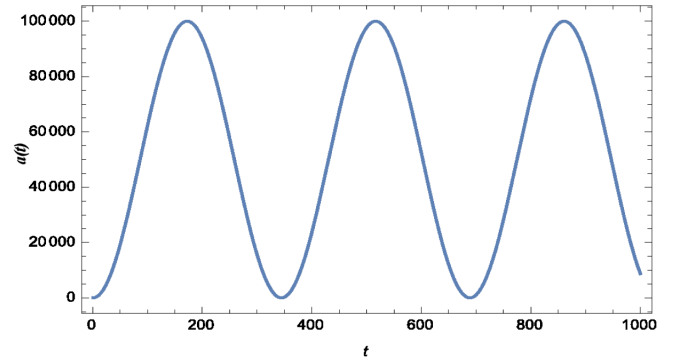


FIG. 12. Evolution in time, t , of the scale factor for an isotropic closed Friedmann universe with negative cosmological constant and domain-wall matter ($p = -2\rho/3$). The domain-wall matter acts like a ghost field to produce smooth nonsingular bounces. No entropy production is included.

The curvature also shows oscillations as before but falls to very small values as the cosmological constant starts to dominate.

B. Negative cosmological constant

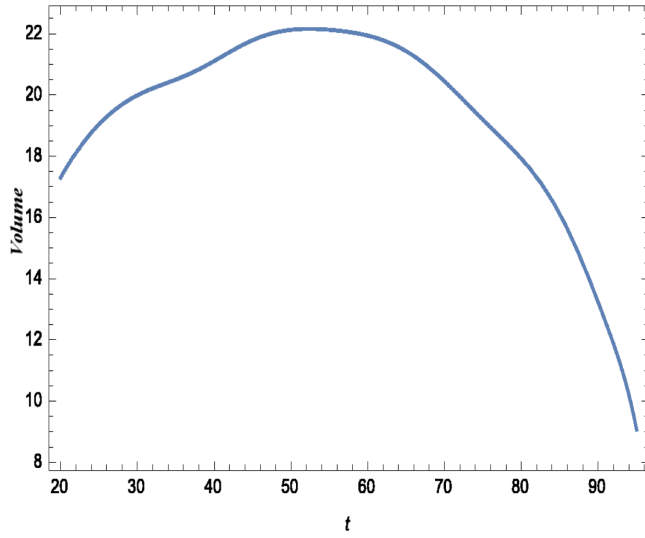
We can also try to look at the case of the negative cosmological constant to find the role of anisotropies in the simple isotropic universe model. This is the closed Friedmann universe with curvature parameter $k = +1$, consisting of “domain-wall matter” which is just matter with an equation of state $p = -2\rho/3 \propto a^{-1}$ and a negative cosmological constant. The Friedmann equation is now ($\Sigma > 0$, constant)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \left(\Lambda + \frac{\Sigma}{a} \right) - \frac{1}{a^2}. \quad (38)$$

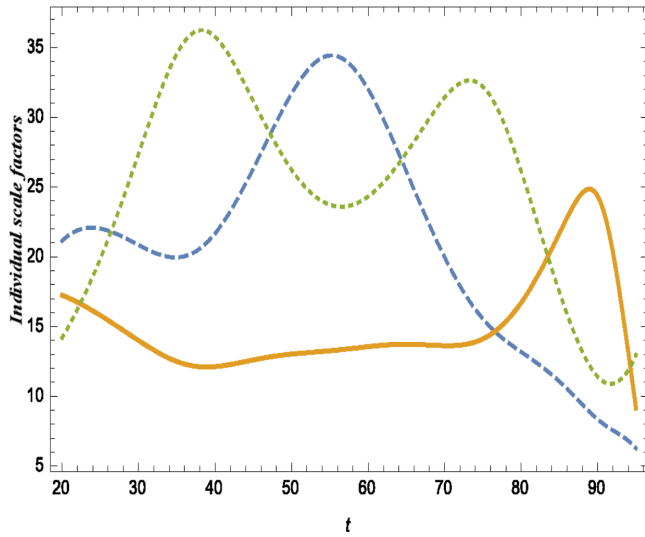
This model has been studied as a simple model of a bouncing universe admitting a simple solution (Fig. 12).

Consider an extension of this model to the anisotropic type IX case. In the absence of a stiff field to smooth out the anisotropies on approach to the bounce, it is expected that the universe on collapse will not be able to reexpand from a singularity. To prevent this, we add an ultrastiff matter field, with positive energy density, as we expect the bounce to be produced by the negative cosmological constant. We find that the volume scale factor undergoes a bounce. The individual scale factors undergo several oscillations with different time periods and different amplitudes. The shear undergoes oscillations with amplitudes that decrease as the volume scale factor expands and starts increasing again as the contraction phase begins. The 3-curvature also shows oscillatory behavior as the expansion phase is followed by the contraction.

On changing the sign of the cosmological constant in the model, with the same initial conditions as we have been using previously, we see that with the injection of entropy



(a)



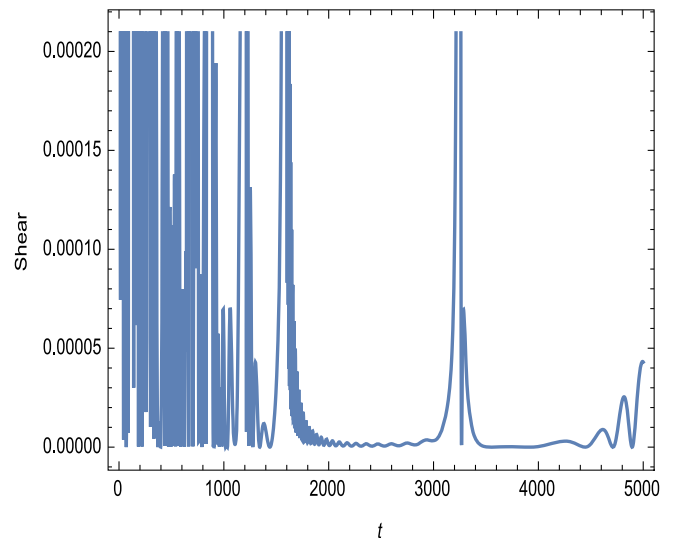
(b)

FIG. 13. Evolution of the (top) volume and (bottom) individual scale factors for a type IX universe with negative Λ , with time, t . The blue dashed, green dotted, and yellow solid lines correspond to $a(t)$, $b(t)$, and $c(t)$, respectively. The model includes an ultrastiff matter field ($p = 5\rho$) with positive energy density in addition to the negative cosmological constant.

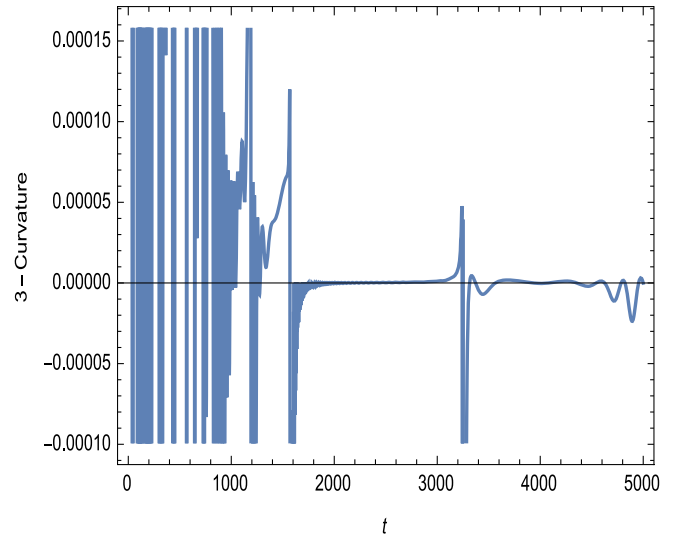
into the radiation field, to facilitate an increase in the expansion maxima, the volume scale factor undergoes irregular oscillations. These can be seen in Fig. 13(a).

The individual scale factors are seen in Fig. 13(b) and are given as before by the blue dashed, green dotted, and solid yellow lines representing the scale factors $a(t)$, $b(t)$, and $c(t)$, respectively. They appear to undergo oscillations as well.

Looking at Figs. 14(a) and 14(b), the shear and the curvature also show several rapid and irregular oscillations as they approach minima. On the collapse of the model, they



(a)



(b)

FIG. 14. (Top) Evolution of the shear and (bottom) the 3-curvature, with negative Λ , vs time, t . The model includes an ultrastiff field ($p = 5\rho$) with positive energy density in addition to a negative cosmological constant.

show no sign of approaching isotropy, blowing up instead. To demonstrate that the long-term evolution is chaotic, we find that on slightly changing the initial conditions by ± 0.001 (where the initial conditions we have chosen for $[x, y, H]$ as well as the densities of the ghost and radiation fields are of order 1) the number of oscillations in the shear and the curvature, as well as their amplitude and shape, drastically change (especially for the initial few cycles of the model). The behavior for the shear for the second set of initial conditions is shown in Fig. 15(a).

Looking at the 3-curvature next, we find a similar situation, where the shape of the oscillations as well as

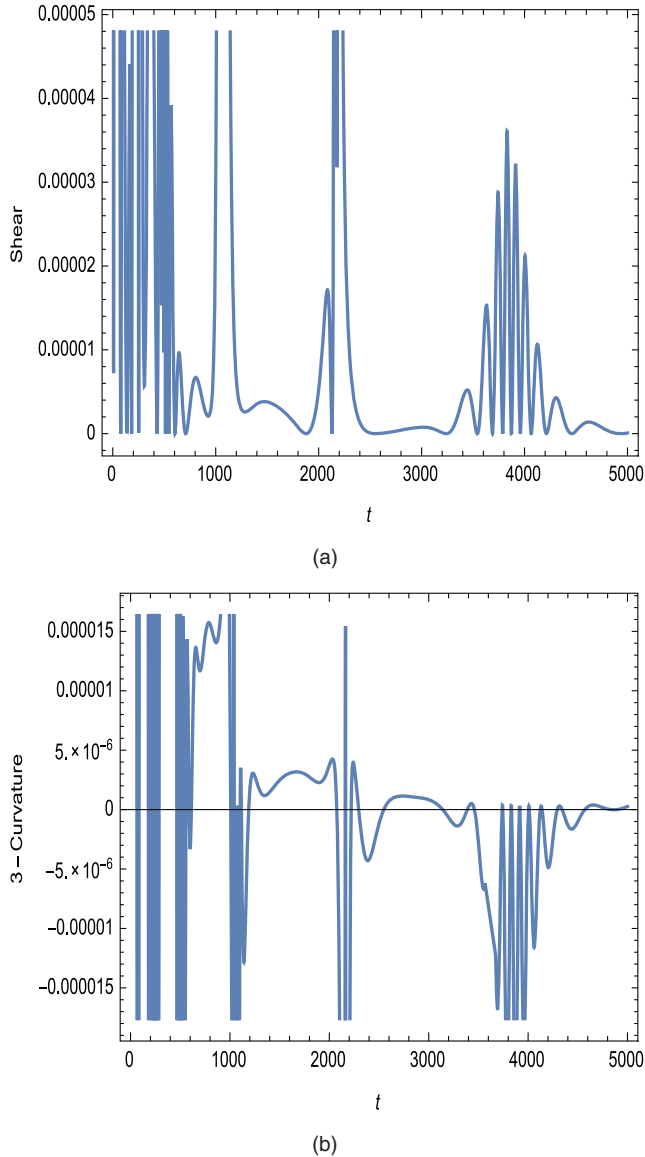


FIG. 15. Evolution of the shear (a) and curvature (b) in the type IX universe with negative Λ and ghost field, with time, with initial conditions differing by ± 0.001 , to illustrate the chaotic sensitivity of the dynamics to small changes in initial data over many cycles of time evolution.

their frequency and amplitude vary greatly with the same slight change in the initial conditions. We see this in the Fig. 15(b).

It is interesting to note that this chaotic behavior from cycle to cycle occurs in t time, rather than in its logarithms as is the case in the chaotic behavior seen on approach to the singularity if bounces do not occur [15,17].

VI. CONCLUSIONS

We have investigated the fate of cyclic universes in the most general spatially homogeneous closed universes of Bianchi type IX. These models are considerably more

general than those previously used to study oscillating universes. We include radiation, matter, and a stiff ghost field with negative density to produce a smooth, non-singular bounce at finite volume at the end of each cycle. A bounce at any finite volume, no matter how small, avoids the issue of chaotic oscillations [17] of the scale factors on approach to the expansion minima. We investigate the effects of an increase in radiation entropy from cycle to cycle, in accord with the second law of thermodynamics, and also the effects of adding a positive or negative cosmological constant to the Einstein equations. We find the following:

A. Flatness and shape evolution

When $\Lambda = 0$, we studied the evolution of type IX universes by analytic approximations and by numerical evolution (starting with Kasner initial conditions). The evolution follows an approximately axisymmetric form in which different scale factors attain their maxima at different times before turning around and collapsing toward their next minima. We found that as the size of the volume maximum increases, the model approaches flatness in the same way that isotropic closed universes do. However, the dynamics do not approach isotropy as the volume maximum is approached or as the ensuing minimum is approached. The successive expansion maxima grow increasingly out of phase. The long-term dynamics are therefore anisotropic and differ significantly from those predicted for inflationary universes.

B. General relativistic effects

The late-time evolution of the type IX universe is dominated by intrinsically general-relativistic effects associated with its 3-curvature anisotropy (for which there is no Newtonian analog). The sign of the 3-curvature scalar, 3R , can change with time. When the type IX universe is significantly anisotropic, 3R is negative, and the dynamics cannot have an expansion maximum. The universe therefore keeps on expanding and eventually becomes close enough to isotropy for 3R to become positive, and then it is able to experience a volume maximum and recollapse.

C. The effects of entropy increase

We injected radiation entropy at each finite expansion minimum to model the effect of increasing entropy. We found that the entropy increase leads to a steady increase in the size of the volume maxima of successive cycles and to their temporal duration, but these maxima are anisotropic.

D. The effect of $\Lambda > 0$

The addition of a cosmological constant is always found to bring the oscillations in the volume of the universe to an end. This occurs no matter how small the value of Λ is. Oscillations of the universe occur and grow anisotropically

as in the case of $\Lambda = 0$ until the size of the maximum grows large enough for Λ to become dynamically important there. Subsequently, after a few scale factor transitory changes, it will dominate before any expansion maximum can occur and accelerate the expansion toward an increasingly de Sitter-like metric evolution. This behavior is in accord with cosmic no-hair theorems even though, technically, they do not apply to the type IX metric because it permits positive 3-curvature, which is excluded by the theorems.

E. The effect of $\Lambda < 0$

The addition of a negative cosmological constant causes any cosmological model to recollapse, regardless of the sign of the 3-curvature. We use the addition of $\Lambda < 0$ to produce a simple bouncing model that experiences a finite nonsingular minimum at the end of each cycle because of the presence of a ghost field. We follow the chaotic evolution of the scale factors, the shear, and the 3-curvature from cycle to cycle. We showed that there is sensitive dependence on initial conditions.

Our analysis introduced some simplifying assumptions. We consider only the diagonal Bianchi IX metric with fluids that possess comoving velocity fields and isotropic pressures.

In a separate study, we will relax these assumptions and show that a similar analysis is possible which leads to similar conclusions. Thus, we have shown that in the most general spatially homogeneous anisotropic cyclic universes in general relativity with $\Lambda = 0$ the growth of entropy leads to never-ending cycles of increasing size and duration, as Tolman first showed for isotropic models. However, although these cycles approach flatness, they do not approach isotropy and do not resemble our observed Universe. If we add $\Lambda > 0$ then, no matter how small the magnitude of Λ is, the growing oscillations always come to an end, and subsequently the dynamics pass through a quasi-isotropic phase before asymptoting toward the isotropic dynamics of a de Sitter metric. These analyses can also be readily extended to other cyclic universe scenarios that are based upon extensions of general relativity [18,37–43].

ACKNOWLEDGMENTS

J. D. B. is supported by the Science and Technology Facilities Council (STFC) of the United Kingdom. C. G. is supported by the Jawaharlal Nehru Memorial Trust Cambridge International Scholarship.

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- [1] A. A. Friedmann, *Z. Phys.* **10**, 377 (1922) [*Gen. Relativ. Gravit.* **31**, 1991 (1999)]; A. Friedmann, *Die Welt als Raum und Zeit* (Verlag Harri Deutsch, Frankfurt am Mein, 2006), translated by G. Singer from 1923 Russian original; English translation: A. A. Friedmann, *The World as Space and Time* (Minkowski Institute Press, Montreal Canada, 2014), translated by S. Andrenenko from 1923 Russian original.
 - [2] R. C. Tolman, *Phys. Rev.* **38**, 1758 (1931); *Relativity, Thermodynamics and Cosmology*, (Oxford University, New York, 1934).
 - [3] J. D. Barrow, G. J. Galloway, and F. J. Tipler, *Mon. Not. R. Astron. Soc.* **223**, 835 (1986); J. D. Barrow, *Classical Quantum Gravity* **21**, L79 (2004).
 - [4] W. B. Bonnor, *Z. Astrophys.* **35**, 10 (1954),
 - [5] H. Zanstra, *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series B* (1957), Vol. 60, p. 285.
 - [6] P. T. Landsberg and D. Park, *Proc. R. Soc. A* **346**, 485 (1975).
 - [7] J. D. Barrow and M. Dąbrowski, *Mon. Not. R. Astron. Soc.* **275**, 850 (1995).
 - [8] Y.-F. Cai, T. Qiu, X. Zhang, Y.-S. Piao, and M. Li, *J. High Energy Phys.* **10** (2010) 071.
 - [9] Y.-F. Cai, E. N. Saridakis, M. R. Setare, and J.-Q. Xia, *Phys. Rep.* **493**, 1 (2010).
 - [10] J. D. Barrow and C. G. Tsagas, *Classical Quantum Gravity* **26**, 195003 (2009).
 - [11] J. D. Barrow and R. A. Matzner, *Phys. Rev. D* **21**, 336 (1980).
 - [12] P. Szekeres, *Commun. Math. Phys.* **41**, 55 (1975).
 - [13] J. D. Barrow and F. J. Tipler, *Mon. Not. R. Astron. Soc.* **216**, 395 (1985).
 - [14] M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University, Princeton, 1975); J. D. Barrow, *Nature (London)* **272**, 211 (1978).
 - [15] J. D. Barrow, *Phys. Rev. Lett.* **46**, 963 (1981); *Phys. Rep.* **85**, 1 (1982); D. Chernoff and J. D. Barrow, *Phys. Rev. Lett.* **50**, 134 (1983); J. D. Barrow, *Classical General Relativity*, edited by W. Bonnor, J. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984), p. 25.
 - [16] Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics* (University of Chicago, Chicago, 1983), Vol. 2, p. 658.
 - [17] C. W. Misner, *Phys. Rev. Lett.* **22**, 1071 (1969); *Phys. Rev.* **186**, 1328 (1970); V. A. Belinskii and I. M. Khalatnikov, *Sov. Phys. JETP* **29**, 911 (1969); V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Sov. Phys. Usp.* **13**, 745 (1971).
 - [18] P. J. Steinhardt and N. Turok, *Phys. Rev. D* **65**, 126003 (2002).
 - [19] J. D. Barrow and K. Yamamoto, *Phys. Rev. D* **82**, 063516 (2010).
 - [20] J. D. Barrow and C. Ganguly, *Classical Quantum Gravity* **33**, 125004 (2016).

- [21] F. J. Tipler, *Astrophys. J.* **209**, 12 (1976).
- [22] J. D. Barrow, *Observatory* **105**, 120 (1985).
- [23] W. B. Bonnor, *Phys. Lett.* **112A**, 26 (1985); N. Pelavas and A. Coley, *Int. J. Theor. Phys.* **45**, 1258 (2006); T. Clifton, G. F. R. Ellis, and R. Tavakol, *Classical Quantum Gravity* **30**, 125009 (2013); J. D. Barrow and S. Hervik, *Classical Quantum Gravity* **19**, 5173 (2002).
- [24] J. D. Barrow and R. A. Matzner, *Mon. Not. R. Astron. Soc.* **181**, 719 (1977).
- [25] L. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th rev. ed. (Pergamon, Oxford, 1975).
- [26] A. D. Doroshkevich, V. N. Lukash, and I. D. Novikov, *Sov. Phys. JETP* **37**, 739 (1973); I. D. Novikov, *IAU Symposium 63*, edited by M. S. Longair (Reidel, Dordrecht, 1974), p. 273.
- [27] V. N. Lukash, *Sov. Astron.* **18**, 164 (1974).
- [28] V. N. Lukash and A. A. Starobinsky, *Sov. Phys. JETP* **39**, 742 (1974).
- [29] J. D. Barrow, *Phys. Rev. D* **55**, 7451 (1997).
- [30] L. P. Grishchuk, A. G. Doroshkevich, and V. M. Yudin, *Sov. Phys. JETP* **42**, 943 (1976).
- [31] R. Wald, *Phys. Rev. D* **28**, 2118 (1983).
- [32] L. G. Jensen and J. A. Stein Schabes, *Phys. Rev. D* **35**, 1146 (1987).
- [33] J. D. Barrow, *Phys. Lett. B* **187**, 12 (1987).
- [34] J. D. Barrow and G. Götz, *Classical Quantum Gravity* **6**, 1253 (1989).
- [35] R. Kantowski and R. K. Sachs, *J. Math. Phys. (N.Y.)* **7**, 443 (1966); E. Weber, *J. Math. Phys. (N.Y.)* **26**, 1308 (1985); J. Yearsley and J. D. Barrow, *Classical Quantum Gravity* **13**, 2693 (1996).
- [36] S. Calogero and J. M. Heinze, *Phys. Rev. D* **81**, 023520 (2010); *Physica (Amsterdam)* **240D**, 636 (2011).
- [37] J. D. Barrow, D. Kimberly, and J. Magueijo, *Classical Quantum Gravity* **21**, 4289 (2004).
- [38] R. Brandenberger and P. Peter, [arXiv:1603.05834](https://arxiv.org/abs/1603.05834).
- [39] J. Haro, J. Amorós, and L. A. Saló, [arXiv:1703.03710](https://arxiv.org/abs/1703.03710).
- [40] D. C. F. Celani, N. Pinto-Neto, and S. D. P. Vitenti, *Phys. Rev. D* **95**, 023523 (2017).
- [41] J. D. Barrow and D. Sloan, *Phys. Rev. D* **88**, 023518 (2013).
- [42] E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky, and S. Yu. Vernov, *J. Cosmol. Astropart. Phys.* **12** (2016) 006.
- [43] R. Moriconi, G. Montani, and S. Capozziello, *Phys. Rev. D* **94**, 023519 (2016).