PHYSICAL REVIEW D **95**, 076021 (2017)

Schwinger's proper time and worldline holographic renormalization

Dennis D. Dietrich[1] and Adrian Koenigstein[1,2]

[1]*Institut für Theoretische Physik, Goethe-Universität, Frankfurt am Main, Germany*
[2]*Frankfurt Institute for Advanced Studies, Frankfurt am Main, Germany*

(Received 6 December 2016; published 26 April 2017)

Worldline holography states that within the framework of the worldline approach to quantum field theory sources of a quantum field theory over Mink_4 naturally form a field theory over AdS_5 *to all orders* in the elementary fields and in the sources of arbitrary spin. (Such correspondences are also available for other pairs of spacetimes, not only $\mathrm{Mink}_4 \leftrightarrow \mathrm{AdS}_5$.) Schwinger's proper time of the worldline formalism is automatically grouped with the physical four spacetime dimensions into an AdS_5 geometry. We show that the worldline holographic effective action in general and the proper-time profiles of the sources in particular solve a renormalization group equation and, reversely, can be defined as solution to the latter. This fact also ensures regulator independence.

DOI: 10.1103/PhysRevD.95.076021

I. INTRODUCTION

Strong interactions are behind a wealth of phenomena but oftentimes beyond our computational abilities. Holographic approaches promise analytic insight and have been applied, for example, to QCD [1–3], extensions of the Standard Model [4,5], condensed-matter physics [6], and the Schwinger effect [7–9]. Holography takes off from the conjectured AdS/CFT correspondence [10–12]. All examples of this correspondence found since, however, hold for theories with a set of symmetries that are not found in nature and that posses a particle content different from QCD. As a result, in practice, one considers deformed bottom-up AdS/QCD descriptions, which describe the QCD hadron spectrum rather well [1,13]. Still, they lack a derivation from first principles. As a consequence, it is very important to comprehend under which circumstances and for which reasons these models represent an acceptable approximation and which features are robust. For some approaches to these questions, see Refs. [14–16].

In this context, Refs. [9,17–19] demonstrated how a quantum field theory over Mink_4 readily turns into a field theory for its sources over AdS_5 in the framework of the worldline formalism [20,21] for quantum field theory. Schwinger's proper time naturally becomes the fifth dimension of an AdS_5 geometry. This result also extends to different pairs of spacetimes including the nonrelativistic case. Reference [18] showed that such an AdS_5 formulation arises *to all orders* in the elementary fields—matter and gauge. Schwinger's proper time represents a length scale (inverse energy scale), which is also the interpretation of the extra dimension in holography [1–3,11,12]. Handling UV divergences of a theory necessitates regularization, which in the worldline formalism is customarily taken care of by proper-time regularization, the introduction of a positive lower bound on the proper time. The proper-time regularization corresponds to the UV-brane regularization in holography [1–3,11,12]. The worldline holographic framework ensures the independence of physical predictions of the unphysical value of this cutoff parameter, which identifies worldline holography as a renormalization group flow.

This central statement of the present paper is treated in Sec. III. Before that, in Sec. II, building on Ref. [18], we reiterate how exactly a quantum field theory over Mink_4 reorganizes into a field theory for its sources over AdS_5 to all orders in the elementary fields as well as the sources and with Schwinger's proper time of the worldline formalism as the fifth dimension. After that, in Sec. III B, we analyze some examples in the free case. First, in Sec. III B 1, we study the holographic renormalization of QED. Subsequently, in Sec. III B 2, we turn to general higher-spin sources. Particularly, we show that the AdS_5 geometry is self-consistent. The penultimate section provides a short summary. We conclude with a discussion and outlook in Sec. V.

II. WORLDLINE HOLOGRAPHY

In order to display the essence of the program, we begin with one massless scalar.[1] flavor and a vector source V combined with the gauge field G in the "covariant derivative" $\mathbb{D} = \partial - \mathrm{i}\mathbb{V}$, where $\mathbb{V} = G + V$. Thus, to all orders, the generating functional for vector correlators reads

$$Z = \langle \mathrm{e}^{w} \rangle = \int [\mathrm{d}G] \mathrm{e}^{w - \frac{i}{4e^2}\int \mathrm{d}^4 x G_{\mu\nu}^2}, \tag{1}$$

where

$$w = -\frac{1}{2}\mathrm{Tr}\ln \mathbb{D}^2. \tag{2}$$

[1]Nothing obstructs the treatment of elementary matter with spin, but, for the sake of simplicity and clarity, here we limit ourselves to spinless elementary matter. Results for fermionic matter are presented elsewhere [22,23].

In the worldline formalism[2] [20,21] after a Wick rotation, w is given by [9,17,19]

$$w = \int \mathrm{d}^4 x_0 \int_{\varepsilon>0}^{\infty} \frac{\mathrm{d}T}{2T^3} \mathcal{L} \equiv \iint_{\varepsilon}^{\infty} \mathrm{d}^5 x \sqrt{g} \mathcal{L}, \tag{3}$$

$$\mathcal{L} = \frac{2\mathcal{N}}{(4\pi)^2} \int_{\mathrm{P}} [\mathrm{d}y] \mathrm{e}^{-\int_0^T \mathrm{d}\tau [\frac{\dot{y}^2}{4} + \mathrm{i}\dot{y}\cdot\mathbb{V}(x_0+y)]}, \tag{4}$$

with the five-dimensional metric g,

$$\mathrm{d}s^2 = g_{MN} \mathrm{d}x^M \mathrm{d}x^N = +\frac{\mathrm{d}T^2}{4T^2} + \frac{\mathrm{d}x_0 \cdot \mathrm{d}x_0}{T}, \tag{5}$$

and the square root of the absolute of its determinant $\sqrt{g}$.[3] The symbol $\cdot$ represents the contraction with the flat four-dimensional metric $\eta_{\mu\nu}$, Wick rotated from mostly plus to all plus, which, simultaneously, turns Eq. (5) from an $\mathrm{AdS}_{4,1}$ (frequently simply referred to as AdS_5; the indices indicate the metric signature) to an $\mathrm{AdS}_{5,0}$ (also referred to as H_5 or EAdS_5) line element. The isometries of the five-dimensional AdS space are the symmetries of the conformal group of the corresponding four-dimensional flat space. T stands for Schwinger's proper time. A factor of T^{-1} in the volume element arises from exponentiating the logarithm in Eq. (2), another factor of T^{-2} from taking the trace. The Lagrangian density $\mathcal{L}$ is made up of a the path integral over all closed paths over the proper-time interval $[0;T]$, i.e., with $y(0) = y(T)$. The $\mathrm{d}^4 x_0$ integral translates these paths to every position in space, $x \equiv y + x_0$. (The translations are the zero modes of the kinetic operator ∂_τ^2. $\dot{y} \equiv \partial_\tau y$. Splitting them off from the rest of the path integral also makes momentum conservation manifest.)[4] Inspection of the free part of the worldline action $\int_0^T \mathrm{d}\tau \frac{(\partial_\tau y)^2}{4} = \int_0^1 \mathrm{d}\hat{\tau} \frac{(\partial_{\hat{\tau}} y)^2}{4T}$, where $\hat{\tau} = \tau/T$, shows that small values of T correspond to short relative distances, i.e., to the UV regime. Thus, the proper-time regularization $T \geq \varepsilon > 0$ (introduced when exponentiating the logarithm 4) is a UV regularization and corresponds to the UV-brane regularization in holography [1–3].

[2]Particle-wave duality. The worldline representation (3) of the functional determinant (2) can actually be seen as the particle dual of the determinant's wave(-function or field) representation as Feynman functional integral

$$w = \ln \int [\mathrm{d}\phi][\mathrm{d}\phi^\dagger] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4 x \phi^\dagger D^2 \phi}.$$

[3]This emergent metric is linked to the metric

$$\mathrm{d}s^2 = (\mathrm{d}z^2 + \mathrm{d}x_0 \cdot \mathrm{d}x_0)/z^2$$

oftentimes used in holography and found, for example, in Refs. [1,2] by $T = z^2$.

[4]For details and more intermediate steps, please see Refs. [17,18,21].

A. Volume elements

Thus, in the worldline formalism, w falls readily into the form of an action (3) over AdS_5. e^w, however, contains all powers of w. Building on the discussion in Ref. [18], for the nth power,

$$w^n = \prod_{j=1}^n \int \mathrm{d}^4 x_j \int_\varepsilon^\infty \frac{\mathrm{d}T_j}{2T_j^3} \mathcal{L}(x_j, T_j). \tag{6}$$

The free part again only depends on the relative positions, and we split off a common absolute coordinate $x_0 = \mathfrak{x}_0(\{x_j\})$, which can be any linear combination of the x_j, e.g., the center of mass $\frac{1}{n}\sum_{j=1}^n x_j$. There remain integrations over $n-1$ four-dimensional relative coordinates, $\mathrm{d}^{4(n-1)}\Delta$,

$$\int \prod_{j=1}^n \mathrm{d}^4 x_j = \int \prod_{j=1}^n \mathrm{d}^4 x_j \int \mathrm{d}^4 x_0 \delta^{(4)}[x_0 - \mathfrak{x}_0(\{x_k\})] \tag{7}$$

$$= \int \mathrm{d}^4 x_0 \int \left(\prod_{j=1}^n \mathrm{d}^4 x_j\right) \delta^{(4)}[x_0 - \mathfrak{x}_0(\{x_k\})] \tag{8}$$

$$= \int \mathrm{d}^4 x_0 \int \mathrm{d}^{4(n-1)}\Delta. \tag{9}$$

Analogously, we introduce an overall proper time $T = \mathfrak{T}(\{T_j\})$ and proper-time fractions $t_j = T_j/T$ using

$$1 = \int \mathrm{d}T \delta[T - \mathfrak{T}(\{T_j\})] \prod_{j=1}^n \left[\int \mathrm{d}t_j \delta\left(t_j - \frac{T_j}{T}\right)\right]. \tag{10}$$

Also here, there exists a continuum of choices for the overall proper time T, all of which allow us to come to the same conclusion below. Introducing additional dimensionful scales would be artificial. In their absence, on dimensional grounds, always $\mathfrak{T}(\{T_j\}) = T \times \mathfrak{T}(\{t_j\})$. A definition invariant under the pairwise exchange of the T_j makes the corresponding symmetry of w^n manifest from the start. The arguably simplest choice with these characteristics would be $\mathfrak{T} = \sum_{j=1}^n T_j$. In any case, physics is invariant under any invertible change of variables. Equation (10) implies

$$\prod_{j=1}^n \int_\varepsilon^\infty \frac{\mathrm{d}T_j}{2T_j^3} = \int \mathrm{d}T \prod_{j=1}^n \left[\int_\varepsilon^\infty \frac{\mathrm{d}T_j}{2T_j^3} \int \mathrm{d}t_j \delta\left(t_j - \frac{T_j}{T}\right)\right] \times \delta[T - \mathfrak{T}(\{T_l\})] \tag{11}$$

$$= \int \frac{\mathrm{d}T}{2T^3} T^{-2(n-1)} \int_{\frac{\varepsilon}{T}}^\infty \left(\prod_{j=1}^n \frac{\mathrm{d}t_j}{2t_j^3}\right) \times 2\delta[1 - \mathfrak{T}(\{t_l\})]. \tag{12}$$

Putting everything together,

$$w^n = \int \mathrm{d}^4x_0 \int \frac{\mathrm{d}T}{2T^3} \int \frac{\mathrm{d}^{4(n-1)}\Delta}{T^{2(n-1)}} \int_{\frac{\varepsilon}{T}}^{\infty} \left[\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \times \mathcal{L}(x_0 + x_j - x_0, Tt_j) \right] 2\delta[1 - \mathfrak{T}(\{t_l\})] \tag{13}$$

$$= \int \mathrm{d}^4x_0 \int \frac{\mathrm{d}T}{2T^3} \int \mathrm{d}^{4(n-1)}\hat{\Delta} \int_{\frac{\varepsilon}{T}}^{\infty} \left[\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \times \mathcal{L}(x_0 + \widehat{x_j - x_0}\sqrt{T}, Tt_j) \right] 2\delta[1 - \mathfrak{T}(\{t_l\})], \tag{14}$$

where $x_j - x_0$ in the argument of $\mathcal{L}_j$ depends only on the relative coordinates Δ and not the absolute coordinate x_0. In the last step, we introduced dimensionless relative coordinates $\hat{\Delta} = \Delta/\sqrt{T}$. This demonstrates that every power w^n takes the form of a Lagrangian density integrated over AdS_5.

B. Contractions

It remains to be shown that all spacetime indices are contracted with AdS metrics. Expressing the vector $\mathbb{V}$ in $\mathcal{L}(x_j, T_j)$ in terms of dimensionless variables, $\widehat{x_j - x_0}$ as well as $\hat{y}_j = y_j/\sqrt{T_j} = y_j/\sqrt{Tt_j}$, and with the help of a translation operator,

$$\mathbb{V}(y_j + x_j) = \mathbb{V}(y_j + x_j - x_0 + x_0) \tag{15}$$

$$= \mathbb{V}\left[\sqrt{T}\left(\hat{y}_j\sqrt{t_j} + \widehat{x_j - x_0}\right) + x_0\right] \tag{16}$$

$$= \mathrm{e}^{\sqrt{T}(\hat{y}_j\sqrt{t_j} + \widehat{x_j - x_0})\cdot\partial_{x_0}}\mathbb{V}(x_0), \tag{17}$$

we obtain from Eq. (4)

$$\mathcal{L}(x_j, T_j) = \frac{2\mathcal{N}}{(4\pi)^2}\int_{\mathrm{P}}[\mathrm{d}y_j]\mathrm{e}^{-\int_0^T \mathrm{d}\tau\left[\frac{\dot{y}^2}{4} + \mathrm{i}\dot{y}\cdot\mathbb{V}(y_j + x_j)\right]} \tag{18}$$

$$= \frac{2\hat{\mathcal{N}}}{(4\pi)^2}\int_{\mathrm{P}}[\mathrm{d}\hat{y}_j]\exp\left\{-\int_0^1 \mathrm{d}\hat{\tau}_j\left[\frac{(\partial_{\hat{\tau}_j}\hat{y}_j)^2}{4} + \mathrm{i}\sqrt{t_jT}(\partial_{\hat{\tau}_j}\hat{y}_j)\cdot\mathrm{e}^{\sqrt{T}\left(\sqrt{t_j}\hat{y}_j + \widehat{x_j - x_0}\right)\cdot\partial_{x_0}}\mathbb{V}(x_0)\right]\right\}, \tag{19}$$

where we also use the dimensionless integration variable $\hat{\tau}_j = \tau_j/T_j = \tau_j/(Tt_j)$. This expression shows that every gradient ∂_{x_0} and every field $\mathbb{V}$, i.e., every four-dimensional spacetime index, is accompanied by one power of $\sqrt{T}$.

The same holds still after integrating out the gauge field G. To see this, we split the Wilson line for $\mathbb{V}$ into one for the sources V and one for the gauge fields G,

$$\mathrm{e}^{\mathrm{i}\oint \mathrm{d}x\cdot\mathbb{V}} = \mathrm{e}^{\mathrm{i}\oint \mathrm{d}x\cdot V}\mathrm{e}^{\mathrm{i}\oint \mathrm{d}x\cdot G}. \tag{20}$$

This is possible because the sources V are gauge singlets and the gauge fields G are flavor singlets and commute as a consequence. Then, with the definition of the gauge-field average from Eq. (1),

$$\left\langle \prod_{j=1}^{n}\mathcal{L}_j \right\rangle = \left(\frac{2\hat{\mathcal{N}}}{(4\pi)^2}\right)^n \prod_{j=1}^{n}\int_{\mathrm{P}}[\mathrm{d}\hat{y}_j]\exp\left\{-\int_0^1 \mathrm{d}\hat{\tau}_j\left[\frac{(\partial_{\hat{\tau}_j}\hat{y}_j)^2}{4} + \mathrm{i}\sqrt{t_jT}(\partial_{\hat{\tau}_j}\hat{y}_j)\cdot\mathrm{e}^{\sqrt{T}(\sqrt{t_j}\hat{y}_j + \widehat{x_j - x_0})\cdot\partial_{x_0}}V(x_0)\right]\right\} \times \left\langle \prod_{l=1}^{n}\mathrm{e}^{\mathrm{i}\oint \mathrm{d}y_l\cdot G(x_l + y_l)} \right\rangle. \tag{21}$$

In the factor on the last line, G is integrated out. Said factor is invariant under reparametrizations of the Wilson line as well as of four-dimensional translations. As a consequence, it is independent of the value of T as well as from x_0. Hence, it only depends on the four-dimensional relative coordinates. Additionally, the factor is a scalar and as such can only depend on the combinations $(\forall l, j)$

$$\eta^{\mu\nu}(y_j + x_j - x_0)_\mu(y_l + x_l - x_0)_\nu = T\eta^{\mu\nu}(\sqrt{t_j}\hat{y}_j + \widehat{x_j - x_0})_\mu(\sqrt{t_l}\hat{y}_l + \widehat{x_l - x_0})_\nu. \tag{22}$$

Taking stock, the powers of $\sqrt{T}$ stay (only) with every ∂_{x_0} and V. Thus, after the $[\mathrm{d}\hat{y}]$ integrations, the result can only contain the combinations $T\eta^{\mu\nu}V_\mu V_\nu$, $T\eta^{\mu\nu}V_\mu\partial_\nu$, and $T\eta^{\mu\nu}\partial_\mu\partial_\nu$, which combine into $g^{\mu\nu}V_\mu V_\nu$, $g^{\mu\nu}V_\mu\partial_\nu$, and $g^{\mu\nu}\partial_\mu\partial_\nu$, respectively[5] Consequently, Eq. (1) can be expressed as an action over AdS_5 for its sources to all orders and to all orders in the elementary fields.

III. THE FIFTH DIMENSION

Previously, Refs. [9,17,18] identified the fifth components of gradients and fields as arising from an induced Wilson (gradient) flow and determined by a variational principle. Here, we show that the identical result is obtained by imposing the independence of the effective action of the ultraviolet proper-time regulator ε. Ultimately, this amounts to a Wilson-Polchinski renormalization condition.

Taking into account the above findings, organized in an expansion with respect to gradients and sources, Eq. (1) can be expressed as

$$Z_\varepsilon = \int \mathrm{d}^4x_0 \int_\varepsilon^\infty \mathrm{d}T\sqrt{g} \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V}(g^{\circ\circ})^{\frac{n_\partial + n_V}{2}}(\partial_\circ)^{n_\partial}[V_\circ(x_0)]^{n_V}. \tag{23}$$

There are only contributions from $n_\partial + n_V$ even. The $\#_{n_\partial, n_V}$ are dimensionless numerical coefficients obtained after integrating out all proper-time fractions t_j and $\hat{\tau}_j$ as well as dimensionless relative coordinates $\hat{\Delta}$. The indices $\circ$

[5]The derivatives always act on some source V.

indicate that the contractions with the five-dimensional (inverse) metric g are only executed in four dimensions. (The addends in the previous expression symbolize the occurring combinations of contractions. Also, not all the derivatives act on all the sources.)

This expression contains the proper-time regulator $\varepsilon > 0$, the value of which possesses *a priori* no physical meaning. As a consequence, Z_ε must not depend on the value of ε, i.e., $Z_{\varepsilon_{\rm old}} \stackrel{!}{=} Z_{\varepsilon_{\rm new}}$ for $\varepsilon_{\rm old} \neq \varepsilon_{\rm new}$. To study the consequences of this requirement, let us try to undo the change $\varepsilon_{\rm old} \to \varepsilon_{\rm new}$ in

$$Z_{\varepsilon_{\rm new}} = \int \mathrm{d}^4 x_0 \int_{\varepsilon_{\rm new}}^{\infty} \frac{\mathrm{d}T}{2T^3} \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (T\eta^{\circ\circ})^{\frac{n_\partial + n_V}{2}} (\partial_\circ)^{n_\partial} [V_\circ(x_0)]^{n_V}. \tag{24}$$

(This is to be accomplished for all configurations V. Therefore, the independence must be enforced order by order, i.e., $\forall n_\partial, n_V$ separately.) To this end, we need to change the integration bound without changing the integrand.

A global rescaling of the integration variables,

$$T \to c_T T \quad \text{as well as} \quad x_0 \to c_x x_0 \tag{25}$$

and consequently

$$\partial_{x_0} \to c_x^{-1} \partial_{x_0}, \tag{26}$$

leads to

$$Z_{\varepsilon_{\rm new}} = \int \mathrm{d}^4 x_0 \int_{c_T \varepsilon_{\rm new}}^{\infty} \frac{\mathrm{d}T}{2T^3} \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} c_x^{4-n_\partial} c_T^{\frac{n_\partial + n_V}{2} - 2} (T\eta^{\circ\circ})^{\frac{n_\partial + n_V}{2}} (\partial_\circ)^{n_\partial} [V_\circ(x_0)]^{n_V}. \tag{27}$$

Restoring the original integration bound requires $c_T = \varepsilon_{\rm old}/\varepsilon_{\rm new}$. Independence from n_∂ necessitates $c_x = c_T^{1/2}$, which also takes care of the n_∂-independent factors from the integration measure. Independence from n_V can only be obtained by also rescaling $V \to c_x^{-1} V$, i.e., like the partial derivative.[6] Thus, regulator independence of Z_ε can be achieved,

$$Z_{\varepsilon_{\rm new}} = \int \mathrm{d}^4 x_0 \int_{\varepsilon_{\rm old}}^{\infty} \frac{\mathrm{d}T}{2T^3} \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (T\eta^{\circ\circ})^{\frac{n_\partial + n_V}{2}} (\partial_\circ)^{n_\partial} [V_\circ(x_0; \varepsilon_{\rm old})]^{n_V}, \tag{28}$$

but the source V must rescale as well and thus depend on the value of the regulator, which is a fifth-dimensional quantity;

$$\varepsilon_{\rm old} V(x_0; \varepsilon_{\rm old}) = \varepsilon_{\rm new} V(x_0; \varepsilon_{\rm new}). \tag{29}$$

A. Using AdS isometries

Given that we had already recognized that Z takes the form of an action over AdS_5 [9,17–19] and that the isometries of AdS_5 coincide with the conformal group over Mink_4, which includes the invariance under scale transformations, the above approach is rather pedestrian. Introducing the aforementioned missing ingredients of fifth-dimensional gradients and components into Eq. (23) completes the field theory over AdS_5, which then features all the isometries of that spacetime.

In the four-dimensional theory, however, there were no fifth-dimensional polarizations. To be allowed to omit them, $\mathcal{V}_T = 0$ must be an admissible gauge condition. That means the extension to five dimensions must feature five-dimensional local invariance under the flavor group. Because of the previously present four-dimensional invariance, this is achieved by[7,8]

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (g^{\bullet\bullet})^{\frac{n_\partial + n_V}{2}} (\nabla_\bullet)^{n_\partial} [\mathcal{V}_\bullet(x_0, T)]^{n_V}, \tag{30}$$

where the indices • indicate the full five-dimensional contraction, and ∇ represents the AdS covariant derivative. Equation (30) does not depend on the value of ε if $\mathcal{V}(x_0, T)$ transforms like a five-dimensional vector. (We would like to point out that we did *not* include an explicit dependence on ε in $\mathcal{V}$.) If we impose the $\mathcal{V}_T = 0$ gauge already at the level of the action, the desired scale invariance is still manifest, as scale transformations do not mix tensor components, while the special conformal transformations do. The full invariance, however, is still present modulo a subsequent local flavor transformation.

In this section, we discuss the example of a theory with classical scale invariance, where the five-dimensional completion of Z yields a scale-invariant $\mathcal{Z}$. For theories

[6]The interaction part in Eq. (4), $\mathrm{e}^{-\mathrm{i}\oint \mathrm{d}y\cdot V}$, being a Wilson loop, is manifestly locally invariant under the transformation $V^\mu \to \Omega[V^\mu + \mathrm{i}\Omega^\dagger(\partial^\mu \Omega)]\Omega^\dagger$, which entails hidden local symmetry [24]. Therefore, an alternative expression only using covariant derivatives is also available [25,26],

$$Z = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \sum_n \#_n (g^{\circ\circ})^n (D_\circ)^{2n},$$

where $D = \partial - \mathrm{i}V$. This corroborates why V must scale like the partial derivative. Moreover, the proper-time regularization keeps this symmetry manifest, at variance with a momentum cutoff.

[7]This is even more clear cut in the representation given in footnote 6, in which one would replace all flavor covariant derivatives by flavor and generally covariant derivatives.

[8]We have chosen the notation $\iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g}$ to be able to collect all pieces of the measure in one piece, while preserving the information on the integration bounds of T.

that depend on one or several intrinsic scales, $\mathcal{Z}$ will depend on as many scales as before the introduction of the regulator. We get back to this point at the end of Sec. III C.

As it stands, $\mathcal{Z}$ is merely a functional of arbitrary source configurations $\mathcal{V}$. The true meaning of an action for its field theory is through the configuration (or configurations) it distinguishes as saddle points, $\check{\mathcal{V}}$. The boundary condition

$$\check{\mathcal{V}}_\mu(x_0, T=\varepsilon) = V_\mu(x_0) \tag{31}$$

communicates the four-dimensional polarizations to the five-dimensional field $\mathcal{V}$ and gives it the same normalization like V, i.e., as the source for *once* the vector current. It is also consistent with the previous findings in the context of worldline holography [9,17,18] that the worldline expressions satisfy a Wilson (gradient) flow equation with this boundary condition.

Accordingly, we must evaluate $\mathcal{Z}$ on the saddle-point configuration in the $\check{\mathcal{V}}_T = 0$ gauge,

$$\check{\mathcal{Z}} = \iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}\sum_{n_\partial,n_V}\#_{n_\partial,n_V}(g^{\bullet\bullet})^{\frac{n_\partial+n_V}{2}}(\nabla_\bullet)^{n_\partial}[\check{\mathcal{V}}_\circ(x_0,T)]^{n_V}. \tag{32}$$

Equation (31) also puts the bare source configuration at the ultraviolet end of the fifth dimension (in the previously discussed sense that small values of T correspond to short four-dimensional distances). This fact together with the requirement that the effective action be independent from the value of the unphysical UV regulator ε, which can also be expressed in differential form,

$$\varepsilon\partial_\varepsilon \ln Z_\varepsilon \stackrel{!}{=} 0, \tag{33}$$

makes this a Wilson-Polchinski renormalization condition [27].

Finally, this is also the boundary condition imposed in holography [1–3,11,12]: the effective action for the four-dimensional side of the holographic duality is described by the five-dimensional action evaluated on its saddle point. Worldline holography identifies Schwinger's proper time with the fifth dimension [9,17–19] and the fifth-dimensional profile of the sources as solution to the renormalization group equation (33).

B. Free case

To flesh out the formalism presented above, we analyze the free case here. It is obtained from Eq. (1) by switching off the coupling to the gauge bosons G. Consequently, here, it is sufficient to study w with $\mathbb{V} = V$.[9]

[9]Based on the observation that at low energies the contributions with the lowest number of exchanged gauge bosons dominate [28–31], these are the kinematically dominant diagrams in that regime.

1. Holographic one-loop renormalization of scalar QED

When identifying the vector source V with a (background) gauge field, w is the QED one-loop effective action. The (logarithmically) UV-divergent piece is given by

$$Z_\varepsilon = \#_{2,2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}g^{\mu\kappa}g^{\nu\lambda}V_{\mu\nu}V_{\kappa\lambda}, \tag{34}$$

where $V_{\mu\nu}$ stands for the (here Abelian) field-strength tensor. (The logarithmic divergence arises in the dT integration, where there is a factor of T^{-3} from the volume element and two factors of T, one from each metric $g^{\mu\nu}$.) For the contribution from $N_f \times N_c$ scalar quarks to QED,

$$\#_{2,2} = \frac{1}{2!}\frac{1}{6}\frac{2N_fN_c}{(4\pi)^2}. \tag{35}$$

Here, the $(4\pi)^{-2}$ part of the normalization, already present in Eq. (4), arises from taking the trace in (2), $\frac{1}{2!}$ is due to the Taylor expansion of the Wilson line to the second order in the source V, and $\frac{1}{6}$ is the result of carrying out the $[\mathrm{d}y]$ path integral as well as the $\mathrm{d}\hat{\tau}_j$ integrations after expanding the sources to second order in four-gradients. For details, see Eqs. (52)–(59) for $L = 1$ in the following subsection. As explained in the last section, the independence from the unphysical value of the regulator ε can be achieved by reconstructing the full five-dimensional expression

$$\mathcal{Z} = \#_{2,2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}g^{MK}g^{NL}\mathcal{V}_{MN}\mathcal{V}_{KL}, \tag{36}$$

where capital indices are summed over all five dimensions. The corresponding classical equations of motion read

$$g^{NL}\nabla_N\check{\mathcal{V}}_{KL}(x_0,T) = 0. \tag{37}$$

Imposing the axial gauge $\check{\mathcal{V}}_T = 0$ automatically implies the Lorenz gauge $\partial\cdot\check{\mathcal{V}} = 0$. For the transverse components, this means in four-dimensional (4D) momentum space

$$\left(\partial_T^2 - \frac{p^2}{4T}\right)\tilde{\check{\mathcal{V}}}^\perp(p,T) = 0. \tag{38}$$

The Fourier transformed boundary condition (31),

$$\tilde{\check{\mathcal{V}}}_\mu(p, T=\varepsilon) = \tilde{V}_\mu(p), \tag{39}$$

identifies p with the 4-momentum of the source. The normalizable solution involves Bessel's K (see Eq. 9.6.1. et seq. in Ref. [32]),

$$\tilde{\check{\mathcal{V}}}^\perp = \tilde{V}^\perp(p)\frac{\sqrt{p^2T}K_1\left(\sqrt{p^2T}\right)}{\sqrt{p^2\varepsilon}K_1\left(\sqrt{p^2\varepsilon}\right)}, \tag{40}$$

for which according to Eqs. 9.6.28 in Ref. [32]

$$\partial_T \tilde{\breve{\mathcal{V}}}^{\perp} = \tilde{V}^{\perp}(p) \frac{p^2 K_0\left(\sqrt{p^2 T}\right)/2}{\sqrt{p^2 \varepsilon} K_1\left(\sqrt{p^2 \varepsilon}\right)}. \tag{41}$$

Next, we have to put this solution back into the action (36) after Fourier transforming it (or, alternatively, we have to transform the solution). Being purely quadratic, $\mathcal{Z}$ on the saddle point amounts to a surface term,

$$\breve{\mathcal{Z}} = 4\#_{2,2} \int \mathrm{d}^4 x_0 \eta^{\nu\lambda} [\breve{\mathcal{V}}_{\nu}^{\perp} \partial_T \breve{\mathcal{V}}_{\lambda}^{\perp}]_{\varepsilon}^{\infty} \tag{42}$$

$$= 4\#_{2,2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \eta^{\nu\lambda} [\tilde{\breve{\mathcal{V}}}_{\nu}^{\perp} \partial_T \tilde{\breve{\mathcal{V}}}_{\lambda}^{\perp *}]_{\varepsilon}^{\infty}, \tag{43}$$

where $\perp$ indicates that only 4D transverse components contribute, and $*$ stands for the complex conjugate. Consequently, using Eq. 9.6.13. from Ref. [32],

$$\breve{\mathcal{Z}} = -2\#_{2,2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \eta^{\nu\lambda} \tilde{V}_{\nu}^{\perp} \tilde{V}_{\lambda}^{\perp *} p^2 K_0\left(\sqrt{p^2 \varepsilon}\right) \tag{44}$$

$$= \#_{2,2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \underbrace{\eta^{\nu\lambda} \tilde{V}_{\nu}^{\perp} \tilde{V}_{\lambda}^{\perp *} p^2}_{\hat{=} |\tilde{V}_{\mu\nu}|^2/2} \{\ln(p^2 \varepsilon) + O[(p^2 \varepsilon)^0]\}. \tag{45}$$

Comparing the UV-divergent contributions to the prefactor of the kinetic term $-(4e^2)^{-1}$ (in our conventions, e is contained in V),

$$\frac{1}{4} \beta_1 \ln(p^2 \varepsilon) \overset{!}{=} \frac{1}{2} \#_{2,2} \ln(p^2 \varepsilon) \Rightarrow \beta_1 = \frac{N_f N_c}{48\pi^2}, \tag{46}$$

which is the known β-function coefficient for the normalization adopted here,

$$\beta_1 = \frac{1}{e^3} \frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = -\frac{\mathrm{d}e^{-2}}{\mathrm{d}\ln\mu^2}. \tag{47}$$

The computation for fermionic elementary matter proceeds in strict analogy, yielding a value for $\#_{2,2}$ that differs by a factor of 4, thus reproducing the corresponding fermionic contribution to the β-function coefficient.[10]

We never required that ε be small. [In Eq. (45), we merely presented the behavior of $\breve{\mathcal{Z}}$ for if it were small.] Originally, ε was introduced to regularize the UV divergence of Z. Hence, at that point, we had in mind to remove the regulator at the end of the computation by sending it to zero in a controlled manner. At finite ε, the renormalization condition (33) ascribes the meaning of a scale to ε. If we wanted to keep ε in its original role, we could introduce counterterms for the divergent piece(s). For Eq. (45), for example,

$$\breve{\mathcal{Z}} = \#_{2,2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \eta^{\nu\lambda} \tilde{V}_{\nu}^{\perp} \tilde{V}_{\lambda}^{\perp *} p^2 \times \{\ln(\mu^2 \varepsilon) + \ln(p^2/\mu^2) + O[(p^2 \varepsilon)^0]\}, \tag{48}$$

the first addend inside the braces, which diverges when $\varepsilon \to 0$, must be cancelled by the introduction of a counterterm with opposite sign in which finite parts can be included as well. The remaining ε independent part depends on the scale μ^2 instead, thereby separating the concepts of regulator and scale.

Taking a step back, we see that for the example in hand, which is classically scale invariant, the completion from Z to $\mathcal{Z}$ restores full scale invariance and ensures the independence of $\mathcal{Z}$ from the value of the regulator ε. Then, when we evaluate $\mathcal{Z}$ on the saddle point, the boundary condition brings in the momentum scale p^2 against which the regulator piles up. Thus, we are able to correctly determine the anomalous breaking of scale invariance.

2. Higher-spin sources/fields

Above, we presented the special case of a rank-1 source V, but sources of any rank contribute to Z_ε. Here, we demonstrate that worldline holography also readily applies to them. More generally, the worldline coupling of a rank-L source $W_{\mu_1\ldots\mu_L}$ symmetric in all indices ($W_{\mu_1\ldots\mu_L} = W_{(\mu_1\ldots\mu_L)}$), traceless ($\eta^{\mu_1\mu_2} W_{\mu_1\ldots\mu_L} = 0$), and transverse ($\partial_\nu \eta^{\nu\mu_1} W_{\mu_1\ldots\mu_L} = 0$) is given by [18]

$$\mathcal{L} = \frac{2\mathcal{N}}{(4\pi)^2} \int_{\mathrm{P}} [\mathrm{d}y] \mathrm{e}^{-\int_0^T \mathrm{d}\tau [\frac{\dot{y}^2}{4} - (-\mathrm{i}\dot{y}\cdot)^L W(x_0+y)/L!]}, \tag{49}$$

where $(\dot{y}\cdot)^L W$ stands for the L-fold contraction of W with $\dot{y}$. After expanding in powers of the sources, gradient expanding the sources, and carrying out the $[\mathrm{d}y]$ as well as $\mathrm{d}\tau$ integrations, we have

$$Z_\varepsilon = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \times \sum_{n_\partial, n_W} \#_{n_\partial, n_W} (g^{\circ\circ})^{\frac{n_\partial + L n_W}{2}} (\partial_\circ)^{n_\partial} [W_{\{\circ\}}(x_0) T^{\frac{1-L}{2}}]^{n_W}, \tag{50}$$

where $\{\circ\}$ ($\{\bullet\}$) indicates that all indices that are not mentioned take values in four (five) dimensions; there are only contributions for $n_\partial + L n_W$ even. Accordingly, the corresponding ε-independent five-dimensionally completed formulation is given by

[10] For the treatment of (nonholographic) renormalization in the framework of the worldline formalism, see Ref. [33].

$$\mathcal{Z} = \iint_{\varepsilon}^{\infty} \mathrm{d}^5x\sqrt{g} \times \sum_{n_\partial, n_W} \#_{n_\partial, n_W} (g^{\bullet\bullet})^{\frac{n_\partial + L n_W}{2}} (\nabla_\bullet)^{n_\partial} [\mathcal{W}_{\{\bullet\}}(x_0, T)]^{n_W}, \tag{51}$$

where $T^{1-L}W(x_0) \to \mathcal{W}(x_0, T)$.

For the sake of concreteness, let us determine the coefficients for the terms up to the second order in fields and gradients in Eq. (49). Up to the second order in the fields,

$$\mathcal{L} \supset \frac{(-\mathrm{i})^{2L}}{2(L!)^2} \frac{2\mathcal{N}}{(4\pi)^2} \int_0^T \mathrm{d}\tau_1 \mathrm{d}\tau_2 \int_{\mathrm{P}} [\mathrm{d}y] \mathrm{e}^{-\int_0^T \mathrm{d}\tau \frac{\dot{y}^2}{4}} (\dot{y}_1 \cdot)^L W_1 (\dot{y}_2 \cdot)^L W_2, \tag{52}$$

where $y_j = y(\tau_j)$. Expanding additionally up to the second order in 4-gradients,

$$W(x_0 + y) = \mathrm{e}^{y\cdot\partial_{x_0}} W(x_0) \approx \left[1 + y\cdot\partial_{x_0} + \frac{1}{2}(y\cdot\partial_{x_0})^2\right] W(x_0),$$

yields

$$\mathcal{L} \supset \frac{(-\mathrm{i})^{2L}}{2(L!)^2} \frac{2\mathcal{N}}{(4\pi)^2} \int_0^T \mathrm{d}\tau_1 \mathrm{d}\tau_2 \int_{\mathrm{P}} [\mathrm{d}y] \mathrm{e}^{-\int_0^T \mathrm{d}\tau \frac{\dot{y}^2}{4}} \{W_0(\cdot\dot{y}_1)^L W_0(\cdot\dot{y}_2)^L + [y_1\cdot\partial_{x_0} W_0(\cdot\dot{y}_1)^L][y_2\cdot\partial_{x_0} W_0(\cdot\dot{y}_2)^L]\}, \tag{53}$$

where $W_0 = W(x_0)$. The first order in the gradients and terms where both gradients act on the same source integrate to zero (also taking into account the tracelessness of W) and are omitted right away. Performing the $[\mathrm{d}y]$ integration yields

$$\mathcal{L} \supset \frac{(-\mathrm{i})^{2L}}{2L!} \frac{2}{(4\pi)^2} \int_0^T \mathrm{d}\tau_1 \mathrm{d}\tau_2 [\ddot{P}_{12}^L W (\eta^{\circ\circ})^L W - \ddot{P}_{12}^L P_{12} \eta^{\mu\nu} (\partial_\mu W)(\eta^{\circ\circ})^L (\partial_\nu W) - L \ddot{P}_{12}^{L-1} \dot{P}_{12}^2 \eta^{\mu\lambda}\eta^{\nu\kappa} (\partial_\mu W_\kappa)(\eta^{\circ\circ})^{L-1}(\partial_\nu W_\lambda)], \tag{54}$$

where, henceforth, we suppress the index 0 to counteract the accumulation of indices. $(\eta^{\circ\circ})^L$ represents the L-fold contraction of the $2L$ indices of the Ws that are not shown explicitly with the inverse flat metric. The worldline propagator $P(\tau_1, \tau_2) \equiv P_{12}$, in the center-of-mass convention $\int_0^T \mathrm{d}\tau y = 0$, where it is manifestly proper-time translationally invariant, and its first two derivatives with respect to its first argument read [21]

$$P_{12} = |\tau_1 - \tau_2| - (\tau_1 - \tau_2)^2/T, \tag{55}$$

$$\dot{P}_{12} = \mathrm{sign}(\tau_1 - \tau_2) - 2(\tau_1 - \tau_2)/T, \tag{56}$$

$$\ddot{P}_{12} = 2\delta(\tau_1 - \tau_2) - 2/T. \tag{57}$$

(The countercharge $-2/T$ on the right-hand side of the last line is required to invert the (one-dimensional) Laplacian ∂_τ^2 on the compact interval $[0; T]$ and is consistent with the center-of-mass convention. See also the derivation in Ref. [9].) Performing the $\mathrm{d}\tau_j$ integrations leads to

$$\mathcal{L} \supset -\frac{2^{L-1}T^{3-L}}{6L!} \frac{2}{(4\pi)^2} \times (\eta^{\mu\nu}\eta^{\kappa\lambda} - L\eta^{\mu\lambda}\eta^{\nu\kappa})(\partial_\mu W_\kappa)(\eta^{\circ\circ})^{L-1}(\partial_\nu W_\lambda). \tag{58}$$

Terms containing P_{12} or $\dot{P}_{12}$ at coincident proper times $\tau_1 = \tau_2$ do not contribute, as $P_{11} = 0 = \dot{P}_{11}$. We regularize powers of δ distributions according to $[\delta(\tau_1 - \tau_2)]^l \to \delta(\tau_1 - \tau_2)/T^{l-1}$ or, equivalently, $\ddot{P}_{12}^L \to (-2/T)^{L-1}\ddot{P}_{12}$. The thus-obtained Lagrangian

$$\mathcal{L} \supset -\frac{2^{L-1}}{6L!} \frac{2}{(4\pi)^2} (g^{\mu\nu}g^{\kappa\lambda} - Lg^{\mu\lambda}g^{\nu\kappa}) \times (\partial_\mu W_\kappa T^{1-L})(g^{\circ\circ})^{L-1}(\partial_\nu W_\lambda T^{1-L}) \tag{59}$$

pertains to a field theory over AdS_5 with all fifth polarizations and gradients zero. Next, we achieve independence from ε by reconstructing the complete five-dimensional and locally invariant theory according to the above rules,

$$\mathcal{Z} \supset -\frac{2^{L-1}}{6L!} \frac{2}{(4\pi)^2} \iint_{\varepsilon}^{\infty} \mathrm{d}^5x\sqrt{g} \times (g^{MN}g^{KJ} - Lg^{MJ}g^{NK}) \times (\nabla_M \mathcal{W}_K)(g^{\bullet\bullet})^{L-1}(\nabla_N \mathcal{W}_J). \tag{60}$$

Expressed with partial instead of covariant derivatives, this amounts to

$$\mathcal{Z} \supset \iint_{\varepsilon}^{\infty} \mathrm{d}^5x\sqrt{g}\{(g^{MN}g^{KJ} - Lg^{MJ}g^{NK}) \times (\partial_M \mathcal{W}_K)(g^{\bullet\bullet})^{L-1}(\partial_N \mathcal{W}_J) + 4(L-1)\mathcal{W}(g^{\bullet\bullet})^L\mathcal{W}\}, \tag{61}$$

where the total contribution from the Christoffel symbols amounts to the term without derivatives. This coincides with the result obtained in Ref. [18]. Varying this effective action with respect to the four-dimensional components, imposing the axial gauge, transversality,[11] and tracelessness, yields the corresponding components of an AdS Frønsdal equation,

$$\left[-T^{1-L}\partial_T T^{L-1}\partial_T - \frac{1}{4}\frac{\Box}{T} + \frac{L-1}{T^2}\right]\mathcal{W}_\perp = 0. \tag{62}$$

Analyzing the small-T behavior by means of a power-law ansatz $\mathcal{W}_\perp \propto T^\alpha$ yields the characteristic equation

[11]Imposing the axial gauge and transversality corresponds to adopting the analog of the five-dimensional radiation gauge.

$$-\alpha(\alpha-2+L)+L-1=0, \tag{63}$$

which is solved by

$$\alpha=1 \,\&\, \alpha=1-L. \tag{64}$$

For scalar elementary matter, this coincides with the result in Refs. [1,14].

The equations obtained by varying with respect to fields with at least one T component are given by

$$g^{MN}(\nabla_M\nabla_N\mathcal{W}_{M_1\dots M_L}-L\nabla_M\nabla_{(M_1}\mathcal{W}_{M_2\dots M_L)N})=0,$$

where we have imposed the axial gauge only *a posteriori* and have not insisted on transversality. Equations with more than two T components vanish identically after imposing the gauge condition. The equation with exactly two T components amounts to $\frac{\partial\ln\mathrm{tr}\mathcal{W}}{\partial\ln T}=$ const.; the equation with exactly one T component states that a linear combination of $\mathrm{div}\mathcal{W}$, $\frac{\partial\mathrm{div}\mathcal{W}}{\partial\ln T}$, and $\mathrm{gradtr}\mathcal{W}$ is zero.[12] Consequently, if $\mathcal{W}$ is transverse as well as traceless on the boundary, it will remain so everywhere in the bulk.[13] Thus, the variation with respect to the fifth-dimensional polarizations (together with the boundary conditions) enforces transversality and tracelessness.[14]

For the scalar source $\mathcal{W}_{L=0}$, Eq. (61) features a tachyonic mass term, which saturates the Breitenlohner-Freedman bound. For the boundary condition $\lim_{T\to 0}(\mathcal{W}_{L=0}T^{L-1=-1})=m^2$, the equation of motion (62) is solved by $\mathcal{W}_{L=0}=m^2T$. This is the tachyon (squared) profile [37] for a free theory of elementary matter with the explicit mass m.

Taking stock, the present framework links a free scalar quantum field theory on Mink_4 with sources of any spin to a field theory for these sources on AdS_5. Such a duality was conjectured to exist [38].

3. *Spin 2 revisited*

For rank-2 sources, the explicit results above correspond to the linearized Einstein equations. In this context, the rank-2 source represented the deviation $h_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$. A straightforward expansion to finite powers of the deviation, however, does not posses full diffeomorphism invariance, but, like for the vector case in footnote, it is possible to devise a fully covariant expansion scheme. In short, while the derivation in the vector case makes use of the Fock-Schwinger gauge to express the vector field in terms of covariant derivatives (and their commutator, the field tensor), the spin-2 case makes use of Riemann normal coordinates, where the full metric $\mathfrak{g}_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ is expressed in terms of curvature tensors constructed from the corresponding Levi-Civitá connection $\nabla[\mathfrak{g}]$. The expansion again takes the form

$$\begin{aligned} Z_\varepsilon &= \int_\varepsilon^\infty \frac{\mathrm{d}T}{2T^3}\int \mathrm{d}^4x_0\sqrt{\mathfrak{g}}\sum_n \#_n(T\mathfrak{g}^{\circ\circ})^n(\nabla_\circ[\mathfrak{g}])^{2n} \\ &= \iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{\bar{\mathfrak{g}}}\sum_n \sharp_n(\bar{\mathfrak{g}}^{\circ\circ})^n(\nabla_\circ[\mathfrak{g}])^{2n}, \end{aligned} \tag{65}$$

where $\bar{\mathfrak{g}}$ stands for the five-dimensional Fefferman-Graham [39] embedding of $\mathfrak{g}$,

$$\bar{\mathfrak{g}}_{MN}\mathrm{d}x^M\mathrm{d}x^N=\natural\left(\frac{\mathrm{d}T^2}{4T^2}+\frac{\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}{T}\right), \tag{66}$$

and $\sharp_n=\#_n\natural^{n-5/2}$. Again, Eq. (65) is not a differential operator. There are no partial derivatives acting to the right. The expansion is to symbolize all occurring combinations; the commutator of two covariant derivatives, for example, yields the Riemann tensor.

The independence (33) from ε can once more be achieved by the five-dimensional completion

$$\mathcal{Z}=\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{\bar{\mathfrak{g}}}\sum_n \sharp_n(\bar{\mathfrak{g}}^{\bullet\bullet})^n(\nabla_\bullet[\bar{\mathfrak{g}}])^{2n} \tag{67}$$

and subsequent evaluation of the action on the saddle point with the boundary condition

$$\breve{\bar{\mathfrak{g}}}_{\mu\nu}(x_0,T=\varepsilon)=\frac{\natural}{\varepsilon}\mathfrak{g}_{\mu\nu}(x_0) \tag{68}$$

and the gauge condition

$$\breve{\bar{\mathfrak{g}}}_{TN}\stackrel{!}{=}\natural g_{TN}\quad\forall\ N, \tag{69}$$

with g from Eq. (5), which corresponds to the absence of deviations with fifth-dimensional polarizations,

$$h_{TN}\stackrel{!}{=}0\quad\forall\ N. \tag{70}$$

Let us study the leading terms. The $\#_n$ are the DeWitt-Gilkey-Seeley coefficients [40]. The first two correspond to a negative cosmological constant and an Einstein-Hilbert term,

---

[12]Technically, this is due to the fact that only the Christoffel symbols with an odd number of components in the T direction are nonzero for Eq. (5): $\Gamma^T{}_{\mu\nu}\stackrel{g}{=}2\eta_{\mu\nu}$, $\Gamma^\mu{}_{T\nu}\stackrel{g}{=}-\frac{1}{2T}\delta^\mu_\nu=\Gamma^\mu{}_{\nu T}$, $\Gamma^T{}_{TT}\stackrel{g}{=}-\frac{1}{T}$.

[13]If we had enforced transversality from the very beginning of the derivation of the effective action (we only used the tracelessness), the (L-dependent) cross-term would be absent. Then, the fifth components of the saddle-point condition would force the trace and the nontransverse components to vanish identically, a fact that had been observed before in Ref. [15].

[14]There are several approaches to obtaining the Frønsdal equations for transverse and traceless fields from an unconstrained or less constrained variational principle, like auxiliary compensator fields [34,35] and the related relaxation to double tracelessness [36].

$$\mathcal{Z} \supset \frac{1}{6(4\pi)^2} \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{\breve{\mathfrak{g}}}(R[\breve{\mathfrak{g}}] + 6). \tag{71}$$

As a consequence, the corresponding Einstein equations admit an AdS_5 solution with the squared AdS curvature radius

$$\natural = R^2_{\mathrm{AdS}} = \frac{(5-1)(5-2)}{6} = 2. \tag{72}$$

Taking into account the boundary and gauge conditions, the solution reads

$$\breve{\mathfrak{g}}_{MN} \mathrm{d}x^M \mathrm{d}x^N = 2\left(\frac{\mathrm{d}T^2}{4T^2} + \frac{\mathrm{d}x_0 \cdot \mathrm{d}x_0}{T}\right). \tag{73}$$

Consequently, at least to this order, an AdS background is self-consistently maintained by the formalism. (To higher orders, a space of constant curvature remains a saddle-point solution, albeit with a different curvature radius.) The isometries of any AdS_5 space, i.e., with any value of the curvature radius, coincide with the conformal group over Mink_4 (just like Mink_4 is Poincaré invariant for any value of the speed of light). Therefore, the value of the AdS radius is of secondary importance insofar as it does not alter the structure of the present result. For example, $\breve{\mathfrak{g}}$

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5 x \breve{\mathfrak{g}}^{1/2} \sum_{n_\partial, n_V} \sharp_{n_\partial, n_V} (\breve{\mathfrak{g}}^{\bullet\bullet})^{\frac{n_\partial + n_V}{2}} (\nabla_\bullet[\breve{\mathfrak{g}}])^{n_\partial} [\mathcal{V}_\bullet(x_0, T)]^{n_V}, \tag{74}$$

where $\sharp_{n_\partial, n_V} = \#_{n_\partial, n_V} \natural^{(n_\partial + n_V - 5)/2}$ is an identical reexpression for Eq. (30), which is independent of $\natural$. Also, the covariant derivatives do not depend on the curvature radius. [Consistently, neither does the (1, 3) Riemann tensor.] Likewise,

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5 x \breve{\mathfrak{g}}^{1/2} \times \sum_{n_\partial, n_W} \sharp_{n_\partial, n_W} (\breve{\mathfrak{g}}^{\bullet\bullet})^{\frac{n_\partial + L n_W}{2}} (\nabla_\bullet[\breve{\mathfrak{g}}])^{n_\partial} [\mathcal{W}_{\{\bullet\}}(x_0, T)]^{n_W}, \tag{75}$$

where $\sharp_{n_\partial, n_W} = \#_{n_\partial, n_W} \natural^{(n_\partial + L n_W - 5)/2}$ is identical to Eq. (51) and independent of $\natural$.

C. Infrared

The worldline holographic framework can also handle variations of infrared scales. Consider

$$Z_{\varepsilon, k^2} = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \rho(k^2 T) \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (g^{\circ\circ})^{\frac{n_\partial + n_V}{2}} (\partial_\circ)^{n_\partial} [V_\circ(x_0)]^{n_V}, \tag{76}$$

where $\rho(k^2 T) \overset{k^2 T \to \infty}{\to} 0$ and $\rho(k^2 T) \overset{k^2 T \to 0}{\to} 1$ [41]. This can, for example, be realized by a sharp proper-time cutoff $\rho(k^2 T) = \theta(k^2 T - 1)$ or by a mass term [21] $\rho(k^2 T) = \exp(-k^2 T)$. We could now analyze the fate of the scale k^2 by repeating the steps from the beginning of Sec. III. From Sec. III A, however, we already know that worldline holography ensures the independence from an overall scale through the isometries of AdS_5 acting on

$$\mathcal{Z}_{\varepsilon, k^2} = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \rho(k^2 T) \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (g^{\bullet\bullet})^{\frac{n_\partial + n_V}{2}} (\nabla_\bullet)^{n_\partial} [\mathcal{V}_\bullet(x_0, T)]^{n_V}. \tag{77}$$

Since we are thus able to eliminate the dependence on such an overall scale, the final result can only depend on the combination $k^2 \varepsilon$. This corresponds to the renormalization condition $\varepsilon \partial_\varepsilon Z_{\varepsilon, k^2} = k^2 \partial_{k^2} Z_{\varepsilon, k^2}$. This is the requirement that the effective action do not change under simultaneous changes of ε and k^2 such that εk^2 remains the same. The renormalization condition is solved by the saddle-point expression

$$\breve{\mathcal{Z}}_{\varepsilon k^2} = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \rho(k^2 T) \times \sum_{n_\partial, n_V} \#_{n_\partial, n_V} (g^{\bullet\bullet})^{\frac{n_\partial + n_V}{2}} (\nabla_\bullet)^{n_\partial} [\breve{\mathcal{V}}_\circ(x_0, T)]^{n_V}. \tag{78}$$

Accordingly, after an introduction of counterterms like in Sec. III B 1 and taking the limit $\varepsilon \to 0$ subsequently, the resulting effective action will only depend on the combination k^2/μ^2. In a UV finite theory, and more generally in all the UV finite terms, ε is not needed in its role as UV regulator, and we can consider the case in which it is zero. Then, the alternative renormalization condition $k^2 \partial_{k^2} \ln Z_{0, k^2} \overset{!}{=} 0$ is solved by $\breve{\mathcal{Z}}_{0, k^2}$. We continue the discussion of infrared scales elsewhere.

One can also read these last few expressions slightly differently. Say k^2 is a physical scale, like, for instance, a mass, possibly already present in the classical action. Then, physical quantities can and should depend on its value. Now, in the worldline formalism, there appears the *a priori* spurious UV regulator ε. As a consequence, there are now two scales in Eq. (77) and not one. The renormalization procedure lifts the meaning of ε to that of a scale but, more importantly for the present argument, restores the scaling behavior to that of one physical scale. This is the standard procedure of doubling a symmetry and breaking it back to the original one, found throughout physics[15] used in

[15] Take the standard model as a concrete example. The Higgs potential is invariant under one copy of the electroweak gauge group, and the gauge-fermion sector is invariant under another copy. The covariant kinetic term for the Higgs breaks this symmetry to the diagonal subgroup, i.e., to simultaneous transformations.

reverse; i.e., first reduce the symmetry even more, and then restore it. This discussion extends directly to the presence of several physical scales. The arguably simplest way to think of it is to pick one scale and express all other scales as multiples of the first.[16]

IV. SHORT SUMMARY

Worldline holography maps a d-dimensional quantum field theory onto a $d+1$-dimensional field theory for the sources of the former, to all orders in the elementary fields and sources of any rank. The metric of the $d+1$-dimensional space is obtained by a Fefferman-Graham embedding [39] of the d-dimensional one. For Mink_d, this gives the known AdS_5. Above, we have shown that worldline holography is the solution to a Wilson-Polchinski renormalization condition (33), which ensures the independence of physical quantities from the ultraviolet regulator. [Infrared scales can be treated analogously (see Sec. III C).] Said renormalization condition serves as (part of) the definition of the worldline holographic framework in general and of the fifth-dimensional profiles of the sources in particular. For the cases studied here, the result is exactly the same as the one in Refs. [17,18] obtained by optimizing a Wilson (gradient) flow [42]. As cross-checks, we holographically reconstructed the leading QED β-function coefficient in Sec. III B 1. In Sec. III B 2, we derived the worldline holographic dual for a free scalar field theory on Mink_4: a field theory for sources turned fields of all integer spins over AdS_5, which was postulated in Ref. [38]. A manifestly diffeomorphism-invariant expansion of the rank-2 case leads to the DeWitt-Gilkey-Seeley coefficients [40]. In Sec. III B 3, this serves to show that AdS_5 is a self-consistent solution of the worldline holographic framework.

V. FURTHER DISCUSSION AND OUTLOOK

Barring anomalies, a quantized version of $\ln \mathcal{Z}$ would bear the necessary isometries to be a solution of the renormalization condition (33) once the appropriate boundary conditions (31) are imposed. The saddle point remains the leading contribution. The first correction is given by the fluctuation determinant.[17] In some situations, like the case discussed in Sec. III B 2, however, the distinction between a "quantum" and a "classical" answer ultimately turns out to be irrelevant, as the quantum contributions from the individual fields of different ranks over AdS cancel when summed over the complete tower of higher-spin states [45].

Furthermore, we are free to again carry out the quantum computation in the worldline formalism. In the course of this computation, g (or $\breve{\mathfrak{g}}$) is Fefferman-Graham embedded into

$$\mathrm{d}s^2 = \frac{\mathrm{d}\Theta^2}{4\Theta^2} + \frac{g_{MN}\mathrm{d}x^M\mathrm{d}x^N}{\Theta}, \tag{79}$$

where Θ is the new proper time. The isometries of the five-dimensional part are preserved, but, while still being conformally flat, this six-dimensional space does not have constant curvature nor enhanced scaling symmetry. Rather, Θ dials through the curvature radius of the five-dimensional part. Additionally, the six-dimensional effective action depends on functions of $\natural\Theta$, which consistently forestall a higher scaling symmetry.

The latter is only one more example for starting worldline holography from a spacetime other than Mink_4, but the pattern already shines through: the lower-dimensional metric will be embedded into a higher-dimensional Fefferman-Graham metric, and additional terms arise in the effective action that depend on the curvature of the original space.

Above, we were predominantly investigating worldline holography linking a quantum field theory over four-dimensional Minkowski space to a field theory for its sources over five-dimensional anti-de Sitter space, more precisely $\mathrm{AdS}_{4,1}$, where the subscripts mark the metric signature. Then again, for the worldline approach, we actually first Wick rotated to Euclidean space and from there found a connection to five-dimensional hyperbolic space H_5, i.e., $\mathrm{AdS}_{5,0}$ (also known as EAdS_5). Analytically continuing the time direction afterward led to $\mathrm{AdS}_{4,1}$. This amounted to changing ϵ_t from -1 to $+1$ in

$$\mathrm{d}s^2 = \epsilon_T\frac{\mathrm{d}T^2}{4T^2} + \frac{\epsilon_t(\mathrm{d}t)^2 + |\mathrm{d}\vec{x}|^2}{T}, \tag{80}$$

for $\epsilon_T = +1$.

Another analytic continuation [46] taking ϵ_T from $+1$ to -1 links five-dimensional de Sitter space dS_5 with H_5 or $\mathrm{AdS}_{2,3}$ with $\mathrm{AdS}_{4,1}$,

$$\begin{tikzcd} \mathrm{dS}_5 \arrow[r, leftrightarrow, "+\epsilon_t-"] \arrow[d, leftrightarrow, "\overset{-}{\underset{+}{\epsilon_T}}"'] & \mathrm{AdS}_{2,3} \arrow[d, leftrightarrow, "\overset{-}{\underset{+}{\epsilon_T}}"] \\ \mathrm{H}_5 \arrow[r, leftrightarrow, "+\epsilon_t-"'] & \mathrm{AdS}_{4,1} \end{tikzcd}$$

All these are holographic pictures of four-dimensional flat spacetimes.

Moreover, in the imaginary-time formalism for thermal field theory, the time t is compactified with the period of the inverse temperature. The straightforward application of worldline holography yields thermal AdS space, i.e., AdS space with a compactified temporal direction. There is, however, a second space with the same boundary topology and identical source configurations on the boundary, the AdS black hole [47]. Both are stationary points of the action (71). The preferred configuration is selected by the relative value of the action. In the present setting, the relative importance of bosonic and fermionic degrees is

[16]Here, an example from the standard model would be the Yukawa couplings.

[17]The use of the worldline formalism allows us to make a link [43] to the Gutzwiller trace formula [44], which describes quantum systems through classical attributes as well.

decisive for which of the two five-dimensional spacetimes is preferred [48].

Furthermore, worldline holographic duals are also available in the nonrelativistic setting [17,18]. Representing the conformal Galilean symmetry of the Schrödinger equation by imposing a fixed light-cone momentum $p^+ = m$ [49] on $4+1$-dimensional Minkowski space, which selects x^+ as normal time, in the worldline approach, we find a six-dimensional line element

$$g_{MN}\mathrm{d}x^M\mathrm{d}x^N = -\frac{\mathrm{d}T^2}{4T^2} + \frac{2(\mathrm{d}x^+)^2}{T^2} + \frac{2\mathrm{d}x^+\mathrm{d}x^- - \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}}{T} \tag{81}$$

with the correct volume element $\sqrt{g} \propto T^{-7/2}$, thus reproducing Refs. [49,50].[18]

So far, we mostly studied two-point functions and kinetic terms. Exceptions are, e.g., the non-Abelian part of the vector kinetic term and covariant expressions for spin-2 backgrounds. Worldline holography also gives a general prescription for determining interactions. Any number of terms can be worked out this way. While this is expected to be possible consistently over an AdS background [52], there are known obstacles over others like Minkowski or de Sitter. In view of the fact that our prescription can yield various $d+1$-dimensional spacetimes (see below), a thorough study of interactions in our framework—in particular, of the higher-spin fields—is an important future task.

In this paper, we have concentrated on scalar elementary matter, in order to not shroud the structure of the worldline holographic framework by carrying along additional degrees of freedom. It is, however, fermionic elementary matter, which is realized in nature. Fermions do not pose any additional fundamental challenges to worldline holography but have a richer phenomenology—especially also from the vantage point of the present framework—due to their spin degree of freedom. The corresponding results are presented elsewhere [22,23].

[18]Reference [50] also allows for cases in which the power of T in the denominator of the $(\mathrm{d}x^+)^2$ term is different from 2. The volume element, however, does not depend on this metric component, as there is no $(\mathrm{d}x^-)^2$ term. With the more general exponent, slices of $x^- =$ const. coincide with the Lifshitz spaces proposed in Ref. [51]. Because of one fewer dimension, the volume element has one negative power of the square root of T less.

ACKNOWLEDGMENTS

The authors would like to thank Stan Brodsky, Guy de Téramond, Luigi Del Debbio, Florian Divotgey, Gerald Dunne, Gia Dvali, Joshua Erlich, Jürgen Eser, Francesco Giacosa, César Gómez, Stefan Hofmann, Paul Hoyer, Johannes Kirsch, Sebastian Konopka, Matti Järvinen, Yaron Oz, Stefan Rechenberger, Dirk Rischke, Ivo Sachs, Andreas Schäfer, Stefan Theisen, and Roman Zwicky for discussions.

[1] A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
[2] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005).
[3] J. Polchinski and M. J. Strassler, arXiv:hep-th/0003136; Phys. Rev. Lett. **88**, 031601 (2002).
[4] D. K. Hong and H.-U. Yee, Phys. Rev. D **74**, 015011 (2006); J. Hirn and V. Sanz, Phys. Rev. Lett. **97**, 121803 (2006); C. D. Carone, J. Erlich, and J. A. Tan, Phys. Rev. D **75**, 075005 (2007); C. D. Carone, J. Erlich, and M. Sher, Phys. Rev. D **76**, 015015 (2007); D. D. Dietrich, M. Jarvinen, and C. Kouvaris, J. High Energy Phys. 07 (2010) 023.
[5] D. D. Dietrich and C. Kouvaris, Phys. Rev. D **78**, 055005 (2008); **79**, 075004 (2009).
[6] S. Sachdev, Annu. Rev. Condens. Matter Phys. **3**, 9 (2012); S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, J. High Energy Phys. 12 (2008) 015.
[7] Y. Sato and K. Yoshida, J. High Energy Phys. 09 (2013) 134; 12 (2013) 051; D. Kawai, Y. Sato, and K. Yoshida, Phys. Rev. D **89**, 101901 (2014); Int. J. Mod. Phys. A **30**, 1530026 (2015); K. Hashimoto and T. Oka, J. High Energy Phys. 10 (2013) 116.
[8] A. S. Gorsky, K. A. Saraikin, and K. G. Selivanov, Nucl. Phys. **B628**, 270 (2002); G. W. Semenoff and K. Zarembo, Phys. Rev. Lett. **107**, 171601 (2011); J. Ambjorn and Y. Makeenko, Phys. Rev. D **85**, 061901 (2012); S. Bolognesi, F. Kiefer, and E. Rabinovici, J. High Energy Phys. 01 (2013) 174; Y. Sato and K. Yoshida, J. High Energy Phys. 04 (2013) 111; 08 (2013) 002.
[9] D. D. Dietrich, Phys. Rev. D **90**, 045024 (2014).
[10] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[11] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[12] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[13] L. Da Rold and A. Pomarol, Nucl. Phys. **B721**, 79 (2005); 01 (2006) 157; J. Hirn, N. Rius, and V. Sanz, Phys. Rev. D **73**, 085005 (2006).
[14] G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **102**, 081601 (2009); **94**, 201601 (2005); S. J. Brodsky and G. F. de Teramond, Phys. Rev. Lett. **96**, 201601 (2006); S. J. Brodsky, G. F. de Teramond, and H. G. Dosch, Phys. Lett. B **729**, 3 (2014); arXiv:1309.4856; S. J. Brodsky, G. F. de Teramond, H. G. Dosch, and J. Erlich, Phys. Rep. **584**, 1

(2015); G. F. de Teramond, H. G. Dosch, and S. J. Brodsky, Phys. Rev. D **91**, 045040 (2015); H. G. Dosch, G. F. de Teramond, and S. J. Brodsky, Phys. Rev. D **91**, 085016 (2015); **92**, 074010 (2015); S. J. Brodsky, G. F. de Tramond, H. G. Dosch, and C. Lorcé, Phys. Lett. B **759**, 171 (2016); Int. J. Mod. Phys. A **31**, 1630029 (2016).

[15] G. F. de Teramond, H. G. Dosch, and S. J. Brodsky, Phys. Rev. D **87**, 075005 (2013).

[16] O. Cata, Phys. Rev. D **75**, 106004 (2007); F. Jugeau, S. Narison, and H. Ratsimbarison, Phys. Lett. B **722**, 111 (2013).

[17] D. D. Dietrich, Phys. Rev. D **89**, 086005 (2014); **89**, 106009 (2014).

[18] D. D. Dietrich, Phys. Rev. D **94**, 086013 (2016).

[19] D. D. Dietrich, Acta Phys. Pol. B Proc. Suppl. **6**, 27 (2013).

[20] R. P. Feynman, Phys. Rev. **80**, 440 (1950).

[21] M. J. Strassler, Nucl. Phys. **B385**, 145 (1992); C. Schubert, Phys. Rep. **355**, 73 (2001).

[22] A. Konigstein, Master's thesis, Institute for Theoretical Physics, Goethe University, 2016.

[23] D. D. Dietrich and A. Koenigstein, arXiv:1702.06955.

[24] M. Bando, T. Kugo, S. Uehara, K. Yamawaki, and T. Yanagida, Phys. Rev. Lett. **54**, 1215 (1985).

[25] M. A. Shifman, Nucl. Phys. **B173**, 13 (1980).

[26] M. G. Schmidt and C. Schubert, Phys. Lett. B **318**, 438 (1993); D. Fliegner, M. G. Schmidt, and C. Schubert, Z. Phys. C **64**, 111 (1994); D. Fliegner, P. Haberl, M. G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) **264**, 51 (1998); The inverse-mass expansion corresponds to a Borel expansion: G. V. Dunne and T. M. Hall, Phys. Rev. D **60**, 065002 (1999); G. V. Dunne and C. Schubert, J. High Energy Phys. 06 (2002) 042.

[27] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); J. Polchinski, Nucl. Phys. **B231**, 269 (1984).

[28] S. Okubo, Phys. Lett. **5**, 165 (1963); G. Zweig, Report No. CERN-TH-401; Report No. CERN-TH-412; J. Iizuka, Prog. Theor. Phys. Suppl. **37**, 21 (1966); C. White *et al.*, Phys. Rev. D **49**, 58 (1994); B. R. Baller *et al.*, Phys. Rev. Lett. **60**, 1118 (1988); J. F. Gunion, S. J. Brodsky, and R. Blankenbecler, Phys. Rev. D **6**, 2652 (1972); P. V. Landshoff, Phys. Rev. D **10**, 1024 (1974).

[29] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 448 (1979); S. Hofmann and T. Rug, Nucl. Phys. **B902**, 302 (2016).

[30] A. De Rujula, H. Georgi, and S. L. Glashow, Phys. Rev. D **12**, 147 (1975); N. Isgur, Phys. Rev. D **13**, 122 (1976); E. Witten, Nucl. Phys. **B149**, 285 (1979); E. Witten, Nucl. Phys. **B156**, 269 (1979).

[31] D. D. Dietrich, P. Hoyer, and M. Jarvinen, Phys. Rev. D **87**, 065021 (2013); **85**, 105016 (2012); P. Hoyer, arXiv:1605.01532; Few Body Syst. **56**, 537 (2015); arXiv:1402.5005; arXiv:1106.1420; Acta Phys. Polon. B **41**, 2701 (2010); *Proc. Sci.*, EPS-HEP2009 (2009) 073.

[32] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).

[33] V. I. Ritus, Zh. Eksp. Teor. Fiz. **69**, 1517 (1975) [Sov. Phys. JETP **42**, 774 (1975)]; V. I. Ritus, Zh. Eksp. Teor. Fiz. **73**, 807 (1977); V. I. Ritus, in *Issues in Intense-Field Quantum Electrodynamics*, edited by V. L. Ginzburg, pp. 1–62; in Sandansky 1998, Frontier tests of QED and physics of the vacuum, 11–28; Zh. Eksp. Teor. Fiz. **75**, 1560 (1978) [Sov. Phys. JETP **48**, 788 (1978)]; S. L. Lebedev and V. I. Ritus, Zh. Eksp. Teor. Fiz. **86**, 408 (1984) [Sov. Phys. JETP **59**, 237 (1984)]; G. V. Dunne, H. Gies, and C. Schubert, J. High Energy Phys. 11 (2002) 032; G. V. Dunne, arXiv:hep-th/0406216.

[34] M. Fierz, Helv. Phys. Acta **12**, 3 (1939).

[35] L. P. S. Singh and C. R. Hagen, Phys. Rev. D **9**, 898 (1974).

[36] C. Fronsdal, Phys. Rev. D **18**, 3624 (1978); J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, MA, 1970), Vol. 1.

[37] F. Bigazzi, R. Casero, A. L. Cotrone, E. Kiritsis, and A. Paredes, J. High Energy Phys. 10 (2005) 012; R. Casero, E. Kiritsis, and A. Paredes, Nucl. Phys. **B787**, 98 (2007); M. Jarvinen, J. High Energy Phys. 07 (2015) 033.

[38] B. Sundborg, Nucl. Phys. B, Proc. Suppl. **102–103**, 113 (2001); E. Witten, John Schwarz 60th Birthday Symposium, LOCATION, 2001 (unpublished), http://quark.caltech.edu/jhs60/; I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **550**, 213 (2002).

[39] C. Fefferman and R. Graham, Astérisque, hors série, 1995, p. 95.

[40] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[41] M. Oleszczuk, Z. Phys. C **64**, 533 (1994); S. B. Liao, Phys. Rev. D **53**, 2020 (1996); **56**, 5008 (1997); D. F. Litim and J. M. Pawlowski, Phys. Lett. B **516**, 197 (2001); Phys. Rev. D **65**, 081701 (2002); **66**, 025030 (2002); H. Gies, Phys. Rev. D **66**, 025006 (2002); Lect. Notes Phys. **852**, 287 (2012).

[42] M. Luscher, Commun. Math. Phys. **293**, 899 (2010); J. High Energy Phys. 08 (2010) 071; *Proc. Sci.*, LATTICE2010 (2010) 015; *Proc. Sci.*, LATTICE2013 (2014) 016; it is a pleasure to thank R. Zwicky and L. Del Debbio for encouragement to investigate this point.

[43] D. D. Dietrich and G. V. Dunne, J. Phys. A **40**, F825 (2007).

[44] M. C. Gutzwiller, J. Math. Phys. (N.Y.) **12**, 343 (1971); R. G. Littlejohn, J. Math. Phys. (N.Y.) **31**, 2952 (1990); P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen, 2012), http://chaosbook.org/; P. Muratore-Ginanneschi, Phys. Rep. **383**, 299 (2003).

[45] S. Giombi and I. R. Klebanov, J. High Energy Phys. 12 (2013) 068; S. Giombi, I. R. Klebanov, and B. R. Safdi, Phys. Rev. D **89**, 084004 (2014).

[46] J. M. Maldacena, J. High Energy Phys. 05 (2003) 013; J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[47] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983); E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).

[48] D. D. Dietrich, arXiv:1509.04294.

[49] D. T. Son, Phys. Rev. D **78**, 046003 (2008).

[50] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008).

[51] S. Kachru, X. Liu, and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).

[52] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. **B291**, 141 (1987).