Discriminating sterile neutrinos and unitarity violation with *CP* invariants

Heinrich Päs[†] and Philipp Sicking[*]

Fakultät für Physik, Technische Universität Dortmund, 44221 Dortmund, Germany

(Received 20 December 2016; published 5 April 2017)

We present a new method to analyze upcoming results in the search for *CP* violating neutrino oscillations. The *CP* violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$ provide parametrization independent observables, which will be accessible by experiments soon. The strong prediction of a unique $\mathcal{A}^{kj}_{\alpha\beta}$ (the Jarlskog invariant) in case of the standard three neutrino model does not hold in models with new physics beyond the standard model. Nevertheless there are still correlations among the amplitudes depending on the specific model. Due to these correlations it is possible to reject specific new physics models by determining only 3 of the *CP* violating amplitudes.

DOI: 10.1103/PhysRevD.95.075004

I. INTRODUCTION

The experimental observation of neutrino oscillations and its interpretation as a consequence of neutrino masses provided the first manifestation of new physics beyond the standard model (SM). The first conclusive evidence of neutrino oscillation by SNO [1,2] and Super-Kamiokande [3] was honored recently by the Nobel Prize of Physics in 2015. With the exception of some anomalies, almost all current data can be well explained by a model of three neutrinos with two mass squared differences, Δm^2_{31} and Δm^2_{21}, three mixing angles θ_{12}, θ_{23}, and θ_{31}, and one *CP* phase δ [4]. All parameters are measured to a relatively high precision, except for the octant of θ_{23}, the mass-ordering, and the *CP* phase. Ongoing and upcoming neutrino experiments will narrow down the viable space for these parameters (see [5] for a review). A first hint for a maximal $\delta = [-3.13, -0.39](\mathrm{NH})$, $[-2.09, -0.74](\mathrm{IH})$ at 90% CL has been reported by T2K [6,7].

This situation cannot be understood as a proof of the minimal three neutrino picture, though. As has been shown by several authors, new physics models can fake a signal at current experiments which look like satisfying the three neutrino paradigm [8–12].

Neutrino oscillation probabilities are described by introducing the mixing matrix U, parametrizing the transformation from neutrino mass to flavor eigenstates, $|\nu_\alpha\rangle = \sum_k U_{\alpha k}|\nu_k\rangle$:

$$P_{\nu_\alpha\to\nu_\beta}(t) = \sum_{k,j} U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j}e^{-i\frac{\Delta m^2_{kj}L}{2E}} \tag{1}$$

$$= \delta_{\alpha\beta} - 4\sum_{k>j}^{N}\mathrm{Re}(U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j})\sin^2\left(\frac{\Delta m^2_{kj}L}{4E}\right) + 2\sum_{k>j}^{N}\mathrm{Im}(U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j})\sin\left(\frac{\Delta m^2_{kj}L}{2E}\right), \tag{2}$$

where $\mathcal{A}^{kj}_{\alpha\beta} = \mathrm{Im}(U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j})$. For antineutrinos the last term switches its sign, so the *CP* violation $P_{\nu_\alpha\to\nu_\beta} - P_{\bar\nu_\alpha\to\bar\nu_\beta}$ depends only on the *CP* violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$. Here, N indicates the number of light neutrinos involved in the oscillation process. If all neutrino mass eigenstates are small compared to the relevant energy scale at production and detection (for instance the pion mass) all eigenstates are involved in the oscillation process and the mixing matrix U is unitary. If, on the other hand, at least one mass eigenstate cannot be produced due to kinematics, or the heavy flavors can be integrated out, the resulting effective mixing matrix U can be nonunitary. Note that in this case in addition to neutrino oscillations zero-distance-effects can arise, which we do not consider in this work. A common approximative parametrization used in the literature is based on a series expansion in $\alpha = \frac{\Delta m^2_{31}}{\Delta m^2_{21}} \ll 1$:

$$P_{\nu_e\to\nu_\mu} \sim A\sin^2\Delta + \alpha\sin\delta B\sin^3\Delta + \alpha\cos\delta C\cos\Delta\sin^2\Delta + \alpha^2 D\sin^2\Delta \tag{3}$$

with $\Delta = \frac{\Delta m^2_{31}L}{4E}$ and A, B, C, D are functions of the standard mixing angles [13]. Equation (3) is only valid in vacuum and for $\alpha\Delta \lesssim 1$ holds.

The *CP* violating term (proportional to $\sin\delta$) is suppressed by α but the unitarity of $U^{3\times 3}$ is implicitly used to derive this formula. Various efforts exist in the literature to improve the above approximation for new, more exact or shorter parametrizations [14–22] or to include matter effects [23–31].

Here we rely on the exact expressions given in Eq. (2) instead, which is invariant under reparametrization. In particular the *CP* violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$ are independent of the parametrization [32,33] and can be determined in various extensions to the SM case. A specific feature which had already been pointed out by Jarlskog [34,35] is that in the case of exactly three flavors and a unitary mixing matrix U, all *CP* violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$ have identical absolute

[*]Corresponding author.
philipp.sicking@tu-dortmund.de
[†]heinrich.paes@tu-dortmund.de

values. This observation was first exploited in the quark sector where the famous Cabibbo-Kobayashi-Maskawa unitarity triangle provides a precise test for unitarity and therefore for the SM itself. Analyses of the lepton sector in terms of unitarity triangles have been worked out in [36–41], but the insights are limited in cases where the triangle does not close, since the source of unitary violation cannot be determined.

Inspired by previous work [32,33] we take a closer look at sums and ratios of the CP violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$ and find useful correlations among them. These correlations depend highly on the specific model and therefore provide a useful test for new physics in CP violating neutrino oscillations.

II. ANALYTIC TREATMENT OF 3+1 ν

A popular extension of the three neutrino model is to add an additional light sterile neutrino [42,43]. This is motivated by the LSND [44], MiniBooNE [45], reactor [46], and gallium anomalies [47] but in conflict with a recent IceCube analysis [48]. In this model the mixing matrix U is now a 4×4 unitary mixing matrix but the 3×3 submatrix is not unitary anymore. Although the resulting amplitudes are no longer unique, they are related due to the unitarity of the complete mixing matrix. By exploiting these relations in the context of the quark sector it has been shown for four flavors that all amplitudes can be reduced to only three independent CP violating amplitudes [49]. In the following we follow these arguments translated to the notation commonly used in neutrino physics. All relations rely on [49] where these relations have been proven for general unitary 4×4 matrices.

In total there exist $4\times 4\times 4\times 4 = 256$ ($\alpha,\beta \in \{e,\mu,\tau,s\}$ and $k,j\in\{1,2,3,4\}$) different CP violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta} = \mathrm{Im}(U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j})$, whereas the number is strongly reduced by the fact that $\mathcal{A}^{kj}_{\alpha\beta} = 0$ for $\alpha=\beta$ or $k=j$ and due to symmetry, $\mathcal{A}^{kj}_{\alpha\beta} = \mathcal{A}^{kj}_{\beta\alpha}$ and $\mathcal{A}^{kj}_{\alpha\beta} = \mathcal{A}^{jk}_{\alpha\beta}$ Therefore it is sufficient to only consider $\mathcal{A}^{kj}_{\alpha\beta}$ where $\alpha<\beta$ and $k>j$. Note that the previous relations hold due to the definition of $\mathcal{A}^{kj}_{\alpha\beta}$ regardless of the underlying U and are not specific for the $3+1\nu$ model. This reduces the number of CP violating amplitudes to 36. These 36 amplitudes are not independent of each other and can be expressed via only nine amplitudes (see Appendix A). Again, these nine amplitudes can be expressed by three remaining amplitudes via the following expression

$$\begin{pmatrix} \mathcal{A}^{32}_{e\mu} \\ \mathcal{A}^{43}_{e\mu} \\ \mathcal{A}^{21}_{\mu\tau} \\ \mathcal{A}^{43}_{\mu\tau} \\ \mathcal{A}^{21}_{\tau s} \\ \mathcal{A}^{32}_{\tau s} \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} \mathcal{R}^{32}_{e\mu}\mathcal{A}^{21}_{e\mu} \\ \mathcal{R}^{43}_{\mu\tau}\mathcal{A}^{43}_{\tau s} \\ \mathcal{R}^{21}_{\mu\tau}\mathcal{A}^{21}_{e\mu} \\ \mathcal{R}^{32}_{\tau s}\mathcal{A}^{43}_{\tau s} \\ (\mathcal{R}^{32}_{\tau\tau} + \mathcal{R}^{32}_{e\mu})\mathcal{A}^{32}_{\mu\tau} \\ (\mathcal{R}^{33}_{\mu\tau} + \mathcal{R}^{32}_{e\mu})\mathcal{A}^{32}_{\mu\tau} \end{pmatrix}, \tag{4}$$

with $\mathbf{M}^{-1}$ defined by the inverse of

$$\mathbf{M} = \begin{pmatrix} -(\mathcal{R}^{22}_{e\mu} + \mathcal{R}^{21}_{e\mu}) & \mathcal{R}^{22}_{e\mu} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}^{43}_{\tau\tau} & 0 & -(\mathcal{R}^{43}_{\tau\tau} + \mathcal{R}^{43}_{\tau s}) & 0 & 0 \\ 0 & 0 & -(\mathcal{R}^{21}_{\mu\mu} + \mathcal{R}^{21}_{e\mu}) & 0 & \mathcal{R}^{21}_{\mu\mu} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{R}^{33}_{\tau s} & -(\mathcal{R}^{33}_{\tau s} + \mathcal{R}^{43}_{\tau s}) \\ \mathcal{R}^{32}_{\tau\tau} & 0 & 0 & 0 & 0 & -\mathcal{R}^{32}_{\mu\tau} \\ 0 & 0 & -\mathcal{R}^{33}_{\mu\tau} & -\mathcal{R}^{32}_{\mu\tau} & 0 & 0 \end{pmatrix}. \tag{5}$$

The amplitudes $\mathcal{R}^{kj}_{\alpha\beta} = \mathrm{Re}(U^*_{\alpha k}U_{\beta k}U_{\alpha j}U^*_{\beta j})$ correspond to the CP conserving amplitudes in neutrino oscillations. These relations therefore provide a connection between the CP violating and the CP conserving processes. To emphasize the differences between 3ν and $3+1\nu$ we want to highlight following relations:

$$\mathcal{A}^{31}_{e\mu} = -\mathcal{A}^{32}_{e\mu} + \mathcal{A}^{43}_{e\mu} \tag{6}$$

$$\mathcal{A}^{21}_{e\tau} = -\mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{43}_{\tau s} \tag{7}$$

$$\mathcal{A}^{31}_{e\tau} = -\mathcal{A}^{32}_{e\tau} - \mathcal{A}^{32}_{\tau s} + \mathcal{A}^{43}_{\tau s} \tag{8}$$

The relations reduce to the 3ν case, if no mixing with the light neutrino takes place. This corresponds to vanishing non diagonal elements in the fourth line and column of U. Consequently, all amplitudes vanish if $\alpha \vee \beta = s$ or $k \vee j = 4$. Due to the expected smallness of mixing with sterile states, the deviations from uniform amplitudes in the 3×3 sector could be treated in a perturbation approach.

III. NUMERIC ANALYSIS OF STERILE NEUTRINOS AND NONUNITARY SCENARIOS

The relations in the previous section rely on the unitarity of the resulting $3 + 1\nu$ model. In general these relations are, if possible, harder to find and more complicated. An easier approach is to use a numeric analysis of the correlations of the different amplitudes for different models. Therefore we pick random numbers for all parameters in the specific model. We adopt a flat distribution for the SM parameters $\sin^2\theta_{12}$, $\sin^2\theta_{31}$, $\sin^2\theta_{32}$, and the CP phase δ_{31} in the range [0, 1] and $[0, 2\pi]$, respectively. This distribution seems to be the most general without any prior assumptions. In any case, the exact shape of the distribution is not expected to affect significantly the results due to the strict experimental constraints for the elements of U. Since the elements of U are independent of the parametrization we are free to choose the standard parametrization from [50]. The ranges for the model specific parameters are shown below.

To check if the generated combination of parameters satisfy current experimental bounds, we compare the entries of the 3×3 submatrix of U with the bounds presented in [51], where a global fit is performed without implying a unitarity of $U^{3\times3}$.

$$|U|^{3\times3}_{3\sigma} = \begin{pmatrix} 0.76 \to 0.85 & 0.50 \to 0.60 & 0.13 \to 0.16 \\ 0.21 \to 0.54 & 0.42 \to 0.70 & 0.61 \to 0.79 \\ 0.18 \to 0.58 & 0.38 \to 0.72 & 0.40 \to 0.78 \end{pmatrix}. \quad (9)$$

For a viable combination of parameters all accessible amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$ are calculated and extracted. For each model we extracted 100,000 viable combinations. To show the correlation we performed a kernel density estimation for different combination of amplitudes, i.e. estimating the underlying probability density function by summing up Gaussian kernels placed on every data point.

We compare 4 different approaches of neutrino physics beyond the three neutrino paradigm:

(i) a model of one additional light sterile neutrino $(3 + 1\nu)$, motivated by LSND [44], MiniBooNE- [45], gallium- [47], and reactor anomaly [46]. Typically the additional mass squared difference lies in the ~1 eV range [42,43]. Due to the low mass the sterile state participates in the oscillation. The sterile neutrino does not interact via SM gauge interactions with other SM particles. The mixing matrix is a 4×4 unitary matrix (see Sec. II for more details). To generate the mixing matrix U we sampled the additional angles $\sin^2\theta_{41}$, $\sin^2\theta_{42}$ and $\sin^2\theta_{43}$ as well as the additional CP phases δ_{41} and δ_{43} from a flat distribution in the range [0, 1] and $[0, 2\pi]$, respectively.

(ii) a model of two additional light sterile neutrinos $(3 + 2\nu)$, similar to model (i) but with an extended parameter space (additional mixing angles and CP phases) due to the additional sterile state. The mixing matrix is a 5×5 unitary matrix. The additional mixing angles and CP phases are drawn analogue to model (i).

(iii) a scenario of nonunitarity without additional constraints (NU). This scenario is realized by modifying the unitary matrix with a lower triangular matrix α

$$U_{NU} = (I - \alpha)U^{3\times3} = \begin{pmatrix} 1 - \alpha_{ee} & 0 & 0 \\ \alpha_{e\mu} & 1 - \alpha_{\mu\mu} & 0 \\ \alpha_{\tau e} & \alpha_{\mu\tau} & 1 - \alpha_{\tau\tau} \end{pmatrix} U^{3\times3} \quad (10)$$

where $|\alpha_{\beta\gamma}| < 1$. The diagonal entries are real and the off-diagonal entries are complex parameters (see for instance [52–54]). In the numerical calculations the absolute values of all elements of α and the phases of the off-diagonal terms are sampled via a flat distribution in the range [0, 1] and $[0, 2\pi]$, respectively, which are the most general distributions with least prior assumptions.

(iv) a scenario of nonunitarity where additional fermions trigger rare decays like $\mu \to e\gamma$. The corresponding constraints from rare decays and electroweak precision observables are presented in [55] ["minimal unitarity violation" (MUV), the nonunitarity is parametrized as in scenario (iii)]

$$\begin{aligned} \alpha_{ee} &< 1.3 \times 10^{-3}, & |\alpha_{\mu e}| &< 6.8 \times 10^{-4}, \\ \alpha_{\mu\mu} &< 2.0 \times 10^{-4}, & |\alpha_{\tau e}| &< 2.7 \times 10^{-3}, \\ \alpha_{\tau\tau} &< 2.8 \times 10^{-3}, & |\alpha_{\tau\mu}| &< 1.2 \times 10^{-3}. \end{aligned} \quad (11)$$

For generating the mixing matrix U we picked all absolute values of the parameters from a flat distribution in the range corresponding to the above constraints (11). The phases of the off-diagonal elements are drawn the same way as in model (iii). Again we use the flat distribution to be as general as possible.

Many new physics models can influence neutrino oscillation in a way described by NU and MUV. For instance heavy right handed neutrinos introduced in seesaw models or nonstandard neutrino interaction (NSI) at production and detection can be described by the MUV and NU scenarios, respectively.

IV. RESULTS

The 95% CL of the generated kernel density estimates (KDE) for oscillations of ν_μ are shown in Figs. 1 and 2. We focus on these modes since the production of ν_μ is well understood and the modes are investigated by several current experiments. We do not consider amplitudes where sterile states are involved due to missing detection mechanisms. We also do not consider amplitudes with additional mass differences beyond the solar and atmospheric Δm^2_{12} and Δm^2_{23} since these are by now not known and current experiments are optimized for the known mass squared differences. As can be seen clearly for the scenarios with additional light neutrinos and nonunitarity without constraints the corresponding parameter spaces allow for significant deviation from the SM prediction of uniform CP violating. The MUV scenario albeit provides only a comparatively small allowed region. The strong constraints for the unitary violating parameters α [see Eq. (11)] as priors strongly restrict deviations from the SM prediction. The allowed regions fulfill all current bounds and display the uncertainties in Eq. (9) and the not yet determined CP phase(s).

The differences between the $3+1\nu$- and $3+2\nu$-model are negligible. Due to invariance under reparametrization the amplitudes in the 3×3 submatrix do not change by rotations in the 4-5-Plane in case of a $3+2\nu$-model. To investigate a difference between $3+1\nu$ and $3+2\nu$ scenarios, amplitudes with sterile states or additional mass squared differences have to be taken into account which are not expected to be accessible experimentally in the near future.

Comparing the models with additional light neutrinos with the scenario of an unconstrained nonunitarity one can find large deviations. The scenario of nonunitarity provides viable parameter sets which are far outside the 95% CL of the models with additional light neutrinos. The MUV scenario provides only a small deviation from the SM due to the strong constraints from electroweak precision observables. The expected deviations are out of reach of current experiments. Therefore a sizable measured deviation from the SM has to have another source than the MUV scenario.

Hence the experimental measurement of the corresponding CP violating amplitudes can be a direct test for the three neutrino paradigm and can also discriminate between different SM extensions: If the experimental values will turn out to lie outside a viable region of $3+1\nu$, $3+2\nu$ or the MUV scenario these models can be ruled out consistently.

Similar plots have been fabricated for all combinations of amplitudes and yield similar results. Whether the best discriminators are provided by the sums or the ratios of amplitudes will turn out once experimental data is available.

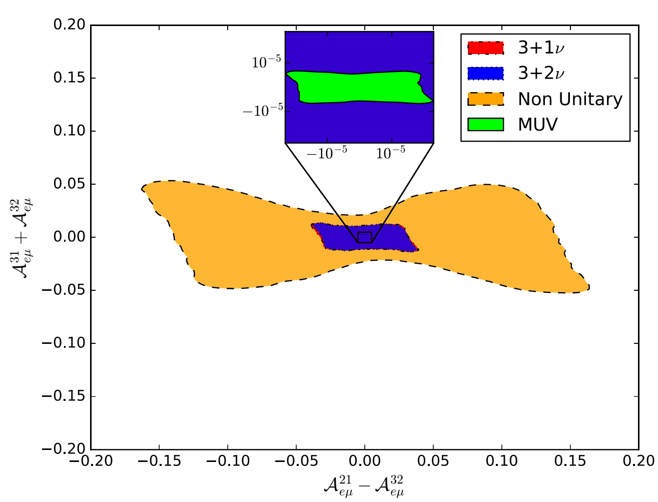

FIG. 1. Kernel density estimates for the different scenarios: $3+1\nu$ in red, $3+2\nu$ in blue, Nonunitarity in yellow and minimal unitarity violation (MUV) in green. Shown is the differences of the 3 different CP violating amplitudes in the $\nu_e\to\nu_\mu$-channel. The colored area corresponds to the 95% CL of the KDE. The three neutrino prediction corresponds to the point at (0,0). Except for numerical effects, the areas for the $3+1\nu$ and the $3+2\nu$ model match each other. A significant deviation between NU and new sterile states can be observed. Due to the strong constraints for MUV, the viable regions are extremely small and deviations from three neutrino prediction will be hard to measure.

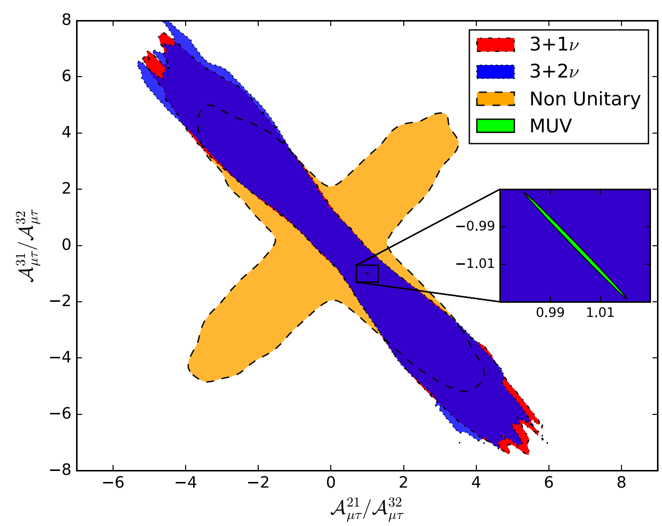

FIG. 2. Kernel density estimates for the different scenarios: $3+1\nu$ in red, $3+2\nu$ in blue, nonunitarity in yellow and ninimal unitarity violation in green. Shown is the ratios of the 3 different CP violating amplitudes in the $\nu_\mu\to\nu_\tau$-channel. The colored area corresponds to the 95% CL of the KDE. The three neutrino prediction corresponds to the point at $(1,-1)$. Except for numerical effects, the areas for the $3+1\nu$ and the $3+2\nu$ model match each other. A significant deviation between NU and new sterile states can be observed. Due to the strong constraints for MUV, the viable regions are extremely small and deviations from three neutrino prediction will be hard to measure.

V. CONCLUSION

In this work we have developed a new method to test and discriminate the standard three neutrino paradigm and several extensions based on the study of various combinations of CP violating amplitudes $\mathcal{A}^{kj}_{\alpha\beta}$. These amplitudes are easily accessible via oscillation experiments searching for CP violation. The amplitudes and the relations among them have been translated into the notation commonly used in the neutrino community. Moreover, the concept has been generalized to scenarios with five neutrinos and nonunitary mixing matrices. Powerful discriminators between different scenarios of physics beyond the SM can be exploited once experiments determine three different amplitudes. In this case it is possible to rule out not only the three neutrino paradigm but also models of additional sterile light neutrinos or the scenario of MUV in large regions of the respective parameter spaces. On the other hand, a determination of a unique amplitude would be in agreement with both the three neutrino model but also with specific parameter combinations of new physics models.

Note, that these calculations rely on the vacuum values of neutrino properties. They are independent of specific mass differences. The most straightforward way to determine values for the amplitudes is to rely on experiments being capable of running both neutrino and antineutrino mode where matter effects can be neglected. This is not the case for current experiments where matter effects have to be taken into account. How to include matter effects is not trivial and will be addressed in future work. Nevertheless the relations from Sec. II hold for a specific energy and baselength even in matter. Therefore an intrinsic validity check of the relations for each energy bin is possible. Currently the experimental uncertainties are too large to deduce a definite statement whether these relations hold but with increasing number of events and better experimental techniques we expect better results in the future.

ACKNOWLEDGMENTS

We thank Dennis Loose for useful discussion regarding the numerical analysis and Mariam Tórtola and Matthias Blennow for useful comments. This project was supported in part by a Fulbright Program grant sponsored by the Bureau of Educational and Cultural Affairs of the United States Department of State and administered by the Institute of International Education. P.S. also thanks Vanderbilt University, Prof. Thomas Kephart and Prof. Thomas J. Weiler for the kind hospitality.

APPENDIX: ANALYTIC RELATIONS OF *CP* VIOLATING AMPLITUDES

The following analytic relations have been taken from [49] and translated into the notations used for this paper. All 36 amplitudes can be reduced to a linear combination of the nine amplitudes $\mathcal{A}^{21}_{e\mu}$, $\mathcal{A}^{32}_{e\mu}$, $\mathcal{A}^{43}_{e\mu}$, $\mathcal{A}^{21}_{\mu\tau}$, $\mathcal{A}^{32}_{\mu\tau}$, $\mathcal{A}^{43}_{\mu\tau}$, $\mathcal{A}^{21}_{\tau s}$, $\mathcal{A}^{32}_{\tau s}$ and $\mathcal{A}^{43}_{\tau s}$.

$$
\begin{aligned}
&\mathcal{A}^{31}_{e\mu} = -\mathcal{A}^{32}_{e\mu} + \mathcal{A}^{43}_{e\mu}, &&\mathcal{A}^{41}_{e\mu} = -\mathcal{A}^{21}_{e\mu} + \mathcal{A}^{32}_{e\mu} - \mathcal{A}^{43}_{e\mu},\\
&\mathcal{A}^{42}_{e\mu} = \mathcal{A}^{21}_{e\mu} - \mathcal{A}^{32}_{e\mu}, &&\mathcal{A}^{21}_{e\tau} = -\mathcal{A}^{21}_{\mu\tau} + \mathcal{A}^{21}_{\tau s},\\
&\mathcal{A}^{31}_{e\tau} = \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{43}_{\mu\tau} - \mathcal{A}^{32}_{\tau s} + \mathcal{A}^{43}_{\tau s}, &&\mathcal{A}^{41}_{e\tau} = \mathcal{A}^{21}_{\mu\tau} - \mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{43}_{\mu\tau} - \mathcal{A}^{21}_{\tau s} + \mathcal{A}^{32}_{\tau s} - \mathcal{A}^{43}_{\tau s},\\
&\mathcal{A}^{32}_{e\tau} = -\mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{32}_{\tau s}, &&\mathcal{A}^{42}_{e\tau} = -\mathcal{A}^{21}_{\mu\tau} + \mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{21}_{\tau s} - \mathcal{A}^{32}_{\tau s},\\
&\mathcal{A}^{43}_{e\tau} = -\mathcal{A}^{43}_{\mu\tau} + \mathcal{A}^{43}_{\tau s}, &&\mathcal{A}^{21}_{es} = -\mathcal{A}^{21}_{e\mu} + \mathcal{A}^{21}_{\mu\tau} - \mathcal{A}^{21}_{\tau s},\\
&\mathcal{A}^{31}_{es} = \mathcal{A}^{32}_{e\mu} - \mathcal{A}^{43}_{e\mu} - \mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{43}_{\mu\tau} + \mathcal{A}^{32}_{\tau s} - \mathcal{A}^{43}_{\tau s}, &&\mathcal{A}^{32}_{es} = -\mathcal{A}^{32}_{e\mu} + \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{32}_{\tau s},
\end{aligned}
$$

$$
\mathcal{A}^{41}_{es} = \mathcal{A}^{21}_{e\mu} - \mathcal{A}^{32}_{e\mu} + \mathcal{A}^{43}_{e\mu} - \mathcal{A}^{21}_{\mu\tau} + \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{43}_{\mu\tau} + \mathcal{A}^{21}_{\tau s} - \mathcal{A}^{32}_{\tau s} + \mathcal{A}^{43}_{\tau s},
$$

$$
\begin{aligned}
&\mathcal{A}^{42}_{es} = -\mathcal{A}^{21}_{e\mu} + \mathcal{A}^{32}_{e\mu} + \mathcal{A}^{21}_{\mu\tau} - \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{21}_{\tau s} + \mathcal{A}^{32}_{\tau s}, &&\mathcal{A}^{43}_{es} = -\mathcal{A}^{43}_{e\tau} + \mathcal{A}^{43}_{\mu\tau} - \mathcal{A}^{43}_{\tau s},\\
&\mathcal{A}^{31}_{\mu s} = -\mathcal{A}^{32}_{e\mu} + \mathcal{A}^{43}_{e\mu} + \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{43}_{\mu\tau}, &&\mathcal{A}^{41}_{\mu\tau} = -\mathcal{A}^{21}_{\mu\tau} + \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{43}_{\mu\tau},\\
&\mathcal{A}^{42}_{\mu\tau} = \mathcal{A}^{21}_{\mu\tau} - \mathcal{A}^{32}_{\mu\tau}, &&\mathcal{A}^{21}_{\mu s} = \mathcal{A}^{21}_{e\mu} - \mathcal{A}^{21}_{\mu\tau},\\
&\mathcal{A}^{31}_{\mu s} = -\mathcal{A}^{32}_{e\mu} + \mathcal{A}^{43}_{e\mu} + \mathcal{A}^{32}_{\mu\tau} - \mathcal{A}^{43}_{\mu\tau}, &&\mathcal{A}^{41}_{\mu s} = -\mathcal{A}^{21}_{e\mu} + \mathcal{A}^{32}_{e\mu} - \mathcal{A}^{43}_{e\mu} + \mathcal{A}^{21}_{\mu\tau} - \mathcal{A}^{32}_{\mu\tau} + \mathcal{A}^{43}_{\mu\tau},\\
&\mathcal{A}^{32}_{\mu s} = \mathcal{A}^{32}_{e\mu} - \mathcal{A}^{32}_{\mu\tau}, &&\mathcal{A}^{42}_{\mu s} = \mathcal{A}^{21}_{e\mu} - \mathcal{A}^{32}_{e\mu} - \mathcal{A}^{21}_{\mu\tau} + \mathcal{A}^{32}_{\mu\tau},\\
&\mathcal{A}^{43}_{\mu s} = \mathcal{A}^{43}_{e\mu} - \mathcal{A}^{43}_{\mu\tau}, &&\mathcal{A}^{31}_{\tau s} = -\mathcal{A}^{32}_{\tau s} + \mathcal{A}^{43}_{\tau s},\\
&\mathcal{A}^{41}_{\tau s} = -\mathcal{A}^{21}_{\tau s} + \mathcal{A}^{32}_{\tau s} - \mathcal{A}^{43}_{\tau s}, &&\mathcal{A}^{42}_{\tau s} = \mathcal{A}^{21}_{\tau s} - \mathcal{A}^{32}_{\tau s}.
\end{aligned}
$$

[1] Q. R. Ahmad *et al.* (SNO), Phys. Rev. Lett. **89**, 011301 (2002).
[2] N. Barros (SNO), Nucl. Phys. B, Proc. Suppl. **237–238**, 107 (2013).
[3] Y. Fukuda *et al.* (Super-Kamiokande), Phys. Rev. Lett. **81**, 1562 (1998).
[4] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, and T. Schwetz, J. High Energy Phys. 01 (2017) 087.
[5] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).
[6] K. Abe *et al.* (T2K collaboration), Phys. Rev. D **91**, 072010 (2015).
[7] K. Abe *et al.* (T2K collaboration), arXiv:1701.00432.
[8] S. Goswami and T. Ota, Phys. Rev. D **78**, 033012 (2008).
[9] O. G. Miranda, M. Tortola, and J. W. F. Valle, Phys. Rev. Lett. **117**, 061804 (2016).
[10] S.-F. Ge, P. Pasquini, M. Tortola, and J. W. F. Valle, Phys. Rev. D **95**, 033005 (2017).
[11] A. de Gouvêa and K. J. Kelly, arXiv:1605.09376.
[12] D. Dutta and P. Ghoshal, J. High Energy Phys. 09 (2016) 110.
[13] M. Freund, Phys. Rev. D **64**, 053003 (2001).
[14] M. C. Gonzalez-Garcia, Y. Grossman, A. Gusso, and Y. Nir, Phys. Rev. D **64**, 096006 (2001).
[15] V. Gupta and X.-G. He, Phys. Rev. D **64**, 117301 (2001).
[16] W.-l. Guo and Z.-z. Xing, Phys. Rev. D **65**, 073020 (2002).
[17] M. Blasone, A. Capolupo, and G. Vitiello, Phys. Rev. D **66**, 025033 (2002).
[18] W. Rodejohann, Phys. Rev. D **69**, 033005 (2004).
[19] N. Li and B.-Q. Ma, Phys. Lett. B **600**, 248 (2004).
[20] G. Dattoli and K. V. Zhukovsky, Eur. Phys. J. C **55**, 547 (2008).
[21] W. Rodejohann and J. W. F. Valle, Phys. Rev. D **84**, 073011 (2011).
[22] M. Huang, D. Liu, J.-C. Peng, S. D. Reitzner, and W.-C. Tsai, arXiv:1108.3906.
[23] K. Kimura, A. Takamura, and H. Yokomakura, Phys. Lett. B **537**, 86 (2002).
[24] E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson, and T. Schwetz, J. High Energy Phys. 04 (2004) 078.
[25] Y.-L. Zhou, Phys. Rev. D **84**, 113012 (2011).
[26] L. J. Flores and O. G. Miranda, Phys. Rev. D **93**, 033009 (2016).
[27] P. B. Denton, H. Minakata, and S. J. Parke, J. High Energy Phys. 06 (2016) 051.
[28] M. B. Johnson and L. S. Kisslinger, arXiv:1604.08233.
[29] M. B. Johnson, E. M. Henley, and L. S. Kisslinger, Phys. Rev. D **91**, 076005 (2015).
[30] L. J. Flores and O. G. Miranda, J. Phys. Conf. Ser. **761**, 012041 (2016).
[31] H. Minakata and S. J. Parke, J. High Energy Phys. 01 (2016) 180.
[32] D. J. Wagner and T. J. Weiler, Phys. Rev. D **59**, 113007 (1999).
[33] T. J. Weiler and D. J. Wagner, AIP Conf. Proc. **444**, 46 (1998).
[34] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).
[35] C. Jarlskog, Phys. Rev. D **36**, 2128 (1987).
[36] Y. Farzan and A. Yu. Smirnov, Phys. Rev. D **65**, 113001 (2002).
[37] Z.-z. Xing and H. Zhang, Phys. Lett. B **618**, 131 (2005).
[38] J. D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. D **74**, 073012 (2006).
[39] Z.-Z. Xing, Chin. Phys. C **36**, 281 (2012).
[40] H.-J. He and X.-J. Xu, Phys. Rev. D **89**, 073002 (2014).
[41] H.-J. He and X.-J. Xu, Phys. Rev. D **95**, 033002 (2017).
[42] J. Kopp, P. A. N. Machado, M. Maltoni, and T. Schwetz, J. High Energy Phys. 05 (**2013**) 050.
[43] K. N. Abazajian *et al.*, arXiv:1204.5379.
[44] A. Aguilar-Arevalo *et al.* (LSND), Phys. Rev. D **64**, 112007 (2001).
[45] A. A. Aguilar-Arevalo *et al.* (MiniBooNE), Phys. Rev. Lett. **105**, 181801 (2010).
[46] G. Mention, M. Fechner, Th. Lasserre, Th. A. Mueller, D. Lhuillier, M. Cribier, and A. Letourneau, Phys. Rev. D **83**, 073006 (2011).
[47] C. Giunti and M. Laveder, Phys. Rev. C **83**, 065504 (2011).
[48] M. G. Aartsen *et al.* (IceCube), Phys. Rev. Lett. **117**, 071801 (2016).
[49] T. Suzuki, J. Math. Phys. (N.Y.) **50**, 123526 (2009).
[50] C. Patrignani *et al.* (Particle Data Group), Chin. Phys. C **40**, 100001 (2016).
[51] S. Parke and M. Ross-Lonergan, Phys. Rev. D **93**, 113009 (2016).
[52] F. J. Escrihuela, D. V. Forero, O. G. Miranda, M. Tortola, and J. W. F. Valle, Phys. Rev. D **92**, 053009 (2015); **93**, 119905(E) (2016).
[53] E. Fernandez-Martinez, M. B. Gavela, J. Lopez-Pavon, and O. Yasuda, Phys. Lett. B **649**, 427 (2007).
[54] S. Antusch, C. Biggio, E. Fernandez-Martinez, M. B. Gavela, and J. Lopez-Pavon, J. High Energy Phys. 10 (2006) 084.
[55] M. Blennow, P. Coloma, E. Fernandez-Martinez, J. Hernandez-Garcia, and J. Lopez-Pavon, arXiv:1609.08637.