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Chiral separation effect in lattice regularization

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We consider the chiral separation effect (CSE) in the lattice-regularized quantum field theory. We discuss two types of regularization: with and without exact chiral symmetry. In the latter case this effect is described by its conventional expression for the massless fermions. This is illustrated by the two particular cases of Wilson fermions and the conventional overlap fermions. At the same time, in the presence of the exact chiral symmetry the CSE disappears. This is illustrated by the naive lattice fermions, when the contributions of the fermion doublers cancel each other. Another example is the modified version of the overlap regularization proposed recently, where there is exact chiral symmetry, but as a price for this the fermion doublers become zeros of the Green function. In this case the contributions to the CSE of zeros and poles of the Green function cancel each other.

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I. INTRODUCTION

The family of nondissipative transport effects has been widely discussed recently, both in the context of high-energy physics and in the context of condensed matter theory [1–8]. The possible appearance of such effects in the recently discovered Dirac and Weyl semimetals has been considered [9–16]. The chiral separation effect (CSE) [17] is one of the members of this family. It manifests itself in the equilibrium theory with a massless Dirac fermion, where in the presence of an external magnetic field (corresponding to the field strength F_{ij}) and the ordinary chemical potential μ the axial current is given by

$$j^k_5 = -\frac{1}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}. \qquad (1)$$

In the context of high-energy physics the possibility to observe the CSE was discussed, in particular, in relation to relativistic heavy-ion collisions (see, for example, Refs. [18–20] and references therein).

At first sight, the CSE has the same origin (the chiral anomaly[1]) as the so-called chiral magnetic effect, which was discussed, for example, in Refs. [18,21–24]. Although it was reported that the possible existence of the chiral magnetic contribution to ordinary conductivity [25] was observed in recently discovered Dirac semimetals [26], it was shown that the original equilibrium [21] version of the CME does not exist. In particular, in Refs. [5–8] using various numerical methods the CME current was investigated in the context of lattice field theory. It was argued that the equilibrium bulk CME does not exist, but close to the boundary of the system the nonzero CME current may appear. It was demonstrated that in the given systems the integrated total CME current remains zero. In the context of condensed matter theory the absence of the CME was reported within the particular model of a Weyl semimetal [27]. Besides, it was argued that the equilibrium CME may contradict the no-go Bloch theorem [28]. However, the way that the author of Ref. [28] tried to extend the Bloch theorem to the field-theoretic systems seems to us nonrigorous. Sufficient analytical proof of the absence of the equilibrium CME (in the systems without superconductivity) was presented by one of us in Refs. [29,30]. This proof relied on the Wigner transformation technique [31–34] applied to the lattice-regularized quantum field theory.

In the present paper we proceed in this line of research and investigate the equilibrium CSE on the same grounds. In the framework of the naive nonregularized quantum field theory, the CSE was discussed recently, for example, in Ref. [3]. Also, it was discussed in the framework of lattice regularization in Ref. [8], where it was argued that the CSE needs no ultraviolet regularization because the expression for the current does not contain ultraviolet divergences. In the present paper, however, we argue that the ultraviolet regularization is important. We demonstrate that without it there is an ambiguity in the calculation of the CSE current. Namely, if the model is considered at small but finite temperatures, then the calculation of the axial current gives the conventional result if the summation over the Matsubara frequencies is performed first, while if the integration over the 3-momenta is performed first the expression for the current remains undefined. We consider this as an indication that the rigorous lattice regularization should be used. Following the formalism developed in Ref. [29], in the present paper we consider the CSE on the basis of the

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[1]It is worth mentioning that Eq. (1) does not follow directly from the general expression for the chiral anomaly. The latter also does not follow from Eq. (1) because in this expression μ is constant. For a value of μ that is dependent on the coordinates, the current j^k_5 should contain a term proportional to $\nabla\mu$, which will also contribute the divergence ∂j_5.

Wigner transformation technique [31,32] applied to the Green functions. Unlike the case of the CME, in the general case the coefficient in the linear response of the axial current to the external magnetic field is not a topological invariant. However, it appears that the coefficient in the CSE current standing at the product of the magnetic field and chemical potential approaches a topological invariant when the mass of the fermion tends to zero. This allows us to derive the conventional expression for the CSE current. Thus, the link between the CSE effect in lattice regularization and momentum-space topology is established.

It is worth mentioning that momentum-space topology is a powerful method that was developed mainly within condensed matter theory. It allows to describe in a simple way, for example, the stability of the Fermi points, the anomalous quantum Hall effect, and the fermion zero modes on vortexes (for a review, see Refs. [35,36]). Recently, certain aspects of momentum-space topology were discussed in the framework of four-dimensional lattice gauge theory (see, for example, Refs. [37–39]).

The paper is organized as follows. In Sec. II, following Refs. [29,30], we describe how the Wigner transformation technique may be applied to the lattice-regularized quantum field theory. In Sec. III we consider the CSE in conventional lattice regularizations (Wilson fermions and overlap fermions). In Sec. IV we consider the CSE in the case of exact chiral symmetry, for the naive lattice regularization with 16 doublers and in the lattice regularization with deformed overlap fermions, where instead of the 15 doublers the zeros of the Green function appear. In Sec. V we demonstrate the ambiguity in the calculation of the CSE current that takes place in the naive continuum theory. In Sec. VI we end with the conclusions.

II. LATTICE FERMIONS IN THE PRESENCE OF AN EXTERNAL $U(1)$ GAUGE FIELD

A. Lattice models in momentum space

In this section we briefly consider the lattice models in momentum space following the methodology of Refs. [29,30]. For a more detailed description of the method and for references, see Refs. [29,30]. In the absence of the external gauge field the partition function of the theory defined on the infinite lattice may be written as

$$Z = \int D\bar{\psi}D\psi \exp\left(-\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|}\bar{\psi}^T(p)\mathcal{G}^{-1}(p)\psi(p)\right), \quad (2)$$

where $|\mathcal{M}|$ is the volume of momentum space $\mathcal{M}$, D is the dimensionality of space-time, and $\bar{\psi}$ and ψ are the Grassmann-valued fields defined in momentum space $\mathcal{M}$. $\mathcal{G}$ is specific for the given system. For example, the model with $3+1$-dimensional Wilson fermions corresponds to a $\mathcal{G}$ that has the form

$$\mathcal{G}(p) = -i\left(\sum_k \gamma^k g_k(p) - im(p)\right)^{-1}, \quad (3)$$

where γ^k are Euclidean Dirac matrices defined in the chiral representation,

$$\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \qquad i = 1,2,3,$$
$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where σ^i is the Pauli matrix. $g_k(p)$ and $m(p)$ are the real-valued functions ($k = 1, 2, 3, 4$) given by

$$g_k(p) = \sin p_k, \qquad m(p) = m^{(0)} + \sum_{a=1,2,3,4}(1-\cos p_a). \quad (4)$$

The fields in coordinate space are related to the fields in momentum space as follows:

$$\psi(r) = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} e^{ipr}\psi(p). \quad (5)$$

At the discrete values of r corresponding to the points of the lattice this expression gives the values of the fermionic field at these points, i.e., the dynamical variables of the original lattice model. However, Eq. (5) allows to formally define the values of fields at any other values of r. The partition function may be rewritten in the form

$$Z = \int D\bar{\psi}D\psi \exp\left(-\sum_{r_n}\bar{\psi}^T(r_n)[\mathcal{G}^{-1}(-i\partial_r)\psi(r)]_{r=r_n}\right). \quad (6)$$

Here the sum in the exponent is over the discrete coordinates r_n. However, the operator $-i\partial_r$ acts on the function $\psi(r)$ defined using Eq. (5). In order to derive Eq. (6) we use the identity

$$\sum_r e^{ipr} = |\mathcal{M}|\delta(p). \quad (7)$$

In the particular case of Wilson fermions we may rewrite the partition function in the conventional way as

$$Z = \int D\bar{\psi}D\psi \exp\left(-\sum_{r_n,r_m}\bar{\psi}^T(r_m)(\mathcal{D}_{r_n,r_m})\psi(r_n)\right), \quad (8)$$

with

$$\mathcal{D}_{x,y} = -\frac{1}{2}\sum_i [(1+\gamma^i)\delta_{x+e_i,y} + (1-\gamma^i)\delta_{x-e_i,y}] + (m^{(0)}+4)\delta_{xy}. \tag{9}$$

Here e_i is the unit vector in the ith direction.

B. Introduction of the gauge field

The gauge transformation of the lattice field takes the form

$$\psi(r_n) \to e^{i\alpha(r_n)}\psi(r_n). \tag{10}$$

In the case of Wilson fermions the $U(1)$ gauge field is typically introduced as the following modification of the operator D:

$$\mathcal{D}_{x,y} = -\frac{1}{2}\sum_i [(1+\gamma^i)\delta_{x+e_i,y} e^{iA_{x+e_i,y}} + (1-\gamma^i)\delta_{x-e_i,y} e^{iA_{x-e_i,y}}] + (m^{(0)}+4)\delta_{xy}. \tag{11}$$

Here $A_{x,y} = -A_{y,x}$ is the gauge field attached to the links of the lattice. In the same way, the gauge field is typically incorporated into the models of solid state physics.

One can easily check that Eq. (8) may be rewritten as

$$Z = \int D\bar{\psi} D\psi \exp\left(-\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}^T(p)\hat{\mathcal{Q}}(i\partial_p, p)\psi(p)\right). \tag{12}$$

Here

$$\hat{\mathcal{Q}} = \mathcal{G}^{-1}(p - A(i\partial_p)), \tag{13}$$

while the pseudodifferential operator $A(i\partial_p)$ is defined as follows. First, we represent the original gauge field $A(r)$ as a series in powers of the coordinate r. Next, the variable r is substituted in this expansion by the operator $i\partial_p$. Besides, in Eq. (13) each product of the components of $p - A(i\partial_p)$ is substituted by the symmetric combination (for details, see Ref. [29]). As it was mentioned above, for the case of Wilson fermions the formulations of Eq. (12) and Eq. (8) are exactly equivalent. For the other regularizations there may be a difference, but it manifests itself in the terms that are proportional to the field strength times a^2 (here a is the lattice spacing). Those extra terms may be neglected in the continuum limit. Therefore, for *any* regularization we accept Eq. (13) as the definition of the model in the presence of an external $U(1)$ gauge field.

C. Electric current

Electric current is defined as the response of the effective action $-\log Z$ to the variation of an external electromagnetic field. This gives [29]

$$j^k(R) = \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \mathrm{Tr}\tilde{G}(R,p)\frac{\partial}{\partial p_k}[\tilde{G}^{(0)}(R,p)]^{-1}, \tag{14}$$

where the Wigner transformation of the Green function is expressed as

$$\tilde{G}(R,p) = \sum_{r=r_n} e^{-ipr} G(R + r/2, R - r/2), \tag{15}$$

while the Green function itself is

$$G(r_1,r_2) = -\frac{1}{Z}\int D\bar{\Psi}D\Psi\bar{\Psi}(r_2)\Psi(r_1) \times\exp\left(-\sum_{r_n}[\bar{\Psi}(r_n)[\mathcal{G}^{-1}(-i\partial_r - A(r))\Psi(r)]_{r=r_n}]\right). \tag{16}$$

At the same time,

$$\tilde{G}^{(0)}(R,p) = \mathcal{G}(p - A(R)). \tag{17}$$

The application of the Wigner transformation technique to the lattice models was developed in Refs. [29,30], following its original formulation specific for the theory in continuous space-time [31–34]. In Ref. [29], the following expression was derived for the linear response of the electric current to an external electromagnetic field:

$$j^{(1)k}(R) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(R), \tag{18}$$

$$\mathcal{M}_l = \int \mathrm{Tr}\nu_l d^4 p \tag{19}$$

$$\nu_l = -\frac{i}{3!8\pi^2}\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\right]. \tag{20}$$

III. LATTICE REGULARIZATION WITH BROKEN CHIRAL SYMMETRY

A. Linear response of the chiral current to an external magnetic field

In this section we consider the linear response of the chiral current to an external electromagnetic field. For the field system in continuous coordinate space this response may easily be calculated using Feynman diagrams. For the field system in the lattice regularization, this response may be calculated following the approach of Refs. [29,30,39] that was briefly described above. In continuum theory the naive expression for the chiral current is $-i\langle\bar{\psi}\gamma^\mu\gamma^5\psi\rangle$. Several different definitions for the particular lattice regularization may give this expression in the naive continuum limit.

The evident choice of the definition of current in lattice regularization is the functional derivative over the axial gauge field of the effective action. The latter field, in turn, may be defined through the covariant derivative, which acts on the left-handed and the right-handed fermions via opposite charges. For the particular choice of the lattice model with exact chiral symmetry (i.e., when $\mathcal{G}$ commutes or anticommutes with γ^5), this definition gives an expression similar to that of Refs. [29,30,39]:

$$j^{5k}(R) = \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \mathrm{Tr}\gamma^5 \tilde{G}(R,p) \frac{\partial}{\partial p_k} [\tilde{G}^{(0)}(R,p)]^{-1}, \quad (21)$$

where

$$\tilde{G}^{(0)}(R,p) = \mathcal{G}(p - \gamma^5 A(R)). \quad (22)$$

Actually, we are able to apply this definition to any lattice theory even without the exact chiral symmetry. One can easily check that in the naive continuum limit this definition gives $-i\langle\bar{\psi}\gamma^k\gamma^5\psi\rangle$.

In order to regularize our expressions for the case of truly massless fermions, let us use the finite-temperature version of the lattice theory. With the periodic boundary conditions in the spatial directions and antiperiodic conditions in the imaginary time direction, the lattice momenta will be

$$p_i \in (0, 2\pi); \qquad p_4 = \frac{2\pi}{N_t}(n_4 + 1/2), \quad (23)$$

where $i = 1, 2, 3$, while $n_4 = 0, \ldots, N_t - 1$. Temperature is equal to $T = 1/N_t$, in lattice units $1/a$, where a is the lattice spacing. Thus the imaginary frequencies are discrete, $p_4 = \omega_n = 2\pi T(n + 1/2)$, where $n = 0, 1, \ldots N_t - 1$, while the axial current (also in lattice units) is expressed via the Green functions as follows:

$$j^{5k} = -\frac{i}{2} T \sum_{n=0}^{N_t-1} \int \frac{d^3 p}{(2\pi)^3} \mathrm{Tr}\gamma^5 (\mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_i} \mathcal{G}^{-1}(\omega_n, \mathbf{p}) \times \partial_{p_j} \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_k} \mathcal{G}^{-1}(\omega_n, \mathbf{p})) F_{ij}. \quad (24)$$

B. The linear response of the chiral current to a chemical potential and external magnetic field

Now let us consider the system without exact chiral symmetry. Recall that the exact chiral symmetry is to be broken if we want to describe one Dirac fermion, which is related to the Nielsen-Ninomiya theorem. For definiteness, we first discuss Wilson fermions. But this is not necessary, and the results of this subsection are valid for any lattice model. We introduce the chemical potential in the standard way, $\omega_n \to \omega_n - i\mu$. The derivative of the current with respect to μ gives

$$j^{5k} = \frac{\mathcal{N}^{ijk}}{4\pi^2} F_{ij} \mu, \quad (25)$$

with

$$\mathcal{N}^{ijk} = -\sum_{n=0}^{N_t-1} \frac{1}{2} T \int \frac{d^3 p}{(2\pi)} \partial_{\omega_n} \mathrm{Tr}\gamma^5 \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_i} \mathcal{G}^{-1}(\omega_n, \mathbf{p}) \times \partial_{p_j} \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_k} \mathcal{G}^{-1}(\omega_n, \mathbf{p}). \quad (26)$$

We assume that the singularities of the Green function (poles or zeros) may appear only at the finite sequence of values of $\omega = \omega^{(0)}, \omega^{(1)}, \ldots$ that do not coincide with the Matsubara frequencies. In the limit $T \to 0$, the sum over Matsubara frequencies becomes an integral that is regularized as follows:

$$\mathcal{N}^{ijk} = \sum_k (-\mathcal{N}_3^{ijk}(\omega^{(k)} + 0) + \mathcal{N}_3^{ijk}(\omega^{(k)} - 0)), \quad (27)$$

where $\mathcal{N}_3(\omega_n)$ is given by

$$\mathcal{N}_3^{ijk}(\omega_n) = -\frac{1}{2} \int \frac{d^3 p}{(2\pi)^2} \mathrm{Tr}\gamma^5 \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_i} \mathcal{G}^{-1}(\omega_n, \mathbf{p}) \times \partial_{p_j} \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_k} \mathcal{G}^{-1}(\omega_n, \mathbf{p}). \quad (28)$$

It is clear that in a model in which there are no poles or zeros of the Green function in the presence of exact chiral symmetry, the linear response of the axial current to a magnetic field is the sum of topological invariants, i.e., it cannot be changed under continuous deformations of the model. However, in a general case when γ^5 does not (anti) commute with the Green function, the terms in this expansion are not topological invariants. We may also rewrite

$$\mathcal{N}^{ijk} = \frac{1}{2} \int_\Sigma \frac{d^3 p}{(2\pi)^2} \mathrm{Tr}\gamma^5 \mathcal{G}(\omega, \mathbf{p}) \partial_{p_i} \mathcal{G}^{-1}(\omega, \mathbf{p}) \times \partial_{p_j} \mathcal{G}(\omega, \mathbf{p}) \partial_{p_k} \mathcal{G}^{-1}(\omega, \mathbf{p}), \quad (29)$$

where Σ is the three-dimensional hypersurface of infinitely small volume that embraces the singularities of the Green function concentrated at the Fermi surfaces (or Fermi points). The advantage of this representation is that Eq. (29) becomes a topological invariant if γ^5 anticommutes with the Green function in a small vicinity of its poles.

It is worth mentioning that the zeros of the Green function seem to decouple if we consider the naive perturbation expansion. However, a more detailed consideration demonstrates that this is not so. More precisely, at $T \to 0$ the contribution to the expression of Eq. (26) of the zero of the Green function is the same as the contribution of the massless excitation. The actual reason for this is the gauge-invariant lattice regularization, which requires the derivative $\partial_k G^{-1}$ at each vertex of the perturbation expansion instead of the matrix γ_k. In turn, this derivative is singular at the position of the zero of the Green function.

C. Regularization with Wilson fermions

For the case of Wilson fermions the singularities of the Green function may appear at $\omega = 0, \pi$ (for $m^{(0)} > 0$ they appear at $\omega = 0$ only). Then, the limit $T \to 0$ gives

$$\mathcal{N}^{ijk} = -\mathcal{N}_3^{ijk}(+0) + \mathcal{N}_3^{ijk}(-0) + \mathcal{N}_3^{ijk}(\pi - 0) - \mathcal{N}_3^{ijk}(\pi + 0). \tag{30}$$

An interesting particular case is when the parameter $m^{(0)}$ vanishes. In this case, at $\mu = 0$ the only Fermi point appears at $\mathbf{p} = 0$ and on Σ we have $\{\gamma^5, \mathcal{G}\} \approx 0$. In this particular case

$$\mathcal{N}^{ijk} = \epsilon^{ijk},$$

which gives the regular expression for the chiral separation effect of Eq. (1). In order to confirm this prediction, we also use numerical methods. Namely, we take Eq. (26) and numerically calculate the integral over 3-momenta for the component $\mathcal{N}^{123}$ and the sum over ω_n using the MAPLE package. It is seen that at $N_t \to \infty$ the answer tends to 1, as it should.

In the presence of a nonzero mass ($m^{(0)} > 0$) the situation is changed, and the poles of the Green function do not appear while $\mu < m^{(0)}$, which gives the vanishing CSE current. At $\mu \geq m^{(0)}$ the Fermi surface appears, and it contributes to the chiral current through Eq. (29). However, in this case γ^5 does not anticommute with $\mathcal{G}$ on Σ. In the continuum limit $m^{(0)} = m^{(0)}_{\text{phys}} a$ and $\mu = \mu_{\text{phys}} a$, where $m^{(0)}_{\text{phys}}$ and μ_{phys} are the parameters of the model in physical units while a is the lattice spacing. At $\mu_{\text{phys}} \gg m^{(0)}_{\text{phys}}$ we recover the conventional result for the CSE of Eq. (1).

Notice that the massive doublers cannot contribute the expression for the response of the chiral current to an external magnetic field at finite values of chemical potential because their physical masses are as large as $1/a$.

D. Overlap fermions

Let us discuss the regularization using overlap fermions [40]. The massless overlap Dirac operator is defined as

$$\mathcal{D}_o = m(\hat{1} + \mathcal{D}(-m)(\mathcal{D}(-m)\mathcal{D}^+(-m))^{-1/2}), \tag{31}$$

where $\mathcal{D}(m^{(0)})$ is the dimensionless Wilson-Dirac operator given by Eq. (9), where we substitute the negative value of the mass parameter $m^{(0)} = -m$. The operator $\mathcal{D}_o$ obeys the Ginsparg-Wilson relation, which may be written in the following form:

$$\{\mathcal{D}_o^{-1}, \gamma^5\} = \frac{\gamma^5}{m}. \tag{32}$$

It is sometimes called "exact" chiral symmetry on the lattice. However, this statement is not precise, and actually the conventional overlap propagator does not obey the exact chiral symmetry, which is $\{\mathcal{D}_o^{-1}, \gamma^5\} = 0$.

In momentum space we have

$$\mathcal{D}_o^{-1} = -i\gamma_\mu C_\mu + \frac{1}{2m}, \tag{33}$$

with

$$C_\mu(p) = \frac{1}{2m}\frac{k_\mu}{\sqrt{k_\mu^2 + A^2} + A}, \qquad A = \frac{\hat{k}_\mu^2}{2} - m, \tag{34}$$

and

$$k_\mu = \sin(p_\mu), \qquad \hat{k} = 2\sin\left(\frac{p_\mu}{2}\right). \tag{35}$$

In this model the fermion Green function

$$\mathcal{G} = \mathcal{D}_o^{-1}$$

only has a pole at $\omega = 0$, $\mathbf{p} = 0$. At $p = (n_1\pi, n_2\pi, n_3\pi, n_4\pi)$ with integer $n_i = 0,\ 1$ such that $n_1 + n_2 + n_3 + n_4 \neq 0$, we have the value $\mathcal{G}(p) = \frac{1}{2m}$.

Again, the above consideration may be applied to the model in this regularization, and we have the expression for the linear response to the magnetic field given by Eq. (25) with $\mathcal{N}$ of Eq. (29). In particular, in the continuum limit when $m \to 0$ and $\mu = 0$ we get $\mathcal{N}^{ijk} = \epsilon^{ijk}$, which results in the usual expression for the CSE current of Eq. (1).

IV. LATTICE REGULARIZATION WITH EXACT CHIRAL SYMMETRY

A. Naive lattice fermions

In this section we consider the case of the lattice model with exact chiral symmetry, when

$$\{\gamma^5, \mathcal{G}\} = 0.$$

The simplest example of such a system is given by the naive lattice fermions with the Green function in momentum space of the form

$$\mathcal{G}(p) = -i\left(\sum_k \gamma^k g_k(p) - im^{(0)}\right)^{-1}, \tag{36}$$

where γ^k are Euclidean Dirac matrices, while $g_k(p)$ are the real-valued functions ($k = 1, 2, 3, 4$) given by

$$g_k(p) = \sin p_k. \tag{37}$$

In this model at $m^{(0)} = 0$, instead of one massless Dirac particle, in the continuum limit there are 16 massless particles. In this case the linear response of the chiral current to an external magnetic field and chemical potential is given by Eq. (25) with $\mathcal{N}$ of Eq. (29). The contributions of the doublers differ due to the orientation of the effective vierbein, i.e., the corresponding low-energy effective theory for massless particle has the one-particle Euclidean Lagrangian

$$\mathcal{L} = |e| e_a^\mu \gamma^a i \nabla_\mu,$$

where

$$|e| e_a^\mu = \begin{pmatrix} (-1)^{n_1} & 0 & 0 & 0 \\ 0 & (-1)^{n_2} & 0 & 0 \\ 0 & 0 & (-1)^{n_3} & 0 \\ 0 & 0 & 0 & (-1)^{n_4} \end{pmatrix}$$

with $n_i = 0, 1$. As a result, the contributions to the CSE current of these 16 doublers cancel each other. Thus, unlike the case of the previous section, in the continuum limit when $m \to 0$ and $\mu = 0$ all doublers contribute the sum, thus giving $\mathcal{N} = 0$, and canceling the overall CSE current.

B. Modified overlap fermions

It was proposed (see, for example, Ref. [40]) to redefine the overlap fermions as follows:

$$\mathcal{G} = \mathcal{D}_o^{-1} - \frac{1}{2m}.$$

In this case the chiral symmetry is exact,

$$\{\mathcal{G}, \gamma^5\} = 0. \tag{38}$$

But the price for this is that at $p = (n_1\pi, n_2\pi, n_3\pi, n_4\pi)$ with $n_1 + n_2 + n_3 + n_4 \neq 0$ we have a vanishing value of the Green function, $\mathcal{G}(p) = 0$.

The zeros of the Green function are in many aspects similar to poles. In particular, they contribute to the CSE in such a way that the total current vanishes. Let us define

$$f(k^2) = \frac{2m(\sqrt{k^2 + A^2} + A)}{k^2}.$$

Then,

$$\mathcal{G} = -i \frac{k_\mu \gamma^\mu}{f(k^2) k^2}. \tag{39}$$

At finite temperatures we have

$$j^{5k} = \frac{-iT}{2\pi} \sum_{n=0}^{N_t - 1} \mathcal{N}_3(\omega_n) \epsilon^{ijk} F_{ij}. \tag{40}$$

At any value of n the functional $\mathcal{N}_3(\omega_n)$ is a topological invariant, i.e, it is not changed under any variation $\mathcal{G} \to \mathcal{G} + \delta\mathcal{G}$ if during such modification the poles or zeros of $\mathcal{G}$ do not appear. It is given by

$$\mathcal{N}_3(\omega_n) = \frac{1}{2 \times 3!} \epsilon^{ijk} \int \frac{d^3 p}{(2\pi)^2} \mathrm{Tr} \gamma^5 \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_i} \mathcal{G}^{-1}(\omega_n, \mathbf{p}) \times \partial_{p_j} \mathcal{G}(\omega_n, \mathbf{p}) \partial_{p_k} \mathcal{G}^{-1}(\omega_n, \mathbf{p}). \tag{41}$$

Therefore, in the model in which there are no poles or zeros of the Green function in the presence of exact chiral symmetry the linear response of the axial current to a magnetic field is the sum of topological invariants, i.e., it cannot be changed under continuous deformations of the model. The pole of the Green function at finite temperature may appear if there exists an integer n such that $\omega_n = \frac{2\pi}{N_t}(n + 1/2) = \pi$. This gives the equation

$$2n + 1 = N_t,$$

which has a solution for odd values of N_t. Therefore, for simplicity in the following we assume that N_t is even.

The ordinary chemical potential cannot cause the appearance of poles or zeros of the Green function, which is seen from the following consideration. Again, let us assume that the chemical potential appears as the imaginary contribution to the Matsubara frequency. In this case the poles or zeros of the Green function may appear if

$$\sin^2(\omega_n - i\mu) + \sum_{l=1}^{3} \sin^2(p_i) = 0. \tag{42}$$

We obtain the following system:

$$\begin{cases} 1 - \cos(2\omega_n)\mathrm{ch}(2\mu) + 2\sum_{l=1}^{3} \sin^2(p_i) = 0, \\ \mathrm{sh}(\mu)\sin(2\omega_n) = 0. \end{cases} \tag{43}$$

The second equation has a solution in real variables. We obtain $\omega_n = \pi/2$ or $\omega_n = \pi$. In the former case the first equation does not have solutions. The latter case is realized for odd values of N_t only, and then the poles of the Green function appear as the solution of the equation

$$1 + 2\sum_{l=1}^{3} \sin^2(p_i) = \mathrm{ch}(2\mu).$$

However, as above we can always choose an even value for N_t, and therefore the poles of the Green function do not appear if we modify the value of μ. Therefore, we are able to calculate $\mathcal{N}_3(\omega_n)$ for vanishing μ, and the result gives the answer for finite μ. This calculation is represented in the Appendix, and as expected it gives the vanishing CSE current.

V. NAIVE CONTINUUM EXPRESSIONS FOR THE CSE CURRENT

A. The integration over 3-momenta before the summation over Matsubara frequencies

Above we considered the CSE using rigorous lattice regularizations. Also, we feel it instructive to present here the discussion of the chiral separation effect in the framework of naive continuous field theory. We will see that there is an ambiguity in this consideration, which is reflected by the lattice constructions with and without exact chiral symmetry.

Let us consider the propagator of massless noninteracting Dirac fermions,

$$G(\omega_n, \mathbf{p}) = \frac{1}{\gamma^\mu p_\mu}. \tag{44}$$

Here $p_\mu = (\omega, \mathbf{p})$. We will consider the case when the magnetic field is directed along the z axis (i.e., $F_{12} = -B$). The expression for the chiral current has the form:

$$j^{5z} = 4Ti \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{\omega_n}{((\omega_n)^2 + p^2)^2} B, \tag{45}$$

and after integration over 3-momenta we arrive at

$$j^{5z} = 4Ti\pi^2 \sum_{n=-\infty}^{\infty} \mathrm{sign}(\omega_n) B. \tag{46}$$

Thus, formally, in the case of massless fermions the chiral current is equal to the sum of the integer numbers. If, as a result of the interaction in medium, G is changed as $\omega_n \to f(\omega_n), p_i \to g(p_i)$, then the result depends on sign f.

We introduce the chemical potential in the standard way, $\omega_n \to \omega_n - i\mu$. In this case we need an analytical continuation of the sign function. We may try to use, for example, the rule

$$\mathrm{sign}(\omega_n - i\mu) = \mathrm{sign}(\mathrm{Re}(\omega_n - i\mu)). \tag{47}$$

Then, from this naive consideration the conclusion may be drawn that the chemical potential does not influence the CSE current. Below we will see that the formal expressions in the continuum theory will lead to a different answer if the summation over the Matsubara frequencies is performed before the integration over momenta.

B. The integration over 3-momenta after the summation over Matsubara frequencies

In the previous section we have shown that the axial current in an external magnetic field is expressed as the sum of the topological invariants multiplied by the field strength. This expression would become the exact result, but only if the theory does not contain divergences.

We may extract another result from the above expressions. Namely, let us first perform the initial summation over the frequencies, and only after that the integration over momenta,

$$j^{5z} = 4Ti \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{\omega_n - i\mu}{((\omega_n - i\mu)^2 + p^2)^2} B \tag{48}$$

$$= 2 \int_C \frac{dz}{2i\pi} \int \frac{d^3p}{(2\pi)^3} \frac{z}{(z^2 - p^2)^2} \mathrm{th}\left(\frac{z-\mu}{T}\right) B, \tag{49}$$

where C is the contour that surrounds poles of the hyperbolic tangent function. We use the relation

$$\mathrm{res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} f(z)(z-z_0)^m \tag{50}$$

to calculate the value of the integral using the theory of residues. We deform the contour C in such a way that it surrounds the points $z = \pm z_0$. After this deformation we have

$$j^{5z} = -2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{z}{(z+p)^2} \frac{d}{dz} \mathrm{th}\left(\frac{z-\mu}{2T}\right)\Big|_{z=p} + \frac{z}{(z-p)^2} \frac{d}{dz} \mathrm{th}\left(\frac{z-\mu}{2T}\right)\Big|_{z=-p} \right]. \tag{51}$$

We can write the equation in this form because

$$\frac{d}{dz} \frac{z}{(z \mp p)^2}\Big|_{z=\mp p} = 0. \tag{52}$$

Thus we can rewrite the integral as

$$j^{5z} = -2B \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{4p} \frac{d}{dp} \mathrm{th}\left(\frac{p-\mu}{2T}\right) - \frac{1}{4p} \frac{d}{dp} \mathrm{th}\left(\frac{p+\mu}{2T}\right) \right), \tag{53}$$

and after the substitution $d^3p = 4\pi p^2 dp$ we find that

$$j^{5z} = -\frac{B}{2\pi^2} \int dp (n_f(p-\mu) - n_f(p+\mu)) = -\frac{B\mu}{2\pi^2}. \tag{54}$$

This expression coincides with the conventional expression (1) (see, for example, Ref. [17]), and it also coincides with the result obtained above using Wilson fermions and conventional overlap fermions. It is still protected from the renormalization of the 3-momentum [$p_i \to g(p_i)$].

However, although the result of the rigorously regularized theory (using, say, lattice Wilson fermions) is reproduced by the approach of the present subsection, we would like to emphasize once again that this approach itself is not self-consistent, and its application to the other problems may be limited. In particular, let us consider a modification of the system that leads to the replacement of $i\omega_n$ by a function $f(i\omega_n)$ such that it tends to $i\omega_n$ at large n. Looking at Eq. (45), we may come to the conclusion that such a modification cannot change the value of the axial current. However, Eq. (49) is not invariant under the substitution $i\omega_n \to f(i\omega_n)$. This demonstrates once again that the rigorous ultraviolet regularization is needed in order to calculate the response of the axial current to the external field strength.

VI. CONCLUSIONS AND DISCUSSIONS

In the present paper we discussed the chiral separation effect in the framework of both the naive continuum nonregularized quantum field theory and the lattice regularized theory. In both cases we also regularized the theory using finite temperatures.

We demonstrated that the naive continuum formulation suffers from ambiguities related to the order of taking the integral over the 3-momenta and the sum over Matsubara frequencies. If the Matsubara frequencies are summed first, then the divergences are not encountered and the conventional expression for the chiral current in the presence of an external magnetic field is reproduced. At the same time, if the 3-momenta are integrated first, then the resulting expression is given by the sum of Eq. (46), where each term is equal to either 1 or -1. This sum is not well defined, but each term in this sum is independent of the chemical potential.

This ambiguity points out that, although certain computational schemes of the CSE current do not encounter the ultraviolet divergences, the theory should be considered in the ultraviolet regularization in order to obtain rigorous results. Therefore, we considered several types of lattice regularization. First of all, we considered the naive lattice regularization, where 16 doublers represent the independent physical excitations. These excitations differ by the orientation of the effective vierbein, and as a result their contributions to the CSE current cancel each other.

Recently, a modification of the regularization using overlap fermions was proposed (see, for example, Ref. [40]) in which a massless physical excitation appears at $\omega = \mathbf{p} = 0$, while at the positions of the other 15 doublers (of naive lattice fermions) the zeros of the Green function appear. As a result the exact lattice chiral symmetry is obeyed, just like in the case of naive lattice fermions. The physical meaning of the zeros of the Green function remains unclear, but it has been discussed in certain publications (mostly in the framework of condensed matter theory). We demonstrated that the contribution of those zeros of the Green function to the CSE cancels the contribution of the physical massless excitation. Thus, in both considered cases of the lattice theory with exact chiral symmetry the CSE does not appear.

Typically, in the lattice models the exact chiral symmetry is broken, which is the price for the elimination of the fermion doublers. We considered two particular cases of such conventional regularization: the case of lattice Wilson fermions and the case of conventional overlap fermions. In both cases the massless excitation appears at $\omega = \mathbf{p} = 0$ only, the other doublers disappear, and there are no zeros of the Green function. The price for this is the absence of the exact chiral symmetry. However, in the case of overlap fermions there is the Ginsparg-Wilson relation instead. In both of these regularizations we observed the emergence of the chiral separation effect. The corresponding current tends to its conventional expression (1) in continuum limit of the model with massless fermions. In the case when the theory describes massive fermions with mass m, the CSE current is absent at $\mu < m$. It appears at $\mu \geq m$, and is given by the same expression of Eq. (1) in the limit $\mu \gg m$.

Actually, our consideration may easily be extended to the other lattice models, including those with interactions. The necessary condition is the presence of the massless Dirac fermions in continuum limit. Therefore, Eq. (1) should be regularization independent. We took the limit $T \to 0$, which allowed us to substitute the sum over Matsubara frequencies by the integral. This consideration also demonstrates that for the noninteracting system the same answer for the CSE current is obtained at finite temperature. This is because the limit $N_t \to \infty$ implies the transition to the continuum limit, and the appropriate tuning of the lattice spacing a allows to treat the final answer as the axial current at finite temperature, $T = 1/(N_t a)$. However, for the interacting system the situation may be different, and at finite temperatures the corrections to the CSE current may appear [41], an effect that we did not discuss here.

We conclude that in the physical regularizations with Wilson and overlap fermions the conventional CSE emerges. At the same time, we suppose that the model with the modified overlap fermions [40] with exact chiral symmetry [Eq. (38)] is unphysical. Although the zeros of the Green function do not contribute to the ordinary perturbation expansion on the same grounds as the physical excitations, they contribute the topological quantities responsible for the CSE in the same way as the fermion doublers. In this respect the modified overlap fermions with Eq. (38) are similar to the naive lattice fermions with 16 doublers, and they do not possess the CSE.

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APPENDIX: AXIAL CURRENT FOR THE MODIFIED OVERLAP FERMIONS

Here we calculate the chiral current for the version of overlap fermions with exact chiral symmetry, $\{\mathcal{G},\gamma^5\}=0$. Let us use the following expression for the axial current:

$$j^{5k}=-\frac{i}{2}\sum_{n=1}^{N_t}\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}(\gamma^5\mathcal{G}(\omega_n,\mathbf{p})\partial_{p_i}\mathcal{G}^{-1}(\omega_n,\mathbf{p})\times\partial_{p_j}\mathcal{G}(\omega_n,\mathbf{p})\partial_{p_k}\mathcal{G}^{-1}(\omega_n,\mathbf{p}))F_{ij}. \quad (A1)$$

We substitute Eq. (39) into this expression:

$$j^{5k}=-\frac{i}{2}\sum_{n=1}^{N_t}\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left(\gamma^5\frac{\gamma^\mu k_\mu}{k^2f(k^2)}\partial^i(\gamma^\nu k_\nu f(k^2))\times\partial^j\left(\frac{\gamma^\lambda k_\lambda}{k^2f(k^2)}\right)(\partial^k\gamma^\rho k_\rho f(k^2))\right)F_{ij}. \quad (A2)$$

It may be written as

$$j^{5k}=-2i\sum_{n=1}^{N_t}\epsilon^{\rho\mu\nu\lambda}\int_M\frac{d^3p}{(2\pi)^3}\frac{f^2(k^2)k_\mu\partial^ik_\nu\partial^jk_\lambda\partial^kk_\rho}{f^2(k^2)k^4}F_{ij}. \quad (A3)$$

We introduce the notation $g^\mu=\frac{k^\mu}{\sqrt{k^2}}$, and the expression for the axial current is given by

$$j^{5k}=-\frac{2i}{3!(2\pi)^3}\sum_{n=1}^{Nt}\epsilon^{\mu\nu\lambda\rho}\int_M g_\mu dg_\nu\wedge dg_\lambda\wedge dg_\rho\epsilon^{ijk}F_{ij}. \quad (A4)$$

To calculate the topological invariant

$$\mathcal{N}_3(\omega_n)=\frac{1}{12\pi^2}\epsilon^{\mu\nu\lambda\rho}\int_M g_\mu dg_\nu\wedge dg_\lambda\wedge dg_\rho, \quad (A5)$$

we will use the method of Ref. [29] and the following parametrization:

$$g_4=\sin(\alpha),\qquad g_i=k_i\cos(\alpha), \quad (A6)$$

where $i=1,2,3$ and $\sum_ik_i^2=1$, while $\alpha\in[\frac{-\pi}{2},\frac{\pi}{2}]$. Thus

$$dg_4=\cos(\alpha)d\alpha,\qquad dg_i=dk_i\cos(\alpha)-k_i\sin(\alpha)d\alpha, \quad (A7)$$

and

$$\begin{aligned}\mathcal{N}_3&=\frac{3}{12\pi^2}\epsilon^{ijk}\int_M\cos^2(\alpha)k_id\alpha\wedge dk_j\wedge dk_k\\&=\frac{3}{12\pi^2}\epsilon^{ijk}\int_Mk_i\left(\frac{1-\cos(2\alpha)}{2}\right)d\alpha\wedge dk_j\wedge dk_k\\&=\frac{3}{12\pi^2}\epsilon^{ijk}\int_Mk_id\left(\frac{\alpha}{2}+\frac{1}{4}\sin(\alpha)\right)\wedge dk_j\wedge dk_k\\&=-\sum_l\frac{3}{12\pi^2}\epsilon^{ijk}\int_{\partial\Omega}k_i\left(\frac{\alpha}{2}+\frac{1}{4}\sin(\alpha)\right)dk_j\wedge dk_k.\end{aligned} \quad (A8)$$

In this expression $\partial\Omega$ is the small vicinity of the point y_l of momentum space where the vector k_l is undefined. The absence of the singularities of g_k implies that $\alpha\to\pm\frac{\pi}{2}$ at such points.

Thus we see that the expression under the integral is the total derivative. We can rewrite it in the form

$$\mathcal{N}_3=-\frac{1}{2}\sum_l\mathrm{sign}(g_4(y_l))\mathrm{Res}(y_l), \quad (A9)$$

where we have used the notation [30]

$$\mathrm{Res}(y_l)=\frac{1}{8\pi}\epsilon^{ijk}\int_{\partial\Omega}g_idg_j\wedge dg_k. \quad (A10)$$

It is worth mentioning that this symbol obeys $\sum_l\mathrm{Res}(y_l)=0$. At each n the value of sign g_4 is constant. Therefore, $\mathcal{N}_3(\omega_n)=0$ for any n.

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