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Composite operators in asymptotic safety

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We study the role of composite operators in the asymptotic safety program for quantum gravity. By including in the effective average action an explicit dependence on new sources, we are able to keep track of operators which do not belong to the exact theory space and/or are normally discarded in a truncation. Typical examples are geometric operators such as volumes, lengths, or geodesic distances. We show that this setup allows us to investigate the scaling properties of various interesting operators via a suitable exact renormalization group equation. We test our framework in several settings including quantum Einstein gravity, the conformally reduced Einstein-Hilbert truncation, and two-dimensional quantum gravity. Finally, we briefly argue that our construction paves the way to approach observables in the asymptotic safety program.

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I. INTRODUCTION

The construction of a well-defined path integral for quantum gravity is at the heart of the asymptotic safety (AS) program [1,2]. Ultimately, however, the construction of this path integral should allow the evaluation of observable quantities, typically expectation values of composite operators. In this work, we make a first step towards the latter goal. So far, the focus of most of the investigations regarding AS has been devoted to probing the existence of a suitable UV fixed point. The framework employed in these investigations involves the effective average action (EAA) and its renormalization group (RG) [3]. In this setting, truncations of increasing complexity have been analyzed including bimetric *Ansätze*, higher derivative terms, and infinite-dimensional truncations; see [4–11] for some of the most recent works.

Our aim in the present work is different. Let us suppose that we have a reasonably well-approximated gravitational EAA. Can we extract all possible information from the EAA alone? We wish to argue that this is not always the case; there are instances in which further efforts are required.

Before entering into the technical aspects, we would like to recall the special status of quantum gravitational theories with respect to nongravitational ones. In particular, observables in a quantum gravitational theory are required to be diffeomorphism invariant, i.e., gauge invariant. In turn, this implies that we cannot think of an observable as depending on a point of the spacetime manifold, since a diffeomorphism transformation would change it. Instead, one may consider quantities integrated over all spacetime. However, such quantities are rather distant from our intuition, which is trained to think in terms of "localized" quantities.

A possible way out of this conceptual dilemma is to recall that in performing a measurement, we actually check for the coincidence of events, like, for instance, that a photon hits our experimental apparatus. The fact that the photon hits the detector is invariant under diffeomorphisms since the statement that the photon and detector are at the same spacetime point remains true after a diffeomorphism transformation is applied.

To implement a consistent description of the system plus the apparatus in the field theoretic language is not an easy task. Following DeWitt [12], one may modify the action functional via $S \to S + \varepsilon A$, where the last term describes the coupling of the system to the detector. As a result, we observe that purely at the quantum field theoretic level, information regarding the new operator A is required. However, in general, this information is not encoded automatically in the EAA.

To properly define observables in quantum gravity also, other approaches have been considered. For instance, one may use scalar fields to localize observables or define correlation functions at fixed geodesic distance; we refer the reader to [12–14] for more details.

With regard to the gravitational EAA formalism, all these approaches have a common feature: they require information about operators which usually are not taken into account in a truncated EAA, at any realistic level of complexity. For instance, it is hard to imagine a truncation for the gravitational EAA to contain information on the geodesic distance of two given points on the spacetime manifolds, a quantity that appears in many observables of practical interest, however [15,16].

In order to obtain information regarding an arbitrary operator in a quantum field theoretic framework, one can couple it to an external source so that it can be inserted into correlation functions by taking functional derivatives with respect to the source. This formalism goes under the name of the *composite operator formalism*. It allows us to define and actually compute correlation functions of not only elementary fields but also of more complicated local operators at a given spacetime point. The main task of this work is to investigate the composite operator formalism and its application within the framework of the gravitational EAA.

The introduction of composite operators is unavoidable also in many other cases. For example, let us consider the correlation function between metric operators at different points in the vielbein formalism. In this case, the metric itself is a composite operator which can be meaningfully defined only via a suitable regularization and renormalization procedure over and above the usual renormalization of couplings in the EAA.

In the present work, we are going to provide the basic framework to properly define this type of operator in the EAA formalism, and we consider some explicit examples that occur in the AS context.

This paper is organized as follows. In Sec. II, we revisit an argument which allows us to define the scaling dimensions of operators straightforwardly in the EAA framework. In Sec. III, we include composite operators into the EAA by coupling them to an external source, discuss possible approximations, and show how to compute the scaling properties of these composite operators.

In Sec. IV, we consider the conformally reduced Einstein-Hilbert (CREH) truncation, a simple model which mimics many features of quantum Einstein gravity (QEG). In this setting, the metric is parametrized by a dynamical conformal factor times a fixed reference metric. The conformal factor is actually a composite operator of the elementary quantum field, and so the metric in the CREH setting can be thought of as a toy model for the composite metric of the vielbein formalism. We discuss the definition of the metric as a composite operator in this framework.

In Sec. V, we investigate the scaling properties of two geometrical objects within QEG: the volume and the length of curves.

Finally, in Sec. VI, we study various composite operators in two-dimensional quantum gravity. The two-dimensional case is interesting for various reasons. First, there is a variety of results coming from other approaches and techniques, such as conformal field theory, to which one may compare the findings given by our framework. Second, two-dimensional asymptotic safety has been recently investigated in detail [17], and, among other things, it has been possible to test the compatibility between the presence of a non-Gaussian fixed point and unitarity in this context. Thus, it is natural to ask what kinds of consequences such a fixed point bears for geometrical objects like the volume operator or the length of a curve. Furthermore, in the Appendix we show as an example how our approach leads to the familiar Knizhnik-Polyakov-Zamolodchikov (KPZ) scaling relations for gravitationally dressed operators.

II. SCALING ARGUMENTS AND FUNCTIONAL RENORMALIZATION GROUP

The EAA is a scale-dependent generalization of the standard effective action [3]. One introduces a scale k below which the integration of momentum modes is suppressed. This is achieved by adding the cutoff term $\Delta S_k = \frac{1}{2}\int \chi \mathcal{R}_k \chi$ to the bare action, with $\mathcal{R}_k$ being a suitable kernel. The scale dependence of the effective average action satisfies the following exact functional RG equation or "FRGE" [3]

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\partial_t \mathcal{R}_k], \tag{2.1}$$

where $\Gamma_k^{(2)}$ is the Hessian of the effective average action Γ_k, and $t \equiv \log k$. This equation can be concretely employed after implementing some approximation scheme.

In this section, we briefly review an argument which allows us to deduce the scaling properties of any operator in the EAA formalism [18]. First let us note that to uniquely solve the flow equation (2.1), a boundary condition must be given. Such a boundary condition is imposed at a certain scale μ, which we call the *floating normalization point*. The dependence of the EAA on the scale μ has been studied in detail in [18], and in this section, we shall revisit the dependence in the framework of the gravitational EAA. In particular, we shall see that the EAA is invariant under suitable changes of the boundary condition. Such invariance properties allow one to write down an equation fully analogous to the Callan-Symanzik equation of standard quantum field theory. This equation, together with simple dimensional analysis, allows one to discuss the scaling properties of the theory straightforwardly.

Let us consider a theory space parametrized by a set of dimensionless couplings $\{\tilde{g}_i\}$. The RG flow is described by a system of differential equations:

$$\partial_t \tilde{g}_i = f_i(\{\tilde{g}_j\}), \tag{2.2}$$

to which one associates boundary conditions like[1]

$$\tilde{g}_i(\mu) = \tilde{g}_i^{(\mathrm{R})}, \tag{2.3}$$

where the "renormalized" couplings $\tilde{g}_i^{(\mathrm{R})}$ are given numbers. By imposing a boundary condition, we select a specific trajectory on theory space. Let us denote this solution by $\tilde{g}_i^{(\mathrm{sol})}(k;\mu,\tilde{g}_i^{(\mathrm{R})})$, where we made explicit its dependence on the boundary values $\tilde{g}_i^{(\mathrm{R})}$ and the scale μ.

Clearly, if one chooses another set of boundary values which, however, still correspond to some point along this trajectory, then the solution of the flow equation will be the very same trajectory again. To cast this simple fact into a mathematical formula, let us consider the specific solution of Eq. (2.2) associated with the boundary condition (2.3). Now we want to change the boundary condition (2.3) to an equivalent boundary condition along the trajectory; i.e., we move μ to some other scale μ' and change the couplings

[1] For lack of a better word, we refer to it as a "boundary" condition even if μ is an inner point of the k interval under consideration.

accordingly. This is achieved by infinitesimally translating $\mu \to \mu' = \mu + \varepsilon$ and $\tilde{g}_i^{(\mathrm{R})} = \tilde{g}_i(\mu) \to \tilde{g}_i^{(\mathrm{R})\prime} = \tilde{g}_i(\mu') = \tilde{g}_i(\mu) + \varepsilon\partial_\mu \tilde{g}_i(\mu)$. The fact that these two boundary conditions are associated to the same solution implies that

$$\begin{aligned}\tilde{g}_i^{(\mathrm{sol})}(k;\mu,\tilde{g}_i^{(\mathrm{R})}) &= \tilde{g}_i^{(\mathrm{sol})}(k;\mu',\tilde{g}_i^{(\mathrm{R})\prime}) \\ &\cong \tilde{g}_i^{(\mathrm{sol})}(k;\mu,\tilde{g}_i^{(\mathrm{R})}) \\ &\quad + \varepsilon\left(\partial_\mu + \partial_\mu \tilde{g}_j(\mu)\frac{\partial}{\partial \tilde{g}_j^{(\mathrm{R})}}\right)\tilde{g}_i^{(\mathrm{sol})}(k;\mu,\tilde{g}_i^{(\mathrm{R})}).\end{aligned}$$

As a consequence, it follows that

$$\boxed{\left(\mu\partial_\mu + \beta_j \frac{\partial}{\partial \tilde{g}_j^{(R)}}\right)\tilde{g}_i^{(\mathrm{sol})}(k;\mu,\tilde{g}_i^{(\mathrm{R})}) = 0,} \tag{2.4}$$

where $\beta_j \equiv \beta_j(\tilde{g}_i^{(\mathrm{R})})$. The same reasoning straightforwardly applies to the entire EAA. Thereby, a wave function renormalization Z_k can be conveniently introduced considering *Ansätze* of the following form[2]:

$$\Gamma_k[\varphi] = \sum_{i=1}^{n} g_i O_i(Z_k^{1/2}\varphi).$$

Here we made explicit the inessential nature of Z_k. The anomalous dimension of the elementary field φ corresponds to $\eta \equiv -Z_k^{-1}\partial_t Z_k$; see [18] for a detailed discussion. One, thus, has an equation which is fully similar to the *Callan-Symanzik equation*:

$$\boxed{\left(\mu\partial_\mu + \beta_j \frac{\partial}{\partial \tilde{g}_j^{(R)}} - \eta\varphi\cdot\frac{\delta}{\delta\varphi}\right)\Gamma_k[\varphi] = 0.} \tag{2.5}$$

Equation (2.5) can be used to deduce scaling properties of correlation functions at a fixed point. To do so, one considers Eq. (2.5) together with a Euler-type differential equation (homogeneity relation) which stems from dimensional analysis.

As an example, let us consider the propagator of a scalar field with mass dimension $[\varphi(x)] = \frac{d-2}{2}$. In the fixed point regime, with $\Gamma = \Gamma_{k\to 0}$ and $\Gamma^{(2)} \equiv \frac{\delta^2\Gamma}{\delta\varphi(q)\delta\varphi(p)}$, we obtain from Eq. (2.5) and dimensional analysis (see [18] for details):

$$\begin{cases} [\mu\partial_\mu - \eta]\Gamma^{(2)} &= 0 \\ [\mu\partial_\mu + p\partial_p + q\partial_q + (d-2)]\Gamma^{(2)} &= 0 \end{cases}. \tag{2.6}$$

[2]Since the total number of running couplings (including the wave function renormalization constant) should be n, we set $g_i = 1$ for some i. For instance, in the case of a scalar field theory, one usually chooses to write the kinetic term $\frac{1}{2}Z_k(\partial\varphi)^2$, with no coupling g_i in front.

Now we eliminate the $\mu\partial_\mu$ term from these two equations. It is convenient to take into account the overall delta function entailing momentum conservation, which has mass dimension $-d$. In particular, we define $\Gamma^{(2)} \equiv \delta(p+q)f(p)$ and obtain

$$[p\partial_p - (2-\eta)]f(p) = 0. \tag{2.7}$$

Remarkably, we note that Eq. (2.7) just derives from the (here assumed) existence of a fixed point. Different fixed point propagators are distinguished by the different values of the anomalous dimension.

One can repeat the same logic for the graviton propagator. Let us remark that in the case of gravitational theories, one can either consider the coordinates dimensionful and the metric dimensionless or vice versa. Either way, the above reasoning leads straightforwardly to a propagator of the type p^4, as it has been already noted in [19–21]. Such propagator can be viewed as a two-dimensional propagator hinting to a dimensional reduction phenomenon [21–23]. The computation of the anomalous dimension in the spirit mentioned above (leading to a propagator of the type $p^{4-\eta}$) has been performed in very few truncations; see, for instance, [4,5,24].

As far as composite operators are concerned, the argument outlined in this section can be straightforwardly generalized and allows us to identify their scaling dimensions. In Sec. III, we shall define the scaling dimension of composite operators and see how they can be estimated.

III. COMPOSITE OPERATORS IN THE FUNCTIONAL RENORMALIZATION GROUP AND ASYMPTOTIC SAFETY

In the functional integral formulation of standard quantum field theory, one deals with composite operators by coupling them to external sources so that one obtains insertions of composite operators in correlation functions by taking suitable functional derivatives of the path integral [25]. In the effective average action formalism, this step is not often made, one of the reasons being that frequently one is interested in the properties of a system at a fixed point, which one describes by the critical exponents associated to the couplings $\{g_i^*\}$ of the operators present in the truncation, O_i. However, there are several situations in which one may wish to couple some operators to their respective sources and carry out the associated renormalization procedure.

First of all, in order to solve Eq. (2.1), a truncation *Ansatz* is typically used. This is one possible reason why not "all" operators are present in the *Ansatz* for the EAA. If one was interested in the scaling of an operator O, which, for any reason, is not present in the truncation *Ansatz* for the EAA, a procedure analogous to the one adopted in standard quantum field theory is very helpful and gives a first estimate of the scaling properties of the operator.

Moreover, there are operators which one is not able to treat directly even in a full-fledged EAA. As an example, let us consider the metric in the vielbein formalism, i.e., $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. If e^a_μ is taken to be an elementary field (possibly together with the spin connection as in the Riemann-Cartan theory), then $g_{\mu\nu}(x)$ is neither an elementary field, it is an operator product of two fields, nor it is an invariant built from elementary fields; i.e., it is not contained even in the exact theory space. In this case, the metric is a composite operator of spin two, and in order to define meaningful correlation functions of the metric, one needs to regularize and renormalize the operator $g_{\mu\nu}$. This begins with coupling $g_{\mu\nu}$ to a spin-two source. Similar considerations hold for many other interesting operators as we shall see later on.

Let us review how one can deal with composite operators in the functional renormalization framework. We denote $\varepsilon(x)$ the source and consider the expectation value[3]

$$\langle O(x)\rangle = \mathcal{N}\int \mathcal{D}\chi O(x) e^{-S} = -\frac{\delta}{\delta\varepsilon(x)}\mathcal{N}\int \mathcal{D}\chi e^{-S-\varepsilon\cdot O}\Big|_{\varepsilon=0},$$

where $\mathcal{N}$ is a suitable normalization constant. Then we define the generating functional $W[J,\varepsilon]$ for the connected Green's functions associated to the modified action $S+\varepsilon\cdot O$:

$$e^{W[J,\varepsilon]} \equiv \int \mathcal{D}\chi e^{-S-\varepsilon\cdot O+J\cdot\chi}.$$

The associated effective action is obtained via a Legendre transform

$$\Gamma[\varphi,\varepsilon] = J\cdot\varphi - W[J,\varepsilon], \qquad \varphi = \frac{\delta W}{\delta J}.$$

It is straightforward to check that

$$\frac{\delta\Gamma}{\delta\varepsilon}[\varphi,\varepsilon] = -\frac{\delta W}{\delta\varepsilon}[J,\varepsilon],$$

which tells us that we can extract the renormalization regarding a single insertion of a composite operator directly considering a single functional derivative with respect to $\varepsilon(x)$ of the EAA. One can repeat the derivation of the FRGE in the case of $\Gamma_k[\varphi,\varepsilon]$. From its ε derivative, we find the following exact flow equation associated to the composite operator [18,26,27]:

$$\partial_t\left(\frac{\delta}{\delta\varepsilon}\Gamma_k[\varphi,\varepsilon]\right)\Big|_{\varepsilon=0} = -\frac{1}{2}\mathrm{Tr}[(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}\frac{\delta\Gamma^{(2)}_k}{\delta\varepsilon}(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}\partial_t\mathcal{R}_k]|_{\varepsilon=0}.$$

We can avoid performing the functional derivative with respect to ε and just compare order by order in ε. Clearly, since we are interested just in a single insertion of the composite operator, we can limit ourselves to consider the case where $\varepsilon^2 = 0$. Furthermore, we denote

$$[O_k]_i \equiv \frac{\delta}{\delta\varepsilon_i}\Gamma_k[\varphi,\varepsilon_j],$$

where k indicates the RG scale, and the subscript i labels n different composite operators. We can rewrite the flow equation for composite operators as [18]

$$\boxed{\partial_t(\varepsilon\cdot[O_k]) = -\frac{1}{2}\mathrm{Tr}[(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}(\varepsilon\cdot[O_k]^{(2)})(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}\partial_t\mathcal{R}_k].} \tag{3.1}$$

To concretely solve Eq. (3.1), some approximation must be implemented. In particular, one may expand the composite operator $[O_k]_i$ in a basis of k-independent operators $\{O_i, i=1,\ldots,n\}$. In this case,

$$[O_k]_i = \sum_{j=1}^{n} Z_{ij}(k) O_j. \tag{3.2}$$

By following the reasoning of Sec. II, one can show that the scaling operators of the theory have dimensions, quantum corrections included, given by the eigenvalues of the matrix

$$d_i\delta_{ij} + (Z^{-1}\partial_t Z)_{ij}, \tag{3.3}$$

where d_i is the (classical) mass dimension of the operator O_i [18].

The crucial matrix $\gamma_{Z,ij} \equiv (Z^{-1}\partial_t Z)_{ij}$ can be directly found manipulating Eq. (3.1). Inserting the *Ansatz* (3.2) and taking a functional derivative with respect to ε_i, we find

$$\sum_j \partial_t(Z_{ij}O_j(x)) = -\frac{1}{2}\mathrm{Tr}\left[(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}\left(\sum_j Z_{ij}O^{(2)}_j(x)\right)\times(\Gamma^{(2)}_k+\mathcal{R}_k)^{-1}\partial_t\mathcal{R}_k\right],$$

which implies the final result for the general case with operator mixing,

[3]Whenever a dot appears in a mathematical expression, e.g., $f\cdot g$, the DeWitt condensed notation is used, meaning that integration and index summation is intended.

$$\sum_j \gamma_{Z,ij} O_j(x) = -\frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}(O_i^{(2)}(x))(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\partial_t \mathcal{R}_k]. \tag{3.4}$$

In the present work, we shall mainly limit ourselves to nonmixing *Ansätze* for the composite operators. This means that we shall consider composite operators approximated by the simple parametrization $[O_k] = Z_O(k)O$. Such an operator acquires an anomalous dimension given by $Z_O^{-1}\partial_t Z_O$, which can be read off from

$$\boxed{\gamma_{Z_O} O(x) = -\frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}(O^{(2)}(x))(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\partial_t \mathcal{R}_k].} \tag{3.5}$$

For the sake of comparison with other results in the literature, it is useful to work out the relation between scaling operators defined by means of explicit introduction of the sources and those found by linearizing the RG flow around the fixed point. Let us consider an *Ansatz* for the EAA expanded in the basis of operators O_i:

$$\Gamma_k = \sum_{i=1}^{n} g_i(k) O_i.$$

Here we consider the basis of operators O_i to be the same that we used previously for the composite operators. Under these approximations, it is straightforward to conclude from the flow equation that

$$\sum_{j=1}^{n} \beta_j O_j = \frac{1}{2}\mathrm{Tr}\left[\left(\sum_{j=1}^{n} g_j O_j^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right].$$

Taking a derivative with respect to the coupling g_i, we find

$$\sum_{j=1}^{n} \partial_{g_i}\beta_j O_j = -\frac{1}{2}\mathrm{Tr}\left[\left(\sum_{j=1}^{n} g_j O_j^{(2)} + \mathcal{R}_k\right)^{-1} \times O_i^{(2)} \left(\sum_{j=1}^{n} g_j O_j^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right]. \tag{3.6}$$

Comparing (3.4) with (3.6) we conclude that at the dimensionful level,

$$\partial_{g_i}\beta_j = \gamma_{Z,ij},$$

which can be rewritten in terms of dimensionless couplings $\tilde{g}_j$ as

$$K_{ia}(d\delta_{ab} + \partial_{\tilde{g}_a}\tilde{\beta}_b)K_{bj}^{-1} = d_i\delta_{ij} + \gamma_{Z,ij}, \tag{3.7}$$

where d is the spacetime dimension and $K_{ij} \equiv k^{d_i}\delta_{ij}$. Thus, under these approximations, the scaling dimensions found by diagonalizing the matrix $d_i\delta_{ij} + \gamma_{Z,ij}$ are exactly the same as those found by linearizing the RG flow and diagonalizing $d\delta_{ij} + \partial_{\tilde{g}_i}\tilde{\beta}_j$. (Recall also that the negative eigenvalues of $\partial_{\tilde{g}_i}\tilde{\beta}_j$ are the fixed point's critical exponents θ_i.)

The usefulness of adopting the composite operator point of view is that there may be cases in which some operators are not included in a truncation, but one would like to have information about their renormalization and scaling properties. As far as the asymptotic safety scenario is concerned, an interesting example is given by the scaling properties of the length of curves, and geodesics, in particular, which usually are not considered as a part of the EAA. Of course, in order to explore gravitational observables, further efforts are required since one needs to identify suitable diffeomorphism invariant operators. Possibly, this can be achieved by having at our disposal further fields which allow us to "localize" quantities in spite of an overall integration over the manifold; see [12,13] for a detailed description. In this work, we shall not pursue this approach further but simply consider the renormalization of possibly interesting composite operators.

Finally, we note that scaling properties of correlation functions involving certain suitable composite operators are also essential in order to compare different approaches to two-dimensional quantum gravity [28,29]. Possibly, one may find similar comparisons between four-dimensional asymptotic safety and other approaches to 4D quantum gravity, like Causal Dynamical Triangulations (CDT), for example. This is a further motivation for the present investigation.

IV. COMPOSITE METRICS IN THE CREH TRUNCATION

In this section, we consider the CREH truncation and evaluate the scaling properties of various operators in this setting. Interestingly, in the CREH truncation, the metric is a composite operator, and, therefore, this framework constitutes an instructive toy model to see which types of computations are required in more-refined cases, such as the composite metric in the vielbein formalism. In Sec. IV A, we briefly recall the CREH truncation, and in Sec. IV B, we treat the composite metric operator in the CREH truncation by means of two different approaches that will turn out equivalent in the end.

A. The CREH action

The CREH truncation is inspired by the classical action functional

TABLE I. The composite conformal factors and volume operators for the distinguished parametrizations in various dimensions.

d	3	4	6
Conformal factor	ϕ^4	ϕ^2	ϕ
Volume operator	ϕ^6	ϕ^4	ϕ^3

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^dx\sqrt{g}(2\Lambda - R) \tag{4.1}$$

evaluated for arguments $g_{\mu\nu}$ which are given by a dynamical conformal factor times a fixed reference metric $\hat{g}_{\mu\nu}$:

$$g_{\mu\nu} = \phi^{2\nu(d)}\hat{g}_{\mu\nu}. \tag{4.2}$$

The conformal factor is written as a power of the elementary dynamical field ϕ, the choice for the exponent being

$$\nu(d) \equiv \frac{2}{d-2}.$$

The exponent 2ν is the integer only in the special dimensions $d = 3$, $d = 4$, and $d = 6$, respectively. (See Table I.) The distinguished parametrization of the conformal factor in (4.2) has the "miraculous" property that, with this choice, the restricted Einstein-Hilbert action $S[\phi] \equiv S[\phi^{2\nu}\hat{g}]$ has a standard quadratic kinetic term for ϕ. The only self-interactions of ϕ are due to the cosmological constant then.

Furthermore, allowing the cosmological and the Newton constants in $S[\phi]$ to be scale dependent ($\Lambda \to \Lambda_k$, $G \to G_k$), this functional reads

$$\Gamma_k[\phi] = -\frac{1}{8\pi\xi(d)G_k}\int d^dx\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\xi(d)\hat{R}\phi^2 - \xi(d)\Lambda_k\phi^{\frac{2d}{d-2}}\right), \tag{4.3}$$

with $\hat{R}$ the curvature scalar of $\hat{g}_{\mu\nu}$, and

$$\xi(d) \equiv \frac{d-2}{4(d-1)}.$$

We shall refer to the action (4.3) as the CREH *Ansatz* for the EAA of conformally reduced gravity.

Despite its simplicity, this model captures many features of full-fledged truncations in QEG with all the modes of the metric retained. In particular, the RG flow is qualitatively identical to that of full QEG displaying, in particular, a nontrivial fixed point (NGFP). It has been studied in detail in [11,30–36].

Note that when the cosmological constant is negligible ($\Lambda_k = 0$), and, correspondingly, we choose a flat background ($\hat{g}_{\mu\nu} = \delta_{\mu\nu}$, $\hat{R} = 0$), the CREH action (4.3) reduces to $\Gamma_k \propto \int(\partial_\mu\phi)^2$. So one could think that we are dealing "only with a free theory" which has no interesting renormalization behavior. But clearly, this is false: In the model at hand, even the most basic operator of physical interest, namely, $g_{\mu\nu}$, is a nontrivial composite operator of the elementary quantum field, ϕ. Hence, there is a large class of physically relevant renormalization effects, namely, those related to operator products, which are not reflected by the running of the EAA in any way.

As a final note, let us remark that the CREH action carries some crucial differences with respect to seemingly similar scalar models used in statistical field theory. We defer a detailed discussion of such differences to the literature [11,30–36]. Let us just remark that different from normally considered statistical field theories, the CREH action has a "wrong sign" kinetic term. Moreover, a full-fledged treatment of the CREH action requires the introduction of an associated scalar background field, which we avoided in our simple setting.

B. Composite metric operators

When using the parametrization (4.2), the metric $g_{\mu\nu}$ becomes proportional to a power of the dynamical, i.e., quantum, field ϕ. Thus, the metric $g_{\mu\nu}$ is a composite operator[4] which must be dealt with by a suitable renormalization procedure.

To see why such a renormalization is needed, let us consider $d = 4$ dimensions where we have $g_{\mu\nu} = \phi^2\hat{g}_{\mu\nu}$. This poses the problem of defining the composite operator ϕ^2. It is instructive to consider explicitly the correlation function $\langle\phi(x)\phi(y)\rangle$ in the EAA formalism and to explore how this two-point function becomes ill-defined in the limit $y \to x$. To properly define $\lim_{x\to y}\langle\phi(x)\phi(y)\rangle$, we shall need a further regularization scheme, this time for the UV, besides the mode suppression built into the EAA. The pertinent (re)normalization procedure will yield the meaningful composite operator ϕ^2 then.

Regarding the connected two-point function $\langle\phi(x)\phi(y)\rangle$, in the EAA formalism, it is most conveniently obtained from the inverse of the Hessian of $\Gamma_k[\phi] + \Delta S_k[\phi] \equiv \tilde{\Gamma}_k[\phi]$:

$$\langle\phi(x)\phi(y)\rangle = \langle x|\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}|y\rangle. \tag{4.4}$$

We assume that we solved the flow equation and found some RG trajectory along which we follow the evolution of the two-point function. For simplicity's sake, we focus on the classical regime of the RG trajectory where we can approximate $G_k = \text{const} \equiv G$, and in addition, we suppose that the cosmological constant can be neglected, $\Lambda_k = 0$. Choosing the flat reference metric $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$, we obtain then in $d = 4$,

[4]For the case $d = 6$ where $g_{\mu\nu} = \phi\hat{g}_{\mu\nu}$ happens to be linear in the quantum field and is special, see [17] for a discussion on this point.

$$\langle\phi(x)\phi(y)\rangle = \langle x|\frac{1}{(-\frac{3}{4\pi G})(-\Box+\mathcal{R}_k(-\Box))}|y\rangle$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{1}{(-\frac{3}{4\pi G})(p^2+\mathcal{R}_k(p^2))}e^{ip(x-y)}. \tag{4.5}$$

Clearly, if we set $x = y$, the above integral diverges, and the limit $\lim_{x\to y}\langle\phi(x)\phi(y)\rangle$ is undefined. In order to arrive at an expression with more regular properties, we consider the RG running of the two-point function and take the limit $x \to y$ only at the level of its scale derivative, which turns out well defined. Differentiating (4.5), we see that the running of the two-point function is given by

$$\partial_t\langle\phi(x)\phi(y)\rangle = \partial_t\langle x|\frac{1}{\tilde{\Gamma}_k^{(2)}}|y\rangle$$
$$= -\langle x|\frac{1}{\tilde{\Gamma}_k^{(2)}}(\partial_t\tilde{\Gamma}_k^{(2)})\frac{1}{\tilde{\Gamma}_k^{(2)}}|y\rangle$$
$$= -\langle x|\frac{1}{\tilde{\Gamma}_k^{(2)}}(\partial_t R_k)\frac{1}{\tilde{\Gamma}_k^{(2)}}|y\rangle. \tag{4.6}$$

Thus, with our approximations $\Lambda_k \approx 0$ and $\partial_t G_k \approx 0$, one obtains

$$\partial_t\langle\phi(x)\phi(y)\rangle = \frac{4\pi G}{3}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{(p^2+\mathcal{R}_k(p^2))^2}\partial_t\mathcal{R}_k(p^2). \tag{4.7}$$

We immediately notice that the function (4.7) is well defined in the limit $x \to y$ thanks to the presence of the k derivative of the cutoff kernel $\mathcal{R}_k$. For example, employing the optimized cutoff [37], one finds explicitly,

$$\partial_t\langle\phi(x)\phi(y)\rangle = \frac{8\pi G}{3}k^2F(k|x-y|), \tag{4.8}$$

with the function F defined by

$$F(k|x-y|) \equiv \int \frac{d^4q}{(2\pi)^4}e^{iq_\mu k(x-y)^\mu}\theta(1-q^2). \tag{4.9}$$

In principle, we can now solve for the evolution equation (4.8) and obtain the k dependence of the two-point function at arbitrary points x and y.

In order to find the composite operator of interest, we set $y = x$ in (4.8) and obtain

$$\partial_t\langle\phi(x)^2\rangle = \frac{8\pi G}{3}k^2F(0) = \frac{1}{12\pi}Gk^2.$$

Integration leads to the following running correlation function of the composite operator ϕ^2:

$$\langle\phi(x)^2\rangle_k - \langle\phi(x)^2\rangle_0 = \frac{1}{24\pi}Gk^2.$$

Recalling $g_{\mu\nu} = \phi^2\hat{g}_{\mu\nu}$ and denoting $\langle\phi(x)^2\rangle_0 \equiv \tau$, we have the final result

$$\boxed{\langle g_{\mu\nu}\rangle_k = \left(1+\frac{1}{\tau}\frac{1}{24\pi}Gk^2\right)\langle g_{\mu\nu}\rangle_0.} \tag{4.10}$$

This simple example makes it quite obvious that, in general, the exploration of the predictions from the same theory requires much more than merely the scale dependence of the couplings in the (truncated) EAA, the reason being that there are physically relevant operators which are not elements of the theory space the EAA lives in, either as a consequence of a truncation, or even at the exact level. As we shall see, this complication is particularly acute in quantum gravity because of the complicated nature of the observables.

The reader may wonder why we considered the equation for the running two-point function instead of using directly the "master equation" (3.1). Indeed, as we shall see in a moment, the same results can be obtained using Eq. (3.1). Employing the two-point function is an instructive alternative though. It may turn out to be cumbersome, however, when considering different operators like ϕ^4 that would require us to consider the coincident limit of a four-point function.

Now let us turn to the master equation (3.1) and find the running of the composite operator ϕ^2. A simple one-loop computation yields

$$\partial_t[\phi^2(x)] = -\int\frac{d^4p}{(2\pi)^4}\frac{1}{\Gamma_k^{(2)}+\mathcal{R}_k}$$
$$\times\frac{1}{\Gamma_k^{(2)}+\mathcal{R}_k}\partial_t\mathcal{R}_k\Big|_{\hat{g}_{\mu\nu}=\delta_{\mu\nu},-\Box\to p^2}. \tag{4.11}$$

Here the factor $1/2$ in the rhs of (3.1) got canceled by the factor 2 coming from the Hessian of ϕ^2. We observe that Eq. (4.11) is equivalent to Eq. (4.7) in the limit $y = x$ once the truncation (4.3) is used. This equivalence, however, is no longer there if one goes beyond the one-loop approximation, simply because these two procedures define different schemes according to which one can renormalize ϕ^2.

Summarizing, this computation shows how one can properly define a composite metric in the FRG framework, using the CREH truncation as an example. A similar reasoning will be applied in the following sections to other composite operators.

V. GEOMETRIC OBSERVABLES: VOLUME AND LENGTH OPERATORS

In this section, we consider the scaling behavior of two geometrical objects: (1) the volume operator $V \equiv \int d^dx\sqrt{g}$, a quantity that one is naturally led to consider as a first possible observable in quantum gravity, and (2) the length of an arbitrary curve. Interestingly, the scaling properties of geometric observables, like the volume and the length, play a central role in the description of 2D quantum gravity and have been widely explored [28,29]. In this section, we consider these geometrical objects in the AS scenario in dimension $d > 2$. We postpone the two-dimensional case to Sec. VI.

In Secs. V A and V B, we study the volume operator in the CREH truncation and in the full-fledged Einstein-Hilbert truncation, respectively. Then, in Sec. V C, we investigate the length of a given spacetime curve in the Einstein-Hilbert truncation.

A. Volume operator in the CREH approximation

As we have already seen, in the conformally reduced setting, the metric is a composite operator. Thus, any operator O depending on the metric is also a composite operator.

In $d > 2$ dimensions, we have the volume element

$$\sqrt{g} = \phi^{d\nu(d)}\sqrt{\hat{g}}.$$

The exponent $d\nu(d)$ is noninteger except in the dimensions reported in Table I. In two dimensions, the exponential parametrization is the distinguished one leading to a free kinetic term and, thus, takes the place of the power-type dependence $\propto \phi^{2\nu(d)}$ [17]; we shall consider the relevant composite operator in Sec. VI.

We have evaluated the anomalous dimensions of the volume operators with integer exponents listed in Table I via Eq. (3.1), i.e., those for $d = 3$, 4, and 6. The calculation makes essential use of Eq. (3.5), and it parallels those described in the previous sections, so that it suffices to comment on the results.

First, let us consider the case $d = 4$, for which $\sqrt{g} = \phi^4\sqrt{\hat{g}}$. The anomalous dimension of the volume operator can be computed expanding the rhs of Eq. (3.5) up to ϕ^4. It is easy to observe that the flow equation induces mixing with infinitely many other operators. We consider a simple nonmixing *Ansatz*, namely, $[\phi^4] = Z_{\phi^4}\phi^4$, and compute the anomalous dimension following the discussion of Sec. III. The Hessian of the action (4.3) in four dimensions reads

$$\Gamma_k^{(2)} = \left(-\frac{3}{4\pi G_k}\right)\left(-\hat{\Box} + \frac{\hat{R}}{6} - 2\Lambda_k\phi^2\right).$$

Inserting the Hessian $\Gamma_k^{(2)}$ and the Hessian of the operators $[\phi^4]$ in Eq. (3.5), one can read off the anomalous dimension. The correction to the classical scaling dimension associated to the volume operator can be found in Table II together with the cases for $d = 3$ and $d = 6$.

From Table II, we note that the anomalous dimensions are proportional to Newton's constant and a certain power of the cosmological constant that renders γ_V dimensionless. The factor of Λ comes from expanding, in the field ϕ, the regularized propagator in the flow equation; the power of Λ is essentially determined by the order in this expansion. The three-dimensional case shows a vanishing anomalous dimension in our approximation.[5]

TABLE II. Volume operators in various dimensions and their one-loop anomalous dimensions according to the CREH model.

d	3	4	6
Volume operator	ϕ^6	ϕ^4	ϕ^3
Anomalous dimension	0	$\frac{2}{\pi}G\Lambda$	$\frac{27}{250\pi^2}G\Lambda^2$

B. Volume in the full Einstein-Hilbert truncation

In this section, we estimate the anomalous dimension of the volume operator via a nonmixing *Ansatz* within the fully fledged Einstein-Hilbert truncation for the gravitational EAA:

$$\Gamma_k[g_{\mu\nu}] = \frac{1}{16\pi G_k}\int d^dx\sqrt{g}(2\Lambda_k - R). \tag{5.1}$$

The metric $g_{\mu\nu}$ is expressed via the sum of a background metric $\bar{g}_{\mu\nu}$ and the dynamical metric $h_{\mu\nu}$, i.e., $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. We equip the *Ansatz* (5.1) with the Feynman–de Donder gauge fixing, which gives a particularly simple Hessian; see, for instance, [38].

We consider now the integrated volume operator $V = \int d^dx\sqrt{g}$, and we do not allow for any mixing with other operators. In order to compute the associated anomalous dimension, and, thus, the scaling properties of V, we employ Eq. (3.5).

Taking into account the presence of the ghosts, the Hessian of the functional $V \equiv V[g] \equiv V[\bar{g} + h]$ has the following block form in field space:

$$V^{(2)} = \begin{pmatrix} \frac{\delta^2 V}{\delta h\delta h} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The simple structure of $V^{(2)}$ in field space also simplifies the trace in Eq. (3.5). Indeed, this latter trace includes not only the gravitons but all the fields, in particular, the ghosts. However, in the present approximation, we see that once the fluctuating fields are set to zero, i.e., $h_{\mu\nu} = c_\mu = \bar{c}^\mu = 0$, the relevant contribution only comes from the gravitons, in particular,

$$\gamma_V V = -\frac{1}{2}\mathrm{Tr}\left[\frac{1}{\Gamma_{k,hh}^{(2)} + \mathcal{R}_{k,hh}}\cdot\left(\frac{\delta^2 V}{\delta h\delta h}\right)\cdot\frac{1}{\Gamma_{k,hh}^{(2)} + \mathcal{R}_{k,hh}}\cdot\partial_t\mathcal{R}_k\right]. \tag{5.2}$$

[5] In the three dimensions, the anomalous dimension vanishes if we parametrize $[\phi^6] = Z_6\phi^6$. A nonzero anomalous dimension occurs upon including the mixing of ϕ^6 with other operators.

The Hessian of the volume operator reads

$$\left(\frac{\delta^2 V}{\delta h \delta h}\right)^{\rho\sigma}_{\mu\nu} = \left(-\frac{1}{2}\sqrt{g}\right)\left(\mathbb{I}^{\rho\sigma}_{\mu\nu} - \frac{d}{2}\mathbb{P}^{\rho\sigma}_{\mu\nu}\right), \tag{5.3}$$

with the matrices $\mathbb{I}^{\rho\sigma}_{\mu\nu} \equiv \frac{1}{2}(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu})$ and $\mathbb{P}^{\rho\sigma}_{\mu\nu} \equiv \frac{1}{d} g_{\mu\nu} g^{\rho\sigma}$. Inserting the operator (5.3) in Eq. (5.2), one finds

$$\gamma_V = \frac{d(d+1)}{2}\frac{16\pi G_k}{2}\left[\frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(\frac{d}{2})}\int_0^\infty dz \frac{z^{d/2-1}}{(z+R_k)^2} \times \left(\partial_t R_k - \frac{\partial_t G_k}{G_k} R_k\right)\right]. \tag{5.4}$$

The integral can be evaluated in terms of the standard threshold functions Φ^p_n and $\tilde{\Phi}^p_n$ from [2]. In terms of the dimensionless couplings $g_k \equiv k^{d-2} G_k$ and $\lambda_k \equiv \Lambda_k / k^2$, and with the anomalous dimension related to Newton's constant $\eta_N \equiv \partial_t G_k / G_k$, we find

$$\boxed{\gamma_V(g,\lambda) = d(d+1)\frac{g}{(4\pi)^{d/2-1}}[\Phi^2_{d/2}(-2\lambda) - \eta_N(g,\lambda)\tilde{\Phi}^2_{d/2}(-2\lambda)].} \tag{5.5}$$

This formula applies to an arbitrary point of the $(g-\lambda)$-theory space. For $\eta_N(g,\lambda)$, one should substitute the standard result from the (full) Einstein-Hilbert truncation.[6]

For the example of the optimized cutoff, Eq. (5.5) becomes

$$\gamma_V(g,\lambda) = \frac{4(d+1)}{(4\pi)^{d/2-1}\Gamma(\frac{d}{2})}\frac{g}{(1-2\lambda)^2}\left(1 - \eta_N(g,\lambda)\frac{1}{d+2}\right). \tag{5.6}$$

At the NGFP (g_*, λ_*) where $\eta_N(g_*, \lambda_*) = 2 - d$, Eq. (5.6) yields for $\gamma^*_V = \gamma_V(g_*, \lambda_*)$:

$$\gamma^*_V = \frac{8d(d+1)}{(4\pi)^{d/2-1}\Gamma(\frac{d}{2})(d+2)}\frac{g_*}{(1-2\lambda_*)^2}. \tag{5.7}$$

For a first orientation, let us focus on $d = 2 + \varepsilon$ dimensions where the Einstein-Hilbert truncation is known [2] to display a non-Gaussian fixed point which has the (universal) coordinate $g_* = \frac{3}{38}\varepsilon$, together with a nonuniversal λ_*, which is likewise of order ε. Since $\Phi^2_1(0) = 1$ for any cutoff shape function, Eq. (5.5) yields in this case

$$\gamma^*_V = 12 g_* + O(\varepsilon^2) = \frac{18}{19}\varepsilon + O(\varepsilon^2). \tag{5.8}$$

This anomalous dimension amounts to a shift of the classical scaling dimension of the volume operator $d_V = -d$, to the corrected value $d_V + \gamma^*_V = -2 - \varepsilon + \frac{18}{19}\varepsilon = -2 - \frac{1}{19}\varepsilon$, which appears to correspond to an effective spacetime dimensionality, which is slightly smaller (larger) than the classical one when $\varepsilon > 0$ ($\varepsilon < 0$).

[6] See Eq. (4.41) with (4.40) in [2].

TABLE III. The anomalous dimensions of the length and volume operators in $d = 4$.

	γ^*_L	γ^*_V
One-loop anomalous dimension	0.0682	2.7273
Full anomalous dimension	0.0997	3.9866

The value of γ_V at the UV fixed point for a four-dimensional spacetime is reported in Table III in Sec. V C. It is worth to notice that the value of this anomalous dimension $\gamma^*_V \approx 3.9866$ is almost equal to the spacetime dimension, i.e., $\gamma^*_V \approx 4$, and that the volume operator has classical *mass* dimension $d_V = -4$. According to the discussion and conventions of Sec. III, the full scaling dimension of the volume operator (and analogous for any other operator) is obtained adding the anomalous dimension to the classical mass dimension: $d^{\text{corrected}}_V = d_V + \gamma^*_V$. In the present case, the quantum contribution almost cancels against the classical value so that the operator V has an almost vanishing scaling dimension, $d_V + \gamma^*_V \approx 0$.

Let us stress that this result may well be an artifact of the truncation and approximations employed so far. Nevertheless, we can possibly make contact with independent results in the literature. First, we recall the connection between the "composite operator formalism" of Sec. III with that of critical exponents θ_i defined by the linearized flow. In particular, Eq. (3.7) allows one to compare our results with those obtained by linearizing the RG around the fixed point.

Now, most of the works in the asymptotic safety literature have produced complex critical exponents θ so far, and, thus, a direct comparison with our present results is far from obvious.

Let us remark that here the scaling dimension can be seen as the exponent which characterizes the volume operator under a rescaling of an appropriate classical length, L_c. In particular, let us consider the volume of a certain domain whose typical size is governed by the typical length L_c. For instance, L_c could be the side length of a cube in the coordinate space. Then we have the behavior

$$\langle V(L_c)\rangle = \langle V(L_{c,0})\rangle \left(\frac{L_c}{L_{c,0}}\right)^{-d_V^{\text{corrected}}}, \tag{5.9}$$

where $d_V^{\text{corrected}}$ is the full mass dimension of the volume operator. Such scaling relations are typical of self-similar fractals, in which case, the similarity dimension can sometimes be used to guess the Hausdorff dimension [39]. With this in mind, the fact of having an almost vanishing scaling dimension could suggest that at very small distance scales (fixed point regime), the spacetime is actually much more "empty" than one would naively expect, like a sort of "dust," i.e., a set of discrete points rather that a continuum.

However, let us stress that ultimately one aims to measure volumes (or other operators) in terms of *physical* lengths (or areas, etc.) rather than the artificial scale L_c. Ideally, one may wish to determine the scaling of volumes with respect to that of expectation values of the geodesic distance operator. This allows us then to define a dimension δ via $V \propto l_{\text{geod}}^{\delta}$, where l_{geod} is a geodesic distance. (It has to be kept in mind, however, that l_{geod} is not diffeomorphism invariant, and a suitable "localization procedure" must be considered somehow.)

Hopefully, the tools introduced in this work can be used in the future to compute this kind of generalized dimension in the asymptotic safety scenario.

C. Quantum average of the length of curves

In this section, we study another geometrical object: the length of curves. Let us denote $x^\mu(s)$ the coordinates of the points visited by a curve as a function the parameter $s \in [0,1]$. The length of this curve on a manifold of fixed metric $g_{\mu\nu}$ is then given by

$$L[x(\cdot), g] \equiv \int_0^1 ds \sqrt{g_{\mu\nu}(x(s))\dot{x}^\mu(s)\dot{x}^\nu(s)},$$
$$\dot{x}^\mu(s) \equiv \frac{dx^\mu(s)}{ds}. \tag{5.10}$$

One is interested in quantum averages of $L[x(\cdot), g]$ over the metrics g realized on the manifold,

$$\left\langle \int_0^1 ds \sqrt{g_{\mu\nu}(x(s))\dot{x}^\mu(s)\dot{x}^\nu(s)} \right\rangle. \tag{5.11}$$

With regard to the quantum metric $g_{\mu\nu}$, or, rather, the fluctuation $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, which is considered an elementary field here, the length $L[x(\cdot), g] \equiv L[x(\cdot), \bar{g} + h]$ is clearly a composite operator. It is, therefore, natural to ask if this operator possesses a nontrivial anomalous dimension which encodes how the length responds to a scale variation.

As usual, we shall compute the anomalous dimension via the master equation (3.1). We need the Hessian of the functional L with respect to the metric for a fixed curve $x^\mu(s)$, on the rhs of Eq. (3.1). It reads explicitly

$$\frac{\delta^2 L[x(\cdot), g]}{\delta h_{\alpha\beta}(x')\delta h_{\gamma\delta}(x'')} = -\frac{1}{4}\int ds \frac{\dot{x}^\alpha(s)\dot{x}^\beta(s)\dot{x}^\gamma(s)\dot{x}^\delta(s)}{[g_{\mu\nu}(x_\lambda)\dot{x}^\mu(s)\dot{x}^\nu(s)]^{3/2}} \times \delta(x' - x(s))\delta(x'' - x(s)). \tag{5.12}$$

We can anticipate that by taking the trace in the flow equation, we are led to contract the indices of the Hessian (5.12) in such a way that the rhs turns out to be proportional to L itself. Considering a flat spacetime background metric, it is rather straightforward to obtain the associated anomalous dimension γ_L from Eq. (3.5).

The final result obtained in this way reads

$$\boxed{\gamma_L(g,\lambda) = \frac{d-3}{d-2}\frac{g}{(4\pi)^{d/2-1}}[\Phi^2_{d/2}(-2\lambda) - \eta_N(g,\lambda)\tilde{\Phi}^2_{d/2}(-2\lambda)].} \tag{5.13}$$

The corresponding one-loop result could be retrieved by neglecting the term proportional to $\eta_N(g,\lambda)$ on the rhs of (5.13) and letting $\lambda \to 0$ in the argument of the threshold functions.

Interestingly enough, the function $\gamma_L(g,\lambda)$ is proportional to $\gamma_V(g,\lambda)$. At any point (g,λ) and for all cutoff functions, we have

$$\gamma_L(g,\lambda) = \frac{d-3}{d(d+1)(d-2)}\gamma_V(g,\lambda). \tag{5.14}$$

Thus, for example, $\gamma_L = \gamma_V$ in $d = 1$, and $\gamma_L = \frac{1}{40}\gamma_V$ in $d = 4$, everywhere in the theory space.

As for the $(2+\varepsilon)$-dimensional case, it is remarkable that the pole proportional to $1/(d-2)$ in (5.14) cancels the linear ε dependence of γ_V^*. Hence, with (5.8) in the leading order in ε,

$$\gamma_L^* = -\frac{1}{6\varepsilon}\gamma_V^* = -\frac{2g_*}{\varepsilon}.$$

This yields a finite, nonzero anomalous dimension in the limit $\varepsilon \to 0$:

$$\gamma_L^* = -\frac{3}{19}.$$

Recall that the anomalous dimension of the volume operator vanishes in this limit, $\gamma_V^* = 0 + O(\varepsilon)$.

The numerical results for the four-dimensional case can be found in Table III. They were obtained with the optimized cutoff [37]. Note that the full anomalous dimensions γ_L^* and γ_V^* do indeed differ by the universal factor $1/40$ predicted above.

VI. QUANTUM GRAVITY IN EXACTLY TWO DIMENSIONS

Liouville field theory is a well-known playground for quantum gravity in (exactly) two dimensions [28,29]. Along a different line of investigations, QEG in $2+\varepsilon$ dimensions has often been used as a theoretical laboratory for asymptotic safety. There, ε is always kept different from zero since, if one employs the Einstein-Hilbert truncation, the (bare) action becomes purely topological at $\varepsilon = 0$.

However, recently it has been shown that if one takes the limit $\varepsilon \to 0$ of the action functional only *after* having already computed the RG flow in $2+\varepsilon$ dimensions, one obtains a nontrivial EAA and fixed point action [17]. The latter action is given by the following manifestly two-dimensional functional, which has the form of the induced gravity action:

$$\Gamma_{k\to\infty} = -\frac{(25-N)}{96\pi}\int d^2x\sqrt{g}R\left(-\frac{1}{\Box}\right)R + \cdots. \tag{6.1}$$

Here the dots stand for a cosmological constant term with a nonuniversal coefficient. This result applies to gravity coupled to N minimally coupled free scalar fields and the exponential parametrization of the metric fluctuations; for the standard linear parametrization, the central charge 25 in Eq. (6.1) would be replaced by 19; see [40].

The 2D functional (6.1) descends from the $(2+\varepsilon)$-dimensional Einstein-Hilbert term alone. Therefore, the total EAA contains further contributions, in particular, the Faddeev-Popov ghosts and the Jacobian leading to a Weyl invariant measure. These contributions change the functional in (6.1), yielding an exactly vanishing total charge of QEG in 2D. (For further details, we refer to [17].) In the following, we shall consider the contribution (6.1) in its own right though.

Inserting metrics of the form $g_{\mu\nu} = e^{2\phi}\hat{g}_{\mu\nu}$, the action (6.1) gives rise to a Liouville theory for ϕ:

$$\Gamma_{k\to\infty} = -\frac{(25-N)}{24\pi}\int d^2x\sqrt{\hat{g}}\{\phi(-\hat{\Box})\phi + \hat{R}\phi + \mu_* k^2 e^{2\phi}\}. \tag{6.2}$$

It describes a RG fixed point[7] on the side of the *effective* action; in particular, $\phi \equiv \langle\chi\rangle$ is the expectation value of the quantum field, χ. Furthermore, in [17,41] also the (re) construction of a well-defined UV-regularized functional integral $\int \mathcal{D}_\Lambda \chi e^{-S_\Lambda[\chi]}$ has been performed, which reproduces the RG trajectories $\Gamma_k[\phi]$.

[7] In the notation of [17], $\mu_* \equiv -2\check{\lambda}_* \equiv -2\lambda_*/\varepsilon$.

Employing the approach outlined in [42] and the UV-regularized measure proposed there, the dependence of the bare action $S_\Lambda[\chi]$ on the UV cutoff scale Λ was deduced from (6.2), with the result

$$S_{\Lambda\to\infty} = \kappa\int d^2x\sqrt{\hat{g}}[\chi(-\hat{\Box})\chi + \hat{R}\chi + \check{\mu}_\Lambda \Lambda^2 e^{2\chi}]. \tag{6.3}$$

Remarkably, the coefficient of the bare kinetic term turned out to be exactly the same as its counterpart at the effective level, namely,

$$\kappa = -\frac{(25-N)}{24\pi}. \tag{6.4}$$

The bare fixed point cosmological constant $\check{\mu}_*$ is different from the effective one μ_* and depends on the precise definition of the measure, $\mathcal{D}_\Lambda\chi$. There exists a normalization such that the bare cosmological constant vanishes. We take advantage of this possibility and, henceforth, set $\check{\mu}_* = 0$. For the details of the reconstruction step, we must refer to the literature [17,41,42].

In this context, Liouville theory comes into play as the exactly two-dimensional limiting case of d-dimensional QEG, as always based on the functional integral $\int \mathcal{D}_\Lambda g^{\mathrm{bare}}_{\mu\nu} e^{-S_\Lambda[g^{\mathrm{bare}}_{\mu\nu}]}$ but now only over metrics of the type

$$g^{\mathrm{bare}}_{\mu\nu} \equiv e^{2\chi}\hat{g}_{\mu\nu}. \tag{6.5}$$

In the present paper, instead we shall not be concerned with the physical origin of the Liouville theory and rather use it as a framework to see the FRGE for composite operators "at work" and to show how it relates to the standard approaches. We shall employ the action (6.3) with an arbitrary value of κ and, for simplicity, $\check{\mu}_* = 0$.

A. Correlators of exponential operators

From the splitting (6.5), it is clear that with respect to the elementary field, the bare conformal factor χ, the metric is a composite operator. We are interested in evaluating its correlation functions, which boils down to evaluating correlators of "vertex operators":

$$\langle e^{2a_1\chi(x_1)}\cdots e^{2a_n\chi(x_n)}\rangle. \tag{6.6}$$

Let us consider a flat background metric implying $\hat{R} = 0$ and take $\mu_\Lambda = 0$.

(1) First we focus on the one-point function $\langle e^{2a_1\chi(x)}\rangle$. The scale dependence of this average could be straightforwardly found employing the techniques used in Secs. IV and V. However, here it turns out more convenient to work directly at the path integral level rather than working out the master equation at higher order in the source ε. Thus, we first consider

$$\langle e^{2a\chi(x)}\rangle_k = \frac{1}{Z_0}\int \mathcal{D}\chi \exp\left\{-\kappa\int d^2y\chi(-\Box+\mathcal{R}_k)\chi + 2a\chi(x)\right\}$$
$$\equiv \frac{1}{Z_0}\int \mathcal{D}\chi \exp\left\{-\kappa\int d^2y\chi(-\Box+\mathcal{R}_k)\chi + \int d^2yJ(y)\chi(y)\right\},$$

with $J(y) \equiv 2a\delta(y-x)$. This is a simple Gaussian integral, and so one obtains

$$\langle e^{2a\chi(x)}\rangle = \exp\left[\frac{a^2}{\kappa}\langle x|\frac{1}{-\Box+\mathcal{R}_k}|x\rangle\right] = \exp\left[\frac{a^2}{\kappa}\mathcal{G}_k(0)\right], \tag{6.7}$$

where $\mathcal{G}_k(0) = \langle x|(-\Box+\mathcal{R}_k)^{-1}|x\rangle$ is the Green's function at coinciding points.

Clearly, $\mathcal{G}_k(0)$ is undefined as it stands, and we need a regularization procedure. Rather than the Green's function *per se*, we determine its scale derivative:

$$\partial_t\mathcal{G}_k(0) = = -\langle x|\frac{\partial_t\mathcal{R}_k}{(-\Box+\mathcal{R}_k)^2}|x\rangle$$
$$= -\int\frac{d^2q}{(2\pi)^2}\frac{\partial_t\mathcal{R}_k(q^2)}{(q^2+\mathcal{R}_k(q^2))^2} \tag{6.8}$$

$$= -\frac{1}{2\pi}\Phi_1^2(0) = -\frac{1}{2\pi}. \tag{6.9}$$

The above result is *universal* in the sense that $\Phi_1^2(0) = 1$ is known to be valid for any cutoff of the type $\mathcal{R}_k = k^2R^{(0)}(-\Box/k^2)$ [2]. By integrating (6.9), we find

$$\mathcal{G}_k(0) - \mathcal{G}_\mu(0) = -\frac{1}{2\pi}\log\frac{k}{\mu}. \tag{6.10}$$

Using this result in Eq. (6.7), we can form the well-defined ratio

$$\frac{\langle e^{2a\chi(x)}\rangle_k}{\langle e^{2a\chi(x)}\rangle_\mu} = \left(\frac{k}{\mu}\right)^{-\frac{1}{2\pi}\frac{a^2}{\kappa}},$$

and so we obtain

$$\langle e^{2a\chi(x)}\rangle_k = \left(\frac{k}{\mu}\right)^{-\frac{1}{2\pi}\frac{a^2}{\kappa}}\langle e^{2a\chi(x)}\rangle_\mu. \tag{6.11}$$

From this relation, we can read off the scaling dimension of the exponential operator:

$$\boxed{\partial_t\langle e^{2a\chi(x)}\rangle_k = -\frac{1}{2\pi}\frac{a^2}{\kappa}\langle e^{2a\chi(x)}\rangle_k.} \tag{6.12}$$

(2) Let us compare the k dependence of the expectation value (6.11) with the one found directly from the composite operator flow equation. We denote Z_O the renormalization constant associated to the operator $O(x) = e^{2a\chi(x)}$. A one-loop computation based upon Eq. (3.5) yields then

$$(Z_O^{-1}\partial_tZ_O)e^{2a\chi(x)} = \mathrm{Tr}\left[-\frac{1}{2}\frac{1}{2\kappa(-\Box+\mathcal{R}_k)}(4a^2e^{2a\chi(x)})\frac{1}{2\kappa(-\Box+\mathcal{R}_k)}2\kappa\partial_t\mathcal{R}_k\right]$$
$$= -\left(\frac{a^2}{\kappa}\right)\frac{1}{2\pi}\Phi_1^2(0)e^{2a\chi(x)}$$
$$= -\frac{1}{2\pi}\left(\frac{a^2}{\kappa}\right)e^{2a\chi(x)}. \tag{6.13}$$

The running found in Eq. (6.13) is the expected result, the same as in Eq. (6.12).

Furthermore, we mention that this approach based upon the master equation (3.1), allows one to obtain the so-called KPZ scaling relations in the FRG framework. As a further illustration of our techniques, we discuss their derivation in the Appendix. For a detailed discussion of Liouville theory and the KPZ scaling in the EAA approach, see, also, [43,44].

(3) Now we generalize (6.7) to the n-point correlation functions, starting out from

$$\langle e^{2a_1\chi(x_1)}\ldots e^{2a_n\chi(x_n)}\rangle_k$$
$$= \frac{1}{Z_0}\int\mathcal{D}\chi\exp\left\{-\kappa\int dy\chi(-\Box+\mathcal{R}_k)\chi + \int dyJ(y)\chi(y)\right\},$$

with the source function $J(y) \equiv \sum_{i=1}^n 2a_i\delta(y-x_i)$. The Gaussian integral yields

$$\langle e^{2a_1\chi(x_1)}\cdots e^{2a_n\chi(x_n)}\rangle_k = \exp\left[\sum_{i=1}^n \frac{a_i^2}{\kappa}\mathcal{G}_k(0) + \sum_{i<j}\frac{2a_ia_j}{\kappa}\mathcal{G}_k(x_i-x_j)\right].$$

While these correlation functions are ill-defined, their flow equation is perfectly regular:

$$\begin{aligned}\partial_t \log\langle e^{2a_1\chi(x_1)}\cdots e^{2a_n\chi(x_n)}\rangle_k &= \sum_{i=1}^n \frac{a_i^2}{\kappa}\partial_t\mathcal{G}_k(0) + \sum_{i<j}\frac{2a_ia_j}{\kappa}\partial_t\mathcal{G}_k(x_i-x_j)\\ &\equiv \frac{1}{\kappa}\left(\sum_{i=1}^n a_i\right)^2\partial_t\mathcal{G}_k(0)\\ &\quad+\frac{2}{\kappa}\sum_{i<j}a_ia_j[\partial_t\mathcal{G}_k(x_i-x_j)-\partial_t\mathcal{G}_k(0)]. \end{aligned}\tag{6.14}$$

Here the regularized propagator occurs at noncoincident points also:

$$\mathcal{G}_k(r) = \int\frac{d^2q}{(2\pi)^2}\frac{e^{iq(x-y)}}{q^2+\mathcal{R}_k}, \qquad r\equiv|x-y|.$$

Considering the example of a masslike cutoff profile, i.e., $\mathcal{R}_k = k^2$, we find

$$\mathcal{G}_k(r)=\frac{1}{2\pi}K_0(kr) \quad\text{and}\quad \partial_t\mathcal{G}_k(r)=-\frac{1}{2\pi}krK_1(kr),\tag{6.15}$$

where K_ν denotes the Bessel function of the second kind. In the limit $kr\to 0$, it gives

$$\begin{aligned}&\mathcal{G}_k(r)\approx -\frac{1}{2\pi}\log(kr) \quad\text{and}\\ &\partial_t\mathcal{G}_k(r)\approx -\frac{1}{2\pi} = \text{const.}\end{aligned}\tag{6.16}$$

In particular, we recover (6.10), i.e.,

$$\mathcal{G}_k(0)-\mathcal{G}_\mu(0)\equiv\lim_{r\to 0}(\mathcal{G}_k-\mathcal{G}_\mu)(r) = -\frac{1}{2\pi}\log\left(\frac{k}{\mu}\right).$$

We also see that

$$\partial_t\mathcal{G}_k(r)-\partial_t\mathcal{G}_\mu(0)\to 0 \quad\text{for } kr\to 0\tag{6.17}$$

is a well-defined limit. Hence, the last term in Eq. (6.14) vanishes when all distances $|x_i-x_j|$ are much smaller than k^{-1}.

Let us consider the case $n=2$, for example. At small distances $|x_1-x_2|\ll k^{-1}$, the RG equation (6.14) yields

$$\partial_t\log\langle e^{2a_1\chi(x_1)}e^{2a_2\chi(x_2)}\rangle_k = -\frac{1}{2\pi\kappa}(a_1+a_2)^2.\tag{6.18}$$

Note that in the limit $k\to 0$, the distances $|x_1-x_2|$ being measured with the fixed metric $\hat{g}_{\mu\nu}=\delta_{\mu\nu}$ all become small in comparison with k^{-1}. Hence, the scaling exponent displayed by the rhs of (6.18) is, indeed, the expected, correct, and universal result [45].

According to (6.18), the two-point correlator equals $k^{-(a_1+a_2)^2/2\pi\kappa}$ multiplied by a k-independent function of $|x_1-x_2|$. To find it, we can start from the formal expression

$$\langle e^{2a_1\chi(x_1)}e^{2a_2\chi(x_2)}\rangle_k = \exp\left[\frac{a_1^2}{\kappa}G_k(0)+\frac{a_2^2}{\kappa}G_k(0)+2\frac{a_1a_2}{\kappa}G_k(x_1-x_2)\right]$$

and obtain the distance dependence by evaluating the following manifestly well-defined ratio for arbitrary $x_1\neq x_2$:

$$\begin{aligned}\frac{\langle e^{2a_1\chi(x_1)}e^{2a_2\chi(x_2)}\rangle_k}{\langle e^{2a_1\chi(x_1)}\rangle_\mu\langle e^{2a_2\chi(x_2)}\rangle_\mu} &= \exp\left[\frac{a_1^2+a_2^2}{\kappa}(G_k(0)-G_\mu(0))\right.\\ &\qquad\left.+2\frac{a_1a_2}{\kappa}G_k(x_1-x_2)\right]\\ &=\left(\frac{k}{\mu}\right)^{-\frac{1}{2\pi}\frac{a_1^2}{\kappa}}\left(\frac{k}{\mu}\right)^{-\frac{1}{2\pi}\frac{a_2^2}{\kappa}}(k|x_1-x_2|)^{-\frac{a_1a_2}{\kappa\pi}}\\ &=\left(\frac{k}{\mu}\right)^{-\frac{1}{2\pi\kappa}(a_1+a_2)^2}(\mu|x_1-x_2|)^{-\frac{a_1a_2}{\kappa\pi}}.\end{aligned}\tag{6.19}$$

From (6.19), it is clear that the IR limit $k\to 0$ can be meaningfully taken only if $a_1=-a_2$. This condition of charge neutrality is a well-known feature of such Coulomb gas calculations; see, for instance, [46].

The problem of the $k\to 0$ limit presents itself differently depending on the sign of κ. If $\kappa>0$, the correlator (6.19) with $a_1\neq -a_2$ diverges for $k\to 0$ at fixed $\mu\neq 0$, while it vanishes in this limit when $\kappa<0$. In the usual Coulomb gas interpretation [47], this leads to the requirement of charge neutrality $\sum a_i = 1$, which we shall not discuss further here since our emphasis was on showing how the FRGE for composite operators relates to the standard methods.

Let us note that imposing the condition of charge neutrality $a_1=-a_2$, one can obtain the power law in (6.19), i.e., the distance dependence $\propto|x_1-x_2|^{\frac{a_1a_2}{\kappa\pi}} = |x_1-x_2|^{\frac{a_1^2}{\kappa\pi}}$, by assuming that the correlation function of two composite operators O_1 and O_2 is determined by their individual (anomalous) scale dimension. In the case at hand, it is given by $-\gamma_{O_1}-\gamma_{O_2} = \frac{a_1^2}{2\kappa\pi}+\frac{a_1^2}{2\kappa\pi}=\frac{a^2}{\kappa\pi}$. This yields the same power law behavior as in (6.19) if the charge neutrality condition holds.

Let us also note that we can use (6.11) in order to eliminate the normalization point μ from (6.19) yielding

$$\langle e^{2a_1\chi(x_1)}e^{2a_2\chi(x_2)}\rangle_k = (k|x_1-x_2|)^{-\frac{a_1a_2}{\kappa\pi}}\langle e^{2a_1\chi(x_1)}\rangle_k\langle e^{2a_2\chi(x_2)}\rangle_k.$$

This relates the product of two expectation values of normal ordered exponentials to the expectation value of their product.

B. Scaling of geodesics and of generic curves

Let us consider the following functional depending on the metric alone:

$$L_g \equiv L[x_g(\cdot); g] = \int_0^1 ds \sqrt{g_{\mu\nu}(x_g(s))\dot{x}_g^\mu(s)\dot{x}_g^\nu(s)}. \quad (6.20)$$

Here, $x_g^\mu(s)$ parametrizes the *geodesic* determined by $g_{\mu\nu}$ and connecting $x_g(0)$ and $x_g(1)$.

Hence, in comparison with (5.10), the length (6.20) has an additional source of metric dependence via the curve considered. As a result, the functional L_g is an even more complicated composite operator built from the quantum metric. We will compute its anomalous dimension which describes how the quantum average of L_g responds to a scale variation.

Let us stress that since $x_g(s)$ occurring in L_g solves the geodesics equation and, thus, depends implicitly on the metric, a variation of $g_{\mu\nu}$ changes the geodesics equation and its associated solution. This is a crucial difference for our computation since it renders the Hessians of the composite operators L_g and L different. Based on this observation, we expect that the scaling dimensions of L_g and L may turn out differently. In general, if one wishes to recover the result regarding L for generic fixed curves, not necessarily geodesics, one just needs to drop from the Hessian of L_g the extra contribution which does not appear in the Hessian of L.

Let us interpret $g_{\mu\nu} \equiv g_{\mu\nu}^{\text{bare}}$ in (6.20) as the bare metric now, and let us parametrize it as $g_{\mu\nu} = e^{2\chi}\delta_{\mu\nu}$ so that

$$L_g = \int_0^1 ds |\dot{x}_g^\mu(s)| e^{\chi(x_g(s))}.$$

The explicit form of L_g can be worked out explicitly in two dimensions, expanding with respect to χ [48,49]. Up to the second order in χ, it reads

$$\begin{aligned} L_g = {}& |x_0 - y_0| \int_0^1 ds \left[1 + \chi(x(s)) + \frac{1}{2}\chi(x(s))^2\right] \\ & - \frac{1}{2}|x_0 - y_0|^3 \int_0^1 du \int_0^1 dv \partial_\perp \chi(x(u)) D_{u,v} \partial_\perp \chi(x(v)), \end{aligned} \quad (6.21)$$

where $|x_0 - y_0|$ is the flat spacetime distance between the two points connected by the geodesics, and

$$\begin{aligned} x^\mu(s) &\equiv x_0^\mu(1-s) + y_0^\mu s, \\ \partial_\perp \chi &\equiv \varepsilon^\mu_\nu \frac{y_0^\nu - x_0^\nu}{|x_0 - y_0|} \partial_\mu \phi, \\ D_{u,v} &\equiv v(1-u)\theta(u-v) + u(1-v)\theta(v-u). \end{aligned}$$

We shall read off the anomalous dimension of L_g, which we denote γ_{L_g}, from Eq. (3.5) by projecting on the monomial L_g in flat spacetime, that is, by setting $\chi = 0$ after having computed the Hessian of L_g. This means that we have to single out the terms proportional to $|x_0 - y_0|$ when we compute the trace on the rhs of (3.5). In this manner, Eq. (3.5) will read

$$\gamma_{L_g} |x_0 - y_0| = -\frac{1}{2} \mathrm{Tr}[\mathcal{G}_k \cdot L_g^{(2)} \cdot \mathcal{G}_k \cdot \partial_t \mathcal{R}_k], \quad (6.22)$$

where $\mathcal{G}_k$ is the regularized inverse propagator $\mathcal{G}_k = (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$, and $L_g^{(2)}$ is the Hessian of the geodesic length operator. Let us observe that, furthermore,

$$\begin{aligned} \gamma_{L_g} |x_0 - y_0| &= -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \langle p | \mathcal{G}_k \cdot L_g^{(2)} \cdot \mathcal{G}_k \cdot \partial_t \mathcal{R}_k | p \rangle \\ &= -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \mathcal{G}_k(p) \mathcal{G}_k(p) \partial_t \mathcal{R}_k(p) \langle p | L_g^{(2)} | p \rangle, \end{aligned} \quad (6.23)$$

where we expressed in flat spacetime the trace as a single momentum integral. As a result, we are left with finding the explicit form of the matrix element $\langle p | L_g^{(2)} | p \rangle$. It can be obtained as follows. After a Fourier transform, $\langle p | L_g^{(2)} | p \rangle$ is seen to consist of three pieces labeled a, b, and c, respectively:

$$\begin{aligned} \langle p | L_g^{(2)} | p \rangle &= \int d^2x' d^2x'' e^{ip\cdot(x' - x'')} \langle x' | L_g^{(2)} | x'' \rangle \\ &\equiv \mathcal{P}_a + \mathcal{P}_b + \mathcal{P}_c. \end{aligned} \quad (6.24)$$

The matrix element $\langle x' | L_g^{(2)} | x'' \rangle \equiv \mathcal{Q}_a + \mathcal{Q}_b + \mathcal{Q}_c$, i.e., the Hessian $\delta^2 L_g / \delta\phi(x')\delta\phi(x'')$ of (6.21), consists of the following three terms:

$$\begin{aligned} \mathcal{Q}_a &\equiv |x_0 - y_0| \int_0^1 ds \delta(x' - x(s)) \delta(x'' - x(s)), \\ \mathcal{Q}_b &\equiv -\frac{1}{2}|x_0 - y_0|^3 \int_0^1 du \\ &\quad \times \int_0^1 dv \partial_\perp \delta(x' - x(u)) D_{u,v} \partial_\perp \delta(x'' - x(v)), \\ \mathcal{Q}_c &\equiv -\frac{1}{2}|x_0 - y_0|^3 \int_0^1 du \\ &\quad \times \int_0^1 dv \partial_\perp \delta(x'' - x(u)) D_{u,v} \partial_\perp \delta(x' - x(v)). \end{aligned} \quad (6.25)$$

(a) The first term $\mathcal{Q}_a$ in Eq. (6.25) is the part of the Hessian which coincides with the one present when we compute the Hessian of the length for a *generic* curve, not necessarily a geodesic. Using Eqs. (6.23) and (6.24), we see that it gives rise to the following contribution to the rhs of (6.23):

$$-\frac{1}{2\pi}\left(\frac{1}{4\kappa}|x_0-y_0|\right). \tag{6.26}$$

(b) Now we evaluate the contribution due to $\mathcal{Q}_b$. Denoting $\xi^\mu \equiv \varepsilon^\mu_\nu(y_0^\nu - x_0^\nu)$, it reads

$$\mathcal{Q}_b = -\frac{1}{2}|x_0-y_0|\int_0^1 du\int_0^1 dv\xi^\mu\frac{\partial}{\partial x'^\mu} \times\delta(x'-x(u))D_{u,v}\xi^\nu\frac{\partial}{\partial x''^\nu}\delta(x''-x(v)). \tag{6.27}$$

Inserting this expression in Eq. (6.24), one obtains

$$\mathcal{P}_b = -\frac{1}{2}|x_0-y_0|\int dx'dx''e^{ip\cdot(x'-x'')}\int_0^1 du \times\int_0^1 dv\xi^\mu(-ip_\mu)\delta(x'-x(u)) \times D_{u,v}\xi^\nu(ip_\nu)\delta(x''-x(v)). \tag{6.28}$$

Integrating over x' and x'', we obtain the following contribution to (6.24):

$$\mathcal{P}_b = -\frac{1}{8}|x_0-y_0|\int_0^1 du \times\int_0^1 dv e^{ip\cdot(x(u)-x(v))}D_{u,v}\xi^\mu\xi^\nu p_\mu p_\nu. \tag{6.29}$$

(c) The third piece $\mathcal{Q}_c$ in Eq. (6.25) gives a result identical to (6.29) but with $x(u)$ and $x(v)$ interchanged. Summing the two contributions, we find

$$\mathcal{P}_b+\mathcal{P}_c = -\frac{1}{8}|x_0-y_0|\int_0^1 du\int_0^1 dv(e^{ip\cdot(x(u)-x(v))} + e^{-ip\cdot(x(u)-x(v))})D_{u,v}\xi^\mu\xi^\nu p_\mu p_\nu. \tag{6.30}$$

Noting that $x^\mu(u)-x^\mu(v) = -(u-v)(x_0^\mu - y_0^\mu)$, the integral over the parameters u and v can be performed now, and we have

$$\begin{aligned}&\int_0^1 du\int_0^1 dv(e^{ip\cdot(x(u)-x(v))}+e^{-ip\cdot(x(u)-x(v))})D_{u,v}\\&= 2\int_0^1 du\int_0^1 dv\cos\left(p\cdot(x(u)-x(v))\right)D_{u,v}\\&= 2\frac{(-2+2\cos\left(p\cdot(x_0-y_0)\right)+(p\cdot(x_0-y_0))^2)}{(p\cdot(x_0-y_0))^4}.\end{aligned} \tag{6.31}$$

This expression shows that the RG flow generates infinitely many monomials proportional to $|x_0-y_0|^n$ on the rhs of the flow equation for the composite operator. To make them explicit, we expand the above term as a power series in $p\cdot(x_0-y_0)$, finding

$$2\frac{-2+2\cos(p\cdot(x_0-y_0))+(p\cdot(x_0-y_0))^2}{(p\cdot(x_0-y_0))^4} = \frac{1}{6}-\frac{1}{180}\left(p\cdot(x_0-y_0)\right)^2+\cdots.$$

So, finally we obtain for (6.30),

$$\mathcal{P}_b+\mathcal{P}_c = -\frac{1}{2}|x_0-y_0| \times\left(\frac{1}{6}-\frac{1}{180}(p\cdot(x_0-y_0))^2+\cdots\right) \times\xi^\mu\xi^\nu p_\mu p_\nu. \tag{6.32}$$

Under our approximation, we must consistently neglect all monomials with (x_0-y_0) dependences different from $|x_0-y_0|$ when evaluating the rhs of Eq. (6.22). As a consequence, Eq. (6.32) implies that $\mathcal{P}_b+\mathcal{P}_c$ gives no contribution to the anomalous dimension γ_{L_g}.

This is seen easily after a symmetric integration over the momenta in (6.23) with (6.32). It replaces $p_\mu p_\nu \to p^2\delta_{\mu\nu}/2$ in the leading term, so that effectively

$$\begin{aligned}\mathcal{P}_b+\mathcal{P}_c &= -\frac{1}{2}|x_0-y_0|\left(\frac{1}{6}+O((x_0-y_0)^2)\right)\xi^\mu\xi^\nu p^2\frac{\delta_{\mu\nu}}{2}\\&= -\frac{1}{2}|x_0-y_0|\left(\frac{1}{6}+O((x_0-y_0)^2)\right)\xi^2\frac{p^2}{2}.\end{aligned}$$

We also observe that $\xi^2 = \varepsilon^\mu_\rho(y_0^\rho - x_0^\rho)\delta_{\mu\nu}\varepsilon^\nu_\sigma(y_0^\sigma-x_0^\sigma) = |x_0-y_0|^2$, since $\varepsilon^\mu_\rho\varepsilon_{\mu\sigma}=\delta_{\rho\sigma}$. This then implies that in (6.32) the lowest order in $|x_0-y_0|$ is proportional to $|x_0-y_0|^3$: $\mathcal{P}_b+\mathcal{P}_c = -\frac{1}{4}|x_0-y_0|^3+\cdots$. As a result, $\mathcal{P}_b+\mathcal{P}_c$ contains no term that matches the linear one on the lhs of (6.23) and could contribute to γ_{L_g}.

Hence, our final conclusion is that the anomalous dimension γ_{L_g} is determined solely by the contribution coming from $\mathcal{P}_a$. It reads

$$\boxed{\gamma_{L_g} = -\frac{1}{2\pi}\frac{1}{4\kappa}.} \tag{6.33}$$

In turn, this demonstrates that the anomalous dimensions for the length of a geodesics γ_{L_g} and for a generic curve, respectively, are equal within the approximation employed.

However, let us stress that, in general, they are likely to be different once the mixing in the running of the geodesic length operator is taken into account. Nevertheless, one may interpret our result as an indication that the anomalous dimensions of the length of a generic curve and of a geodesic are not too different (at least in two dimensions).

We emphasize that the geodesic length enters in many potentially observable correlation functions. For instance, given two local operators O_1 and O_2 it would be interesting to compute [15,16]

$$G(r) \equiv \left\langle \int d^d x \sqrt{g(x)} \int d^d y \sqrt{g(y)} O_1(x) \times O_2(y) \delta(r - L_g(x,y)) \right\rangle.$$

Clearly, L_g being a nontrivial composite operator, the scaling analysis of $G(r)$ is affected by the presence of the delta function involving L_g. Once the full scaling dimensions of the operators involved are known, it is straightforward to invoke scaling arguments for this type of correlation function; see, for instance, [44].

VII. CONCLUSIONS AND OUTLOOK

In this work, we considered the role of composite operators in the asymptotic safety program. We argued that the introduction of composite operators via suitable sources is convenient in a number of cases. In particular, our framework makes it possible to consider geometrical objects, like the length of an arbitrary curve or of a geodesic, whose quantum properties would hardly be seen in any realistic truncation for the EAA. Moreover, we demonstrated that particular operators, like a composite metric in the vielbein formalism, require a careful regularization and renormalization procedure, on top of that related to the EAA, to be meaningfully defined. Within the FRG setting, we systematized this procedure for arbitrary composite operators. In general, the introduction of composite operators is useful whenever one wishes to investigate the quantum properties of operators that are not contained in the (exact) EAA or in the truncation considered.

In Secs. II and III, we reviewed the inclusion of composite operators in the EAA formalism and discussed a method which allows us to identify the scaling properties of the composite operators at the fixed point. In Sec. IV, we considered the CREH truncation and studied the case of composite metrics in this setting. The CREH example made it explicit that a dedicated regularization and subsequent (re)normalization is necessary in order to define the metric whenever the latter is a composite field. As such, the composite metric analyzed in Sec. IV can be viewed as a toy model for the composite metric in the vielbein formalism.

The CREH model also illustrates nicely that by choosing different field parametrizations, quantum corrections can be changed crucially both in the EAA and the composite operator. As an extreme example, with $g_{\mu\nu} = \phi^2 \delta_{\mu\nu}$ the metric is a composite operator, whereas with the alternative parametrization $g_{\mu\nu} = \psi \delta_{\mu\nu}$, it is not. But in the latter case, also the CREH *Ansatz* (4.3) acquires a different form, the kinetic term $\propto (\partial_\mu \psi)^2/\psi$ is no longer bilinear in the dynamical field, and so the running of the couplings involved will be different.

In Sec. V, we tested our framework further by computing the anomalous dimensions of the volume and the length operators. Finally, in Sec. VI, we considered Liouville theory in two dimensions. In particular, we computed the correlation functions of composite metrics and the anomalous dimension of the geodesic distance.

In general, since in our computations the approximations and *Ansätze* were too simple still, we do not expect our results to be quantitatively precise. However, it is important to note that the framework introduced in this work has allowed us for the first time to give an estimate of the quantum properties of geometrical objects, like the length of a curve, that have never been considered before for the case of asymptotically safe quantum gravity.

We would like to remark that the final purpose of explicitly keeping track of selected composite operators is making contact with quantum gravitational observables. Clearly, this is beyond the scope of the present paper, but we made a first step towards this goal. Indeed, as we argued in the Introduction, in order to consider certain types of observables, it is unavoidable to introduce further operators on top of those present in the gravitational EAA. In this sense, it would be natural to follow the logic of the two-dimensional case where fixed-volume and fixed-geodesic distance functionals have been discussed in detail. Ultimately, indeed, we believe that a comparison between the different attempts to define a well-defined gravitational path integral can only be made by considering observable quantities.

Summarizing, we believe that the framework developed in this work opens the door to new avenues in comparing different approaches to quantum gravity and gives a viable road to access observables in the asymptotic safety scenario for quantum gravity.

APPENDIX: LIOUVILLE THEORY: KPZ SCALING

Scaling arguments in quantum gravity have been particularly fruitful in two dimensions. Here we shall derive the so-called KPZ relations [28,29,50] following the notation adopted in [51]. The partition function can be written as follows:

$$Z = \int \mathcal{D}\phi \exp\left[-\left(\frac{25 - c}{48\pi} \right) \times \int d^2 x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + R\phi \right) \right]. \quad \text{(A1)}$$

Now, the bare action in the exponent of (A1) enjoys the trivial symmetry $g_{\mu\nu} \to e^{\sigma(x)} g_{\mu\nu}$, $\phi \to \phi - \sigma(x)$ since it can be rewritten as a functional of the product $e^{\phi} g_{\mu\nu}$ only. This means that the bare ("classical") action is annihilated when one acts on it with the operator

$$\mathcal{L} \equiv \left(g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} - \frac{\delta}{\delta \phi(x)} \right).$$

We require that this invariance is enjoyed also at the quantum level by observables. In particular, we shall construct the diffeomorphism invariant operator

$$O \equiv \int d^2x\sqrt{g}e^{\alpha\phi}.$$

One notices that for $\alpha \neq 1$, the classical functional $O \equiv O[\phi; g]$ is not invariant under the σ transformation. The reason for considering a general parameter α is that we will determine its value such that

$$\mathcal{L}\langle O\rangle = 0, \tag{A2}$$

which is to say that the operator $\mathcal{L}$ annihilates O at the quantum level. Indeed, quantum corrections to the naive scaling properties will force us to fix α to some specific value different from unity.

Let us turn to the scaling properties of the operator $e^{\alpha\phi(x)}$. At the quantum level, $e^{\alpha\phi(x)}$ will acquire an anomalous dimension γ which we will compute later on. In the fixed point regime, the anomalous dimension enters the corresponding Callan-Symanzik equation as follows:

$$(\mu\partial_\mu + \gamma)\langle e^{\alpha\phi}\rangle = 0. \tag{A3}$$

As usual, to deduce the scaling properties of $\langle e^{\alpha\phi}\rangle$, we need to eliminate the μ derivative from Eq. (A3). This can be done by means of simple dimensional analysis. When working in curved spacetime, one may choose either the coordinates or the metric to be dimensionful. In our case, it is natural to take the coordinates dimensionless, as they are merely variables devoid of any particular meaning. Moreover, we note that $e^{\alpha\phi}$ is classically dimensionless. Then, dimensional analysis implies

$$\left(\mu\partial_\mu - 2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\right)\langle e^{\alpha\phi}\rangle = 0. \tag{A4}$$

Eliminating the $\mu\partial_\mu$ term from Eqs. (A3) and (A4) yields

$$\left(2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \gamma\right)\langle e^{\alpha\phi}\rangle = 0. \tag{A5}$$

Now let us determine the action of $\mathcal{L}$ on $\langle O\rangle$:

$$\begin{aligned}\mathcal{L}\langle O\rangle &= \left(g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} - \frac{\delta}{\delta\phi}\right)\langle O\rangle \\ &= \left(g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} - \frac{\delta}{\delta\phi}\right)\left\langle \int d^2x\sqrt{g}e^{\alpha\phi}\right\rangle \\ &= \left(g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} - \frac{\delta}{\delta\phi}\right)\int d^2x\sqrt{g}\langle e^{\alpha\phi}\rangle,\end{aligned}$$

where we brought the volume element outside the average $\langle\cdot\rangle$ since the latter is not dynamical. Now we note that the functional derivative with respect to the metric acts obviously on $\sqrt{g}$ but also on the implicit dependence of $\langle e^{\alpha\phi}\rangle$ on the metric. This latter dependence is easily obtained from Eq. (A5). Therefore, we have the following contributions:

$$\begin{aligned}g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\sqrt{g}(x') &= \sqrt{g}\delta(x-x'), \\ g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\langle e^{\alpha\phi(x')}\rangle &= -\frac{\gamma}{2}\langle e^{\alpha\phi}\rangle\delta(x-x') \\ -\frac{\delta}{\delta\phi(x)}\langle e^{\alpha\phi(x')}\rangle &= -\alpha\langle e^{\alpha\phi}\rangle\delta(x-x').\end{aligned}$$

Summing all the terms, we can finally write Eq. (A2) as

$$1 - \alpha - \frac{\gamma}{2} = 0. \tag{A6}$$

We shall see in a moment that this is the celebrated KPZ relation.

Finally, we come to our point, the actual computation of γ. According to the discussion in Sec. III, we need to evaluate $\gamma = Z^{-1}_{e^{\alpha\phi}}\partial_t Z_{e^{\alpha\phi}}$ via Eq. (3.5). It is sufficient and particularly convenient to set $g_{\mu\nu} = \delta_{\mu\nu}$. Doing so, one obtains

$$\begin{aligned}(Z^{-1}_{e^{\alpha\phi}}\partial_t Z_{e^{\alpha\phi}})e^{\alpha\phi(x)} &= -\frac{1}{2}\mathrm{Tr}\left[\frac{1}{(\frac{25-c}{48\pi})(-\Box + R_k)}(\alpha^2 e^{\alpha\phi(x)})\right. \\ &\qquad \left.\times \frac{1}{(\frac{25-c}{48\pi})(-\Box + R_k)}\left(\frac{25-c}{48\pi}\right)\partial_t R_k\right] \\ &= -\frac{12\alpha^2}{25-c}e^{\alpha\phi(x)}.\end{aligned}$$

Inserting this value for γ in Eq. (A6), we obtain

$$\boxed{1 - \alpha + \frac{6\alpha^2}{25-c} = 0.} \tag{A7}$$

This relation is a well-known result in Liouville gravity, the basis, in particular, for the "gravitational dressing" of arbitrary matter field operators; see, e.g., [51].

We can rephrase our arguments also in the following way: The operator $e^{\alpha\phi}$ has a vanishing classical mass dimension. However, quantum corrections afflict $e^{\alpha\phi}$ with an anomalous dimension equal to γ. Under a Weyl rescaling, an operator with mass dimension γ is transformed by an overall factor $(e^{-\frac{\sigma}{2}})^\gamma$.[8] Therefore, at the quantum level, the operator $\sqrt{g}e^{\alpha\phi}$ gets transformed by an overall factor $e^{\sigma}e^{-\alpha\sigma}e^{-\frac{\sigma}{2}\gamma}$, where the first exponential comes from the determinant of the metric, while the other exponentials come from the classical and quantum scaling properties of the operator $e^{\alpha\phi}$, respectively. We note that the exponent of the overall scaling factor $e^{\sigma(1-\alpha-\gamma/2)}$ is precisely the lhs of

[8] We recall that in the conventions adopted in this appendix, the metric transforms via $g_{\mu\nu} \to e^{\sigma(x)}g_{\mu\nu}$ under a Weyl transformation.

Eq. (A6), which we require to vanish in order to satisfy the invariance under the operator $\mathcal{L}$ at the quantum level.

Equations fully analogous to (A7) can also be obtained in a similar manner, and they determine the parameters usually indicated with α_n for the gravitational dressing of other operators. Eventually, one can follow the arguments given in the literature; see, e.g., [51] to recover the known results. For further details on the EAA approach to Liouville theory and KPZ scaling, we refer to [43].

However, let us stress that in the main body of the paper, the attitude towards composite operators is different than in this appendix. In particular, here we applied the following logic: By a suitable choice of the parameter α the composite operator is tuned in the UV so that the property (A2) is satisfied at the full quantum level, i.e., at $k = 0$. On the contrary, in the main body of the paper, we did not ask for any special property at the quantum level. For this reason, a straightforward comparison between the results of this appendix and those given in the main text is not possible.

Furthermore, we recall that KPZ arguments are normally used to compute the scaling property of heat kernel related observables, which are different composite operators with respect to the ones considered in the main text; hence, also for this reason, the results cannot be compared straightforwardly.

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