Cosmological Einstein-Skyrme solutions with nonvanishing topological charge

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Time-dependent analytic solutions of the Einstein-Skyrme system—gravitating Skyrmions—with topological charge are analyzed in detail. In particular, the question of whether these Skyrmions reach a spherically symmetric configuration for $t \rightarrow +\infty$ is discussed. It is shown that there is a static, spherically symmetric solution described by the Ermakov-Pinney system, which is fully integrable by algebraic methods. For $\Lambda > 0$, this spherically symmetric solution is found to be in a "neutral equilibrium" under small deformations, in the sense that under a small squashing it would neither blow up nor disappear after a long time, but it would remain finite forever (plastic deformation). Thus, in a sense, the coupling with Einstein gravity spontaneously breaks the spherical symmetry of the solution. However, in spite of the lack of isotropy, for $t \rightarrow \infty$ (and $\Lambda > 0$), the spacetime is locally flat and the anisotropy of the Skyrmion only reflects the squashing of spacetime.

I. INTRODUCTION

The Skyrme system [1] is one of the most useful models in nuclear and particle physics due to its close relationship to low-energy QCD [2]. A remarkable feature of the Skyrme action is that it allows for the existence of solitons (Skyrmions) that behave as Fermionic degrees of freedom, in spite of the fact that the basic fields are scalar. Furthermore, Skyrmions describe nucleons both theoretically and phenomenologically (see, e. g., [2–14]), where the identification of the winding number of the Skyrmion with the baryon number in particle physics [2] plays a crucial role. Following [3,7], the possibility of treating the Skyrme solitons as fermions was extended to curved spaces as well [15,16], opening the possibility for applying this theory to general relativity and astrophysics.

The above reasons imply that the Einstein-Skyrme system might be relevant for astrophysics from a phenomenological point of view. From a more theoretical angle, numerical computations following earlier results in [17,18] indicate the existence of spherically symmetric black-hole solutions with a nontrivial Skyrme field (Skyrme hair) [19,20]. These were the first counterexamples to the black hole no-hair conjecture, and, moreover, the stability against spherical linear perturbations was shown in [21]. Regular particlelike configurations [22] and dynamical properties of the system have also been investigated numerically [23]. Even in the sector with vanishing topological charge, the cosmological consequences of the Skyrme model are quite interesting [24–26].

Thus, having analytic solutions of the Einstein-Skyrme system with nontrivial topological charges would be extremely useful. In particular, the gravitational implications of the discreteness of the topological charge together with the fact that such topological objects have a characteristic size, deserve an in-depth investigation. An especially compelling case is the time-dependent situation in which the coupling of the Skyrme system with gravity could reveal unexpected departures from the "natural" spherical symmetry of configurations with winding number $W = 1$.

At first glance, the possibility of finding nontrivial analytic solutions of the Einstein-Skyrme system may seem hopeless. Until a few years ago, no analytic solutions of the Skyrme model in flat space had been found. Quite recently, however, the generalized hedgehog ansatz (introduced in [27] and its generalizations in [28–35]) allowed for the construction of exact multi-Skyrmion configurations as well as the first analytic gravitating Skyrmions [34]. Moreover, these approaches also work in the Yang-Mills case [35].

In [34], the full Einstein-Skyrme field equations, in the Bianchi IX case and the $W = 1$ sector, reduce to a system of two autonomous second-order ODEs for two scale factors, where the Skyrme field equations, which are usually the difficult part of the problem, are automatically satisfied in this ansatz. Such a system allows addressing the question of whether or not the $W = 1$ Skyrmion—which is known to be spherically symmetric in flat space—retains this symmetry when coupled to gravity. A preliminary analysis in the reference [36] suggests that the answer should depend on the value of the cosmological constant and not just on its sign. Here we generalize the analysis of

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[36], confirming that the cosmological constant is one of the relevant parameters of the dynamical evolution. Moreover, we also clarify in which sense the $t \to \infty$ evolution of the system is "asymptotically" spherically symmetric, that is, "asymptotically isotropic." This paper is organized as follows.

The action integral for the Einstein-Skyrme model with cosmological constant is presented in Sec. II, where we introduce the self-gravitating Skyrmion model in the background geometry of a locally rotational Bianchi IX universe. The remaining equations are those of general relativity in which the energy momentum tensor is produced by the Skyrmion. In Sec. III, the field equations are shown to describe a mechanical system of two degrees of freedom. In the limit in which the Bianchi IX spacetime is isotropic corresponding to an Einstein-Skyrme system with $W = 1$, the integrability of the Ermakov-Pinney system provides a solution, including a special solutions for the static Einstein universe. In Sec. IV, we analyze the stability of the isotropic $W = 1$ solution by studying the first-order perturbations around it and show that it is not stable. However, for a positive cosmological constant, we show that the final universe is approximately isotropic. The discussion of our results and our conclusions are given in Sec. V.

II. THE ACTION INTEGRAL

We are interested in self-gravitating Skyrmions for the $SU(2)$ group described by the action

$$I[g, U] = \int d^4 x \sqrt{-g} \left(\frac{R - 2\Lambda}{2\kappa} + \frac{K}{4} \mathrm{Tr} \left[A^\mu A_\mu + \frac{\lambda}{8} F_{\mu\nu} F^{\mu\nu} \right] \right). \tag{1}$$

Here A_μ is a shorthand for the Maurer-Cartan form $U^{-1} \nabla_\mu U$, with $U \in SU(2)$ and $F_{\mu\nu} = [A_\mu, A_\nu]$; $A_\mu = A_\mu^j t_j$ where $t_j = -i\sigma_j$ are the $SU(2)$ generators, and σ_j are the Pauli matrices. In our conventions $c = \hbar = 1$, the spacetime signature is $(-, +, +, +)$ and greek indices run over spacetime. Moreover, R is the Ricci scalar, Λ is the cosmological constant and κ is the gravitational constant. Here K and λ are (positive) coupling constants, related to the experimentally determined phenomenological parameters F_π and e through [11]

$$K = \frac{1}{4} F_\pi^2, \qquad K\lambda = \frac{1}{e^2},$$

$$F_\pi = 186 \text{ MeV}, \qquad e = 5.45.$$

The Skyrme equation, obtained by varying (1) with respect to U, together with Einstein's equations are

$$\nabla^\mu A_\mu + \frac{\lambda}{4} \nabla^\mu [A^\nu, F_{\mu\nu}] = 0, \tag{2a}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2b}$$

where $G_{\mu\nu}$ is the Einstein tensor and the energy-momentum tensor for the Skyrme field is

$$T_{\mu\nu} = -\frac{K}{2} \mathrm{Tr} \left[A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A^\alpha A_\alpha + \frac{\lambda}{4} \left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right]. \tag{3}$$

A. Static self-gravitating Skyrmion

The spacetime geometry for the static solutions of the coupled system (2) is the product $R \times S^3$,

$$ds^2 = -dt^2 + \frac{\rho_0^2}{4} [(d\gamma + \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2], \tag{4}$$

where $0 \leq \gamma \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ are the coordinates on the 3-sphere of constant radius ρ_0.

Following [27–30,32,34], we adopt the standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$ as

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu) \mathbb{I} \pm Y^i(x^\mu) t_i, \quad (Y^0)^2 + Y^i Y_i = 1, \tag{5}$$

where $\mathbb{I}$ is the 2×2 identity matrix. The unit vector $Y^A = (Y^0, Y^i)$ defines the embedded three sphere, which is naturally given by

$$Y^0 = \cos\alpha, \qquad Y^i = n^i \sin\alpha, \tag{6a}$$

$$n^1 = \sin\Theta \cos\Phi, \quad n^2 = \sin\Theta \sin\Phi, \quad n^3 = \cos\Theta. \tag{6b}$$

With this information, one can solve (2a) for α, Θ, and Φ as functions of γ, θ, and φ. It can be directly checked that the configuration

$$\Phi = \frac{\gamma + \varphi}{2}, \qquad \tan\Theta = \frac{\cot(\frac{\theta}{2})}{\cos(\frac{\gamma - \varphi}{2})},$$

$$\tan\alpha = \frac{\sqrt{1 + \tan^2\Theta}}{\tan(\frac{\gamma - \varphi}{2})}, \tag{7}$$

identically satisfies the Skyrme equations (2a) in the background metric (4). This was already noted long ago by Manton and Ruback [37] (see also [38]). Those authors, however, did not produce a consistent solution taking into account the backreaction of the Skyrmion on the geometry. In other words, they did not attempt to solve the Einstein equations (2b) with the stress-energy tensor (3) generated by a Skyrmion U of the form (5), (6), (7). Plugging (7) into (6) and (5), the only nonvanishing components of T^ν_μ are found to be

$$T^t_t = -\frac{3K(\lambda + \rho_0^2)}{2\rho_0^4}, \qquad T^\gamma_\gamma = T^\theta_\theta = T^\varphi_\varphi = \frac{K(\lambda - \rho_0^2)}{2\rho_0^4}. \tag{8}$$

It can be observed that although the solution U explicitly depends on the angles γ, θ and φ, the energy-momentum tensor does not, which means that the backreaction should not upset the isometries of the background geometry (4). Solving Einstein's equations with the energy-momentum tensor (8) algebraically fixes the radius of the three-dimensional sphere and the cosmological constant in terms of the remaining parameters in the action

$$\rho_0^2 = \frac{2\lambda\kappa K}{2 - \kappa K}, \qquad \Lambda = \frac{3(2 - \kappa K)^2}{8\lambda\kappa K}. \qquad (9)$$

Hence, the metric (4) together with the static Skyrmion (5), (6), and (7) define a self-consistent solution of the full Einstein-Skyrme system (2) provided the conditions (9) are satisfied. Note that this requires λ, $(2 - \kappa K)$ and λ to have the same sign, which we take tentatively positive. This solution is the self-gravitating generalization of the Skyrmions in [37]. It is useful to stress here that the above constraint is only needed if one wants a static solution with $a(t) = 1$. On the other hand, all rest of the analysis of the present paper will hold for generic values of the coupling constants and cosmological constant.

Our result can also be seen as a generalization of the hedgehog ansatz discussed in [27] that allows for the construction of exact multi-Skyrmion configurations composed by elementary spherically symmetric Skyrmions with nontrivial winding number in four dimensions [29,30].

On any three-dimensional constant time hypersurface, the winding number for the configuration is

$$w = \frac{-1}{24\pi^2} \int \mathrm{Tr}[\epsilon^{ijk} A_i A_j A_k] = +1, \qquad (10)$$

which implies that this Skyrmion cannot be continuously deformed to the trivial $SU(2)$ vacuum, $U = 1$ [5].

B. Bianchi-IX self-gravitating Skyrmions

Remarkably, the above static Skyrmion can be promoted to a time-dependent solution in which the spacetime metric is of the Bianchi type-IX described by the metric

$$ds^2 = -dt^2 + \frac{\rho(t)^2}{4}[a^2(t)(d\gamma + \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2], \qquad (11)$$

where $\rho(t)$ is a global scaling factor and $a(t)$ is a squashing coefficient. As can be directly verified, a Skyrmion of the same form as before (5), with Y^0 and Y^i still given by (6), still identically satisfies the Skyrme field equations in a time-dependent background geometry of the form (11). The technical reason why this happens is that the scale factor ρ and the squashing parameter a depend only on time, while the Skyrme ansatz depends only on the spatial coordinates. This is actually consistent with an ansatz for the Skyrmion in which the full Skyrme system is consistently reduced to a

single scalar equation for the profile [27,28]. The Skyrmion in this case still has baryon charge $+1$.

III. THE TIME-DEPENDENT SYSTEM

The full Einstein-Skyrme field equations (2) with the metric (11) reduce to

$$2a\rho^2(2\rho\dot{a} + 3a\dot{\rho})\dot{\rho} - 2a^2\rho^2(\Lambda\rho^2 + a^2 - 4)$$
$$- \kappa K[(2\rho^2 + \lambda)a^2 + \rho^2 + 2\lambda] = 0, \qquad (12a)$$

$$2a^2\rho^2(2\rho\ddot{\rho} + \dot{\rho}^2) - 2a^2\rho^2(\Lambda\rho^2 + 3a^2 - 4)$$
$$- \kappa K[(2\rho^2 + \lambda)a^2 - \rho^2 - 2\lambda] = 0, \qquad (12b)$$

$$a\rho^3(\rho\ddot{a} + 3\dot{\rho}\,\dot{a}) + (a^2 - 1)[\kappa K(\lambda + \rho^2) + 4a^2\rho^2] = 0. \qquad (12c)$$

The function $a(t)$ describes the deviations from spherical symmetry. For $a(t) = \pm 1$, the spatial sections are three-spheres and so the solution has full spherical symmetry (which is expected for a gravitating soliton of charge 1 which, on a flat background, has spherical symmetry). Thus, an interesting question would be whether or not the solutions of the above system of equations have the property that

$$\lim_{t \to +\infty} a(t) = \pm 1, \qquad (13)$$

which would mean that the solutions approach the "most symmetric configuration." Alternatively, when this condition is violated, spherical symmetry is "spontaneously" broken. The flat Skyrmion of charge ± 1 in flat spacetime is isotropic (see, for instance, [5]), whereas if Eq. (13) does not hold, the gravitating Skyrmion is not spherically symmetric.

As seen in [34], assuming $a(t) = \pm 1$ turns (12a), (12b), and (12c) into a consistent one-dimensional dynamical system for $\rho(t)$, which can be solved explicitly, as discussed in the following sections. A preliminary analysis of the interesting properties of this system for generic $a(t)$ was presented in [36]. In the present paper, we will generalize the analysis of [36] clarifying the issue of the final state of the dynamical system. In particular, we address the question of whether (13) holds and in which sense this is a stable condition. The integrability properties of the reduced dynamical system for $a(t) = \pm 1$ will also be analyzed.

A. Minisuperspace Lagrangian and Hamiltonian

It is convenient to write the dynamical system made of Eqs. (12a), (12b), and (12c) using Hamiltonian formalism. The first step is to observe that Eqs. (12b) and (12c) follow from the variational principle of the following Lagrange function,

$$L(x^k, \dot{x}^k) = L_{\mathrm{GR}} + V_\Lambda + V_{Sk}, \qquad (14)$$

where L_{GR} is the Lagrangian of general relativity (GR) in the mini-superspace geometries of the form (11), i.e.

$$L_{GR}(a, \dot{a}, \rho, \dot{\rho}) = (2\rho^2 \dot{a}\dot{\rho} + 3a\rho\dot{\rho}^2) + (a^2 - 4)a\rho. \quad (15)$$

It can be checked that varying L with respect to a and ρ yields (12b), (12c), where V_Λ and V_{Sk} are the potential terms which correspond to the cosmological constant and to the Skyrmion field,

$$V_\Lambda(a, \rho) = \Lambda a \rho^3,$$

$$V_{Sk}(a, \rho) = \frac{\kappa K(a^2(2\rho^2 + \lambda) + \rho^2 + 2\lambda)}{2a\rho}. \quad (16)$$

Since Lagrangian (14) describes an autonomous system invariant under time translations generated by ∂_t, Noether's theorem implies energy conservation, which turns out to be the left-hand side of (12a). The fact that the energy vanishes reflects the fact that, in general relativity, it is constrained to be zero by invariance under time reparametrizations, $t \to \tau(t)$. In a generic time choice, the metric (11) is

$$ds^2 = -N^2(\tau)d\tau^2 + \frac{\rho^2(\tau)}{4}[a^2(\tau)(d\gamma + \cos\theta d\varphi)^2$$
$$+ d\theta^2 + \sin^2\theta d\varphi^2], \quad (17)$$

where $N(\tau)d\tau = dt$. In this parametrization, the Lagrangian is

$$\bar{L}(N, a, \dot{a}, \rho, \dot{\rho}) = \frac{1}{N}(2\rho^2\dot{a}\dot{\rho} + 3a\rho\dot{\rho}^2)$$
$$- N(a^2 - 4)a\rho + NV_\Lambda + NV_{Sk}. \quad (18)$$

Here it is manifest that the only dynamical degrees of freedom of the system are metric coefficients ρ and a, and the Skyrmion does not bring in new dynamical variables. Then, varying with respect to the variables N, a, and ρ yields equations (12a), (12b) and (12c), respectively. The corresponding Hamiltonian for this system is

$$\mathcal{H} \equiv N\left[\frac{p_a p_\rho}{2\rho^2} - \frac{3a}{4\rho^3}p_a^2 - (a^2 - 4)a\rho - V_\Lambda - V_{Sk}\right], \quad (19)$$

and the Legendre transformation from a, ρ, N to p_a, p_ρ, π_N reads

$$p_a = \frac{2\rho^2}{N}\dot{\rho}, \qquad p_\rho = \frac{2\rho^2}{N}\dot{a} + \frac{3a}{\rho}\dot{\rho}, \qquad \pi_N = 0. \quad (20)$$

B. Isotropic spacetime and the Ermakov-Pinney equation

For the spherically symmetric spacetime $a^2 = 1$, (12c) is identically satisfied, while (12a) and (12b) reduce to the following system [34]:

$$\dot{\rho}^2 = \frac{\Lambda}{3}\rho^2 + \frac{\lambda\kappa K}{2\rho^2} + \frac{\kappa K - 2}{2}, \quad (21)$$

$$\ddot{\rho} = \frac{\Lambda}{3}\rho - \frac{\lambda\kappa K}{2\rho^3}. \quad (22)$$

As noted before, (21) is the vanishing energy constraint, while (22) is a particular case of the well-known Ermakov-Pinney (EP) equation[1] [40,41], which is also found in various physical systems (see, for instance, [39,42,47]). One of its features is that it is invariant under a larger than expected symmetry, $SL(2, R)$ in this case. The representation of the symmetry algebra depends on whether $\Lambda \lessgtr 0$. Specifically, the generators of the $SL(3, R)$ Lie algebra are the autonomous symmetry $\Gamma^1 = \partial_t$, and the two generators Γ^2 and Γ^3 with representations

$$\Gamma^2_{(\Lambda>0)} = \frac{2}{\omega}\sinh(\omega t)\partial_t + \cosh(\omega t)\rho\partial_\rho,$$

$$\Gamma^3_{(\Lambda>0)} = \frac{2}{\omega}\cosh(\omega t)\partial_t + \sinh(\omega t)\rho\partial_\rho \quad (23)$$

for positive cosmological constant, where $\omega^2 := 4|\Lambda|/3$, or

$$\Gamma^2_{(\Lambda<0)} = \frac{2}{\omega}\sin(\omega t)\partial_t + \cos(\omega t)\rho\partial_\rho,$$

$$\Gamma^3_{(\Lambda<0)} = \frac{2}{\omega}\cos(\omega t)\partial_t - \sin(\omega t)\rho\partial_\rho \quad (24)$$

for negative cosmological constant, while when $\Lambda = 0$, the generators take the simple form

$$\Gamma^2_{(\Lambda=0)} = 2t\partial_t + \rho\partial_\rho, \Gamma^3_{(\Lambda=0)} = t^2\partial_t + t\rho\partial_\rho. \quad (25)$$

The solution of the EP equation (22) can be expressed, using a generic solution of the associated linear equation $\ddot{\rho} = \frac{\Lambda}{3}\rho$ [39,41], as

$$\omega^2\rho^2 = -(K-2) + (K - 2 + \rho_0^2\omega^2)\cosh(\omega t)$$
$$\pm \omega\sqrt{2\rho_0^2(K-2) + 2\kappa K\lambda + \rho_0^4\omega^2}\sinh\omega t (\Lambda > 0) \quad (26)$$

for $\Lambda > 0$ and

$$\omega^2\rho^2 = K - 2 + (-(K-2) + \rho_0^2\omega^2)\cos(\omega t)$$
$$\pm \omega\sqrt{2\rho_0^2(K-2) + 2\kappa K\lambda - \rho_0^4\omega^2}\sin\omega t (\Lambda < 0), \quad (27)$$

for $\Lambda < 0$, where $\rho_0 = \rho(0)$ and the second integration constant have been eliminated by the constraint equation (21).

Furthermore, for $\Lambda = 0$, the solution is a power law,

$$\rho^2 = \rho_1(t - t_0)^2 + \rho_0, \quad (28)$$

where $\rho_0 = \frac{\lambda\kappa K}{2 - \kappa K}$ and $\rho_1 = \frac{\kappa K - 2}{2}$.

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[1]The EP equation has the form $\ddot{u} + \omega^2 u + bu^{-3} = 0$ and admits exact solutions $u = F(y_1, y_2)$ where y_1, y_2 are the independent solutions of the associated problem $\ddot{y} + \omega^2 y = 0$ [39].
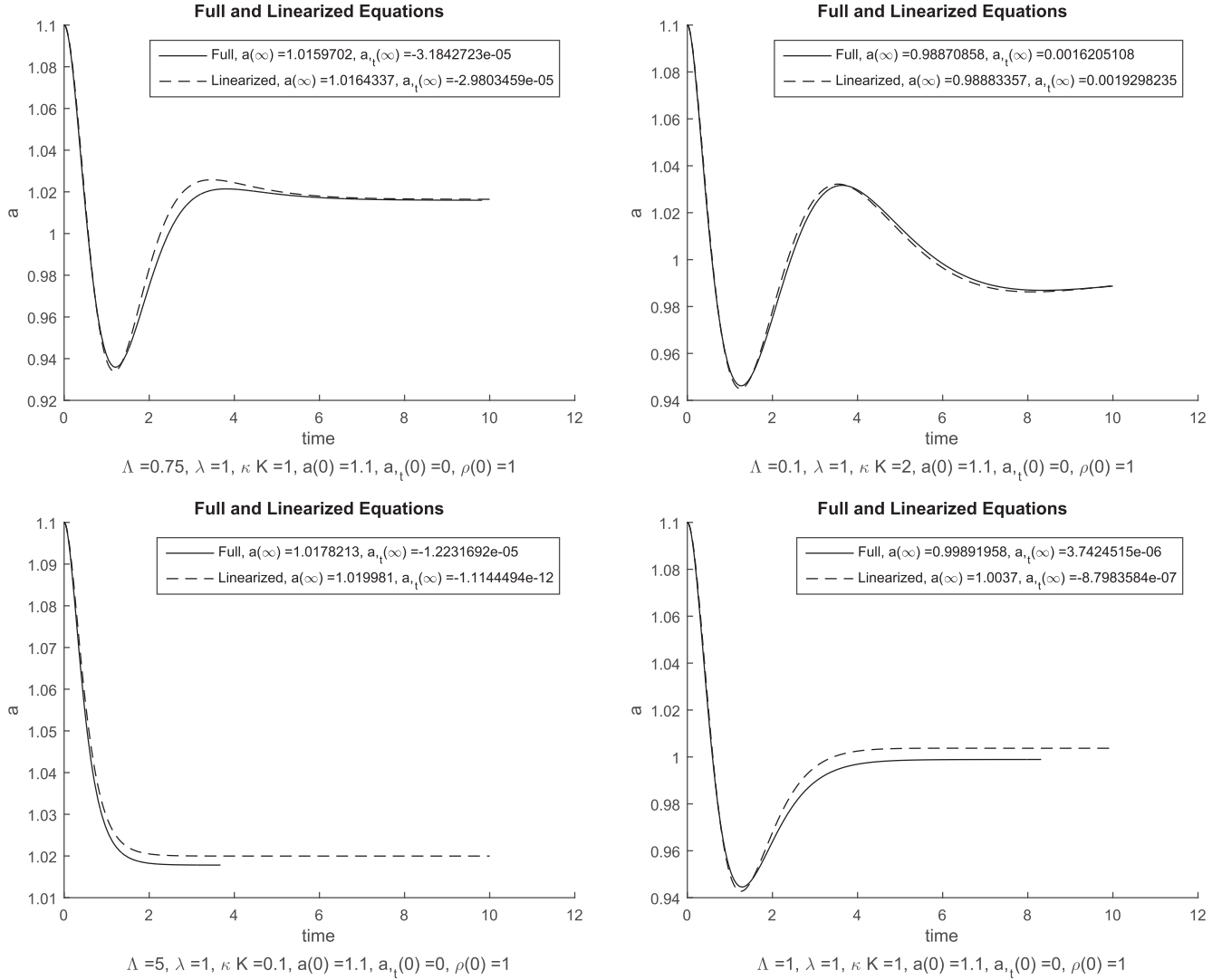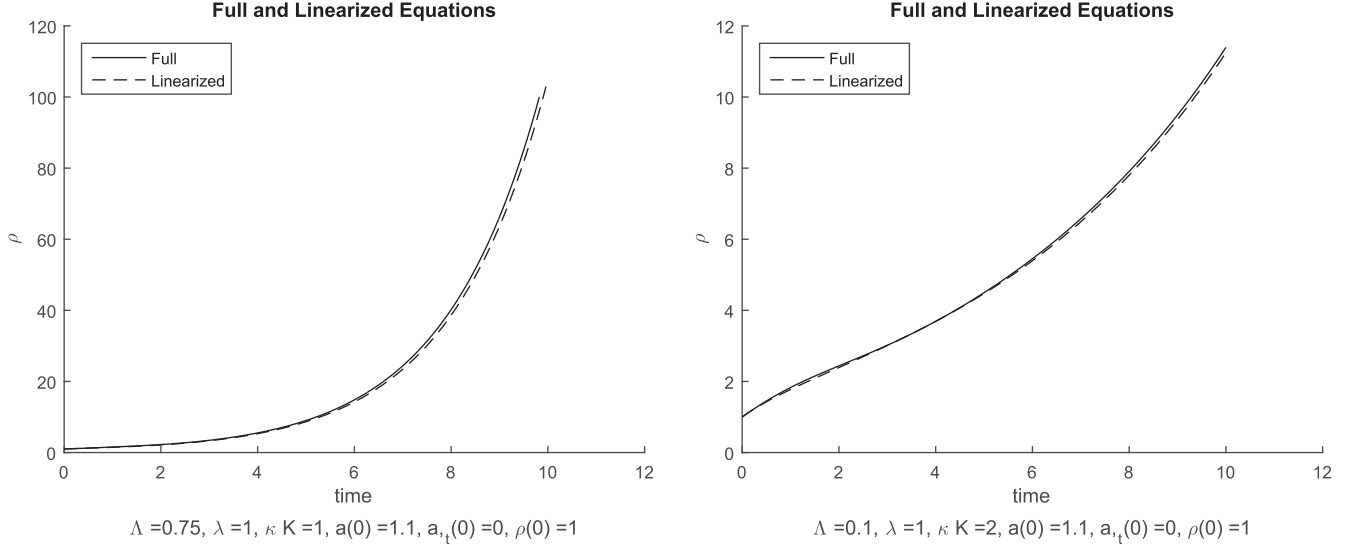
FIG. 1.   Qualitative behavior of the general solution perturbed around the stable solution $a = 1$ which is given by the Ermakov-Pinney equation. The plots are for the function $a(t)$ which follow from the total system or the linearized system and for various values of the free parameters where $\Lambda > 0$.

We note that the functional forms of the exact solutions are related with the representation of the corresponding admitted $\mathbf{sl}(2, R)$ Lie algebra. From the exact solutions in which $a^2(t) = 1$ we observe that for positive cosmological constant the spacetime (11) has a de Sitter evolution, while for negative cosmological constant the scale factor $\rho(t)$ is periodic with frequency ω. Finally, for zero cosmological constant and for $t \rightarrow \infty$ the spacetime (11) describes the Milne universe.

C. Einstein static universe

In order to examine the stability properties of the static Einstein universe around the isotropic solutions (26) and (27), let us consider the critical points for the field equations (12a)–(12c). The critical points of the Hamiltonian (19) are given by the conditions

$$\frac{\partial V_{\text{eff}}}{\partial a} = 0 \quad \text{and} \quad \frac{\partial V_{\text{eff}}}{\partial \rho} = 0, \tag{29}$$

where $V_{\text{eff}} = -(a^2 - 4)a\rho - V_\Lambda - V_{Sk}$. Taking into account the additional constraint (12a)—which reduces to $V_{\text{eff}} = 0$, the critical points in the (ρ, a) plane are identified as[2]

$$P_{\pm}: \rho_c = \left[\frac{3(2 - \kappa K)}{4\Lambda}\right]^{1/2} = \left[\frac{3\lambda\kappa K\Lambda}{2}\right]^{1/4}, \qquad a_c = \pm 1. \tag{30}$$

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[2]For $\kappa K < 0$ and $\lambda < 0$ there would be an additional possible critical point with $a_0 \neq 0$ at, $\tilde{P}_0 : \tilde{\rho}_c = \left[\frac{8 - \kappa K}{2\Lambda}\right]^{1/2} = \left[-\lambda\frac{a_0^2 + 4}{a_0^2 + 2}\right]^{1/2}$ with $a_c = a_0 \neq 0$ and $\kappa K = -2a_0^2(a_0^2 + 4)$. The critical point $\tilde{P}_0$ can be neglected in the standard situations where $\kappa K \geq 0$.

FIG. 2.   Qualitative behavior of the general solution perturbed around the stable solution $a = 1$ which is given by the Ermakov-Pinney equation. The plots are for the function $\rho(t)$ which follow from the total system or the linearized system and for various values of the free parameters where $\Lambda > 0$.

Observe that for $\kappa K > 2$, the critical points $P_\pm$ exist provided both Λ and λ are negative, while the opposite happens if $\kappa K < 2$ ($\lambda > 0$, $\Lambda > 0$). Last but not least, for zero cosmological constant, $P_\pm$ exist if and only if $kK = 2$ and $\lambda = 0 = \Lambda$.

Finally, we note that these critical points in momentum space are located at $(p_a, p_\rho) = (0,0)$ and therefore they correspond to static configurations. It should be noted that the critical points $P_\pm$ are exact solutions of the field equations and describe isotropic Einstein static spacetimes 43,44]] and therefore perturbing around them is a meaningful test for the stability of the solutions. In the next section, we examine the stability of the critical points $P_\pm$ in the linearized approximation of the time-dependent field equations.

IV. STABILITY OF THE SPHERICALLY SYMMETRIC SKYRMION

Let us now study the evolution of an infinitesimal perturbation around the classical solution near the critical point for $a = 1$,

$$a := 1 + u(t), \qquad \rho := \rho_E + v(t), \tag{31}$$

where ρ_E stands for the exact solution of the EP equation (22), and u and v are the small perturbations. Substituting this into (12) and keeping up to first order in u and v, one finds (from now on we drop the label E from the exact solution ρ_E)

$$0 = \ddot{u} + 3\frac{\dot{\rho}}{\rho}\dot{u} + 2[K\kappa + 4 + K\kappa\lambda\rho^{-2}]\rho^{-2}u, \tag{32a}$$

$$0 = \ddot{v} + \frac{\dot{\rho}}{\rho}\dot{v} + \rho^{-2}[4 + 4(\dot{\rho})^2 - 4\kappa K + 12\ddot{\rho}\rho - 8\Lambda\rho^2]v$$

$$+ \frac{1}{2}\rho^{-3}[\kappa K - 2\Lambda\rho^4 - \kappa K\lambda - 4\rho + 2(\dot{\rho}\rho)^2 - 2\kappa K\rho^2 + 4\rho^3\ddot{\rho}]u$$

$$+ \frac{1}{2}[1 + (\dot{\rho})^2 - \kappa K + 2\ddot{\rho}\rho - \Lambda\rho^2]\rho^{-1} + \frac{\kappa K}{4}(-\lambda + 2\Lambda + 1)\rho^{-3}, \tag{32b}$$

$$0 = \frac{\dot{\rho}}{\rho}\dot{u} + \left[\frac{2}{\rho^2} + 3\left(\frac{\dot{\rho}}{\rho}\right)^2 - \frac{\kappa K\lambda}{2\rho^4} - \frac{\kappa K}{\rho^2} - \Lambda\right]u + 3\frac{\dot{\rho}}{\rho}\frac{\dot{v}}{\rho} + \left[\frac{3}{\rho^2} + 3\left(\frac{\dot{\rho}}{\rho}\right)^2 - \frac{3\kappa K}{\rho^2} - 2\Lambda\right]\frac{v}{\rho}$$

$$+ \left[\frac{3}{2\rho^2} + 3\left(\frac{\dot{\rho}}{\rho}\right)^2 - \frac{3\kappa K}{4\rho^2} - \frac{\Lambda}{2}\right] - \frac{\kappa K}{4\rho^4}(2\Lambda + \lambda). \tag{32c}$$

Since the solution $\rho(t)$ for (22) is explicitly known, Eq. (32a) is an ODE for $u(t)$ that can be directly solved. If $\Lambda > 0$, (26) implies $\rho \sim \rho_0 e^{(\omega/2)t}$ for $t \to \infty$. In this limit, Eq. (32a) reduces to $\ddot{u} + (3\omega/2)\dot{u} = 0$, whose solution is
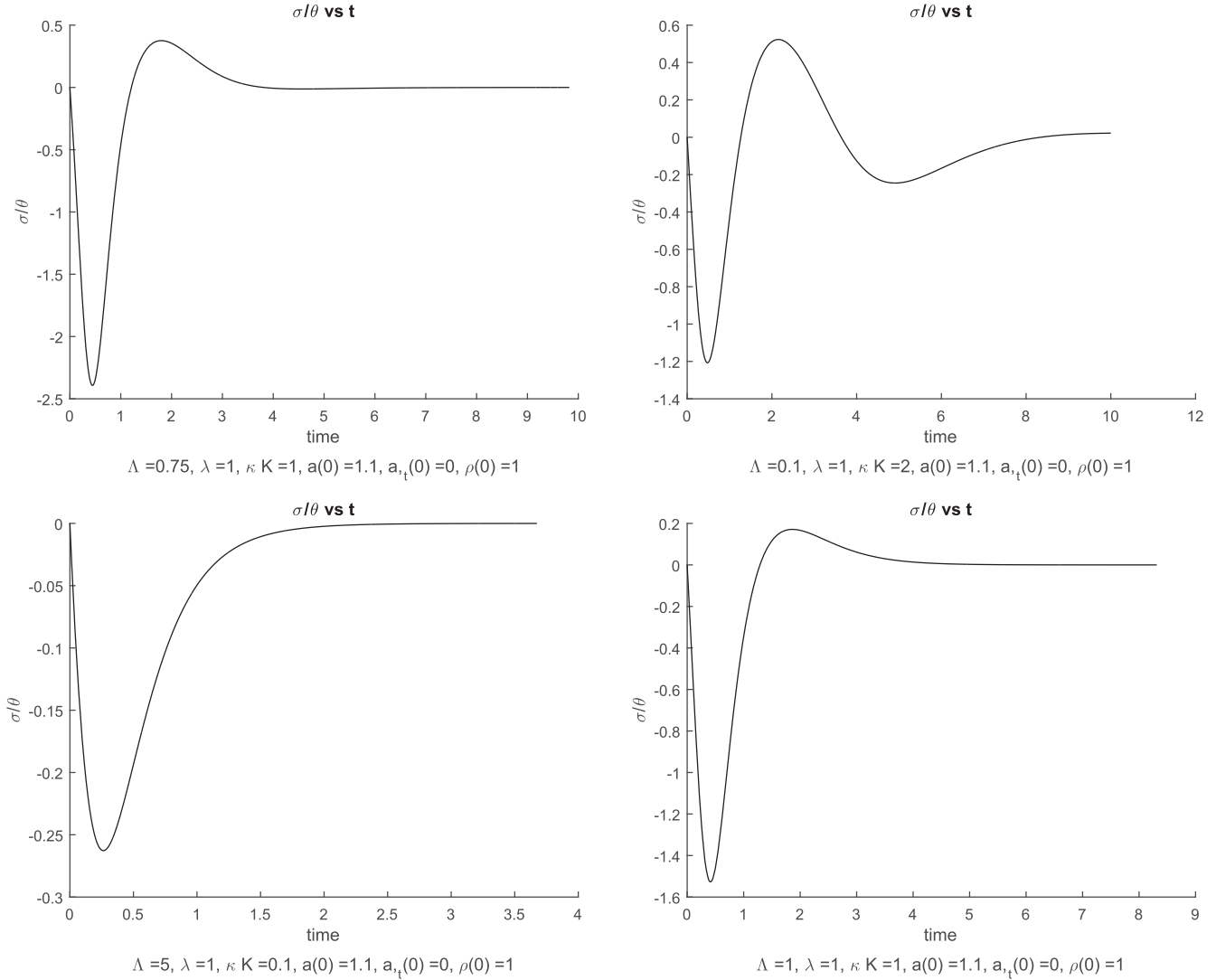
FIG. 3.   Qualitative behavior of the general solution perturbed around the stable solution $a = 1$ which is given by the Ermakov-Pinney equation. The plots are for the function $\sigma(t)/\theta(t)$ which follow from the solution of the field equations with various values of the free parameters where $\Lambda > 0$. The initial conditions are that of fig 1.

$$u(t) = u_0 e^{-(3\omega/2)t} + c, \qquad (33)$$

where u_0 and c are arbitrary constants fixed by the initial conditions of the perturbations. This means that for $t \to \infty$, a can approach any constant value $1 + c$ and there is nothing special about $a = 1$ or $a \neq 1$. In fact, Eq. (32a) has the form of a damped oscillator driven by an effective harmonic potential $u^2[K\kappa + 4 + K\kappa\lambda\rho^{-2}]\rho^{-2}$, which vanishes exponentially for $t \to \infty$, as well as all of its derivatives. This is a case of the so-called "neutral equilibrium" [45].

Having found u, Eq. (32b) can now be solved for v. Substituting the asymptotic expression for ρ, (32b) takes the form

$$0 = \ddot{v} + \frac{\omega}{2}\dot{v} - 2\omega^2 v, \qquad (34)$$

whose solution is

$$v(t) = v_0 e^{mt} \qquad (35)$$

with $m = (-1 \pm \sqrt{33})\omega/4$. This means that $v(t)$ either vanishes or blows up for large t. Which of the two branches actually occurs is decided by the constraint equation (32c). This last equation is identically satisfied by the exponentially decaying perturbation and is grossly violated by the unstable branch. It is, therefore, verified that under a small perturbation around the critical point $\{\rho = \rho_E, a = 1\}$, the solution settles to $\{\rho = \rho_E, a = 1 + c\}$.

Numerical simulations of the system (32) and of the original equations (12) with initial conditions around $a = 1$ are summarized in Figs. 1–4. For $\Lambda > 0$, Fig. 1 shows the scalar factor $a(t)$ while 2 describes the behavior of $\rho(t)$. These figures show that for large t, $\rho(t) \to +\infty$ and $a(t) \to a_0 = \pm 1 + c$, where c is the constant of (33) that

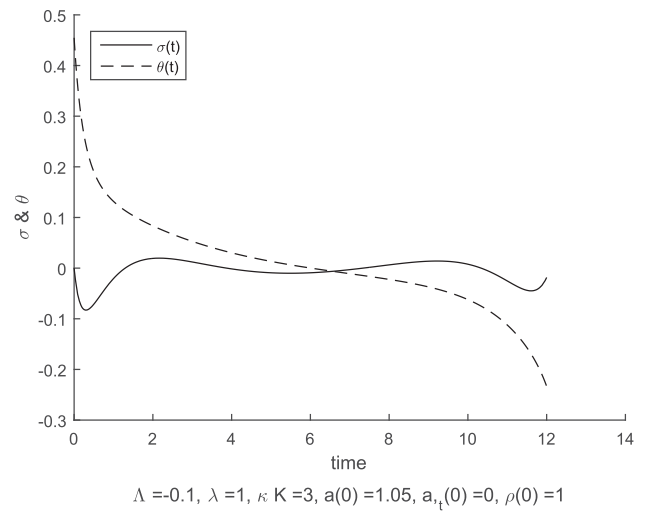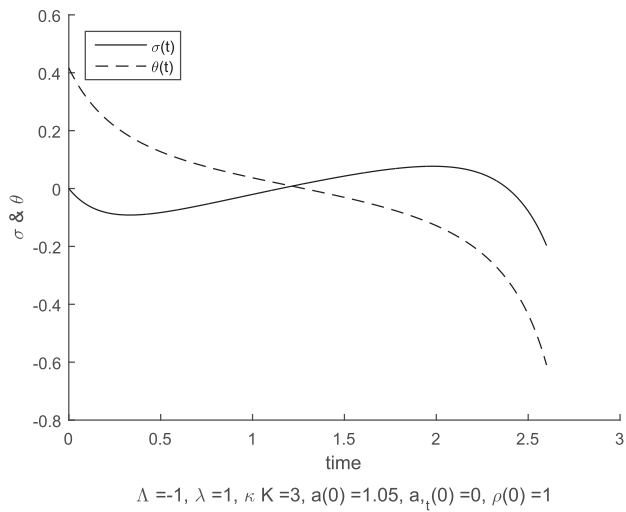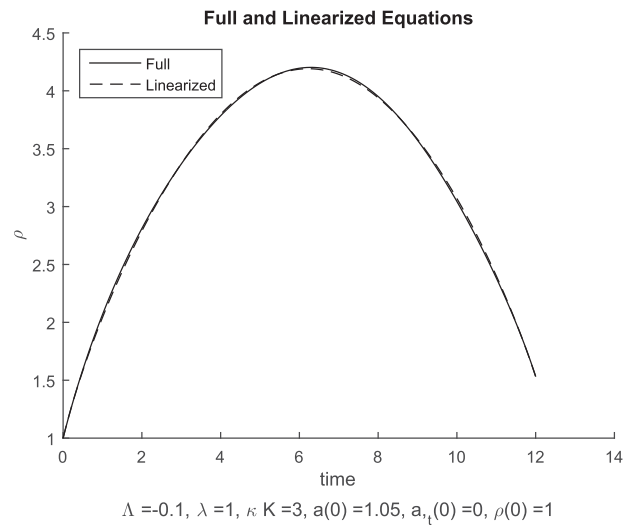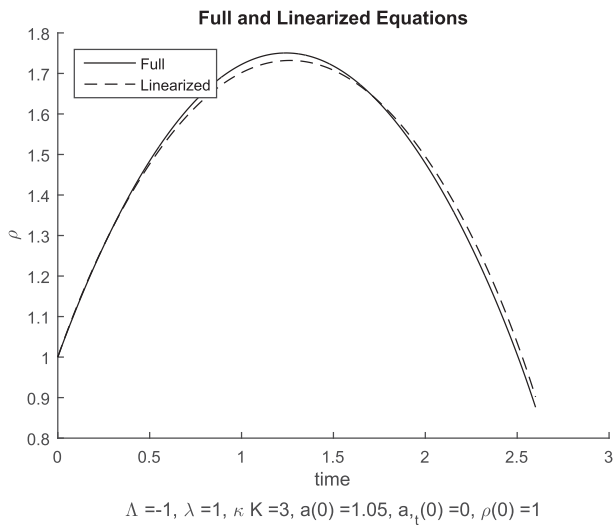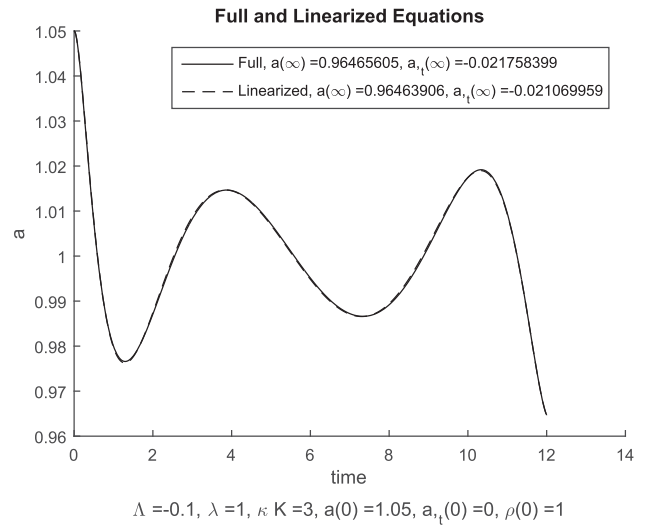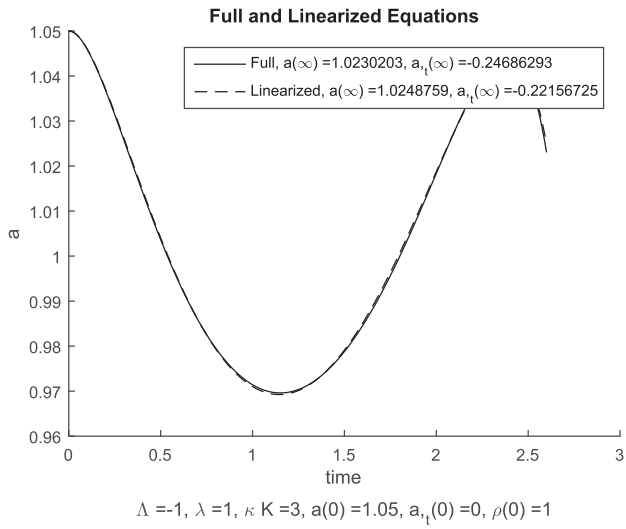FIG. 4. Qualitative behavior of the general solution perturbed around the stable solution $a = 1$ which is given by the Ermakov-Pinney equation. The plots are for the function $a(t), \rho(t)$ and $\sigma(t)/\theta(t)$ which follow from the total system or the linearized system and for various values of the free parameters where $\Lambda < 0$.
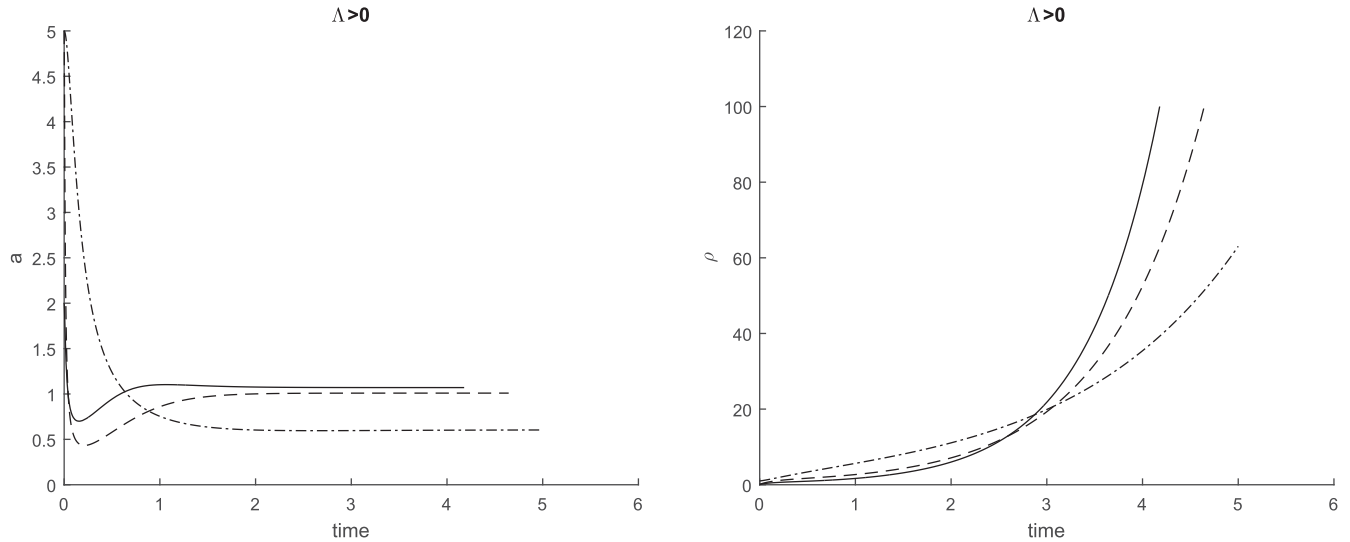
FIG. 5.   Qualitative behavior of the solution of the field equations (12a)–(12c) for initial conditions far for $a(0) = 1$. The left plot is for the scale factor $a(t)$, while the right plot is for the scale factor $\rho(t)$. The lines are for various values of the free parameters, where $\Lambda > 0$.

can take any value depending on the initial conditions. Although the solution is not strictly stable around $a^2(t) = 1$, the spacetime for $t \to \infty$ is an infinitely large squashed sphere and therefore to a good approximation, locally indistinguishable from a sphere. The main reason is that, when $\Lambda > 0$, the terms in the dynamical system which lead to the instability of the isotropic solution are suppressed for $t \to +\infty$ so that, effectively, such "destabilizing" terms only *act for a finite time* after which the value of $a(t)$ becomes constant as we will see in the next Section. The peculiar neutral equilibrium feature of the present system means that if the initial data are close to $a^2 = 1$, for later times $a^2(t)$ approach a_0^2 in the vicinity of 1.

A numerical simulation for the case $\Lambda < 0$ is shown in Fig. 4. In this case $\rho(t)$ is periodic and may vanish for
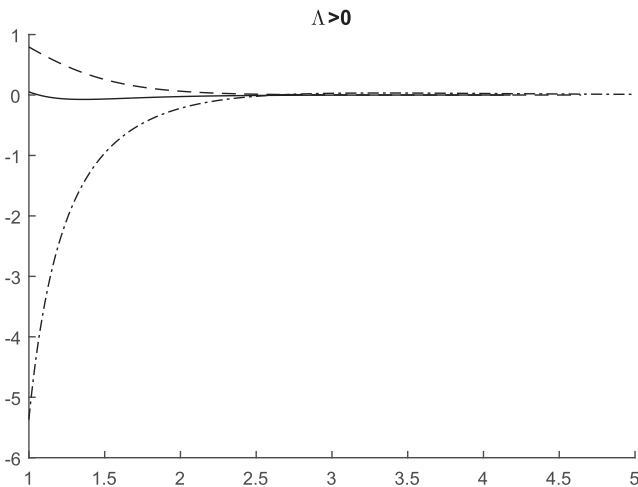

FIG. 6.   Qualitative behavior of the anisotropic parameter $\sigma(t)/\theta(t)$, for the solutions of Fig. 5.

specific initial conditions. In that case, the solution $u(t)$ from (32a) reaches a singularity for which $\ddot{u}(t) \to \infty$. It is straightforward to see that in general $u(t)$ is not a decreasing function which means that the EP solution is unstable.

A. Asymptotically isotropic spacetime

Let us now examine the isotropization of spacetime for large t. According to [46], if a solution of the field equations (12a)–(12c), in the limit $t \to +\infty$, satisfies the conditions: (a) the global scale factor $\rho(t)$ is going to infinity, i.e. $\rho(t) \to +\infty$, (b) the anisotropic parameter $a(t)$ becomes constant, $a(t) \to a_0$, (c) the weak energy condition is not violated $T^{00} > 0$, while it holds $T^{0i}/T^{00} \to 0$ and (d) the ratio of the shear σ with the expansion rate θ vanishes, i.e. $\frac{\sigma}{\theta} \to 0$, then the spacetime (11) will be asymptotically isotropic. $T^{\mu\nu}$ is the energy momentum tensor, the kinematic quantities σ and θ are defined by the observer $u^\mu = \delta_t^\mu (u^\nu u_\nu = -1)$, such as $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$, where $\sigma_{\mu\nu} = u_{(\kappa;\lambda)}(h_\mu^\kappa h_\nu^\lambda - \frac{1}{3}\theta h_{\mu\nu})$ and $\theta = u_{(\mu;\nu)}h^{\mu\nu}$ in which $h_{\mu\nu}$ is the projective tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

In Fig. 3, the evolution of the anisotropy parameter σ/θ is presented from where we can see that the ration vanishes.

For $\Lambda > 0$, conditions (a), (b) and (c) are satisfied. Figure 5 shows the evolution of $a(t)$, $\rho(t)$ for the system (12a)–(12c) with $\Lambda > 0$ and initial conditions far from the point $a_0 = 1$. We observe that as $t \to +\infty$, conditions (a) and (b) are satisfied, while Fig. 6 shows that condition (d) is also satisfied, because $\sigma = \frac{\sqrt{6}}{3}|\frac{\dot{a}}{a}|$, which implies that spacetime is asymptotically isotropic. On the other hand, for $\Lambda < 0$, condition (c), $T^{00} > 0$, can be violated which means that the "isotropization" is not guaranteed.

The present analysis shows that in general, the exact solution (26) with $a(t) = \pm 1$ is unstable. However the spacetime is asymptotically isotropic for large values of t. That means that in the late-time the only fluid-term which survives is that of the cosmological constant. That result revises the previous analysis of [36].

V. CONCLUSIONS

We have analyzed the gravitating, time-dependent analytic solutions of the Einstein-Skyrme system with topological charge introduced in [34]. In particular, we have shown that these solutions—whose analogs in flat spacetimes would be spherically symmetric—reach an isotropic asymptotic state for $t \to +\infty$. This question was also analyzed numerically in [36]. In addition, we have shown that the isotropic solution, given by the Ermakov-Pinney equation, itself is not stable configuration, but a state of neutral equilibrium, like a spontaneously broken vacuum. Thus, the isotropy of the charge 1 Skyrmion on flat spaces may be broken by the coupling with Einstein gravity. However, despite this fact, the asymptotic solutions for $\Lambda > 0$ of the dynamical system describing the time-dependent gravitating Skyrmion are asymptotically isotropic in large scales. The main reason is that, when $\Lambda > 0$,

the "destabilizing" terms in the dynamical system (leading to the instability of the isotropic solution) are suppressed for $t \to +\infty$. Consequently, such terms only act for a finite amount of time after which the value of $a(t)$ freezes. To the best of the authors' knowledge, this is the first explicit example of a symmetry breaking induced by the coupling with Einstein gravity of a topological soliton (which on flat spaces would be isotropic) in a realistic theory such as the Skyrme model. Moreover, we have discussed in detail the integrability of the isotropic solution in terms of the Ermakov-Pinney system.

ACKNOWLEDGMENTS

[1] T. Skyrme, Proc. R. Soc. A **260**, 127 (1961); **262**, 237 (1961); Nucl. Phys. **31**, 556 (1962).

[2] E. Witten, Nucl. Phys. **B223**, 422 (1983); **B223**, 433 (1983).

[3] D. Finkelstein and J. Rubinstein, J. Math. Phys. (N.Y.), **9**, 1762 (1968).

[4] H. Weigel, *Chiral Soliton Models for Baryons* (Lecture Notes in Physics) (Springer, New York, 2008).

[5] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2007).

[6] David I. Olive and Peter C. West (Editors), *Duality and Supersymmetric Theories* (Cambridge University Press, Cambridge, England, 1999).

[7] D. Giulini, Mod. Phys. Lett. A **08**, 1917 (1993).

[8] A. P. Balachandran, H. Gomm, and R. D. Sorkin, Nucl. Phys. **B281**, 573 (1987).

[9] A. P. Balachandran, A. Barducci, F. Lizzi, V. G. J. Rodgers, and A. Stern, Phys. Rev. Lett. **52**, 887 (1984).

[10] A. P. Balachandran, F. Lizzi, V. G. J. Rodgers, and A. Stern, Nucl. Phys. **B256**, 525 (1985).

[11] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[12] E. Guadagnini, Nucl. Phys. **B236**, 35 (1984).

[13] Adam, J. Sanchez-Guillen, and A. Wereszczynski, Phys. Lett. B **691**, 105 (2010).

[14] C. Adam, T. Klahn, C. Naya, J. Sanchez-Guillen, R. Vazquez, and A. Wereszczynski, Phys. Rev. D **91**, 125037 (2015).

[15] D. Auckly and M. Speight, Commun. Math. Phys. **263**, 173 (2006).

[16] S. Krusch and M. Speight, Commun. Math. Phys. **264**, 391 (2006).

[17] H. Luckock and I. Moss, Phys. Lett. B **176**, 341 (1986).

[18] S. Droz, M. Heusler, and N. Straumann, Phys. Lett. B **268**, 371 (1991).

[19] S. Droz, M. Heusler, and N. Straumann, Phys. Lett. B **268**, 371 (1991).

[20] M. Heusler, S. Droz, and N. Straumann, Phys. Lett. B **285**, 21 (1992).

[21] S. Droz, M. Heusler, and N. Straumann, Phys. Lett. B **271**, 61 (1991).

[22] N. K. Glendenning, T. Kodama, and F. R. Klinkhamer, Phys. Rev. D **38**, 3226 (1988); B. M. A. G. Piette and G. I. Probert, Phys. Rev. D **75**, 125023 (2007); G. W. Gibbons, C. M. Warnick, and W. W. Wong, J. Math. Phys. (N.Y.) **52**, 012905 (2009); S. Nelmes and B. M. Piette, Phys. Rev. D **84**, 085017 (2011).

[23] P. Bizon and T. Chmaj, Phys. Rev. D **58**, 041501 (1998); P. Bizon, T. Chmaj, and A. Rostworowski, Phys. Rev. D **75**, 121702 (2007); S. Zajac, Acta Phys. Pol. B **40**, 1617 (2009); **42**, 249 (2011).

[24] F. Canfora, A. Giacomini, and S. A. Pavluchenko, Phys. Rev. D **90**, 043516 (2014).

[25] L. Parisi, N. Radicella, and G. Vilasi, Phys. Rev. D **91**, 063533 (2015).

[26] A. Paliathanasis and M. Tsamparlis, J. Geom. Phys. **114**, 1 (2017).

[27] F. Canfora and P. Salgado-Rebolledo, Phys. Rev. D **87**, 045023 (2013).

[28] F. Canfora and H. Maeda, Phys. Rev. D **87**, 084049 (2013).

[29] F. Canfora, Phys. Rev. D **88**, 065028 (2013).

[30] F. Canfora, F. Correa, and J. Zanelli, Phys. Rev. D **90**, 085002 (2014).

[31] S. Chen, Y. Li, and Y. Yang, Phys. Rev. D **89**, 025007 (2014).

[32] F. Canfora, M. Kurkov, M. Di Mauro, and A. Naddeo, Eur. Phys. J. C **75**, 443 (2015).

[33] S. Chen and Y. Yang, Nucl. Phys. **B904**, 470 (2016).

[34] E. Ayon-Beato, F. Canfora, and J. Zanelli, Phys. Lett. B **752**, 201 (2016).

[35] F. Canfora and G. Tallarita, Phys. Rev. D **94**, 025037 (2016).

[36] S. A. Pavluchenko, Phys. Rev. D **94**, 044046 (2016).

[37] N. S. Manton and P. J. Ruback, Phys. Lett. B **181**, 137 (1986); N. S. Manton, Commun. Math. Phys. **111**, 469 (1987).

[38] L. Bratek, Nonlinearity **16**, 1539 (2003).

[39] P. G. L. Leach and K. Andriopoulos, Appl. Anal. Discrete Math. **2**, 146 (2008).

[40] V. Ermakov, Univ. Isz. Kiev Series III **9** (1880) (translated by A O Harin).

[41] E. Pinney, Proc. Am. Math. Soc. **1**, 681 (1950).

[42] F. L. Williams and P. G. Kevrekidis, Classical Quantum Gravity **20**, L177 (2003).

[43] J. D. Barrow, G. Ellis, R. Maartens, and C. Tsagas, Classical Quantum Gravity **20**, L155 (2003).

[44] J. D. Barrow and K. Yamamoto, Phys. Rev. D **85**, 083505 (2012).

[45] V. I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, Vol. 60 (Springer, New York, 1989).

[46] C. B. Collins and S. W. Hawking, Astrophys. J. **180**, 317 (1973).

[47] M. Tsamparlis and A. Paliathanasis, J. Phys. A **45**, 275202 (2012).