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3d fermion-boson map with imaginary chemical potential

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We study the three-dimensional $U(N)$ Gross-Neveu and CP^{N-1} models in the canonical formalism with fixed $U(1)$ charge. For large N, this is closely related to coupling the models to Abelian Chern-Simons fields in a monopole background. We show that the presence of the imaginary chemical potential for the $U(1)$ charge makes the phase structure of the models remarkably similar. We calculate their respective large N free energy densities and show that they are mapped into each other in a precise way. Intriguingly, the mapping of the free energies involves the Bloch-Wigner function and its generalizations introduced by Zagier. We expect that our results are connected to the recently discussed 3d bosonization.

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I. INTRODUCTION

Physics in three dimensions is fascinating, physically relevant, and largely experimentally testable. It is therefore very useful to identify possible universal patterns in it, which eventually might better organize our understanding of the huge web of condensed matter systems. One such recurrent pattern is 3d bosonization [1], via statistical transmutation [2,3]. Remarkably, but perhaps expectedly, 3d bosonization has been very recently [4–7] connected to another recurrent theme of $2+1$ dimensional physics, particle-vortex duality, e.g., [8]. At finite temperature, various dualities between fermionic and bosonic matter theories coupled to non-Abelian Chern-Simons have also been recently discussed (see, for example, [9,10] and references therein). Those works present a remarkable progress in our understanding of three-dimensional physics, and its possible holographic higher-spin duals.

However, if the fermion-boson map is a fundamental property of three-dimensional physics one may wonder whether the presence of non-Abelian Chern-Simons gauge fields is necessary to probe it. With this idea in mind, we revisit the finite temperature phase structure of two three-dimensional systems; the fermionic $U(N)$ Gross-Neveu model and the bosonic CP^{N-1} model. We study both systems in the canonical formalism. This can be elegantly done by introducing an imaginary chemical potential for the $U(1)$ charge. Nevertheless, it seems that we cannot completely forget about the Chern-Simons. The canonical partition function of the systems is intimately related to the partition function of the same systems coupled to an Abelian Chern-Simons gauge field expanded around a monopole background. For that, one arguably needs to assume a suitable mean field approximation [11] as well as a large N expansion. In such a case, the system's $U(1)$ charge density is associated to the Chern-Simons level.

It is well known that the above two models exhibit interesting patterns of symmetry breaking at finite temperature T. The Gross-Neveu model has a parity broken phase at low temperatures, which disappears for a critical temperature. On the other hand, while the CP^{N-1} model exhibits the usual continuous symmetry breaking pattern at zero temperature, the broken phase ceases to exist for $T>0$ in accordance to the Mermin-Wagner-Colleman theorem that forbids continuous symmetry breaking at finite T for two-dimensional systems. What instead happens is that when the coupling is tuned to its critical value at $T=0$, then a finite temperature scaling regime with a nonzero thermal mass for the scalars emerges. Nevertheless, although these models appear to have completely different finite temperature phase structure, we show that this situation changes in the presence of the imaginary chemical potential. For that, we evaluate the canonical partition function of the models for large N by a saddle point expansion to obtain the two gap equations for the symmetry breaking order parameter and for the charge density. We then observe that the two sets of gap equations can be mapped into each other. As a consequence, the fermionic system acquires a finite temperature scaling regime, where a nonzero fermion condensate is possible when the coupling is tuned to its critical value at $T=0$. On the other hand, the bosonic system obtains an symmetry-breaking phase at finite temperature. We claim that the presence of the latter phase is not inconsistent with the Mermin-Wagner-Coleman theorem since the system has been now effectively anyonized and hence the broken symmetry is quite possibly discrete.

Of particular interest are our results for the large N $U(1)$ charge densities of the systems. These are given in terms of the remarkable Bloch-Wigner function [12], and they turn out to be purely imaginary except at two special values of the chemical potential where they vanish. The latter property gives rise to two different critical and $U(1)$-neutral vacua for each one of the models. The values of the

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free energy densities at these vacua exhibit a precise fermion-boson duality, in accordance to expectations.

Finally, we calculate the on-shell large N free energy densities of the systems and show that they can be written in terms of a generalized Bloch-Wigner function introduced by Zagier [13] (see also [14] for a recent discussion). The results reveal the existence of a precise map between them. We find that the sum of the free energy densities for values of the imaginary chemical potential that differ by $i\pi$ is given *exactly* by the Bloch-Wigner function.

In Sec. II we discuss some general aspects of three-dimensional systems at imaginary chemical potential and their relation to Abelian Chern-Simons theories coupled to matter. In Sec. III we present our calculations for the gap equations, the phase structure, and the free energy density of our models. We conclude and raise a number of open questions on Sec. IV. Finally, technical details can be found in the Appendixes.

II. IMAGINARY CHEMICAL POTENTIAL FOR $U(1)$ CHARGE AND CHERN-SIMONS THEORIES

A. Canonical formalism and imaginary chemical potential

Consider a system at finite temperature $T = 1/\beta$ with a global $U(1)$ charge operator $\hat{Q}$. Its canonical partition function can be formally calculated as the thermal average over states of fixed $\hat{Q}$ as

$$Z_c(\beta, Q) = \mathrm{Tr}[\delta(Q - \hat{Q})e^{-\beta\hat{H}}]. \tag{1}$$

We can normalize Z_c by dividing with the thermal partition function so that in the absence of charged states $Z_c(\beta, 0) = 1$. One generally expects that the eigenvalues Q of the charge operator $\hat{Q}$ are integers, in which case an explicit representation for (1) can be written as

$$\begin{aligned} Z_c(\beta, Q) &= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta Q} \mathrm{Tr}[e^{-\beta\hat{H} - i\theta\hat{Q}}] \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta Q} Z_{gc}(\beta, \mu = -i\theta/\beta), \end{aligned} \tag{2}$$

with $Z_{gc}(\beta, \mu)$ being the grand canonical partition function with imaginary chemical potential μ. The latter function exhibits, in general, certain periodicity properties with respect to θ that are intimately connected to the physics of the underlying theory. For example, [15], in QCD-like systems with a $SU(N)$ non-Abelian gauge symmetry and $\hat{Q}$ the fermion number operator, one expects that in the confining phase the spectrum contains only color singlets. In this case Q is a multiple of N and $Z_{gc}(\beta, \mu)$ will be periodic with a θ-period $2\pi/N$. If however there is a phase, e.g., at high temperatures, where fundamental particles turn up in the spectrum, one might expect to find instead a $2\pi\theta$-period. Indeed, although the $\mathbb{Z}_N$ symmetry of the pure $SU(N)$ Yang-Mills action appears to enforce the $2\pi/N$ periodicity, one generically finds a more complicated structure at high temperature, which may be attributed to a deconfining transition [16].

Integrals like (2) may be evaluated by a saddle point analysis. The saddle point equation is

$$iQ - \beta \frac{\partial}{\partial\theta} F_{gc}(\beta, -i\theta/\beta) = 0, \tag{3}$$

where the grand canonical potential (i.e., free energy) is $\beta F_{gc}(\beta, -i\theta/\beta) = -\ln Z_{gc}(\beta, -i\theta/\beta)$. However, in most of the physically relevant situations (i.e., charge conjugation, CP invariance, etc.), the grand canonical partition function is an even function of μ and hence of θ [17]. Therefore, real solutions for Q would require imaginary θ and one returns to the usual case of a real chemical potential.

Nevertheless, an interesting situation can arise if $F_{gc}(\beta, -i\theta/\beta)$ has one (or more) extrema for some real θ_*. In such a case, the canonical partition function of the system in the absence of charged excitations ($Q = 0$) is given to leading order in some approximation scheme such as large N by the grand canonical partition function of the same system at fixed imaginary chemical potential $\mu_* = -i\theta_*/\beta$,

$$Z_c(\beta, 0) \approx e^{-\beta F_{gc}(\beta, -i\theta_*/\beta)}. \tag{4}$$

This is not inconsistent with the normalization of Z_c since fixing the imaginary chemical potential in a system at finite temperature is tantamount to statistical transmutation [18,19]. Despite the absence of charged modes, the system described by (4) is, in general, different from the initial one, as its elementary degrees of freedom would obey different statistics.

B. Chern-Simons coupled to scalars and fermions in a monopole background

It is well known [20] that when scalars and fermions are coupled to a gauge field A_μ at finite temperature the temporal component A_0 is formally equivalent to having an imaginary chemical potential. For example, consider Dirac fermions in three Euclidean dimensions [23] coupled to an Abelian Chern-Simons field at level k. The finite temperature partition function is

$$Z_f(\beta, k) = \int [\mathcal{D}A_\mu][\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp\left[-S_f(\bar{\psi}, \psi, A_\mu)\right], \tag{5}$$

$$\begin{aligned} S_f(\bar{\psi}, \psi, A_\mu) = -\int_0^\beta d\tau \int d^2x \Big[&\bar{\psi}(\not{\partial} - i\not{A})\psi \\ &+ i\frac{k}{4\pi}\epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \cdots \Big], \end{aligned} \tag{6}$$

where the dots denote the possible presence of fermionic self interactions. We expand the CS field around a static (i.e., τ-independent) monopole configuration $\bar{A}_\mu$ [24]

$$A_\mu = \bar{A}_\mu + \alpha_\mu, \qquad \bar{A}_\mu = (0, \bar{A}_1(x), \bar{A}_2(x)),$$
$$\alpha_\mu = (\alpha_0(\tau), \alpha_1(\tau, x), \alpha_2(\tau, x)), \tag{7}$$

normalized as [25]

$$\frac{1}{2\pi}\int d^2x \bar{F}_{12} = 1, \qquad \bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\nu. \tag{8}$$

Hence, (6) describes the attachment of k units of monopole charge to the fermions. One then finds

$$S_f(\bar{\psi}, \psi, A_\mu) = -\int_0^\beta d\tau \int d^2x \Big[\bar{\psi}(\partial\!\!\!/ - i\gamma_i \bar{A}_i - i\gamma_\mu \alpha_\mu)\psi + i\frac{k}{4\pi}\epsilon_{\mu\nu\rho}\alpha_\mu \partial_\nu \alpha_\rho + \cdots\Big] - ik\int_0^\beta d\tau a_0. \tag{9}$$

At this point, we can think of performing the path integral over the CS fluctuations projecting to a sector with fixed total monopole charge. Within this sector, we assume the existence of a mean field approximation such that the spatial CS fluctuations compensate for the magnetic background $\langle \alpha_i \rangle = -\bar{A}_i$ [11]. Probably, the validity of such an approximation requires also a suitable large N limit, and it would be interesting to clarify it further [26]. We then obtain

$$\begin{aligned} Z_f(\beta, k) &= \int [\mathcal{D}\alpha_0][\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp \\ &\quad \times \Big[\int_0^\beta d\tau \int d^2x [\bar{\psi}(\partial\!\!\!/ - i\gamma_0\alpha_0)\psi + \cdots] \\ &\quad + ik\int_0^\beta d\tau \alpha_0\Big] \\ &= \int (\mathcal{D}\theta) e^{ik\theta} Z_{gc,f}(\beta, -i\theta/\beta), \end{aligned} \tag{10}$$

where in the second line we have defined $\theta = \int_0^\beta d\tau \alpha_0(\tau)$ and compared with the standard formulas in [21,22].

Similarly, the thermal partition function of a complex scalar ϕ coupled to Abelian CS at level k may be written as

$$Z_b(\beta, k) = \int [\mathcal{D}A_\mu][\mathcal{D}\bar{\phi}][\mathcal{D}\phi] \exp\left[-S_b(\bar{\phi}, \phi, A_\mu)\right], \tag{11}$$

$$S_b(\bar{\phi}, \phi, A_\mu) = \int_0^\beta d\tau \int d^2x \Big[|(\partial_\mu - iA_\mu)\phi|^2 - i\frac{k}{4\pi}\epsilon_{\mu\nu\rho}A_\mu \partial_\nu A_\rho + \cdots\Big], \tag{12}$$

where again with the dots we have allowed for the presence of a nontrivial scalar potential. Expanding as in (7), and (8) and assuming a similar mean field and large N approximation, we find

$$\begin{aligned} Z_b(\beta, k) &= \int [\mathcal{D}\alpha_0][\mathcal{D}\bar{\phi}][\mathcal{D}\phi] \exp \\ &\quad \times \Big[-\int_0^\beta d\tau \int d^2x [|(\partial_\tau - i\alpha_0)\phi|^2 + |\partial_i \phi|^2 + \cdots] + ik\int_0^\beta d\tau a_0\Big] \\ &= \int [\mathcal{D}\theta] e^{ik\theta} Z_{gc,b}(\beta, -i\theta/\beta), \end{aligned} \tag{13}$$

where we have used the same definition for θ as above and compared with the standard formulas giving the grand canonical partition function for charged scalars [21,22]. This discussion shows that the partition function of fermions and charged scalars coupled to Abelian CS in a monopole background are intimately related to their respective canonical partition functions at fixed total $U(1)$ charge.

Now we can also understand the role of the imaginary chemical potential for statistical transmutation. Consider, for example, the fermionic theory (10). One notices that the presence of the imaginary chemical potential can be cancelled by the following Abelian gauge transformation for the fermions:

$$\begin{aligned} \psi(\tau, x) &\mapsto \psi'(\tau, x) = e^{i\int_0^\tau d\tau' \alpha_0(\tau')}\psi(\tau, x), \\ \bar{\psi}(\tau, x) &\mapsto \bar{\psi}'(\tau, x) = e^{-i\int_0^\tau d\tau' \alpha_0(\tau')}\bar{\psi}(\tau, x). \end{aligned} \tag{14}$$

However, at finite temperature, the fermions obey antiperiodic boundary conditions on the thermal circle

$$\psi(\beta, \bar{x}) = -\psi(0, \bar{x}), \qquad \bar{\psi}(\beta, \bar{x}) = -\bar{\psi}(0, \bar{x}). \tag{15}$$

We then see that the gauge transformed fields would satisfy

$$\psi'(\beta, x) = -e^{i\theta}\psi'(0, x), \qquad \bar{\psi}'(\beta, x) = -e^{-i\theta}\bar{\psi}'(0, x). \tag{16}$$

Hence, the antiperiodic boundary conditions are preserved only if $\theta = 2\pi n$, $n \in \mathbb{Z}$. Other values of θ would "twist" the boundary conditions and change the statistical properties of the underlying system. A similar argument goes through for bosonic systems such as (13), as well where the complex scalars satisfy periodic boundary conditions on the thermal circle. The twisting of the thermal boundary conditions is the main underlying mechanics behind the possible statistical transmutation in systems whose grand canonical potential is extremized at nontrivial values of the imaginary chemical potential.

III. THE FERMION-BOSON MAP AT IMAGINARY CHEMICAL POTENTIAL

We have argued above that the imaginary chemical potential and its corresponding charge density may be viewed as mean field fluctuations of Abelian Chern-Simons around a monopole background. This point of view raises the crucial issue at the core of our analysis. One might think that calculating the canonical partition function in systems with global $U(1)$ charges is, in principle, agnostic to the underlying microscopic structure, e.g., whether the elementary degrees of freedom forming the charge operator are bosonic or fermionic. From the outset, the only additional piece of information one has at hand is the periodicity of the grand canonical partition function as explained in (2). What then might be a salient property that distinguishes the above two cases?

In many ways, the situation resembles a quantum mechanical system in a periodic potential. Indeed if we think of θ as a periodic coordinate, then (2) has a strong resemblance to the calculation of the overlap between two Bloch wave functions that differ by lattice momentum Q (see, e.g., Eq. (2) of [27]). Although one generally cannot go very far without using a particular microscopic model at hand, one can use certain topological properties of a single band, like the eigenvalues of the Zak phase [28], to study physical properties of the system such as the polarization. We do not pursue this line of ideas any further here, but if there is a lesson to be learned it is that there should be some universality in generic canonical partition function calculations for bosonic and fermionic 3d systems. This is exploited below by considering two explicit three dimensional models: the $U(N)$ fermionic Gross-Neveu and the bosonic CP^{N-1} model.

A. The $U(N)$ Gross-Neveu model at imaginary chemical potential

To calculate the canonical partition function of the $U(N)$ Gross-Neveu model in the presence of imaginary chemical potential $\mu = -i\alpha$, we use the Euclidean action [18,19]

$$S_{GN} = -\int_0^\beta d\tau \int d^2x \left[\bar{\psi}^a(\partial\!\!\!/ - i\gamma_0\alpha)\psi^a + \frac{G}{2N}(\bar{\psi}^a\psi^a)^2 + iQ\alpha\right], \quad a = 1, 2, \ldots N, \tag{17}$$

where Q is the eigenvalue of the fermion number density [29] operator $\hat{Q} = \psi^\dagger\psi$. Introducing an auxiliary scalar field σ, the canonical partition function is given by

$$Z_f(\beta, Q) = \int (\mathcal{D}\alpha)(\mathcal{D}\sigma)e^{-S_{f,\text{eff}}}, \tag{18}$$

$$S_{f,\text{eff}} = iQ\int_0^\beta d\tau \int d^2x\alpha - \frac{N}{2G}\int_0^\beta d\tau \int d^2x\sigma^2 + N\text{Tr}\ln(\partial\!\!\!/ - i\gamma_0\alpha + \sigma)_\beta. \tag{19}$$

To evaluate (18), we look for constant saddle points α_* and σ_*. At large N, these are given by the gap equations

$$\frac{\sigma_*}{G} = \frac{2\sigma_*}{\beta}\sum_{n=-\infty}^{\infty}\int^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{p^2 + (\omega_n - \alpha_*)^2 + \sigma_*^2}, \tag{20}$$

$$i\frac{Q}{N} = \lim_{\epsilon\to 0}\frac{2}{\beta}\int^\Lambda \frac{d^2p}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\frac{e^{i\omega_n\epsilon}(\omega_n - \alpha_*)}{p^2 + (\omega_n - \alpha_*)^2 + \sigma_*^2}, \tag{21}$$

where the fermionic Matsubara sums are over the discrete frequencies $\omega_n = (2n+1)\pi/\beta$. As explained in the Appendix, we have used the parameter ϵ to regulate the sum before performing the integral. The latter is also regulated using the UV cutoff Λ.

At this moment, is it important to recall the physics of the model in the absence of chemical potential [30,31]. Setting $\alpha_* = 0$ into the first gap equation (20), one finds

$$2\sigma_*\left[-\frac{\mathcal{M}}{4\pi} + \frac{\sigma_*}{4\pi} + \frac{1}{2\pi\beta}\ln(1 + e^{-\beta\sigma_*})\right] = 0, \tag{22}$$

where

$$\frac{\mathcal{M}}{4\pi} = \frac{1}{2G_*} - \frac{1}{2G}, \qquad \frac{1}{2G_*} = \int^\Lambda \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}. \tag{23}$$

In writing (22) we have taken the cutoff to infinity. The arbitrary mass scale $\mathcal{M}$ quantifies the distance of the bare coupling G from the zero temperature critical coupling G_*. A nonzero solution for σ_* (22) would break parity by giving mass to the elementary fermions. This is only possible if $\mathcal{M} > 0$, which requires $G > G_*$. But even then, the value of σ_* goes to zero if the temperature reaches a critical value $T_c = 1/\beta_c = \mathcal{M}/2\ln 2$. At $G \le G_*$, parity is restored since the only solution of (22) is $\sigma_* = 0$. Exactly at the critical point $G = G_*$, one can evaluate the (subtracted) free energy density of the system as

$$f_f(\infty) - f_f(\beta) \equiv \Delta f_f(\beta) = N\frac{3}{2}\frac{\zeta(3)}{2\pi\beta^3}, \tag{24}$$

where $f(\beta) = -\ln Z(\beta)/(\beta V_2)$. The result (24) is identical with the corresponding thermal free energy density for N free massless Dirac fermions in three dimensions reviewed in the Appendix. The latter result of course corresponds to setting $G, \sigma_* = 0$ in the Gross-Neveu model.

The physics of the system for real nonzero chemical potential μ was studied in various works in the past, e.g., [32]. A real chemical potential lowers the critical temperature T_c which eventually becomes zero at some critical value μ_c. Moving on to imaginary chemical potential, however, we discover some new features [19]. Firstly, Eq. (20) now becomes

$$2\sigma_*\left[-\frac{\mathcal{M}}{4\pi}+\frac{\sigma_*}{4\pi}+\frac{1}{4\pi\beta}\ln\left(1+2\cos(\beta\alpha_*)e^{-\beta\sigma_*}+e^{-2\beta\sigma_*}\right)\right]=0. \tag{25}$$

At the critical point where $\mathcal{M}=0$, the gap equation (25) has two real nonzero solutions for σ_* inside the window [33] $2\pi/3<|\beta\alpha_*|<\pi$. These are the roots of

$$x^2+(2\cos(\beta\alpha_*)-1)x+1=0, \qquad x=e^{-\beta\sigma_*}. \tag{26}$$

The above chemical potential window corresponds to a novel scaling regime for the fermionic system. In contrast to the case of zero or real chemical potential, now, when the system is tuned to its zero temperature critical coupling G_*, it can break parity since $\sigma_*\neq 0$, while being at the same time a thermal CFT. This resembles the finite temperature behavior of a three-dimensional bosonic CFT as we later review [18,34].

The second gap equation (21) is evaluated as [35]

$$\frac{Q}{N}=-\frac{i}{\pi\beta^2}D(-e^{-\beta\sigma_*-i\beta\alpha_*}) =\frac{i}{2\pi\beta^3}[Cl_2(2\beta\alpha_*)+Cl_2(2\phi-2\beta\alpha_*)-Cl_2(2\phi)]. \tag{27}$$

Here, we notice the remarkable appearance of the Bloch-Wigner function $D(z)$ [12] in the result. This function is defined as

$$D(z)=\mathrm{Im}[Li_2(z)]+\ln|z|\mathrm{Arg}(1-z) \tag{28}$$

and can be expressed in terms of the Clausen function $Cl_2(z)$ (see p. 8 of [36]) with

$$\phi=\arctan\left[\frac{e^{-\beta\sigma_*}\sin(\beta\alpha_*)}{1+e^{-\beta\sigma_*}\cos(\beta\alpha_*)}\right]. \tag{29}$$

It is well known [12] that the Bloch-Wigner function $D(z)$ gives the volume of an ideal tetrahedron in Euclidean hyperbolic space $\mathcal{H}_3$ when the four vertices of the former lie in $\partial\mathcal{H}_3$ at the points 0, 1, ∞, and z (z is a dimensionless cross ratio here). Here, $D(z)$ is single-valued, real, and analytic for all $z\in\mathbb{C}$, except at $z=0$, 1, where it is nondifferentiable. We conclude that in this model there are no real solutions for Q, in accordance with our generic expectations mentioned in Sec. II.

The maximal value of $D(z)$ is obtained on the unit circle for $z_\pm=(1\pm i\sqrt{3})/2$. These numbers correspond to the end points of the interval discussed below (25) $\beta\alpha_*=\pm 2\pi/3$ where $\sigma_*=0$. We obtain

$$\frac{1}{N}Q_{\max}\left(\pm\frac{2\pi}{3}\right)=\mp\frac{i}{\pi\beta^2}Cl_2\left(\frac{\pi}{3}\right). \tag{30}$$

Here, $Cl_2(\pi/3)=-Cl_2(-\pi/3)$ are the maximal (minimal) values of the Clausen function.

Recall now that for $\alpha_*=0$ and $\sigma_*=0$, we return to the zero chemical potential case, with $Q=0$ and the free energy density given by (24). However, at the middle point of the allowed interval $\beta\alpha_*=\pi$, $\phi=0$, and consequently, we have $Q=0$. At this point, the critical ($\mathcal{M}=0$) gap equation becomes

$$2\sigma_*\left[\frac{\sigma_*}{4\pi}+\frac{1}{2\pi\beta}\ln\left(1-e^{-\beta\sigma_*}\right)\right]=0. \tag{31}$$

Interestingly, (31) can be written as

$$\frac{\sigma_*}{\pi\beta}D_1(e^{-\beta\sigma_*})=0, \tag{32}$$

where $D_1(z)$ is the first in the series of the odd-indexed generalized Bloch-Wigner functions $D_m(z)$, $m=0,1,2,3\ldots$ introduced by Zagier in [13]. Here, $D_1(z)$ has the real positive root

$$\sigma_*=\sigma_g\equiv\frac{2}{\beta}\ln\left(\frac{1+\sqrt{5}}{2}\right), \tag{33}$$

which can be interpreted as a parity breaking mass term for the fermions. Notice also that when $\beta\alpha_*=\pi$ we have a bosonization of the Matsubara frequencies in (20) and (21). At this value, we find for the free energy density of the model

$$\Delta f_f(\beta)=-N\frac{8}{5}\frac{\zeta(3)}{2\pi\beta^3}. \tag{34}$$

This is two times *minus* the large N free energy density of the bosonic $O(N)$ vector model at its nontrivial critical point [18,34]. The minus sign in (34) implies that the theory at $\beta\alpha_*=\pi$ is nonunitary, and it probably requires an analytic continuation to be matched with the usual $O(N)$ bosonic model.

B. The CP^{N-1} model at imaginary chemical potential

The Euclidean action for the CP^{N-1} model [37,38] with an imaginary chemical potential is

$$S_{CPN}=\int_0^\beta d\tau\int d^2x\left[|(\partial_\tau-i\alpha)\phi^a|^2+|\partial_i\phi|^2+i\sigma\left(\bar{\phi}^a\phi^a-\frac{N}{g}\right)+iq\alpha\right], \quad a=1,2,\ldots,N, \tag{35}$$

where the auxiliary scalar field σ enforces the constraint $|\phi|^2 = N/g$ and q is the eigenvalue density of the $U(1)$ charge density operator $\hat{q} = -ig\bar{\phi}^a \overleftrightarrow{\partial}_0 \phi^a$. The model has a global $SU(N)$ symmetry, as well as a global $U(1)$ symmetry that can be trivially gauged by the introduction of a nonpropagating Abelian gauge field. Integrating out the scalar fields, we obtain the canonical partition function as

$$Z_b(\beta, q) = \int (\mathcal{D}\alpha)(\mathcal{D}\sigma) e^{-S_{b,\text{eff}}}, \tag{36}$$

$$S_{b,\text{eff}} = iq \int_0^\beta d\tau \int d^2x \alpha + iN \frac{1}{g} \int_0^\beta d\tau \int d^2x \sigma - N \text{Tr} \ln \left(-(\partial_0 - i\alpha)^2 - \partial^2 + i\sigma\right)_\beta. \tag{37}$$

Again, we evaluate (36) at constant saddle points $i\sigma_* \equiv m_*^2$ and α_*. The latter are determined by the gap equations

$$\frac{1}{g} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + (\omega_n - \alpha_*)^2 + m_*^2}, \tag{38}$$

$$i\frac{q}{N} = -\lim_{\epsilon \to 0} \frac{2}{\beta} \int \frac{d^2p}{(2\pi)^2} \sum_{-\infty}^{\infty} \frac{e^{i\omega_n \epsilon} (\omega_n - \alpha_*)}{p^2 + (\omega_n - \alpha_*)^2 + m_*^2}, \tag{39}$$

where the bosonic frequencies are $\omega_n = 2\pi n/\beta$.

As before, we briefly recall the phase structure of the model at zero chemical potential [38,39]. We firstly point out that the *free* CP^{N-1} model corresponds to sending $g \to 0$ and $m_* = 0$. In this case the free energy density is found to be *twice* the result given in (B6) for the free massless $O(N)$ model. Of course, this is the expected result since now we have complex scalars. For $g \neq 0$, the gap equation (38) in that case reads

$$-\frac{\mathcal{M}}{4\pi} + \frac{m_*}{4\pi} + \frac{1}{2\pi\beta} \ln\left(1 - e^{-\beta m_*}\right) = 0, \qquad m_* > 0, \tag{40}$$

where

$$\frac{\mathcal{M}}{4\pi} = \frac{1}{g_*} - \frac{1}{g}, \qquad \frac{1}{g_*} = \int^\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}. \tag{41}$$

The gap equation (22) diverges for $m_* = 0$ implying the absence of a finite temperature phase transition for the continuous $SU(N)$ global symmetry, in accordance to the Mermin-Wagner-Coleman theorem. At the critical coupling $g = g_*$, however, $\mathcal{M} = 0$ and (40) has a real solution given by the "golden mean" value shown in (33). At this point, the free energy density is evaluated to be

$$f_b(\infty) - f_b(\beta) \equiv \Delta f_b(\beta) = N \frac{8}{5} \frac{\zeta(3)}{2\pi\beta^3}. \tag{42}$$

To recapitulate, in the absence of a chemical potential, the fermionic GN model has a parity breaking finite temperature transition but it does not exhibit a nontrivial scaling regime (except at some physically obscure imaginary mass [40]). The bosonic CP^{N-1} model, on the other hand, does not have a finite temperature transition, but it exhibits a nontrivial finite temperature scaling regime. One should therefore have guessed by now that the presence of the imaginary chemical potential, or equivalently of a Chern-Simons field expanded around a monopole background, catalyzes an interpolation between these two theories. Let us explicitly see how this happens from the bosonic side this time.

The gap equation (38) gives

$$-\frac{\mathcal{M}}{4\pi} + \frac{m_*}{4\pi} + \frac{1}{4\pi\beta} \ln\left(1 - 2\cos(\beta\alpha_*) e^{-\beta m_*} + e^{-2\beta m_*}\right) = 0. \tag{43}$$

At the critical coupling where $\mathcal{M} = 0$, the equation above has one real positive solution for m_* in the window $-\pi/3 < \beta a_* < \pi/3$. This is the root of

$$x^2 - (2\cos(\beta\alpha_*) - 1)x + 1 = 0, \qquad x = e^{-\beta m_*}. \tag{44}$$

Notice that this range is the mirror image with respect to the vertical axis of the allowed range for a nonzero fermionic mass discussed after (25). For example, the middle point in the fermionic case (for $\beta\alpha_* = \pi$) is mapped into the middle point here which is at $\beta\alpha_* = 0 \pmod{2\pi}$.

The second gap equation (39) is now evaluated to

$$\frac{q}{N} = \frac{i}{\pi\beta^2} D(e^{-\beta\sigma_* - i\beta\alpha_*}) = -\frac{i}{2\pi\beta^3} [Cl_2(2\beta\alpha_*) + Cl_2(2w - 2\beta\alpha_*) - Cl_2(2w)], \tag{45}$$

where

$$w = \arctan\left[\frac{e^{-\beta\sigma_*} \sin(\beta\alpha_*)}{e^{-\beta\sigma_*} \cos(\beta\alpha_*) - 1}\right]. \tag{46}$$

As previously, the maximal value of q is also reached at the end points of the above interval where $m_* = 0$. We find

$$\frac{1}{N} q_{\max}\left(\pm\frac{\pi}{3}\right) = \mp \frac{i}{\pi\beta^2} Cl_2\left(\frac{\pi}{3}\right). \tag{47}$$

For $\alpha_* = 0$, we return to the zero chemical potential case where $q = 0$, the $SU(N)$ symmetry is unbroken, the mass of the scalars is given by (33), and the free energy is (42). But now, for $\pi/3 \leq |\beta\alpha_*| \leq \pi$, the only allowed solution for the gap equation (43) is $m_* = 0$. This might appear to imply the finite temperature breaking of the continuous $SU(N)$ symmetry and the violation of the Mermin-Wagner-Coleman theorem. However, we do not believe that this is the case. The underlying theory has now been effectively fermionized, and

the broken symmetry must be a discrete one. It would be important nevertheless to further clarify this point. At the middle point of this window, we have $\beta\alpha_* = \pi$, $w = 0$, and hence, $q = 0$. The theory is now fermionized, and the calculation of the free energy density yields

$$\Delta f_b(\beta) = -N\frac{3}{2}\frac{\zeta(3)}{2\pi\beta^3}. \tag{48}$$

This is exactly *minus* the fermionic free energy result (34).

C. The free energy map

We have seen that in the presence of an imaginary chemical potential the gap equations of the fermionic and bosonic models map into each other. As a consequence, there is a corresponding map between the physical properties of the two models. The map can be translated into a precise statement about the corresponding free energy densities.

The fermionic free energy density is given by

$$\begin{aligned}\frac{1}{N}\Delta f_f(\beta) &= -\int\frac{d^3p}{(2\pi)^3}\ln p^2 + iQ\alpha_* + \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2\bar{p}}{(2\pi)^2}\ln(\bar{p}^2 + (\omega_n - a_*)^2 + \sigma_*^2) - \frac{\sigma_*^2}{2G}\\ &= \int\frac{d^3p}{(2\pi)^3}\left[\ln\left(\frac{p^2+\sigma_*^2}{p^2}\right) - \frac{\sigma_*^2}{p^2+\sigma_*^2}\right] + \frac{\alpha_*}{\pi\beta^2}D(-z_*) + \frac{\sigma_*^2}{2\pi\beta}Re\{\ln(1+z_*)\}\\ &\quad + \frac{1}{\pi\beta}\int_{\sigma_*}^{\infty}xdxRe\{\ln(1+e^{-\beta x - i\beta\alpha_*})\},\end{aligned} \tag{49}$$

where

$$z_* = e^{-\beta\sigma_* - i\beta\alpha_*}, \tag{50}$$

and in the second line we have substituted the solutions of the gap equations. A few more details on the calculation are given in the Appendix. The second line of (49) brings into the result the cutof- dependent critical coupling $1/2G_*$. This has to be subtracted in order to obtain a finite result for the critical theory. Then, quite remarkably, the finite result of the second line in (49) combined with the third line gives *exactly* the generalized Bloch-Wigner-Zagier function $D_3(z)$ defined as [13]

$$\begin{aligned}D_3(z) &= Re\{Li_3(z)\} - \ln|z|Re\{Li_2(z)\}\\ &\quad - \frac{1}{2}\ln^2|z|Re\{\ln(1-z)\} + \frac{\ln^3|z|}{12}.\end{aligned} \tag{51}$$

Our final result reads

$$\frac{1}{N}\Delta f_f(\beta) = \frac{\alpha_*}{\pi\beta^2}D(-z_*) - \frac{1}{\pi\beta^3}D_3(-z_*). \tag{52}$$

The analogous calculation for the bosonic theory yields

$$\frac{1}{N}\Delta f_b(\beta) = -\frac{\alpha_*}{\pi\beta^2}D(z_*) + \frac{1}{\pi\beta^3}D_3(z_*), \tag{53}$$

where we have set $m_* = \sigma_*$. We see that the free energies are mapped into each other when $\beta\alpha_* \mapsto \beta\alpha_* \pm \pi$. The exact relation reads

$$\frac{1}{N}\Delta f_f(\beta)\Big|_{\beta\alpha_*\pm\pi} + \frac{1}{N}\Delta f_b(\beta)\Big|_{\beta\alpha_*} = \pm\frac{1}{\beta^3}D(z_*). \tag{54}$$

An intriguing geometric interpretation of our result is that the bosonic and fermionic partition functions at imaginary chemical potential correspond to partial volumes of an ideal tetrahedron. Their sum gives the entire volume.

IV. DISCUSSION

Motivated by the recent revival of 3d bosonization, we studied the fermionic $U(N)$ Gross-Neveu and the bosonic CP^{N-1} models at finite temperature and imaginary chemical potential for a $U(1)$ charge. We started by noticing that if the charge density vanishes for nontrivial values of the chemical potential, the underlying system could effectively undergo a statistical transmutation. We have also pointed out that the canonical partition function of the systems is intimately related to the thermal partition function of Abelian Chern-Simons fields coupled to matter in a monopole background, when a suitable mean-field approximation is assumed. One may think of the latter as the regime where the $U(1)$ charge density essentially corresponds to the monopole charge.

One of our main results is that the phase structures of the Gross-Neveu and CP^{N-1} models are altered in the presence of the imaginary chemical potential. A novel scaling phase opens up for the Gross-Neveu model, while a pseudobroken phase of a continuous symmetry appears in the CP^{N-1} model. The latter situation may appear to violate the Mermin-Wagner-Coleman theorem, but we believe that this is not the case since the system has been fermionized and the broken symmetry must be discrete. Thus, the phase structures appear to be mapped into each other. Another result of our analysis states that the charge densities for both models are purely imaginary and given by the Bloch-Wigner function. This is rather tantalizing. The Bloch-Wigner function gives the volume of ideal tetrahedra in $\mathcal{H}_3$, and apart from its great importance for the classification of 3-manifolds, it also gives

the imaginary part of a complex Chern-Simons action. This fits rather nicely with our association of the $U(1)$ charge density to the Chern-Simons level. Morever, the second gap equation can be identified with the $D_1(z)$ function introduced by Zagier in [13]. Finally, we have calculated the free energy densities of our two models and found, again rather remarkably, that they are given in terms of the generalized Bloch-Wigner function $D_3(z)$ [13]. Our results allowed us to make a precise duality statement between the free energies of the models in (54). Again, the Bloch-Wigner function $D(z)$ plays an important role here as it gives the sum of the fermionic and bosonic free energy densities at imaginary chemical potentials that differ by $i\pi$.

Our work raises numerous questions that could shed more light into the nature of 3d fermion-boson duality. Firstly, we note that our approach to the canonical formalism closely resembles the physics of a quantum mechanical system in a periodic potential. Indeed, the imaginary chemical potential may be thought of as the periodic coordinate and the charge density as the quasimomentum. Such a quantum mechanical systems exhibit a band structure which can be studied by restricting the quasimomentum to the first Brillouin zone. One cannot go very far without a detailed model at hand; however, there are certain topological properties like, e.g., the Zak phase [28], which only depend on the band symmetry. It is then tempting to associate the fermion-boson duality with some symmetry properties of the band.

Our results were derived in the large N limit. It is therefore rather interesting to test whether there are remnants of the fermion-boson map at subleading orders in the $1/N$ expansion. After all, the above bosonization is probably an exact statement even for a single boson or fermion [4–6]. To this effect, the calculation of the $1/N$ corrections to the free energies of the bosonic and fermionic models is relevant.

Another interesting point is the surprising, to us, relevance of the hyperbolic geometry for the fermion-boson map. The bosonic and fermionic systems have a very different phase structure for real chemical potential when the charge density is real for both systems. In that case we are not aware of a manifestation of the boson-fermion map. Introducing the imaginary chemical potential the charge density becomes purely imaginary and apparently evaluates the volume of ideal tetrahedra in $\mathcal{H}_3$. Since the latter is related to the imaginary part of the complex $SL(2,\mathbb{C})$ Chern-Simons action, our results seem to suggest that the canonical partition function for fermions and bosons in three dimensions is related to the full partition function of the above complex group. It would be important to quantify such a possible relationship, as well as to understand the meaning of the charge density extrema. Finally, the appearance of the $D_1(z)$ function in the gap equation and the $D_3(z)$ function in the free energy density is another intriguing result. There is a large mathematical literature for symmetry properties of these functions (also related to Nielsen's generalized polylogarithms, see, e.g., [41]), which consequently should be inherited by the free energy densities of fermions and bosons in three dimensions. Finally, there are strong indications [42] that analogous results, namely, the appearance of higher $D_m(z)$ functions, are relevant in studies of higher-dimensional fermionic and bosonic critical systems such as the finite temperature extensions of the models discussed in [43].

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APPENDIX A: NOTATION AND USEFUL RESULTS

Following [22], we use two-component Euclidean Dirac spinors and a Hermitian representation for the gamma matrices as $\gamma_\mu = \sigma_\mu$, where $\mu = 0, 1, 2$. Here, σ_μ are the usual Pauli matrices with the definition $\gamma_0 = \sigma_0 \equiv \sigma_3$. Latin indices run as $i = 1, 2$.

A standard integral used in the text is

$$\int^{\Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{p^2+\sigma_*^2} = \frac{\Lambda}{2\pi^2} - \frac{\sigma_*}{4\pi} + O(\sigma_*/\Lambda). \quad \text{(A1)}$$

The Matsubara sums can be done with the help of the Poisson sum formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx e^{-i2\pi kx} f(k). \quad \text{(A2)}$$

As an application we consider the sum that appears in the fermionic charge density gap equation (21)

$$\lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \frac{e^{i\omega_n\epsilon}(\omega_n - \alpha_*)}{p^2 + (\omega_n-\alpha_*)^2 + \sigma_*^2}. \quad \text{(A3)}$$

Without the introduction of the convergence factor $e^{i\omega_n\epsilon}$, $\epsilon > 0$, the sum would be undetermined [44]. Doing then the integral in the right-hand side of (A2) term by term, we first note that the $n = 0$ term vanishes, and we obtain

$$\lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \frac{e^{i\omega_n\epsilon}(\omega_n - \alpha_*)}{p^2 + (\omega_n-\alpha_*)^2 + \sigma_*^2} = i\frac{\beta}{2}\left(\frac{1}{1+e^{\beta\sqrt{\bar{p}^2+\sigma_*^2}+i\beta\alpha_*}} - \frac{1}{1+e^{\beta\sqrt{\bar{p}^2+\sigma_*^2}-i\beta\alpha_*}}\right). \quad \text{(A4)}$$

Integrating (A4) over the spatial momenta $\bar{p}$ and using the definition of the dilogarithm

$$Li_2(z) = -\int_0^z \frac{dw}{w}\ln(1-w), \tag{A5}$$

we obtain (27). The vanishing of the $n=0$ mode above should be contrasted with the sum that appears in the first gap equation (20), which is evaluated to

$$\begin{aligned}\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+(\omega_n-\alpha_*)^2+\sigma_*^2} &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+\sigma_*^2}-\frac{1}{2}\int\frac{d^2\bar{p}^2}{(2\pi)^2}\frac{1}{\sqrt{\bar{p}^2+\sigma_*^2}} \\ &\times\left(\frac{1}{1+e^{\beta\sqrt{\bar{p}^2+\sigma_*^2}+i\beta\alpha_*}}+\frac{1}{1+e^{\beta\sqrt{\bar{p}^2+\sigma_*^2}-i\beta\alpha_*}}\right).\end{aligned} \tag{A6}$$

The divergent first term in the right-hand side of the above, which is needed for the coupling constant renormalization, comes from the $n=0$ term in the sum.

APPENDIX B: THERMAL PARTITION FUNCTION FOR FREE SCALARS AND FERMIONS IN $d=3$

We review here, in some detail, the free field theory results for the free energy density of scalars and fermions in $d=3$. Consider the $O(N)$ invariant action of free massive scalars $\phi^i(x)$, $a=1,2,\ldots,N$ in $d=3$

$$\mathcal{I}_b = \int d^3x\left(\frac{1}{2}\partial_\mu\phi^a\partial_\mu\phi^a+\frac{1}{2}m_b^2\phi^a\phi^a\right). \tag{B1}$$

We put the theory in Euclidean $S_1\times\mathbb{R}^2$, where S_1 has radius $L=\beta=1/T$, and impose periodic boundary conditions as

$$\begin{aligned}p_\mu &= (\omega_n,\bar{p}), \qquad \bar{p}=(p^2,p^2),\\ \omega_n &= \frac{2\pi}{\beta}n, \qquad n=0,\pm1,\pm2,\ldots\end{aligned} \tag{B2}$$

The thermal free energy density $f_b(\beta)$ is defined as

$$Z_b = \int(\mathcal{D}\phi^i)\exp^{-\frac{1}{2}\int_0^\beta dx^3\int d^2x[\phi^i(-\partial^2)\phi^i+m_b^2\phi^i\phi^i]} \equiv e^{-\beta V_2 f_b(\beta)}, \tag{B3}$$

with V_2 the volume of $\mathbb{R}^2$. The interesting quantity is the difference

$$\begin{aligned}f_b(\infty)-f_b(\beta) &\equiv \Delta f_b(\beta)\\ &= \frac{N}{2}\int\frac{d^3p}{(2\pi)^3}\ln p^2\\ &\quad -\frac{N}{2L}\sum_{n=-\infty}^{\infty}\int\frac{d^2\bar{p}}{(2\pi)^2}\ln(\bar{p}^2+m_b^2+\omega_n^2),\end{aligned} \tag{B4}$$

which is expected to be positive in a stable theory since $f_b(\beta)=-\mathcal{P}_b(\beta)$ is the bosonic pressure density at temperature $T=1/\beta$. We find (we give the result for general d)

$$\begin{aligned}\frac{1}{N}\Delta f_b(\beta) &= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\ln\left(\frac{p^2+m_b^2}{p^2}\right)\\ &\quad -\frac{1}{\beta}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\ln\left(1-e^{-\beta\sqrt{\bar{p}^2+m_b^2}}\right)\\ &= -\frac{\pi S_d}{2d(2\pi)^d}\left[\frac{m_b^d}{\sin\frac{\pi d}{2}}-4d\frac{S_{d-1}}{S_d}\frac{1}{\beta^d}\right.\\ &\quad \left.\times\int_0^{e^{-\beta m_b}}(\ln^2x-m_b^2\beta^2)^{\frac{d-3}{2}}\ln x\ln(1-x)\frac{dx}{x}\right],\end{aligned} \tag{B5}$$

with $S_d=2\pi^{d/2}/\Gamma(d/2)$ the d-dimensional solid angle. The integral above can be evaluated in closed form for d odd, in terms of Nielsen's generalized polylogarithms [41,45] (and eventually as a finite sum of polylogarithms which turn into a linear combination of Bloch-Wigner-Zagier functions after the introduction of a gauge field or an imaginary chemical potential [42]). For $d=3$ and $m_b=0$, we have

$$f_b(\infty)-f_b(L) = \frac{N}{2\pi L^3}\zeta(3). \tag{B6}$$

This is *half* the value of the free energy for density for the free CP^{N-1} model.

For N Dirac fermions ψ^i, $\bar{\psi}^a$, $a=1,2,\ldots,N$ in three Euclidean dimensions, we have

$$\mathcal{I}_f = -\int d^3x(\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i+m_f\bar{\psi}^i\psi^i). \tag{B7}$$

The corresponding thermal free energy calculation for fermions yields

$$\begin{aligned}\frac{1}{N\mathrm{Tr}\mathbb{I}}\Delta f_f(\beta) &= -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\ln p^2\\ &\quad +\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2\bar{p}}{(2\pi)^2}\ln(\bar{p}^2+m_b^2+\omega_n^2),\end{aligned} \tag{B8}$$

where the fermionic frequencies are $\omega_n = \frac{\pi}{\beta^2}(2n+1)$. We obtain

$$\frac{1}{N\mathrm{Tr}\mathbb{I}}\Delta f_f(\beta) = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\ln\left(\frac{p^2+m_f^2}{p^2}\right) + \frac{1}{\beta}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\ln\left(1+e^{-L\sqrt{\vec{p}^2+m_f^2}}\right)$$
$$= \frac{\pi S_d}{2d(2\pi)^d}\left[\frac{m_f^d}{\sin\frac{\pi d}{2}} - 4d\frac{S_{d-1}}{S_d}\frac{1}{\beta^d}\int_0^{e^{-\beta m_f}}(\ln^2 x - m_f^2\beta^2)^{\frac{d-3}{2}}\ln x \ln(1+x)\frac{dx}{x}\right]. \tag{B9}$$

For $d=3$ and $m_f = 0$, we obtain

$$\Delta f(\beta) = \frac{1}{2\pi L^3}N\mathrm{Tr}\mathbb{I}\frac{3}{4}\zeta(3). \tag{B10}$$

Since $\mathrm{Tr}\mathbb{I} = 2$ here, the fermionic result is $3/2$ times the bosonic one.

[1] E. H. Fradkin and F. A. Schaposnik, Phys. Lett. B **338**, 253 (1994).
[2] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
[3] A. M. Polyakov, Mod. Phys. Lett. A **03**, 325 (1988).
[4] A. Karch and D. Tong, Phys. Rev. X **6**, 031043 (2016).
[5] J. Murugan and H. Nastase, arXiv:1606.01912.
[6] N. Seiberg, T. Senthil, C. Wang, and E. Witten, Ann. Phys. (Amsterdam) **374**, 395 (2016).
[7] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, Phys. Rev. D **94**, 085009 (2016).
[8] M. E. Peskin, Ann. Phys. (N.Y.) **113**, 122 (1978).
[9] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, Eur. Phys. J. C **72**, 2112 (2012).
[10] O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena, and R. Yacoby, J. High Energy Phys. 03 (2013) 121.
[11] M. Barkeshli and J. McGreevy, Phys. Rev. B **89**, 235116 (2014).
[12] D. Zagier, In *Frontiers in Number Theory, Physics and Geometry II*, edited by P. Cartier, B. Julia, P. Moussa, and P. Vanhove (Springer-Verlag, Berlin, 2007), Chapter 1.
[13] D. Zagier, Math. Ann. **286**, 613 (1990).
[14] O. G. E. D'Hoker, M. B. Green, and P. Vanhove, arXiv:1512.06779.
[15] A. Roberge and N. Weiss, Nucl. Phys. **B275**, 734 (1986).
[16] G. Aarts, J. Phys. Conf. Ser. **706**, 022004 (2016).
[17] F. Karbstein and M. Thies, Phys. Rev. D **75**, 025003 (2007).
[18] A. C. Petkou and M. B. Silva Neto, Phys. Lett. B **456**, 147 (1999).
[19] H. R. Christiansen, A. C. Petkou, M. B. Silva Neto, and N. D. Vlachos, Phys. Rev. D **62**, 025018 (2000).
[20] See for example the standard references [21,22].
[21] J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications* (Cambridge University Press, Cambridge, 2011).
[22] J. Zinn-Justin, Int. Ser. Monogr. Phys. **113**, 1 (2002).
[23] Our notations follow [22] and are briefly presented in the Appendix.
[24] C. D. Fosco, G. L. Rossini, and F. A. Schaposnik, Phys. Rev. D **59**, 085012 (1999).
[25] For example, one may consider the theory on $S^1 \times S^2$.
[26] We wish to thank Ofer Aharony for an illuminating communication on this point.
[27] T. Li, L. Duca, M. Reitter, F. Grusdt, E. Demler, M. Endres, M. Schleier-Smith, I. Bloch, and U. Schneider, Science **352**, 1094 (2016).
[28] J. Zak, Phys. Rev. Lett. **62**, 2747 (1989).
[29] With respect to the spatial volume V_2.
[30] B. Rosenstein, B. Warr, and S. H. Park, Phys. Rep. **205**, 59 (1991).
[31] M. Moshe and J. Zinn-Justin, Phys. Rep. **385**, 69 (2003).
[32] S. Hands, A. Kocic, and J. B. Kogut, Nucl. Phys. **B390**, 355 (1993).
[33] We take $\beta\alpha_* \in [-\pi, \pi]$.
[34] S. Sachdev, Phys. Lett. B **309**, 285 (1993).
[35] The following formulas correct some misprints in [19].
[36] A. N. Kirillov, Prog. Theor. Phys. Suppl. **118**, 61 (1995).
[37] I. Ya. Arefeva and S. I. Azakov, Nucl. Phys. **B162**, 298 (1980).
[38] A. D'Adda, M. Luscher, and P. D. Vecchia, Nucl. Phys. **B146**, 63 (1978).
[39] G. Murthy and S. Sachdev, Nucl. Phys. **B344**, 557 (1990).
[40] A. C. Petkou and G. Siopsis, arXiv:hep-th/0008198.
[41] J. M. Borwein and A. Straub, J. Approx. Theory **193**, 74 (2015).
[42] V. Filothodoros, A. C. Petkou, and N. D. Vlachos, arXiv:1608.07795v1.
[43] L. Fei, S. Giombi, and I. R. Klebanov, Phys. Rev. D **90**, 025018 (2014).
[44] F. Nogueira, M. Silva-Neto, and N. F. Svaiter, Phys. Lett. B **441**, 339 (1998).
[45] S. K. Kolbig, SIAM J. Math. Anal. **17**, 1232 (1986).