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Finiteness of two- and three-point functions and the renormalization group

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Two- and three-point functions of composite operators are analyzed with regard to (logarithmically) divergent contact terms. Using the renormalization group of dimensional regularization it is established that the divergences are governed by the anomalous dimensions of the operators and the leading UV behavior of the $1/\epsilon$ coefficient. Explicit examples are given by the $\langle G^2 G^2\rangle$, $\langle\Theta\Theta\rangle$ (trace of the energy momentum tensor) and $\langle\bar{q}q\bar{q}q\rangle$ correlators in QCD-like theories. The former two are convergent when the $1/\epsilon$ poles are resummed but divergent at fixed order implying that perturbation theory and the $\epsilon \to 0$ limit do not generally commute. Finite correlation functions obey unsubtracted dispersion relations which is of importance when they are directly related to physical observables. As a by-product the R^2 term of the trace anomaly is extended to next-to-next-to-leading order $[\mathcal{O}(a_s^5)]$, in the minimal subtraction scheme, using a recent $\langle G^2 G^2\rangle$ computation.

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I. INTRODUCTION

In this paper divergences are investigated which arise when (composite) operators approach each other. These ultraviolet (UV) divergences are necessarily local and expressed in terms of delta functions and derivatives thereof [i.e. contact terms (CTs)]. This requires renormalization in addition to the parameters of the theory and the composite operators themselves.

These CTs play an important role as they manifest themselves as anomalies in correlation functions of composite operators, the chiral anomaly serving as a primary example,[1] and the perspective on other anomalies continues to evolve [8–10]. On the other hand CTs are not important when studying the spectrum of two-point functions (e.g. QCD sum rules [11]) or lattice QCD [12]) since they bear no relation to the infrared (IR) spectrum. In lattice simulations of correlation functions CTs require additional renormalization conditions, a problem for which the $4+1$-dimensional gradient flow offers new perspectives [13–15].

Our work originates from the observation that the leading logarithm (LL) ϵ poles of the field strength tensor correlation function sums to an expression

$$\int d^4x e^{ix\cdot p}\langle[G^2(x)][G^2(0)]\rangle|_{\rm LLpoles} \sim p^4\frac{1}{\epsilon+\beta_0 a_s} = p^4\frac{1}{\epsilon}\left(1-\frac{\beta_0 a_s}{\epsilon}+\frac{(\beta_0 a_s)^2}{\epsilon^2}+\mathcal{O}({a_s}^3)\right), \tag{1}$$

which is finite for $\epsilon \to 0$ but divergent at each fixed order in perturbation theory. Using the renormalization group (RG) of dimensional regularization (DR) the absence of potential logarithmic divergences is systematized in various ways. Firstly, simple criteria for convergence, involving RG quantities, are established of generic two-point functions. The discussion is extended to include the nonperturbative condensate terms, multiple couplings and three-point functions. Using the local quantum action principle (QAP) a closed integral expression for the R^2 anomaly is given in terms of the first pole of the correlation function (1).

The paper is organized as follows. In Sec. II the finiteness criteria for two-point functions are discussed, followed by the explicit examples of $\langle G^2 G^2\rangle$ and $\langle\Theta\Theta\rangle$ correlators in QCD-like theories in Sec. III. Implications for dispersion integrals, RG-scale dependence (physicality) and the R^2 anomaly are elaborated on in Secs. III B 1, III B 2 and III D, respectively. The 1-coupling case of the two-point function is generalized to multiple couplings and three-point functions in Secs. IV A and IV B. The paper ends with a summary and conclusions in Sec. V. Appendix A contains details about the $\langle G^2 G^2\rangle$-correlation function computation and Appendix B discusses the convergence of the $\langle\bar{q}q\bar{q}q\rangle$- and $\langle J^5_\mu J^5_\nu\rangle$-correlation functions. The β-function conventions are given in Appendix C.

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[1]Early analyses centered around configuration space singularities in correlation functions, without particular emphasis on perturbation theory, of the chiral and trace anomalies can be found in [1,2] and [2–4] and reviewed in [5], respectively. Recently CTs in three-point correlation functions were the center of discussion on whether in $d=4$ nontrivial unitary scale but not conformal field theories exist [6,7].

II. TWO-POINT FUNCTION IN MOMENTUM SPACE

We consider a renormalizable theory [i.e. with a UV fixed point (FP)] in four dimensions with no explicit mass scales and a nontrivial flow. The Euclidean two-point functions of marginal operators are parametrized as[2] follows[3]:

$$\Gamma_{AB}(p^2) = \int d^4x e^{ip\cdot x}\langle[O_A(x)][O_B(0)]\rangle_c = \mathbb{C}^{\mathbb{1}}_{AB}(p^2)p^4, \tag{2}$$

where c stands for the connected component, $\langle\ldots\rangle$ for the vacuum expectation value (VEV), $[O_{A,B}]$ are renormalized scalar (composite) operators of mass dimension 4 and $\mathbb{C}^{\mathbb{1}}_{AB}$ are dimensionless functions. Such a divergence might be thought of as the Wilson coefficient of the identity operator. In an asymptotically free (AF) theory the coefficient $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ is potentially logarithmically divergent by power counting. In coordinate space this divergence results from singular behavior as $x\to 0$. The latter can be removed by local counterterms within the standard renormalization program. The renormalized correlation function $\Gamma^{\mathcal{R}}_{AB}$ is obtained from the bare one Γ_{AB} by splitting the bare Wilson coefficient $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ into renormalized $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2)$ and a counterterm $L^{\mathbb{1},\mathcal{R}}_{AB}$ part

$$\mathbb{C}^{\mathbb{1}}_{AB}(p^2) = \mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2) + L^{\mathbb{1},\mathcal{R}}_{AB}. \tag{3}$$

Above, the letter L either stands for local and R denotes a renormalization scheme. To be clear we wish to add that $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}$ is finite whereas $L^{\mathbb{1},\mathcal{R}}_{AB}$ is generally not despite the R label. We are going to be careful as to which statements are generic for any scheme [i.e. a specific split in (3)] and what is valid when $\mathcal{R}$ stands for the minimal subtraction (MS) scheme.

In coordinate space this translates into

$$\hat{\Gamma}^{\mathcal{R}}_{AB}(x^2) = \underbrace{\langle[O_A(x)][O_B(0)]\rangle_c}_{\equiv\hat{\Gamma}_{AB}(x^2)} - L^{\mathbb{1},\mathcal{R}}_{AB}\Box^2\delta(x), \tag{4}$$

where $\Box = \partial_\mu\partial^\mu$ and $\delta(x)$ is the four-dimensional Dirac delta function throughout. With slight abuse in notation we refer to $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ as the bare correlation function despite its dependence on the RG scale through the renormalization of the composite operators $[O_{A,B}]$. The renormalized correlation function $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2)$ and the counterterm $L^{\mathbb{1},\mathcal{R}}_{AB}$ are in general RG-scale dependent even if $[O_{A,B}]$ are not. This particular RG-scale dependence of course cancels in the sum and consists of CTs since $L^{\mathbb{1},\mathcal{R}}_{AB}$ is local.

A. Two-point function in dimensional regularization with one coupling

At first we restrict ourselves to one coupling $a_s = a_s(\mu)$ whose scale dependence will frequently be suppressed throughout. In the MS scheme with DR ($d = 4-2\epsilon$) the counterterm

$$L^{\mathbb{1},\mathrm{MS}}_{QQ}(\mu) \equiv \sum_{n\geq 1}\frac{r^{\mathbb{1}(n)}_{QQ}(a_s(\mu))}{\epsilon^n},$$
$$r^{\mathbb{1}(1)}_{QQ}(a_s) = r^{\mathbb{1}(1,0)}_{QQ} + r^{\mathbb{1}(1,1)}_{QQ}a_s + \mathcal{O}(a_s^2), \tag{5}$$

is given by a Laurent series. The residues $r^{\mathbb{1}(n)}_{QQ}$ are dimensionless and functions of the running coupling only. Since in this work we use the MS scheme in all practical computation we do not indicate this circumstance with a further label. We proceed to derive a RG equation (RGE) for $L^{\mathbb{1}}_{QQ}$. The starting point is (3), which in DR

$$\mathbb{C}^{\mathbb{1}}_{QQ}(p^2)p^{-2\epsilon} = (\mathbb{C}^{\mathbb{1},\mathcal{R}}_{QQ}(p^2) + L^{\mathbb{1},\mathcal{R}}_{QQ})\mu^{-2\epsilon}. \tag{6}$$

The renormalized Wilson coefficient $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{QQ}(p^2)$ is finite for $\epsilon\to 0$ in the sense of being analytic in ϵ (in particular no poles). Suppose that $[O_Q]$ can be made RG invariant by premultiplying by a finite factor $\kappa_Q(a_s)$, i.e.

$$\frac{d}{d\ln\mu}\kappa_Q Z_{QQ} = 0, \qquad Z_{QQ}O_Q = [O_Q] + \cdots, \tag{7}$$

where the dots correspond to equation of motion (EOM) operators which do not contribute to structure we are discussing. Using $\frac{d}{d\ln\mu}\kappa_Q^2\mathbb{C}^{\mathbb{1}}_{QQ}(p^2) = 0$ and the finiteness of the renormalized Wilson coefficient one deduces[4]

$$\left(2\hat{\gamma}_Q + \frac{d}{d\ln\mu} - 2\epsilon\right)L^{\mathbb{1},\mathcal{R}}_{QQ} = -\chi^{\mathcal{R}}_{QQ}, \tag{8}$$

where

$$\hat{\gamma}_Q = \frac{d}{d\ln\mu}\ln\kappa_Q, \tag{9}$$

[2]Various extensions of this setup will be discussed: condensate corrections in the language of the operator product expansion (OPE) [1], nondiagonal correlation functions, multiple couplings and three-point functions are discussed in Secs. III C, IV A and IV B, respectively. An extension to operators with spin is possible and we refer the reader to [16] where this is done in the context of QCD for diagonal correlation functions.

[3]The correlation function $\langle[O_A(x)][O_B(0)]\rangle_c$ is formally defined by the connected part of a regularized path-integral representation $\int D\phi[O_A(x)][O_B(0)]e^{-S[\phi]}$.

[4]In the case where O_Q is marginal (and not $m\bar{q}q$) $\kappa_Q = \hat{\beta}^Q$ and $\hat{\gamma}_Q = 2\partial_{\ln a_s}\hat{\beta}^Q$, the non-$\epsilon$ part becomes Lie derivative, $\mathcal{L}_\beta = 2\hat{\gamma}_Q + \hat{\beta}^P\partial_P$, acting on the 2-tensor $L^{\mathbb{1},\mathcal{R}}_{QQ}$. This circumstance is put into evidence in the multiple coupling section IV which reveals the structure more systematically.

with $\hat{\beta} = -\epsilon + \beta$ and $\hat{\gamma}_Q = \gamma_Q - \xi_Q \epsilon$ being the d-dimensional β function and anomalous dimension, respectively.[5] The quantity $\chi^{\mathcal{R}}_{QQ}$ follows from the requirement of finiteness and is given in the MS scheme by

$$\chi^{\rm MS}_{QQ} = 2(a_s \partial_{a_s}(a_s r^{\mathbb{1}(1)}_{QQ}) + \xi_Q r^{\mathbb{1}(1)}_{QQ}). \tag{10}$$

The ordinary differential equation (8) is solved by

$$L^{\mathbb{1},\mathcal{R}}_{QQ}(\mu) = \int_{\ln\mu}^{\infty} \chi^{\mathcal{R}}_{QQ}(a_s(\mu')) I_{QQ}(\mu,\mu') \left(\frac{\mu}{\mu'}\right)^{2\epsilon} d\ln\mu', \tag{11}$$

which shows the MS property that all higher pole residues of $L^{\mathbb{1}}_{QQ}$ follow from the first one [encoded in χ_{QQ} (10)]. Above $(\mu/\mu')^{2\epsilon} I_{QQ}$ is an integrating factor with

$$I_{QQ}(\mu,\mu') = \exp\left(2\int_{\ln\mu}^{\ln\mu'} \hat{\gamma}_Q(a_s(\mu''))d\ln\mu''\right). \tag{12}$$

Generally it is the function I_{QQ} and the power behavior of χ_{QQ} which decide on whether or not the integral diverges for $\mu' \to \infty$ and $(\mu/\mu')^{2\epsilon}$ serves as a potential UV regulator. A more refined analysis is required to distinguish whether the UV FP is of the AF type $a_s^{\rm UV} \equiv a_s(\infty) = 0$ or asymptotically safe (AS) type $a_s^{\rm UV} \neq 0$.

1. Asymptotically free theory

For the asymptotic analysis it is convenient to change the variable to the RG time $t \equiv \ln\mu/\mu_0$. In the asymptotic regime a LL analysis is sufficient. Assuming $\hat{\beta}(a_s) = -\epsilon + \beta = -\epsilon - \beta_0 a_s + \mathcal{O}(a_s^2)$ the LL relation is given by[6]

$$a_s(t) = \frac{\epsilon a_s(\mu) e^{-2\epsilon t}}{\epsilon + \beta_0 a_s(\mu)(1-e^{-2\epsilon t})} = \frac{a_s(\mu)}{1+2\beta_0 a_s(\mu) t} + \mathcal{O}(\epsilon), \tag{13}$$

with slight abuse in notation and initial value $a_s(t=0) = a_s(\mu)$ (for $\epsilon \to 0$) and UV value $a_s(t\to\infty) = 0$. The anomalous dimension is parametrized by $\gamma_Q = a_s \gamma_{Q,0} + \mathcal{O}(a_s^2)$ implying the asymptotic behavior $I_{QQ}(t) \sim t^\eta$ with $\eta = \gamma_{Q,0}/\beta_0$. Assuming $\chi_{QQ} \sim t^{-n_{QQ}}$ to be perturbative for $t \to \infty$ with $n_{\rm QQ} \geq 0$ [$n_{QQ} = 0$, i.e. $\chi_{QQ} = \mathcal{O}(a_s^0)$, being the nominal case for a nontrivial unitary theory] the condition for UV finiteness is[7]

$$1 + \frac{\gamma_{Q,0}}{\beta_0} < n_{QQ} \Leftrightarrow L^{\mathbb{1},\mathcal{R}}_{QQ} \overset{\epsilon/(\beta_0 a_s)\to 0}{\longrightarrow} \bar{L}^{\mathbb{1},\mathcal{R}}_{QQ} = [\text{finite}]. \tag{14}$$

This result is presumably scheme independent since, as it is well known, both β_0 and $\gamma_{Q,0}$ are scheme independent.

Leading behavior in the MS scheme.—The leading behavior of $L^{\mathbb{1},\rm MS}_{QQ}$ is obtained explicitly by using the one-loop expressions for β, γ_Q and the LO of $\chi^{\rm MS}_{QQ}$ involves hypergeometric functions and is given in (A6) in Appendix A 2. For $\xi = 0$ (A6) becomes simpler which can be expanded in a_s:

$$\begin{aligned} L^{\mathbb{1},\rm MS}_{QQ}(\mu) &\simeq 2 r^{\mathbb{1}(1,0)}_{QQ} \int_0^\infty e^{-2\epsilon t} \epsilon^{-\frac{\gamma_{Q,0}}{\beta_0}} (\epsilon + \beta_0 a_s (1-e^{-2\epsilon t})^{\frac{\gamma_{Q,0}}{\beta_0}}) dt \\ &= r^{\mathbb{1}(1,0)}_{QQ} \frac{(1+\frac{a_s\beta_0}{\epsilon})^{1+\frac{\gamma_{Q,0}}{\beta_0}} - 1}{a_s(\beta_0+\gamma_{Q,0})} \\ &= r^{\mathbb{1}(1,0)}_{QQ} \left(\frac{1}{\epsilon} + \frac{\gamma_{Q,0} a_s}{2\epsilon^2} \right. \\ &\quad \left. + \frac{(-\beta_0\gamma_{Q,0} + (\gamma_{Q,0})^2) a_s^2}{6\epsilon^3} + \mathcal{O}(a_s^3)\right), \end{aligned} \tag{15}$$

where the μ dependence arises from $a_s = a_s(\mu)$. There are divergent terms at each order in the a_s expansion. Provided (14) is met for $n_{QQ} = 0$ the $\epsilon \to 0$ limit is finite and gives

$$\bar{L}^{\mathbb{1},\rm MS}_{QQ} = -\frac{r^{\mathbb{1}(1,0)}_{QQ}}{a_s(\beta_0 + \gamma_{Q,0})}. \tag{16}$$

Two important remarks are in order. First when (15) is expanded in powers of a_s then $1/\epsilon$ poles appear irrespective of whether condition (14) is obeyed or not. This is an example of where fixed order perturbation theory gives the wrong indication about convergence. Secondly, even though convergent the $\epsilon \to 0$ followed by $a_s \to 0$ limit does not exist for the $\langle O_Q O_Q \rangle$-correlation function. That is to say that in general the a_s expansion (fixed order) and $\epsilon \to 0$ limit do *not* commute. In the cases where the correlation function is related to a physical observable, such as the trace of the energy moment tensor (TEMT) correlation function, there are a_s-dependent prefactors which ensure a smooth limit.

2. Asymptotically safe theory

The nontrivial FP is characterized by generally nonvanishing anomalous dimensions $\gamma_Q = \gamma_Q^* + (a_s - a_s^{\rm UV})\gamma_{Q,0} + \cdots$. The integrating factor assumes the form $I_{QQ}(t) \sim e^{2\gamma_Q^* t}$. The exponential behavior dominates

[5]Note that the γ_Q's refer to the anomalous dimensions of the operators and not to the κ_Q parameters ($\gamma_Q = -\gamma_{\kappa_Q}$). Using that for a 1 coupling theory and a mass-independent scheme $\frac{d}{d\ln\mu} = 2\hat{\beta}\partial_{\ln a_s}$, (8) can be written as $((\epsilon - \hat{\gamma}_Q) - \hat{\beta}\partial_{\ln a_s}) L^{\mathbb{1},\mathcal{R}}_{QQ} = \chi^{\mathcal{R}}_{QQ}/2$.

[6]For $\hat{\beta}(a_s) = -\epsilon + \beta = -\epsilon - \beta_0 a_s^r + \mathcal{O}(a_s^{r+1})$ this leads to $a_s(t) = (a_s(\mu) e^{-2\epsilon t} \epsilon^{1/r})/(\epsilon + \beta_0 a_s^r(\mu)(1-e^{-2r\epsilon t}))^{1/r} \to a_s(\mu)(1+2r\beta_0 a_s^r(\mu) t)^{-1/r}$ for $\epsilon \to 0$ provided $r > 0$. In the case where $\gamma_Q(a_s) \sim a_s^r$ the formula (14) still applies. For more generic cases we leave it to the reader to work out the relevant formula from Eqs. (11) and (12).

[7]In the case of a nondiagonal correlation function, as in (2), with a single coupling theory the criterion (14) generalizes to $1 + \frac{\gamma_{A,0}+\gamma_{B,0}}{2\beta_0} < n_{AB}$ where the operator basis has been assumed to be diagonalized at LO. An example is given by QCD with the topological term $O_1 = G\tilde{G}$ and $O_2 = \partial_\mu \bar{q}\gamma^\mu\gamma_5 q$ which do mix with each other $Z_1^2 = 12 C_F a_s \frac{1}{\epsilon} + \mathcal{O}(a_s^2)$.

over the polynomial behavior of $\chi^{\mathcal{R}}_{QQ}$. Hence the sign of γ^*_Q determines the convergence

$$\gamma^*_Q < 0 \Rightarrow L^{\mathbb{1},\mathcal{R}}_{QQ} \overset{\epsilon/a_s \to 0}{\longrightarrow} [\text{finite}]. \tag{17}$$

If $\gamma^*_Q > 0$, $L^{\mathbb{1},\mathcal{R}}_{QQ}$ diverges and if $\gamma_Q = 0$, then the analysis of the AF case in the previous section applies.

B. Summary and contemplation

In summary the presence or absence of UV divergences depends on the anomalous dimension γ_Q and the leading power behavior of the χ_{QQ}. A more detailed comparison is instructive. In the AF case (14) the condition depends on both quantities mentioned above whereas in the AS case (17) it only depends on the anomalous dimension at the FP. The polynomial behavior of χ_{QQ} is overruled by the exponential behavior of the anomalous dimension. This is reminiscent of marginal flows requiring specific analysis in order to determine whether or not they are relevant or exactly marginal, whereas relevant and irrelevant flows are settled from the start. The behavior of the AS case is similar to the case of a scale or conformally invariant field theory. The two-point function of operators, of scaling dimension $\Delta_O = d_O + \gamma_O$, is given by $\langle O(x)O(0)\rangle \sim (x^2)^{-\Delta_O}$. In our case $d_O = 4$ and the Fourier transform of the p^4 structure is convergent provided $\gamma_O < 0$ in accordance with the criteria for an AS theory (17).

A priori the divergent structure of two-point function of dimension-four operators in momentum space reads ($d = 4$)

$$\Gamma_{AB} \sim a\Lambda^4_{\rm UV} + bp^2\Lambda^2_{\rm UV} + cp^4 \ln \Lambda_{\rm UV} + [\text{finite}], \tag{18}$$

for a cutoff regularization. Above a, b, c are dimensionless functions of $\Lambda_{\rm UV}/\mu_0$ where μ_0 is some reference scale. In this section it was shown under what conditions $c_{\rm DR}(\Lambda_{\rm UV}/\mu_0)\ln\Lambda_{\rm UV} = [\text{finite}]$ holds for $\Lambda_{\rm UV} \to \infty$ in DR (symbolically $\ln\Lambda_{\rm UV} \leftrightarrow 1/\epsilon$). Since DR is defined only in perturbation theory one might question as to whether the result holds outside this framework. An argument in favor is that perturbation theory is trustworthy in the UV and that the LL approximation should therefore be sufficient. One assumption though is that the UV divergences can be captured as a Laurent expansion in powers of $1/\epsilon$. Whether or not this is valid outside perturbation theory is unknown since DR is only defined perturbatively. It is well known that DR is blind to power divergences since no explicit scale is introduced into the integral regularization other than the prefactor $\mu^{-2\epsilon}$. Hence $a_{\rm DR} = b_{\rm DR} = 0$ is built into DR rather than being a result.[8]

In Secs. IV A and IV B the results are generalized straightforwardly to the case of multiple couplings and three-point functions. First, we illustrate the findings in the familiar setting of QCD-like gauge theories including an extension of the R^2 anomaly to one order higher. This will clarify the meaning of the quantity χ_{QQ} as being related to trace anomaly of the external sources of the corresponding operators; cf. [18].

III. QCD-LIKE GAUGE THEORY AS AN EXAMPLE

We consider a QCD-like gauge theory, i.e. N_f massless fermions in a fundamental representation coupled to gluons in the adjoint representation for a $SU(N_c)$ gauge group. This implies in particular a nontrivial RG flow. In Sec. III A finiteness of the $\langle G^2G^2\rangle$ and the closely related $\langle\Theta\Theta\rangle$ correlators is established,[9] followed by a discussion of the physical consequences: unsubtracted dispersion relation (Sec. III B 1) and observability of the bare correlation function (Sec. III B 2). In Sec. III C the discussion is extended to include condensates through the OPE.

A. Gauge theory correlation functions

Correlation functions of the field strength tensor.—We consider the two-point function of the field strength correlation function, with

$$[O_g] = \left[\frac{1}{g_0^2}G^2\right], \tag{19}$$

where $G^2 = G^2_{\mu\nu}$ is the usual field strength tensor $G_{\mu\nu} = -i[D_\mu, D_\nu]$ squared with covariant derivative $D_\mu = (\partial + iA)_\mu$. From (7) $\kappa_g = \hat{\beta}$ and therefore $\gamma_g = 2\partial_{\ln a_s}\hat{\beta}$ follows. This leads to a simple form of the integrating factor (12):

$$I_{gg}(\mu,\mu') = \left(\frac{\hat{\beta}(\mu')}{\hat{\beta}(\mu)}\right)^2. \tag{20}$$

The corresponding Laurent series (11), changing variables to $d\ln\mu' = du/(2u\hat{\beta}(u))$, takes on the form

$$L^{\mathbb{1},\mathcal{R}}_{gg}(\mu,\epsilon) = -\frac{1}{2\hat{\beta}^2(a_s(\mu))}\int_0^{a_s(\mu)} \chi^{\mathcal{R}}_{gg}(u)\hat{\beta}(u)\left(\frac{\mu(a_s)}{\mu(u)}\right)^{2\epsilon}\frac{du}{u}, \tag{21}$$

where the factor $(\ldots)^{2\epsilon}$ will be specified further below. This expression is convergent as $\gamma_{g,0} = -2\beta_0$ and $\chi^{\mathcal{R}}_{gg}(a_s) \sim \mathcal{O}(a_s^0)$ obey the inequality (14) with 1–2 < 0. This means that

[8] Let us mention, in passing, that it has been argued by Bardeen [17] that cutoff regularizations are not a natural choice for renormalizable theories. For example when a theory exhibits a global chiral symmetry one would preferably use a chirally invariant regularization as otherwise the Ward identities need to be fixed by adding local counterterms.

[9] The $\langle\bar{q}q\bar{q}q\rangle$ and $\langle J^5_\mu J^5_\nu\rangle$-correlation functions are discussed in Appendixes B 1 and B 2, respectively. Its convergence depends on the number of flavors and colors.

$$L_{gg}^{\mathbb{1},\mathcal{R}}(\epsilon) \overset{\epsilon/\beta\to 0}{\longrightarrow} \bar{L}_{gg}^{\mathbb{1},\mathcal{R}} = [\text{finite}]. \tag{22}$$

It is instructive to consider this constant explicitly at LL in the MS scheme:

$$L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} = \frac{r_{gg}^{\mathbb{1}(1,0)}}{\epsilon+\beta_0 a_s} \overset{\epsilon/(\beta_0 a_s)\to 0}{\longrightarrow} \bar{L}_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} \equiv \frac{r_{gg}^{\mathbb{1}(1,0)}}{\beta_0 a_s} = [\text{finite}], \tag{23}$$

as it becomes apparent that the correlation function is not finite for $a_s \to 0$. One should keep in mind that the field strength correlation function is not a physical quantity unlike the closely related correlation function of the TEMT discussed below. Before doing so let us emphasize that when expanding (21) in a_s divergent terms appear; cf. (15). We indeed reproduce the divergent terms in [19,20] at next-to-leading order and next-to-next-to-leading order (NNLO), respectively. To obtain agreement it is important to expand the term to the power 2ϵ in the integrand $(\mu(a_s)/\mu(u))^{2\epsilon} = \exp(\epsilon \int_u^{a_s} du'/(u'\hat{\beta}(u'))) = u/a_s(\mu) + \mathcal{O}(1/\epsilon)$.

Correlation functions of the trace of the energy momentum tensor.—The TEMT decomposes as follows:

$$\langle T^{\rho}{}_{\rho}\rangle = (-\delta_{s(x)})\ln\mathcal{Z} = \langle\Theta\rangle + \langle\Theta_{\mathrm{gravity}}\rangle + \langle\Theta_{\mathrm{eom}}\rangle + \langle\Theta_{\mathrm{gf}}\rangle, \qquad \Theta = \frac{\hat{\beta}}{2}[O_g], \tag{24}$$

where Θ, $\Theta_{\mathrm{gravity}}$, Θ_{eom}, and Θ_{gf} are the operator, curvature-dependent, EOM and gauge-fixing part of the TEMT, respectively. The Θ_{gf} part does not contribute to physical observables, $\Theta_{\mathrm{gravity}}$ vanishes in flat space, and Θ_{eom} contributes to the $(p^2)^0$ structure. We can therefore concentrate on Θ.[10] Adapting the notation $[O_\theta] = \Theta$ in analogy with (19), Eq. (24) implies a relation between the two Laurent series:

$$L_{\theta\theta}^{\mathbb{1},\mathcal{R}} = \frac{\hat{\beta}^2}{4} L_{gg}^{\mathbb{1},\mathcal{R}} + [\text{finite}]. \tag{25}$$

An expression for $L_{\theta\theta}^{\mathbb{1},\mathrm{MS}}$ is obtained from (21) by multiplying by $\hat{\beta}^2/4$, partial integration and subtracting the finite constant in (25):

$$L_{\theta\theta}^{\mathbb{1},\mathrm{MS}} = \frac{1}{4}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right) u\left(\left(\frac{\mu(a_s)}{\mu(u)}\right)^{2\epsilon} - \frac{u}{a_s}\right) r_{gg}^{\mathbb{1}(1)}(u)du. \tag{26}$$

The limiting expression $\bar{L}_{\theta\theta}^{\mathbb{1},\mathrm{MS}}$ is manifestly finite and well behaved in the limit $\beta \to 0$ and $a_s \to 0$. For instance, the LL expression is given by $\bar{L}_{\theta\theta}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} = (r_{gg}^{\mathbb{1}(1,0)}/4)\cdot\beta_0 a_s$. In passing we note that the criteria (14) for the $\langle\Theta\Theta\rangle$ correlator is obeyed with $\gamma_\theta = 0$ and $n_{\theta\theta} = 2$. Finiteness of the trace of the $\langle\Theta\Theta\rangle$ correlator and the $\langle G^2G^2\rangle$ correlator in AF gauge theories has been noted elsewhere [16,21–24].

We conclude this section by stating that both $\mathbb{C}_{gg}^{\mathbb{1}}(p^2)$ and $\mathbb{C}_{\theta\theta}^{\mathbb{1}}(p^2)$ are finite for $\epsilon \to 0$ and that $\mathbb{C}_{gg}^{\mathbb{1}}(p^2)$, being proportional to $1/\beta$ (21), cannot be expanded in a_s. An AF theory is therefore different from a conformal field theory (CFT), of which a free theory is a special case, in that the correlator of marginal operators is finite in the former but not the latter case. In a CFT $\Gamma_{XX}(x^2) \sim 1/x^8$, with $\Delta_{O_X} = 4$, which diverges upon Fourier transformation whereas $\Gamma_{gg}(x^2) \sim 1/x^8 f(\ln(\mu^2 x^2))$ converges in the AF case.

B. Consequences of finiteness of $\mathbb{C}_{gg}^{\mathbb{1}}(p^2)$ and $\mathbb{C}_{\theta\theta}^{\mathbb{1}}(p^2)$

There are three points connected to the finiteness of $L_{gg}^{\mathbb{1},\mathcal{R}}$ and $L_{\theta\theta}^{\mathbb{1},\mathcal{R}}$ which we would like to discuss. First, since the bare Wilson coefficients $\mathbb{C}_{gg}^{\mathbb{1}}(p^2)$ and $\mathbb{C}_{\theta\theta}^{\mathbb{1}}(p^2)$ are both finite, they satisfy a dispersion relation which does not require subtractions (i.e. no regularization). Note that if regularization was necessary, then the $\epsilon \to 0$ limit would not exist contrary to our findings. An explicit dispersion representation is given at LL in Sec. III B 1. Second, since $\mathbb{C}_{\theta\theta}^{\mathbb{1}}(p^2)$ is finite and scale independent (since bare) it may be related to a physical observable, which is indeed the case; cf. Sec. III B 2. A third aspect is the R^2-trace anomaly associated with the $\langle\Theta\Theta\rangle$ correlator. Since anomalies *can be* interpreted as originating from UV divergences one might wonder whether UV finiteness means that the R^2 anomaly (related to $\langle\Theta\Theta\rangle$; cf. Sec. III D) is an artifact of perturbation theory only. The answer to this question is no, at least in the MS scheme since it is the $\ln\mu$ term which is the true signal of the anomaly. Finiteness though means that one can choose a scheme [25] where the R^2 anomaly is absent or absorbed into the renormalization of the dynamical operators.

1. *Explicit unsubtracted dispersion representation for leading logarithms*

We introduce $P^2 \equiv -p^2$, where P^2 might be thought of as a Minkowski momentum allowing us to write the dispersion relation in the usual way. The starting point is the LL expression (23). The associated logarithms are $1/\epsilon^n \leftrightarrow -\ln^n(1/\mu^2)$ (which is derived in Appendix A 1 from the bare correlation function) and by dimensional analysis this implies $1/\epsilon^n \leftrightarrow -\ln^n(-P^2/\mu^2)$. At LL the expression can be written as follows:

$$\begin{aligned}\mathbb{C}_{gg}^{\mathbb{1}}(p^2)|_{\mathrm{LL}} &= \mathbb{C}_{gg}^{\mathbb{1},\mathrm{MS}}(p^2)|_{\mathrm{LL}} + (L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}})_{\epsilon\to 0} \\ &= (L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}})_{\epsilon^{-n}\to -\ln^n(-\frac{P^2}{\mu^2})} + (L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}})_{\epsilon\to 0} \\ &= -\frac{r_{gg}^{\mathbb{1}(1,0)}\ln(-P^2/\mu^2)}{1+a_s\beta_0\ln(-P^2/\mu^2)} + \frac{r_{gg}^{\mathbb{1}(1,0)}}{a_s\beta_0} \\ &= \frac{r_{gg}^{\mathbb{1}(1,0)}}{a_s\beta_0}x(P^2),\end{aligned} \tag{27}$$

[10] In the case where the fermions are massive $\Theta \to \Theta + N_f m_f(1+\gamma_m)\bar{q}q$.

with

$$x(P^2) = \frac{1}{1 + a_s\beta_0 \ln(-P^2/\mu^2)}, \tag{28}$$

and it should be kept in mind that the ϵ^{-n} replacement rule is to be applied to the renormalized part only. We refer the reader to [26] for a next-to-leading logarithmic expression but the reason we content ourselves with LL is that it is sufficient for the asymptotic behavior. Since $x(P^2)$ is finite for $P^2 \to \infty$, it obeys an unsubtracted dispersion relation of the form

$$x(P^2) = \frac{1}{2\pi i}\int_\Gamma \frac{x(s)}{s - P^2}, \tag{29}$$

where Γ is such that no singularities are crossed. The singularities of $x(P^2)$ are a branch cut at $P^2 \geq 0$ and a pole in the Euclidean domain at $P^2 = P_0^2 \equiv -\mu^2 \exp(-1/(\beta_0 a_s))$ on which we comment in the next section. It is convenient to split the dispersion representation into the pole part

$$x(P^2) = \frac{-1}{1 - P^2/P_0^2} + \hat{x}(P^2) \tag{30}$$

and the integration over the cut

$$\hat{x}(P^2) = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}[x(s)]}{s - P^2 - i0} = \int_0^\infty \frac{ds}{s - P^2 - i0} \frac{1}{(1 + a\beta_0 \ln(s/\mu^2))^2 + (a\beta_0\pi)^2}. \tag{31}$$

Above it was used that $x(s) \to 0$ for $s \to \infty$ as otherwise the arc at infinity would contribute to the dispersion integral. This is the formal solution and it is easily seen that for finite P^2 the integrand behaves $\int_0^\infty ds/(s\ln(s/\mu^2)^2) < 0$ which is finite. The integral (31) is explicitly evaluated in Appendix A 3 to reproduce the expression in (30). The dispersion relation for the TEMT part is simply given by $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)|_{\mathrm{LL}} = \beta_0^2/4a_s^2 \mathbb{C}^{\mathbb{1}}_{gg}(p^2)|_{\mathrm{LL}}$.

2. Finiteness of $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ implies observability

Generally physical quantities are RG-scale independent and finite. Bare correlation functions with renormalized composite operators, such as $\Gamma_{AB}(p^2)$ (2), are RG-scale independent but, in the case where they are not finite, do not qualify as physical observables. Since $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ is finite the situation changes and the bare function is observable. For example $\Delta\bar{b}$, the difference of the flow of $\Box R$ term of the Weyl anomaly, is related by $\Delta\bar{b} = \frac{1}{8}\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0)$ [25].

Below we illustrate the scale independence of $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ (and the analogous case of the bare $m^2\langle\bar{q}q\bar{q}q\rangle$ correlator is discussed in Appendix B 1). In a one-scale theory with one external momentum any quantity reads $\varphi(p^2/\mu^2, a_s(\mu/\mu_0))$ where μ_0 is a reference scale, e.g. Λ_{QCD}, which we suppress further below. In the case where φ is a physical quantity, and therefore independent of the renormalization scale μ, the functional dependence simplifies to

$$\frac{d}{d\ln\mu}\varphi(p^2/\mu^2, a_s(\mu^2)) = 0 \Leftrightarrow \varphi = \tilde{\varphi}(a_s(p^2)). \tag{32}$$

This is indeed the case for $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ at LL. Starting with (27) one gets

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)|_{\mathrm{LL}} = \frac{r_{gg}^{\mathbb{1}(1,0)}}{4}\frac{a_s(\mu^2)\beta_0}{1 + a_s(\mu^2)\beta_0 \ln(p^2/\mu^2)} = \frac{r_{gg}^{\mathbb{1}(1,0)}}{4}\beta_0 a_s(p^2) + \mathcal{O}(\beta_1), \tag{33}$$

a function which depends on $a_s(p^2)$ only. Note that if we were guided by fixed order perturbation theory, then we would resort to the renormalized $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{\theta\theta}(p^2)$ which is scale dependent, $\frac{d}{d\ln\mu}\mathbb{C}^{\mathbb{1},\mathcal{R}}_{\theta\theta}(p^2) = -\lim_{\epsilon\to 0}(\frac{d}{d\ln\mu} - 2\epsilon)L^{\mathbb{1},\mathcal{R}}_{\theta\theta} = \chi^{\mathcal{R}}_{\theta\theta}$ with the last equality following from (11). This is why it is sometimes stated that only $p^2\frac{d}{dp^2}\mathbb{C}^{\mathbb{1},\mathcal{R}}_{\theta\theta}(p^2) = p^2\frac{d}{dp^2}\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ (e.g. [20]) is physical whereas we advocate that the bare term $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)$ is physical and should be stated. An example being the previously mentioned $\Box R$ flow: $\Delta\bar{b} = \frac{1}{8}\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0)$. The scheme-dependent splitting of the bare function into a counterterm and a renormalized part defines a flow for $\bar{b}$ connecting the UV and IR values [25].

The pole discussed in the previous section is the Landau pole of the gauge coupling. It has no direct physical meaning and contradicts the analytic structure of the spectral representation. It is precisely this pole that is removed in the approach of analytic perturbation theory by enforcing a physical singularity structure on the amplitude [27]. In the fully nonperturbative version this pole disappears. The correlation function satisfies an unsubtracted dispersion ($P^2 = -p^2$):

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}(P^2) = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}[\mathbb{C}^{\mathbb{1}}_{\theta\theta}(s)]}{s - P^2 - i0} + \omega_0, \tag{34}$$

consistent with the Källén-Lehmann representation. Above ω_0 is an arbitrary, scale-independent, finite constant which can be added by changing the theory by a local term in the UV. The addition of this term is more than a choice of scheme; it corresponds to changing the theory by a local term. We have therefore silently assumed $\omega_0 = 0$ which is automatic in the conventional setup. This constant, being arbitrary, should not impact on any physical predictions. In the above mentioned formula of the $\Box R$-flow anomaly this is ensured by the implicit boundary condition $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty) = 0$. If this boundary condition is generic $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty) = \omega_0$, as in (34), then the formula simply changes to $\Delta\bar{b} = \frac{1}{8}(\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0) - \omega_0)$ [25].

C. OPE extension with condensates

This section may be considered as a minor digression and the reader may or may not want to directly proceed to Sec. IV. The discussion below has some overlap with Ref. [28] but goes beyond it in the emphasis on finiteness. So far we have treated the correlation function (2) within the framework of perturbation theory. Equation (2) is a good approximation for large p^2 and preasymptotic effects for $p^2 \gg \Lambda_{\mathrm{QCD}}^2$ can be parametrized in terms of vacuum condensates $\langle [O_C] \rangle \sim \mathcal{O}(\Lambda_{\mathrm{QCD}}^4)$ with the framework of the OPE [1]. The vacuum condensates appear as power suppressed with respect to perturbation theory (i.e. the $\langle \mathbb{1} \rangle = 1$ term):

$$\Gamma_{AB}(p^2) = \mathbb{C}_{AB}^{\mathbb{1}}(p^2) p^4 \langle \mathbb{1} \rangle + \sum_C \mathbb{C}_{AB}^C(p^2) \langle [O_C] \rangle. \tag{35}$$

Anticipating Sec. IV A we include several operators which correspond to the several coupling case. In the formula above we have assumed all perturbations $[O_C]$ to be of dimension four, i.e. marginal. The OPE has been shown to hold in perturbation theory [29,30] and has enjoyed phenomenological success in the nonperturbative regime where it is expected to hold [31].

It is our aim to investigate whether or not the Wilson coefficient $\mathbb{C}_{gg}^g(p^2)$, in analogy to $\mathbb{C}_{gg}^{\mathbb{1}}(p^2)$,

$$\Gamma_{gg}(p^2) = \mathbb{C}_{gg}^{\mathbb{1}}(p^2) p^4 + \mathbb{C}_{gg}^g(p^2) \langle [O_g] \rangle, \tag{36}$$

is finite or not. To do so the QAP proves useful. The key idea of the QAP is that differentiation of the finite partition function with respect to finite renormalized parameters leads directly to finite well-defined quantities. The QAP might be regarded as a scheme for the renormalization of composite operators. For convenience we employ the local QAP (e.g. [28,32]) where the couplings g^A are promoted to local functions $g^A(x)$ which then become sources for the corresponding local operators

$$\langle [O_A(y)] \rangle = (-\delta_{A(y)}) \ln \mathcal{Z}, \qquad \delta_{A(x)} \equiv \frac{\delta}{\delta g^A(x)}, \tag{37}$$

where the corresponding Lagrangian in bare quantities reads $\mathcal{L} = g_0^A O_A + \cdots$. The principle also applies to higher point functions such as

$$\begin{aligned}\Gamma_{AB}^{\mathcal{R}}(x-y) &= (-\delta_{B(y)})(-\delta_{A(x)}) \ln \mathcal{Z} = (-\delta_{B(y)}) \langle [O_A(x)] \rangle \\ &= \langle [O_A(x)][O_B(y)] \rangle_c - \langle (\delta_{B(y)}[O_A(x)]) \rangle \\ &= [\text{finite}]. \end{aligned} \tag{38}$$

Since the right-hand side is finite this means that the local divergences of the unrenormalized two-point function,

$$\mathbb{C}_{AB}^C(p^2) - L_{AB}^{C,\mathcal{R}} = [\text{finite}], \tag{39}$$

ought to cancel against corresponding divergencies in $L_{AB}^{C,\mathcal{R}}$ coming from the variation of the operator renormalization constants

$$\begin{aligned}\langle (\delta_{B(y)}[O_A(x)]) \rangle = &\sum_C L_{AB}^{C,\mathcal{R}} \langle [O_C] \rangle \delta(x-y) \\ &+ \mu^{d-4} L_{AB}^{\mathbb{1},\mathcal{R}} \langle \mathbb{1} \rangle \Box^2 \delta(x-y) + [\text{finite}]. \end{aligned} \tag{40}$$

The quantities $L_{AB}^{C,\mathcal{R}}$ are given in terms of the RG mixing matrix $\mathbb{Z}_A{}^I$:

$$L_{AB}^{C,\mathcal{R}} = (\partial_B \mathbb{Z}_A{}^I)(\mathbb{Z}^{-1})_I^C, \qquad \partial_Q \equiv \frac{\partial}{\partial g^Q}, \tag{41}$$

where the scheme dependence comes from the scheme-dependent $\mathbb{Z}_A{}^I|_{\mathcal{R}} = \mathbb{Z}_A^I$ which we suppress throughout in order to lighten the notation. For our example with a Yang-Mills Lagrangian $\mathcal{L} = 1/4 O_g$ with O_g defined in (19) the renormalized composite operator follows from $\langle [O_g(x)] \rangle = (-\frac{4}{g^2} \delta_{1/g^2(x)}) \ln \mathcal{Z} = (2\delta_{\ln g(x)}) \ln \mathcal{Z}$. The renormalization of $[O_g]$ is given by [33,34]

$$\begin{aligned}&[O_g] = \mathbb{Z}_g{}^g O_g + \cdots, \\ &\mathbb{Z}_g{}^g = \left(1 + \frac{\partial \ln Z_g}{\partial \ln g}\right) = \frac{(d-4)}{2\hat{\beta}},\end{aligned} \tag{42}$$

with the dots standing for EOM and gauge-dependent terms which vanish on physical states and are therefore immaterial for the current discussion. From Eqs. (40) and (42) one gets

$$\begin{aligned}L_{gg}^{g,\mathcal{R}_1} &= 2\partial_{\ln g} \ln \mathbb{Z}_g{}^g \\ &= 2\left(1 + \frac{\partial \ln Z_g}{\partial \ln g}\right)^{-1} \frac{\partial^2 \ln Z_g}{\partial (\ln g)^2} \\ &= -\frac{2\partial_{\ln g}\hat{\beta}}{\hat{\beta}} \overset{\epsilon/\beta \to 0}{\longrightarrow} [\text{finite}]. \end{aligned} \tag{43}$$

The term $L_{gg}^{g,\mathcal{R}_1}$ differs from $L_{gg}^{g,\mathrm{MS}}$ in finite terms since the latter is a power series in $1/\epsilon$ by definition. From (39) it then follows that $\mathbb{C}_{gg}^g(p^2)$ is finite in the limit $\epsilon \to 0$ but divergent at each order in perturbation theory.[11] It is noted

[11]To compare with the literature we need the CTs of G^2 as opposed to $[O_g]$. These can be obtained by accounting for the factor $1/g_0^2$ in the definition of O_g [see (19)], which leads to the following modification of (43): $2\partial_{\ln g} \ln \mathbb{Z}_g{}^g / g_0^2 = L_{gg}^{g,\mathcal{R}_1} - 2(d-4)/\hat{\beta}$. Expanding this expression in powers of a_s results in $-4(a_s^2[\frac{\beta_1}{\epsilon}] + a_s^3[-\frac{\beta_0\beta_1}{\epsilon^2} + \frac{2\beta_2}{\epsilon}] + a_s^4[\frac{\beta_0^2\beta_1}{\epsilon^3} - \frac{\beta_1(\beta_1+2\beta_0)}{\epsilon^2} + \frac{3\beta_3}{\epsilon}])$ up to corrections of the order of $\mathcal{O}(a_s^5, \epsilon^0)$. This is identical to $\frac{1}{4} Z_{11}^L / Z_{11}$ in [35,36] [Eq. (4.7)] up to finite terms. However, the observation of finiteness and its possible implications were not made in [35,36].

though that $L^{\theta,\mathcal{R}}_{\theta\theta}(p^2) = \hat{\beta}/2L^{g,\mathcal{R}}_{gg} + [\text{finite}] = -\partial_{\ln g}\hat{\beta} + [\text{finite}]$ is finite, but even finite in each order in perturbation theory provided every quantity, e.g. beta function, is treated consistently in d dimensions. The same applies therefore to the corresponding bare Wilson coefficient $\mathbb{C}^{\theta}_{\theta\theta}(p^2) = \hat{\beta}/2\mathbb{C}^{g}_{gg}(p^2)$. Hence one can write down convergent dispersion relations for both $\mathbb{C}^{g}_{gg}(p^2)$ and $\mathbb{C}^{\theta}_{\theta\theta}(p^2)$ as done in Sec. III B 1. The coefficient $\mathbb{C}^{\theta}_{\theta\theta}(p^2)$ is scale independent.

It is again instructive to write down the LL expression

$$\mathbb{C}^{g}_{gg}(p^2)|_{\mathrm{LL}} = \frac{-1}{1 + a_s(\mu^2)\beta_0 \ln(p^2/\mu^2)},$$
$$\mathbb{C}^{\theta}_{\theta\theta}(p^2)|_{\mathrm{LL}} = \frac{\beta_0 a_s(\mu^2)}{2(1 + a_s(\mu^2)\beta_0 \ln(p^2/\mu^2))} = \frac{\beta_0}{2} a_s(p^2) + \mathcal{O}(\beta_1), \tag{44}$$

which happens to be proportional to $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)|_{\mathrm{LL}}$ and $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2)|_{\mathrm{LL}}$ (33), respectively.

D. Extending the R^2 anomaly to $\mathcal{O}(a_s^5)$

1. R^2 anomaly from the $\langle\Theta\Theta\rangle$ correlator and the quantum action principle

It has been known for a long time that trace anomalies in curved space time (e.g. [37]) are related to correlation functions of the TEMT in flat space. The link is provided by the local QAP.

In four dimensions the VEV of the TEMT in curved space reads (e.g. [38])

$$\langle T^{\rho}{}_{\rho}\rangle = -(\beta^{\mathrm{IR}}_a E_4 + \beta^{\mathrm{IR}}_c W^2 + \beta^{\mathrm{IR}}_b H^2) + 4\bar{b}^{\mathrm{IR}}\Box H + d\Lambda^{\mathrm{IR}},$$
$$H \equiv \frac{1}{(d-1)} R, \tag{45}$$

with E_4, W^2, R and Λ being the Euler, the Weyl squared, the Ricci scalar and the cosmological constant term, respectively, and $\beta_a(\mu) \overset{\mu\to 0}{\to} \beta^{\mathrm{IR}}_a$, etc.[12] The Euler term is topological and analogous to the $G\tilde{G}$ term of the chiral anomaly known as a type A anomaly [39]. The W^2 and H^2 arise from the introduction of a scale in the process of regularization and $\Box H$ is the variation of a local term. These are known as type B anomalies [39].

We write the gravitational counterterm as follows:

$$\mathcal{L}_{\mathrm{gravity}} = -(a_0 E_4 + c_0 W^2 + b_0 H^2), \tag{46}$$

with $b_0 = \mu^{d-4}(b^{\mathcal{R}} + L^{\mathcal{R}}_b)$ and notation that largely follows Shore's review [32]. A double variation of the Weyl factor $s(x)$ $(g_{\mu\nu} \to e^{-2s(x)} g_{\mu\nu})$ is finite since both the partition function and the metric are finite. When the latter is Fourier transformed and projected on the p^4 structure one obtains

$$\int d^d x e^{ip\cdot x}((-\delta_{s(x)})(-\delta_{s(0)}) \ln \mathcal{Z})|_{p^4}$$
$$= \int d^d x e^{ip\cdot x}\langle T^{\rho}{}_{\rho}(x) T^{\rho}{}_{\rho}(0)\rangle|_{p^4} + 8b_0$$
$$= \int d^d x e^{ip\cdot x}\langle \Theta(x)\Theta(0)\rangle|_{p^4} + 8b_0$$
$$= \mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2) p^{-2\epsilon} + 8b_0 = [\text{finite}], \tag{47}$$

where in passing to the third line we used the fact that EOM terms contribute to the $(p^2)^0$ structure only and assumed that neither virial currents nor nonimprovable scalars are present. This is correct for QCD-like theories in the conformal window used in the next section. In the parameterization of (48) this implies the nontrivial, known [40], relation

$$L^{\mathcal{R}_1}_b = -\frac{1}{8} L^{\mathbb{1},\mathcal{R}_2}_{\theta\theta} + [\text{finite}], \tag{48}$$

which translates into $L^{\mathrm{MS}}_b = -\frac{1}{8} L^{\mathbb{1},\mathrm{MS}}_{\theta\theta}$ for the MS scheme. We end this section with three slightly disjoint points.

(i) One application of the finiteness of $L^{\mathcal{R}}_b$, i.e. p^4 structure of the $\langle\Theta\Theta\rangle$ correlator, is that one can choose a scheme where $\beta^{\mathcal{R}}_b$ vanishes [25]. The contribution is absorbed into the operators appearing in the trace anomalies e.g. G^2 in the case of QCD-like theories. Below β_b is given in the MS scheme for which is nonzero outside the FPs.

(ii) Remark on the sign conventions and the specifics of the gravity counterterms. First a_0, b_0 and c_0 are taken to be independent of the scale in accordance with Refs. [37,40,41] but differing from the classic work [18] where b_0 is reduced to $\mu^{d-4} L^{\mathrm{MS}}_b$. We refer the reader to Appendix B of our paper [25] for further comments. Our approach determines $b^{\mathcal{R}}$ in $b_0 = \mu^{d-4}(b^{\mathcal{R}} + L^{\mathcal{R}}_b)$ up to a scale-independent constant e.g. [37] which incidentally is the ω_0 in (34). The sign convention of b_0 is such that β_a decreases along the flow leading to the counterterm with the opposite sign in (47) as compared to (4), which explains the sign difference in (48).

(iii) Equation (48) allows us to elucidate the quantity $\chi^{\mathcal{R}}_{QQ}$ (8) in the context where $Q = \theta$. Since the TEMT is physical, $\gamma_\theta = 0$, and from (11) it follows that $[\beta^{\mathcal{R}}_b \equiv -(\frac{d}{d\ln\mu} - 2\epsilon)L^{\mathcal{R}}_b]$

$$\beta^{\mathrm{MS}}_b = -\frac{1}{8}\chi^{\mathrm{MS}}_{\theta\theta}, \tag{49}$$

$\chi^{\mathrm{MS}}_{\theta\theta}$ and the $\beta^{\mathrm{MS}}_b R^2$ anomaly (45) are related. It seems worthwhile to mention that the link between $\chi^{\mathcal{R}}_{AB}$ and the TEMT generalizes for local source couplings other than $s(x)$. Instead of geometric terms like R^2, the $\chi^{\mathcal{R}}_{AB}$ appear in front of covariant

[12] The latter is defined by $\langle T_{\alpha\beta}\rangle = g_{\alpha\beta}\Lambda^{\mathrm{IR}}$ and may or may not be canceled by adding a suitable counterterm to the UV action.

expression in the source couplings $g_{A,B}(x)$; cf. Eqs. (2.5) and (2.8) in [18]. The concept also generalizes to higher point functions both at the level of gravity terms (e.g. E_4 is related to three-point functions) as well as covariant coupling terms.

2. Application to QCD-like theories

The generally valid relations (48) and (49) are applied to QCD-like theories in this section. From (48) and $L_{\theta\theta}^{\mathbb{1},\mathrm{MS}}$ in (26) it is then observed that ($\epsilon \to 0$ implied)

$$L_b^{\mathrm{MS}}(a_s(\mu)) = -\frac{1}{32}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right)u\left(1-\frac{u}{a_s}\right)r_{gg}^{\mathbb{1}(1)}(u)du, \tag{50}$$

which allows us to write an explicit formula for the β_b-anomaly term:

$$\begin{aligned}\beta_b^{\mathrm{MS}} &= -\left(\frac{d}{d\ln\mu}-2\epsilon\right)L_b^{\mathrm{MS}}\\ &= \frac{1}{16}\frac{\beta(a_s)}{a_s}\int_0^{a_s}\partial_u\left(\frac{\beta(u)}{u}\right)u^2 r_{gg}^{\mathbb{1}(1)}(u)du. \end{aligned}\tag{51}$$

Hence from the $r_{gg}^{\mathbb{1}(1)}$ counterterm of the TEMT correlation function one can deduce the R^2-anomaly term. The $r_{gg}^{\mathbb{1}(1)}$ term can be found in the recent computation [36] up to NNLO. We extract

$$\begin{aligned} r_{gg}^{\mathbb{1}(1,0)} &= \frac{n_g}{4\pi^2},\\ r_{gg}^{\mathbb{1}(1,1)} &= r_{gg}^{\mathbb{1}(1,0)}\left(\frac{17}{2}C_A-\frac{10}{3}N_F T_F\right),\\ r_{gg}^{\mathbb{1}(1,2)} &= 4r_{gg}^{\mathbb{1}(1,0)}\left(C_A^2\left(\frac{11}{6}\zeta_3+\frac{22351}{1296}\right)\right.\\ &\quad - C_A N_F T_F\left(\frac{14}{3}\zeta_3+\frac{799}{81}\right)\\ &\quad \left.+ n_g N_F T_F\left(\zeta_3-\frac{107}{18}\right)+\frac{49}{81}T_F^2N_F^2\right), \end{aligned}\tag{52}$$

with ζ_3 being the Riemann zeta function at the value 3 and $n_g = C_A C_F/T_F|_{SU(N_c)} = N_c^2-1$ being the number of gluons (dimension of the adjoint representation). The $\mathcal{O}(a_s^3)$ contribution agrees with [40] [Eq. (7.7)] at the level of β_0 and β_1 which straightforwardly extends to QCD-like theories.

From (51) one then obtains β_b up to $\mathcal{O}(a_s^5)$. We give the result in terms of the first pole in $L_b^{\mathrm{MS}} = \sum_{n\geq 1} b_n(a_s)\epsilon^{-n}$ by the MS-type relation (with an extra factor of 2 in the first equality with respect to [37,40] from the $d = 4-2\epsilon$ versus $d = 4-\epsilon$ convention)

$$\begin{aligned}\beta_b^{\mathrm{MS}} &= 2\partial_{a_s}(a_s b_1)\\ &= 8b_{1,3}a_s^3 + 10b_{1,4}a_s^4 + 12b_{1,5}a_s^5 + \mathcal{O}(a_s^6),\end{aligned}\tag{53}$$

where $[b_n = b_{n,0} + b_{n,1}a_s + \mathcal{O}(a_s^2)]$

$$\begin{aligned} b_{1,3} &= \frac{1}{24\cdot 16}\beta_0\beta_1 r_{gg}^{\mathbb{1}(1,0)},\\ b_{1,4} &= \frac{1}{120\cdot 16}((4\beta_1^2+6\beta_0\beta_2)r_{gg}^{\mathbb{1}(1,0)}+3\beta_0\beta_1 r_{gg}^{\mathbb{1}(1,1)}),\\ b_{1,5} &= \frac{1}{720\cdot 16}(50\beta_0\beta_2 r_{gg}^{\mathbb{1}(1,0)}+(24\beta_0\beta_2+15\beta_1^2)r_{gg}^{\mathbb{1}(1,1)}\\ &\quad +12\beta_0\beta_1 r_{gg}^{\mathbb{1}(1,2)}) \end{aligned}\tag{54}$$

follows from (51). Comparing with [18,37,40] we find agreement with [37,40] to the computed order of $\mathcal{O}(a_s^3)$ and with [18] to order $\mathcal{O}(a_s^4)$. The fact that β_b^{MS} is proportional to the β function is consistent with β_b being zero in CFTs [42,43]. In fact a stronger statement can be made since formula (51) shows that β_b^{MS} vanishes for one-loop β functions consistent with all coefficients above involving a β_n with $n \geq 1$.

At least let us make a general observation which makes use of the finiteness of $\bar{L}_b^{\mathrm{MS}} = \lim_{\epsilon\to 0}L_b^{\mathrm{MS}}$. Firstly, we observe that one may take the $\epsilon \to 0$ limit in (51) directly and replace $\frac{d}{d\ln\mu} \to \beta\frac{\partial}{\partial\ln g}$ leading to $\beta_b^{\mathrm{MS}} = -\beta\frac{\partial}{\partial\ln g}\bar{L}_b^{\mathrm{MS}}$. This result generalizes to multiple couplings as follows:

$$\beta_b^{\mathrm{MS}} = -\beta^A\partial_A\bar{L}_b^{\mathrm{MS}}, \tag{55}$$

where $\bar{L}_b^{\mathrm{MS}}$ is well defined in the limit of vanishing β function as will be shown in Sec. IVA. This result is accordance with $\beta_b^{\mathrm{CFT}} = 0$ [42,43].

IV. EXTENSIONS OF THE ONE-COUPLING TWO-POINT FUNCTION CASE

A. Multiple couplings and finiteness of TEMT correlators

In this section we proceed to show the finiteness of the $\langle\Theta\Theta\rangle$ correlator for a general field theory with a UV FP. Consider a RG flow generated by a deformation $\delta\mathcal{L} = \sum_A g_0^A O_A$. The induced trace anomaly reads[13]

$$\Theta = \hat{\beta}^A[O_A], \tag{56}$$

where here the β functions for the couplings g^A are given by

$$\hat{\beta}^A = \frac{d}{d\ln\mu}g^A = \beta^A - \epsilon g^A\xi^A. \tag{57}$$

[13]Three possible structures are neglected. EOM terms can be omitted for the same reasons as before. It is assumed that no virial currents $\Theta = \partial\cdot V + \cdots$ are present implicit in the assumption that the UV FP is conformal (no nontrivial unitary scale but not conformally invariant theories are known to date). Terms of the form $\Theta = -\Box\phi^2 + \cdots$ originating from nonconformally coupled scalars can be improved *à la* Callan, Coleman and Jackiw [44]. An exception is the chirally broken phase but since the term is relevant in the IR and not the UV we do not need to consider it for the purposes of this section.

The ξ^A are an artifact of going from four to d dimensions (e.g. [18] whose notation is adapted here). Note that in Sec. III, unlike here, the logarithmic β function was used and that in QCD $\xi^g = 1$. The generalization of (6) to the nondiagonal case is straightforward and given by

$$\mathbb{C}^{\mathbb{1}}_{AB}(p^2)p^{-2\epsilon} = (\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2) + L^{\mathbb{1},\mathcal{R}}_{AB})\mu^{-2\epsilon}, \tag{58}$$

where $\mathbb{C}^{\mathbb{1}}_{AB}$ is again finite in the sense that there are no poles upon expanding in ϵ as long as no expansion in the couplings g^Q is attempted. The multiple coupling generalization of (8) reads

$$(\mathcal{L}_\beta - 2\epsilon)L^{\mathbb{1},\mathcal{R}}_{AB} = -\chi^{\mathcal{R}}_{AB}, \tag{59}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative on a 2-tensor in coupling space

$$\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{AB} = \partial_A\hat{\beta}^C L^{\mathbb{1},\mathcal{R}}_{CB} + \partial_B\hat{\beta}^C L^{\mathbb{1},\mathcal{R}}_{AC} + \hat{\beta}^C\partial_C L^{\mathbb{1},\mathcal{R}}_{AB} \tag{60}$$

[∂_A defined in (40)] since $\hat{\gamma}_A{}^B$ is

$$\begin{aligned}\hat{\gamma}_A{}^B &= \partial_A\hat{\beta}^B \\ &= \partial_A\beta^B - \delta_A{}^B\xi^A\epsilon = \gamma_A{}^B - \delta_A{}^B\xi^A\epsilon,\end{aligned} \tag{61}$$

which follows from $\frac{d}{d\ln\mu}\langle\Theta\rangle = 0$ in flat space. The above equation is the analogue of $\gamma_g = \hat{\gamma}_g = 2\partial_{\ln a_s}\hat{\beta}$ stated below (19). The reason for $\gamma_g = \hat{\gamma}_g$ is that we used the logarithmic β function for QCD-like theories for which the $\mathcal{O}(\epsilon)$ term is coupling independent. The quantity χ^{MS}_{AB} generalizing (8) is then given by

$$\begin{aligned}\chi^{\mathrm{MS}}_{AB} &= 2\left(1 + \frac{1}{2}(\xi^A + \xi^B) + \frac{1}{2}\xi^Q g^Q\partial_Q\right)r^{\mathbb{1}(1)}_{AB}, \\ L^{\mathbb{1},\mathrm{MS}}_{AB} &= \sum_{n\geq 1}\frac{r^{\mathbb{1}(n)}_{AB}}{\epsilon^n}.\end{aligned} \tag{62}$$

The RGE (59) can be solved by the method of characteristics in terms of the anomalous dimension matrices γ_A^B:

$$L^{\mathbb{1},\mathcal{R}}_{AB}(\mu) = \int_{\ln\mu}^{\infty} I_A{}^C(\mu,\mu')\chi^{\mathcal{R}}_{CD}(\mu')I_B{}^D(\mu,\mu')\left(\frac{\mu}{\mu'}\right)^{2\epsilon} d\ln\mu', \tag{63}$$

where

$$I_A{}^B(\mu,\mu') = \left(\exp\left(\int_{\ln\mu}^{\ln\mu'}\hat{\gamma}(\mu'')d\ln\mu''\right)\right)_A{}^B. \tag{64}$$

It can be shown that[14]

$$\hat{\beta}^A(\mu)I(\mu,\mu')_A{}^B = \hat{\beta}^B(\mu'). \tag{65}$$

As previously $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p^2) = \hat{\beta}^A\hat{\beta}^B\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ and the generalization of (25) reads

$$L^{\mathbb{1},\mathcal{R}}_{\theta\theta}(\mu) = \int_{\ln\mu}^{\infty}\hat{\beta}^A(\mu')\hat{\beta}^B(\mu')\chi^{\mathcal{R}}_{AB}(\mu')\left(\frac{\mu}{\mu'}\right)^{2\epsilon} d\ln\mu' + [\text{finite}]. \tag{66}$$

For the asymptotic analysis it is, again, more convenient to use the variable $t = \ln\mu'$. We will now argue that the $\epsilon\to 0$ limit can be safely taken. Assuming $\chi_{AB} = \mathcal{O}(t^{-n_{AB}})$ with $n_{AB}\geq 0$ the integrand of (66) is controlled by the β functions for large t which tend to 0 by the UV-FP assumption.

Note that the criteria for finiteness are easily generalized. Finiteness of $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ and $L^{\mathbb{1},\mathcal{R}}_{AB}$ is easily established. For the cases of AF and AS, of Secs. II A 1 and II A 2, respectively, $\beta^Q_{\mathrm{AF}} = -\beta^Q_0/(4\pi)^2(g^Q)^{1+r_Q} + \cdots$ and $r_Q > 0$ and $\beta^Q_{\mathrm{AS}} = |a_Q|((g^Q)^{\mathrm{UV}} - g^Q) + \cdots$. Expressed in the RG time variable t this reads

$$\beta^Q_{\mathrm{AF}} \sim \frac{1}{t^{(1+\frac{1}{r_Q})}}, \qquad \hat{\beta}^Q_{\mathrm{AS}} \sim e^{-|a_Q|t}. \tag{67}$$

This means that the terms in the integrands in (63) and (66) vanish at least as $t^{-(2+\frac{1}{r_A}+\frac{1}{r_B})}$ or are exponentially suppressed which guarantees convergence of the t integral. Hence the $\epsilon\to 0$ limit can be taken safely and $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ and $L^{\mathbb{1},\mathcal{R}}_{AB}$ are finite which is the aimed result. Note that $L^{\mathbb{1},\mathcal{R}}_{AB}$ is the analogue of $L^{\mathbb{1},\mathcal{R}}_{gg}$ in the QCD-like case. If the operators are part of the dynamics i.e. present in the trace anomaly, then the case has to be reconsidered.

Condensate corrections.—As before we proceed to discuss the finiteness in the presence of vacuum condensates. The local coupling formalism of Sec. III C applies. From (39) and (56) it follows that

$$L^{Q,\mathcal{R}}_{\theta\theta} = \hat{\beta}^A\hat{\beta}^B L^{Q,\mathcal{R}}_{AB} + [\text{finite}]. \tag{68}$$

Along with (41) one deduces

$$\hat{\beta}^A\hat{\beta}^B L^{Q,\mathcal{R}}_{AB} = \hat{\beta}^A\hat{\beta}^B(\partial_B\mathbb{Z}_A{}^P)(\mathbb{Z}^{-1})_P{}^Q = -\hat{\beta}^A\hat{\gamma}_A{}^Q = [\text{finite}] \tag{69}$$

because of the boundedness of the anomalous dimension matrix $\hat{\gamma}_A{}^Q$. The scheme dependence on the left-hand side arises from $(\mathbb{Z}^{-1})_P{}^Q = (\mathbb{Z}^{-1})_P{}^Q|_{\mathcal{R}}$ which we suppress. Finiteness of $L^{Q,\mathcal{R}}_{\theta\theta}$ follows from (68) and completes the task of this paragraph.

B. Finiteness criteria for three-point functions

Another extension of interest are higher point functions. In general they consist of kinematic structures which are

[14]This follows by writing $\hat{\beta}^A(\mu)I(\mu,\mu')_A{}^B = f^B(\mu')$ which satisfies the differential equation $\partial_{\ln\mu'}f^B = f^C\hat{\gamma}_C{}^B(\mu')$ with initial condition $f^B(\mu) = \hat{\beta}^B(\mu)$. It is easy to show using (61) that $f^B(\mu') = \hat{\beta}^B(\mu')$ is the unique solution to to the initial value problem.

sensitive to the anomalous dimension of all operators in the correlation function *and* structures which are governed by lower-dimensional point functions. The latter can be identified by setting one or more of the external momenta to zero. A comprehensive analysis in CFTs can be found in [10] whereas we focus on theories with a nontrivial flow.

We introduce the following notation:

$$\begin{aligned}\Gamma_{ABC}(p_A^2,p_B^2,p_C^2) \\ &= \int d^4x d^4y e^{i(p_A\cdot x+p_B\cdot y)}\langle[O_A(x)][O_B(y)][O_C(0)]\rangle_c \\ &= \mathbb{C}^{\mathbb{1}}_{(A)BC}(p_Q^2)p_A^4+\mathbb{C}^{\mathbb{1}}_{ABC}(p_Q^2)P_{BC}+\text{cyclic},\end{aligned} \tag{70}$$

where cyclic permutation over A, B and C is implied, $p_A+p_B+p_C=0$, $Q=A,\ B,\ C$ and $P_{BC}=p_A^4-p_A^2(p_B^2+p_C^2)$ are kinematic structures which vanish when *any* of the three external momenta $p_{A,B,C}$ is set to zero. Hence the $\mathbb{C}^{\mathbb{1}}_{(A)BC}$ coefficients are the two-point function structures. By applying

$$\partial_B\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AC}(p_Q^2)+\partial_C\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p_Q^2)-\partial_A\mathbb{C}^{\mathbb{1},\mathcal{R}}_{BC}(p_Q^2)=[\text{finite}], \tag{71}$$

using the global version of (40) and noting ∂_A corresponds to a zero momentum insertion of $[O_A]$, the following equation for the Laurent series emerges:

$$\begin{aligned}L^{\mathbb{1},\mathcal{R}}_{(A)BC}&=L^{Q,\mathcal{R}}_{BC}L^{\mathbb{1},\mathcal{R}}_{QA}-\frac{1}{2}(\partial_BL^{\mathbb{1},\mathcal{R}}_{AC}+\partial_CL^{\mathbb{1},\mathcal{R}}_{AB}-\partial_AL^{\mathbb{1},\mathcal{R}}_{BC}) \\ &\quad+[\text{finite}].\end{aligned} \tag{72}$$

One infers that $L^{\mathbb{1},\mathcal{R}}_{(A)BC}$ is determined by two-point functions only and finiteness follows from the finiteness of the two-point functions. This implies that the truly three-point CT information is encoded in the $L^{\mathbb{1},\mathcal{R}}_{ABC}$ terms. The results of Sec. IV A apply straightforwardly. The RGE assumes the form

$$(\mathcal{L}_\beta-2\epsilon)L^{\mathbb{1},\mathcal{R}}_{ABC}=-\chi^{\mathcal{R}}_{ABC}, \tag{73}$$

where

$$\begin{aligned}\chi^{\rm MS}_{ABC}&=2\left(1+\frac{1}{2}(\xi^A+\xi^B+\xi^C)+\frac{1}{2}\xi^Qg^Q\partial_Q\right)r^{\mathbb{1}(1)}_{ABC}, \\ L^{\mathbb{1},\rm MS}_{ABC}&=\sum_{n\geq1}\frac{r^{\mathbb{1}(n)}_{ABC}}{\epsilon^n},\end{aligned} \tag{74}$$

and $\mathcal{L}_\beta$ denotes the Lie derivative, acting on a 3-tensor, as in the previous section:

$$\begin{aligned}\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{ABC}&=\partial_A\hat\beta^DL^{\mathbb{1},\mathcal{R}}_{DBC}+\partial_B\hat\beta^DL^{\mathbb{1},\mathcal{R}}_{ADC}+\partial_C\hat\beta^DL^{\mathbb{1},\mathcal{R}}_{ABD} \\ &\quad+\hat\beta^D\partial_DL^{\mathbb{1},\mathcal{R}}_{ABC},\end{aligned} \tag{75}$$

where $\partial_A\hat\beta^D=\hat\gamma_A{}^D$ is the anomalous dimension matrix (61). The finiteness of $L^{\mathbb{1},\mathcal{R}}_{ABC}$ and $L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}$ follows from the same arguments as in Sec. IV A and we caution the reader that the couplings $g^{A,B,C}$ refer to couplings governing the dynamics as otherwise the refined conditions apply.

All operators being the same is an interesting special case leading to the expected reduction in the kinematics:

$$\Gamma_{ABC}(p_A^2,p_B^2,p_C^2)=\mathbb{C}^{\mathbb{1}}_{(A)BC}(p_Q^2)P_3+\mathbb{C}^{\mathbb{1}}_{ABC}(p_Q^2)\lambda_3, \tag{76}$$

with complete symmetry in A, B and C, where

$$\begin{aligned}\lambda_3&=p_A^4+p_B^4+p_C^4-2(p_A^2p_B^2+p_A^2p_C^2+p_B^2p_C^2), \\ P_3&=p_A^4+p_B^4+p_C^4\end{aligned} \tag{77}$$

are the important kinematic Källén function and the two-point function structure, respectively. Finiteness of the three-point function of TEMT follows from similar arguments as in Sec. IV A.

V. SUMMARY AND CONCLUSIONS

In this work we have investigated the logarithmically divergent CTs of two- and three-point functions. Using the d-dimensional renormalization group, convergence criteria have been given for asymptotically free and safe UV FPs in Eqs. (14) and (17), respectively. This is followed by an explicit discussion of the $\langle G^2G^2\rangle$ and $\langle\Theta\Theta\rangle$ correlators and the $\langle\bar qq\bar qq\rangle$ correlators in QCD-like theories in Sec. III and Appendix B 1. By taking into account all orders the former two were shown to be finite but divergent at fixed order perturbation theory implying that the $\epsilon\to0$ and the perturbation expansion do not commute in general. Hence fixed order results can give the wrong indication about convergence. Finiteness implies that the bare correlators satisfy unsubtracted dispersion relations and are in principle observable since they are finite and scale independent; cf. Secs. III B 1 and III B 2. An application of the latter is given by the flow of the $\Box R$ anomaly which is related to the zero momentum limit of the p^4 structure of the TEMT correlation function [25]. Using a recent computation and the quantum action principle the R^2 anomaly was extended in the MS scheme to NNLO ($\mathcal{O}(a_s^5)$) in Sec. III D. Generalizations of the finiteness conditions to several couplings, assuming a conformal UV FP, and three-point functions were presented in Secs. IV A and IV B, respectively.

In what follows we discuss specific applications of the finiteness and the possibility of adding finite CTs to the UV action. A crucial point is that correlation functions which are finite with operators without anomalous scaling are RG-scale independent and therefore can serve as observables. In this view the finiteness of the $\langle\Theta\ldots\Theta\rangle$ correlators is our most important result. From it follows that the R^2 anomaly (45) is always proportional to β functions of the couplings (55). Furthermore the finiteness serves as the basis for showing that the difference of the UV and IR $\Box R$ anomaly (45) is flow independent[15] as well as the existence of a scheme for which the R^2 anomaly vanishes along the entire RG flow [25].

Hence generally CTs are meaningful only when they are related to observables. Otherwise they are arbitrary (i.e. scheme dependent) since one cannot forbid adding local terms to the UV action in general. It is though our opinion that the local terms are not arbitrary to the point that they ought to be RG-scale independent since the bare partition function is scale independent. Concerning observables one may distinguish the following two cases. Either the CTs drop out in the observable(s) or not. In the latter case they might either be fixed by some other principle or they need to be determined experimentally. The first two examples mentioned in the previous paragraph are of the first type mentioned. The bare contact term vanishes when taking the scale derivative (55) or in the difference of the UV and IR $\Box R$ value. The third example of the R^2 anomaly concerns a scheme-dependent question and corresponds to a reorganization of terms in the trace of the energy momentum tensor. Let us mention two well-known examples where (divergent) CTs can be handled by symmetry. For the vacuum polarization, the correlation function of two electromagnetic currents, the dispersion integral needs to be subtracted once but the value of the subtraction is fixed by gauge symmetry (zero photon mass). The chiral anomaly, which can be regarded as coming from a divergent contact term, cannot be removed by a local term while maintaining gauge symmetry. Another example where this ambiguity is settled by a symmetry, namely chiral symmetry, is the low energy constant L_{10} from chiral perturbation theory. The quantity L_{10} is related to a dispersion relation of the correlation function of left- and right-handed octet currents [$L_{10} \sim \Pi_{\rm LR}(q^2=0)$ with pion pole subtracted]. Chiral symmetry in the UV forbids us to add a contact term to $\Pi_{\rm LR}$ since the latter is sensitive to chiral symmetry breaking.

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Note added.—As compared to our v1 on the arXiv we believe to have improved the notation by adapting $2\chi^{\mathcal{R}}_{QQ} \to \chi^{\mathcal{R}}_{QQ}$ [defined in (8)] $\delta\mathbb{C}^{\mathbb{1}}_{xx} \to L^{\mathbb{1}}_{xx}$ [e.g. (3)] where the latter stands for local and we changed $\mathbb{C}^{\mathbb{1}}_{ss} \to \mathbb{C}^{\mathbb{1}}_{\theta\theta}$ when referring to the operator part [cf. (24) for a more generic decomposition] $\Theta = \beta^A[O_A]$ of the trace of the energy momentum tensor. In addition we have indicated the scheme with labels in more consequence since scheme independence is a key feature for orientation and consistency.

APPENDIX A: SOME ADDITIONAL FORMULAS FOR RELEVANT TO LEADING LOGARITHM

1. Form of leading logarithms of $\mathbb{C}^{\mathbb{1}}_{gg}$

The leading terms in the bare correlation function take the form

$$\int d^dx e^{ip\cdot x}\langle 0|O_g(x)O_g(0)|0\rangle_{\rm LL} = k\sum_{n\geq 0}\frac{(\beta_0 a_{s0})^{n-1}}{\epsilon^n}\left(\frac{\mu^2}{p^2}\right)^{n\epsilon}, \tag{A1}$$

with k being a constant which is immaterial for the argument. Upon renormalizing the operator $[O_g] = Z_{G^2}O_g$ and the coupling $a_{s0} = a_s Z_{a_s}$ with $Z_{G^2} = Z_{a_s}$ in the LL approximation one finds

$$\int d^dx e^{ip\cdot x}\langle 0|[O_g(x)][O_g(0)]|0\rangle_{\rm LL} = k\sum_{n\geq 0}\frac{f_n(\beta_0 a_s)^{n-1}}{\epsilon^n}, \tag{A2}$$

where

$$f_n = \sum_{j=0}^{n-1}(-1)^j\left(\frac{\mu^2}{p^2}\right)^{(n-j)\epsilon}\binom{n}{n-j}. \tag{A3}$$

This sum evaluates to

$$\begin{aligned} f_n &= \sum_{j=0}^{n-1}(-1)^j\binom{n}{n-j} \\ &\quad + \frac{\epsilon^n}{n!}\ln^n\left(\frac{\mu^2}{p^2}\right)\sum_{j=0}^{n}(-1)^j(n-j)^n\binom{n}{n-j} \\ &= (-1)^{n+1} + \epsilon^n\ln^n\left(\frac{\mu^2}{p^2}\right), \end{aligned} \tag{A4}$$

confirming the rule $\epsilon^{-n} \leftrightarrow -\ln^n(p^2/\mu^2)$ used in Sec. III B 1.

Note that nonlocal divergent terms in (A4) are avoided since the sum, somewhat magically,

[15]Finiteness is also a crucial ingredient to the a theorem as it allows one to establish positivity via an unsubtracted dispersion relation [24,45].

$$\sum_{j=0}^{n-1}(-1)^j (n-j)^l \binom{n}{n-j} = 0, \qquad 0 < l < n, \tag{A5}$$

only contributes for $l=0$ and $l=n$. We note that such nonlocal terms could not be eliminated by local counterterm in perturbation theory.

2. The leading poles of the counterterm $L_{QQ}^{\mathbb{1},\mathrm{MS}}$ for $\xi_Q \neq 0$

The leading poles of the counterterm $L_{QQ}^{\mathbb{1},\mathrm{MS}}$ (11) for an AF theory in the MS scheme for $\xi_Q \neq 0$ is given by

$$L_{QQ}^{\mathbb{1},\mathrm{MS}}(\mu) \simeq 2(1+\xi_Q) r_{QQ}^{\mathbb{1}(1,0)} \int_0^\infty e^{-2(1+\xi_Q)\epsilon t} \epsilon^{-\frac{\gamma_{Q,0}}{\beta_0}} (\epsilon + \beta_0 a_s (1-e^{-2\epsilon t})^{\frac{\gamma_{Q,0}}{\beta_0}}) dt$$
$$= r_{QQ}^{\mathbb{1}(1,0)} \left(\frac{(1+\frac{a_s\beta_0}{\epsilon})^{1+\frac{\gamma_{Q,0}}{\beta_0}} (-1-\frac{\epsilon}{\beta_0 a_s})^{\xi_Q} \Gamma(1+\xi_Q)\Gamma(-\frac{\gamma_{Q,0}}{\beta_0}-\xi_Q)}{a_s(\beta_0(1+\xi_Q)+\gamma_{Q,0})\Gamma(-\frac{\gamma_{Q,0}}{\beta_0})} + \frac{2\beta_0(-\epsilon)^{-\frac{\gamma_{Q,0}}{\beta_0}} {}_2F_1[-\frac{\gamma_{Q,0}}{\beta_0}, -1-\frac{\gamma_{Q,0}}{\beta_0}+\xi_Q, -\frac{\gamma_{Q,0}}{\beta_0}+\xi, \frac{a\beta_0+\epsilon}{a\beta_0}]}{\epsilon(\beta_0(1+\xi_Q)+\gamma_{Q,0})} \right), \tag{A6}$$

where ξ_Q is the difference of the d-dimensional and four-dimensional anomalous dimension $\hat{\gamma}_Q = \gamma_Q - \xi_Q \epsilon$. For $\xi_Q \to 0$ the formula simplifies considerably and is given in Sec. II A 1 in (15).

3. Explicit evaluation of the dispersion integral

As a check the integral (31) is integrated explicitly. This is best done by changing variables to $s = \mu^2 e^y$ which results in an integral (recall $P^2 = -p^2$)

$$\hat{x}(P^2) = \int_{-\infty}^\infty dy \frac{dy e^y}{e^y - P^2/\mu^2} \frac{1}{(1+a_s\beta_0 y)^2 + (a\beta_0\pi)^2}, \tag{A7}$$

with a pole at $y_\pm = -1/(a_s\beta_0) \pm i\pi$ and a series of poles $y_{n_\pm} = \ln(-P^2/\mu^2) \pm i\pi(2n+1)$ for $n \geq 0$. The integration contour can, for example, be closed in the upper half plane. The y_+ pole result in the pole term in (30) and the series of poles $y_{n_+} = \ln(-P^2/\mu^2) + i\pi(2n+1)$ for $n \geq 0$ leads to a series

$$\hat{x}(P^2) = \frac{1}{1-P^2/P_0^2} - (2\pi i) \sum_{n\geq 0} \frac{1}{1 + a_s\beta_0(\ln(-P^2/\mu^2) + i\pi(2n+1))^2 + (a\beta_0\pi)^2}$$
$$= \frac{1}{1-P^2/P_0^2} + x(P^2), \tag{A8}$$

which can be resummed into an analytic form. The final result is consistent with Eq. (30) which was the aim of this Appendix.

APPENDIX B: QUARK CURRENT CORRELATORS

1. The $\langle \bar{q}q\bar{q}q \rangle$ correlator in QCD-like gauge theories

Finally we consider the bifermion scalar operator

$$[O_M] = [\bar{q}q], \qquad \kappa_M = m, \tag{B1}$$

for which $m[\bar{q}q] = m_0 \bar{q}q$ is a RG invariant. The parameter m does not enter the dynamics and is regarded as a source term only. The relevant input to criteria (14) is given by $\gamma_{M,0}, \chi_{MM}$ and β_0. The leading order of the mass anomalous dimension is given by ($\gamma_{\bar{q}q} = \gamma_M = -\gamma_m$ and $\hat{\gamma}_m = \gamma_m$ since $\bar{q}q$ is a kinetic operator)

$$\gamma_M = \gamma_{M,0} a_s + \mathcal{O}(a_s^2), \qquad \gamma_{M,0} = -6C_F, \tag{B2}$$

where $r_{MM}^{\mathbb{1}(1)}(a_s) = r_{MM}^{\mathbb{1}(1,0)} + \mathcal{O}(a_s)$ and β_0 and C_F are given in (C2) and (C3). With $\chi_{MM}^{\mathcal{R}} = \mathcal{O}(a_s^0)$ (i.e. $n_{MM} = 0$) condition (14) reads

$$-\frac{\gamma_{M,0}}{\beta_0}\bigg|_{SU(N_c)} = \frac{3(N_c^2-1)/(N_c)}{11/3N_c - 2/3N_f} > 1 \Leftrightarrow L_{MM}^{\mathbb{1}} = [\text{finite}]. \tag{B3}$$

This criteria is satisfied for $N_f > (9+2N_c^2)/(2N_c)$ which for $N_c = 3$ leads to convergence for $N_f > 4.5$.

The leading pole contribution (15) is given by

$$L_{MM}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} = r_{MM}^{\mathbb{1}(1,0)} \frac{(1+\frac{a_s\beta_0}{\epsilon})^{1+\frac{\gamma_{M,0}}{\beta_0}} - 1}{a_s(\beta_0+\gamma_{M,0})} \overset{(\mathrm{B.3})}{\to} \bar{L}_{MM}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}}$$
$$= -\frac{r_{MM}^{\mathbb{1}(1,0)}}{a_s(\beta_0+\gamma_{M,0})}, \tag{B4}$$

where we have assumed (B3) to obeyed. For QCD with three massless flavors $N_f = 3$ and $N_c = 3$ the expression is divergent. Presumably this means that the constant part of the Γ_{MM} correlator is not directly related to a physical quantity. Expanding in a_s one obtains (15)

$$L^{\mathbb{1},\mathrm{MS}}_{MM}|_{\mathrm{LL}} = r^{\mathbb{1}(1,0)}_{MM}\left(\frac{1}{\epsilon} + \frac{\gamma_{M,0}a_s}{2\epsilon^2} + \frac{(-\beta_0\gamma_{M,0} + (\gamma_{M,0})^2)a_s^2}{6\epsilon^3} + \mathcal{O}(a_s^3)\right), \quad \text{(B5)}$$

from where the leading poles in [46,47] are recovered. For the sake of illustration let us quote the LL result, obtained by replacing $\frac{1}{\epsilon} \to -\ln(\frac{p^2}{\mu^2})$:

$$\Gamma^{\mathrm{MS}}_{MM}|_{LL}(p^2) = \int d^4x e^{ip\cdot x}\langle 0|[\bar{q}q(x)][\bar{q}q(0)]|0\rangle_{\mathrm{LL}} = -p^2 r^{\mathbb{1}(1,0)}_{MM}\frac{(1 + a_s\beta_0\ln(\frac{p^2}{\mu^2}))^{1+\frac{\gamma_{M,0}}{\beta_0}} - 1}{a_s(\beta_0 + \gamma_{M,0})} + \cdots, \quad \text{(B6)}$$

where the dots stand for condensate contributions. Expanding in $a_s\ln(\frac{p^2}{\mu^2})$ the $\mathcal{O}(a_s^3)$-LL expression matches the result in [46].

Following Sec. III B 2 we explicitly demonstrate at LL that the bare correlator, multiplied by $\kappa_M^2 = m^2(\mu)$, is μ independent in the following sense:

$$m^2(\mu)\Gamma_{MM}(p^2,\mu) = \mu_0^4 f(a_s(\mu^2/\mu_0^2), m/\mu_0, p^2/\mu_0^2) = p^2 m^2(p^2)F(a_s(p^2/\mu_0^2)), \quad \text{(B7)}$$

and μ_0 being an arbitrary reference scale. First we note that the renormalized correlator

$$m^2(\mu)\Gamma^{\mathrm{MS}}_{MM}|_{\mathrm{LL}}(p^2) = p^2\left(-r^{\mathbb{1}(1,0)}_{MM}\frac{m^2(p^2)}{a_s(p^2)(\beta_0 + \gamma_{M,0})} + r^{\mathbb{1}(1,0)}_{MM}\frac{m^2(\mu)}{a_s(\mu)(\beta_0 + \gamma_{M,0})}\right) \quad \text{(B8)}$$

splits into a μ-independent nonlocal and a μ-dependent local term. If we now restrict to the convergent case satisfying (B3), then the second term is equal to (B4) and in the $\epsilon \to 0$ limit

$$m^2(\mu)\Gamma_{MM}(p^2) = m^2(\mu)\Gamma^{\mathrm{MS}}_{MM}(p^2) + \bar{L}^{\mathbb{1},\mathrm{MS}}_{MM} \overset{\mathrm{LL}}{=} -p^2 r^{\mathbb{1}(1,0)}_{MM}\frac{m^2(p^2)}{a_s(p^2)(\beta_0 + \gamma_{M,0})}, \quad \text{(B9)}$$

which satisfies (B7) in analogy with (33).

2. The $\langle J^5_\mu J^5_\nu\rangle$ correlator

The axial current two-point function in an AF theory has been studied by Shore [28] and is worthwhile to be captured language of this paper. The correlator decomposes into

$$\int d^4x^{ix\cdot p}\langle J^5_\mu(x)J^5_\nu(0)\rangle = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\mathbb{C}^{\mathbb{1},T}_{J_5J_5}(p^2) + p_\mu p_\nu\mathbb{C}^{\mathbb{1},L}_{J_5J_5}(p^2), \quad \text{(B10)}$$

a transversal (T) and a longitudinal (L) part. Since $\gamma_{J_5,0} = 0$, the criteria (14) implies convergence for $n^{T,L}_{J_5J_5} > 1$, where $\chi^{T,L}_{J_5J_5} \sim a_s^{n^{T,L}_{J_5J_5}}$ is defined in analogy to (8). In the case of massless fermion considered here the axial current correlation function is identical to the vector current correlation function (vacuum polarization). Hence the important ingredient to the analysis is the conservation of the vector current which implies that the transverse part contributes at LO $n^T_{J_5J_5} = 0$ and further implies that $n^L_{J_5J_5} > 0$. Thus the well-known LO divergent contact term of the vacuum polarization is not resummed to a finite expression. Yet in the longitudinal part the chiral anomaly itself contributes at NNLO, with $n^L_{J_5J_5} = 2$, which then implies convergence and a scaling of the type $\mathbb{C}^{\mathbb{1},L}_{J_5J_5}(p^2) \sim a_s$ in analogy to the TEMT correlator (33). This result is consistent with Eq. (6.36) of Shore's work [28].

APPENDIX C: CONVENTIONS FOR β FUNCTION

In this work the bare β function $\hat{\beta}$ of DR is defined as

$$\hat{\beta} = \frac{d\ln g}{d\ln\mu} = \frac{(d-4)}{2} + \beta = -\epsilon + \beta. \quad \text{(C1)}$$

We draw the reader's attention to the fact that the logarithmic β function (C1) is used throughout in order to keep the formulas more compact. Explicitly

$$\beta = -\beta_0 a_s - \beta_1 a_s^2 - \beta_2 a_s^3 - \beta_3 a_s^4 + \cdots, \qquad a_s = \frac{\alpha_s}{4\pi} = \frac{g^2}{(4\pi)^2}, \quad \text{(C2)}$$

where β_{0-3} in $\overline{\mathrm{MS}}$ scheme can be found in Ref. [48]. The first two coefficients, which are universal in mass-independent schemes, read

$$\beta_0 = \left(\frac{11}{3}C_A - \frac{4}{3}N_F T_F\right), \qquad \beta_1 = \left(\frac{34}{3}C_A^2 - \frac{20}{3}N_c N_F T_F - 4C_F T_F N_F\right),$$

where C_F and C_A are the quadratic Casimir operators of the fundamental (quark) and adjoint (gluons) representations, respectively, N_F is the number of quarks and $\mathrm{tr}[T^aT^b] = T_F\delta^{ab}$ is a Lie algebra normalization factor of the fundamental representation. For $SU(N_c)$ these factors are given by

$$C_A = N_c, \qquad C_F = \frac{N_c^2 - 1}{2N_c}, \qquad T_F = \frac{1}{2}. \quad \text{(C3)}$$

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