Adiabatic regularization for gauge fields and the conformal anomaly

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Adiabatic regularization for quantum field theory in conformally flat spacetime is known for scalar and Dirac fermion fields. In this paper, we complete the construction by establishing the adiabatic regularization scheme for the gauge field. We show that the adiabatic expansion for the mode functions and the adiabatic vacuum can be defined in a similar way using Wentzel-Kramers-Brillouin-type (WKB-type) solutions as the scalar fields. As an application of the adiabatic method, we compute the trace of the energy momentum tensor and reproduce the known result for the conformal anomaly obtained by the other regularization methods. The availability of the adiabatic expansion scheme for the gauge field allows one to study various renormalized physical quantities of theories coupled to (non-Abelian) gauge fields in conformally flat spacetime, such as conformal supersymmetric Yang Mills, inflation, and cosmology.

I. INTRODUCTION

The study of the dynamics of quantum field theory (QFT) in curved spacetime is not only relevant for the understanding of a number of important physical problems such as inflation or Hawking radiation, but it is also rather challenging. See, for example, the books [1–3] for a general exposition. One of the challenges is the determination of the vacuum. In fact, as time is not a diffeomorphic invariant concept, neither is the vacuum. The observer dependent nature of the vacuum is therefore intrinsic to QFT in curved spacetime. Even after one fixes a choice of time, the vacuum in perturbation theory is generally still not unique. For metric with isometries, it is often preferred to choose the vacuum to respect the symmetries. But still one may not be able to get a unique one. For example, for the de Sitter metric, the alpha-vacua give a one parameter family of vacua which are invariant under the de Sitter isometries, and one is able to single out the Bunch-Davies vacuum only if the Hadamard property is also imposed.

After one decided on the vacuum, one could then proceed to study various quantum properties of the system using traditional tools of quantum field theory in flat spacetime. However, extra care must be exercised to take into account the effects of particle creation which is a simple consequence of the fact that, in general for a time dependent background, a vacuum at time t may not be a vacuum anymore at a different time t'. As a result, instead of the S matrix, it is more sensible to consider correlation functions of operators [4–7] for QFT in curved spacetime.

Historically, the conformal (Weyl) anomaly of the energy momentum tensor was one of the first quantities studied and computed for a QFT in curved spacetime. And various methods have been developed to regularize the UV divergence found in the energy momentum tensor. These include, for example, the Dewitt-Schwinger geodesic point-splitting method [8], the zeta-function regularization [9,10], and the adiabatic regularization method [11].

Adiabatic regularization [11,12] is a useful and simple method to obtain physically meaningful renormalized results from the formally UV divergent quantities, e.g., the vacuum expectation value of the energy momentum tensor, in an expanding universe such as a conformally flat spacetime. The most studied example is the adiabatic regularization for scalar fields [11–14] (see also [15] for a recent review and references therein). Recently the adiabatic regularization for fermion has been established [16,17]. On the other hand, as far as we know, adiabatic regularization for gauge fields has never been considered in the literature. One of the motivations of this paper is to fill this gap.

At the first glance, one may think that the adiabatic regularization for the gauge field is rather straightforward since the theory of a massless gauge field in a four-dimensional conformally flat spacetime is conformally invariant. One may then infer that the mode function of a gauge field A_μ can be collectively written in the same form, up to some overall scaling factor, as that of a massless conformally coupled scalar field, and thus the adiabatic expansion for the gauge field can be performed exactly in the same way as that for a massless conformally coupled scalar field. This is actually wrong, and one would get the wrong result for the conformal anomaly. The reason for the mistake is that one has missed a very important nontrivial issue related to the gauge fixing of the theory, the latter of which is essential to the setting up of the perturbation theory.

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The conformal anomaly for the gauge field has been obtained using other regularization schemes before [18–25] and is given by[1]

$$\langle T^\mu_\mu \rangle = \frac{1}{2880\pi^2}\left[62\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + d\Box R\right], \quad (1)$$

where it is known that there is a discrepancy in the coefficient d of $\Box R$ among different schemes: dimensional regularization gives 12 [21,22], while the DeWitt-Schwinger point-splitting expansion gives -18 [9]. In fact, it is well understood that this term is regularization dependent since it can be expressed as the variation of a local action [21,22],

$$\sqrt{-g}\Box R = \frac{1}{6}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^2, \quad (2)$$

and so the value of d can be shifted to any arbitrary value by using an appropriate counter term. Regularization dependence of the $\Box R$ term has also been discussed recently in [27]. It has also been pointed out [24,25] that the coefficient of $\Box R$, at least in the DeWitt-Schwinger regularization scheme, is gauge dependent. We are interested not only in computing the conformal anomaly of the gauge field using the adiabatic method but also in comparing our results, especially the gauge dependence of the $\Box R$ term, with those obtained in the other regularization methods.

Recently the $\mathcal{N} = 4$ superconformal Yang-Mills theory on de Sitter space [28] has been introduced, and it has been proposed [29] to be the holographic dual of the type IIB string theory on $AdS_5 \times S^5$ background with certain boundary conditions. Furthermore, the holographic duality suggests that the de Sitter space superconformal Yang-Mills theory has a number of rather interesting quantum properties similar to that of the maximal superconformal Yang-Mills theory on flat spacetime. To check, a consistent framework of evaluating the quantum loop contributions in the conformally flat spacetime is necessary. Compared to the other regularization schemes, the adiabatic regularization scheme is practical and particularly useful for perturbative quantum field theory computation in a conformally flat metric as it has taken full advantage of the homogeneity of the metric. As a result, the mode expansion of the field can be greatly simplified, and one simply obtains an oscillator with a time dependent frequency, whose solution can be obtained via an adiabatic expansion in terms of slowness of the temporal change of the metric. However, while the adiabatic regularization schemes for the scalar field and fermion field are available, the adiabatic scheme for the gauge field has not been constructed before. The

main motivation of this work is indeed to develop such a scheme for the gauge field so that one has available a practical and complete framework that one can use to handle the UV divergences and study the renormalization of the theory.

The next section is devoted to a brief review of the adiabatic regularizations for a scalar field and for a Dirac fermion. In Sec. III, we consider the adiabatic expansion for $U(1)$ gauge theory. In Sec. IV, we compute the conformal anomaly for the $U(1)$ gauge theory in the adiabatic regularization. We summarize our result in Sec. V.

Our convention of the Minkowski metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and the Riemann and Ricci tensors are given by

$$\begin{aligned}R^\rho_{\ \sigma\mu\nu} &= \partial_\mu\Gamma^\rho_{\ \nu\sigma} - \partial_\nu\Gamma^\rho_{\ \mu\sigma} + \Gamma^\rho_{\ \alpha\mu}\Gamma^\alpha_{\ \sigma\nu} - \Gamma^\rho_{\ \alpha\nu}\Gamma^\alpha_{\ \sigma\mu},\\ R_{\sigma\nu} &= R^\rho_{\ \sigma\rho\nu}.\end{aligned} \quad (3)$$

Note that sign convention on the signature of the metric affects the signs of the d'Alembertian operator, and the convention of the Riemann and Ricci tensors affects the sign of the scalar curvature R. In the conformal anomaly the overall sign of the $\Box R$ term is thus convention dependent. For example, $\Box R$ in our convention has the same sign as those in [22,23] and the opposite sign as those in [9,24].

II. ADIABATIC REGULARIZATIONS FOR SCALAR FIELD AND DIRAC FERMION IN CONFORMALLY FLAT SPACETIME

To see the basic strategy of the adiabatic method, in this section we give a brief review of the adiabatic expansions and regularizations for a scalar field and for a Dirac fermion in a conformally flat spacetime.

A. Conformally coupled scalar field

We consider a conformally flat spacetime with metric

$$g_{\mu\nu} = C(\tau)\eta_{\mu\nu}, \qquad x^\mu = (\tau, x^i), \qquad i = 1, 2, 3, \quad (4)$$

where $C(\tau) \equiv a(\tau)^2$ and $a(\tau)$ is the cosmological scale factor. To perform the adiabatic expansion, we need to introduce a mass m to the scalar field and take a zero mass limit in the end of calculation. This mass will play a role in capturing the effect of the background gravitational field. The action is given by

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(m^2 + \frac{R}{6}\right)\phi^2\right), \quad (5)$$

with the field equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \left(m^2 + \frac{R}{6}\right)\phi = 0, \quad (6)$$

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[1]The general form of the conformal anomaly in arbitrary dimensions was obtained by momentum space calculation in [26].

where ∇_μ is the covariant derivative associated with the background metric. This gives the energy momentum tensor

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{2}{3}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi$$
$$- \frac{1}{2}g_{\mu\nu}m^2\phi^2 - \frac{1}{3}\phi\nabla_\mu\nabla_\nu\phi$$
$$+ \frac{1}{3}g_{\mu\nu}\phi g^{\rho\sigma}\nabla_\rho\nabla_\sigma\phi + \frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\phi^2, \qquad (7)$$

and the trace

$$T^\mu{}_\mu = -m^2\phi^2, \qquad (8)$$

where the field equation (6) has been used to obtain (8). The field equation (6) can be solved with the Fourier expansion

$$\phi(x) = \frac{1}{\sqrt{C}}\int\frac{d^3k}{(2\pi)^3}(a_{\vec{k}}\varphi(\tau,k)e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \qquad (9)$$

where $k = |\vec{k}|$ and the mode function $\varphi(\tau,k)$ satisfies a second order differential equation which is precisely that of a harmonic oscillator with a time dependent frequency

$$(\partial_0^2 + \omega^2)\varphi(\tau,k) = 0, \qquad \omega^2 = k^2 + m^2C. \quad (10)$$

The operators $a_{\vec{k}}$ satisfy the commutation relation of creation and annihilation operators

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \qquad [a_{\vec{k}}, a_{\vec{k}'}] = 0 \quad (11)$$

iff the mode function $\varphi(\tau,k)$ satisfies the normalization condition

$$\varphi(\tau,k)\partial_0\varphi^*(\tau,k) - \partial_0\varphi(\tau,k)\varphi^*(\tau,k) = i. \quad (12)$$

The vacuum of the theory is defined to be a state annihilated by the operators $a_{\vec{k}}$. However, this depends on the choice of the mode function as different choices of the mode functions determine different sets of annihilation operators $a_{\vec{k}}$, and hence the vacuum states. In general, solving (10) analytically is impossible. A useful observation is that the normalization condition (12) can be conveniently solved by [11]

$$\varphi(\tau,k) = \frac{1}{\sqrt{2W(\tau)}}(\alpha e^{-i\int^\tau W(\tau')d\tau'} + \beta e^{i\int^\tau W(\tau')d\tau'}), \quad (13)$$

where α, β are constant coefficients satisfying

$$|\alpha|^2 - |\beta|^2 = 1 \qquad (14)$$

and $W(\tau)$ is an arbitrary function. The differential equation (10) for φ becomes the differential equation for W,

$$W^2 = \omega^2 - \left(\frac{W''}{2W} - \frac{3(W')^2}{4W^2}\right), \qquad (15)$$

where the prime denotes the differential with respect to the conformal time, $' \equiv \partial_0 = \partial/\partial\tau$. This equation is complicated and, again, impossible to solve analytically in general. However, if one considers the background to be slowly changing and parametrizes the time variation by a small parameter $\epsilon \ll 1$: $\partial_0 \to \epsilon\partial_0$, then Eq. (15) can be solved iteratively to give rises to an expansion in powers of time derivatives

$$W = W_{(0)} + \epsilon^2 W_{(2)} + \epsilon^4 W_{(4)} + \cdots. \qquad (16)$$

Here $W_{(n)}$ contains n orders of time derivatives. The expansion (16) is a Wentzel-Kramers-Brillouin-type (WKB-type) expansion and defines the adiabatic expansion of the mode function of the scalar field, with n being called the order of the adiabatic expansion. The first few terms of the expansion are

$$W_{(0)} = \omega, \qquad (17)$$

$$W_{(2)} = \frac{3}{8}\frac{(\omega')^2}{\omega^3} - \frac{1}{4}\frac{\omega''}{\omega^2}, \qquad (18)$$

$$W_{(4)} = -\frac{297}{128}\frac{(\omega')^4}{\omega^7} + \frac{99}{32}\frac{(\omega')^2\omega''}{\omega^6} - \frac{13}{32}\frac{(\omega'')^2}{\omega^5}$$
$$- \frac{5}{8}\frac{\omega'\omega'''}{\omega^5} + \frac{1}{16}\frac{\omega''''}{\omega^4}. \qquad (19)$$

In [11] it was argued that, for a sufficiently slow and smooth expansion, it is the choice $\beta = 0$ for the mode function

$$\varphi(\tau,k) = \frac{1}{\sqrt{2W}}e^{-i\int^\tau W(\tau')d\tau'}, \qquad (20)$$

which give rises to operators $a_{\vec{k}}$ that correspond to physical particles. This choice of the vacuum

$$a_{\vec{k}}|0\rangle_A = 0 \qquad (21)$$

is called the adiabatic vacuum. Note that for a time independent metric, all higher order terms vanish and $W = \omega$. In this case the mode function $\varphi(\tau,k)$ is the ordinary positive frequency solution, and the adiabatic vacuum reduces to the standard Minkowski vacuum.

In the adiabatic regularization, the renormalized energy momentum tensor is given by

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \langle T_{\mu\nu}^{(m=0)}\rangle - \lim_{m\to 0}{}_A\langle 0|T_{\mu\nu}|0\rangle_A. \qquad (22)$$

As the adiabatic expansion becomes more accurate for large k, the second adiabatic subtraction term has the same UV divergent structure as that of the first term, and so $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is UV finite. It is known that the adiabatic regularization of the energy momentum tensor is equivalent to renormalizing the gravitational coupling constants in the Einstein equation [14]. As a matter of fact, the adiabatic regularization is a method for renormalization rather than that for regularization of divergent momentum integrals, since the adiabatic subtraction term precisely cancels mode by mode the contribution from large momenta to the first term in (22), and the result $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is thus completely finite. To remove all the divergences in the expectation value of the energy momentum tensor, the adiabatic expansion should be performed up to the fourth order, i.e., the same order as the mass dimension of $T_{\mu\nu}$, the physical quantity being considered.

For our theory, substituting $\omega = k^2 + m^2 C$ to (16), the adiabatic expansion for W up to the fourth adiabatic order is given by

$$W = \omega - \frac{m^2 C''}{8\omega^3} + \frac{5m^4 (C')^2}{32\omega^5} + \frac{m^4 C''''}{32\omega^5}$$
$$- \frac{m^4}{128\omega^7}(28 C''' C' + 19(C'')^2)$$
$$+ \frac{221 m^6 C''(C')^2}{256\omega^9} - \frac{1105 m^8 (C')^4}{2048\omega^{11}}, \qquad (23)$$

where we have absorbed back the formal expansion parameter ϵ into the time derivatives, effectively setting $\epsilon = 1$. The conformal anomaly in the classically conformally invariant theory is determined by the massless limit of the adiabatic subtraction term

$$\langle T^\mu{}_\mu\rangle_{\text{ren}} = -\lim_{m\to 0}{}_A\langle 0|T^\mu{}_\mu|0\rangle_A$$
$$= \lim_{m\to 0}\frac{m^2}{4\pi^2 C}\int_0^\infty dk\, \frac{k^2}{W_k}$$
$$= \lim_{m\to 0}\frac{m^2}{4\pi^2 C}\int_0^\infty dk\, k^2 \left[\frac{1}{\omega} + \frac{m^2 C''}{8\omega^5} - \frac{5m^4 (C')^2}{32\omega^7} - \frac{m^4 C''''}{32\omega^7}\right.$$
$$\left. - \frac{m^4}{128\omega^9}(28 C''' C' + 21(C'')^2) - \frac{231 m^6 C''(C')^2}{256\omega^{11}} + \frac{1155 m^8 (C')^4}{2048\omega^{13}}\right]$$
$$= \frac{1}{960\pi^2}\left(5\frac{(C')^4}{C^6} - 11\frac{(C')^2 C''}{C^5} + 3\frac{(C'')^2}{C^4} + 4\frac{C' C'''}{C^4} - \frac{C''''}{C^3}\right)$$
$$= \frac{1}{2880\pi^2}\left[\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \Box R\right], \qquad (24)$$

where we have used (A2) in the last equality. In the above computation only the fourth adiabatic order terms survive to contribute. The reason is that by introducing an UV momentum cutoff $k = a(\tau)\Lambda$ where Λ is the physical momentum cutoff, the first and second adiabatic order terms give contributions that are proportional to m^4 and m^2, respectively, and thus they vanish by taking $m \to 0$ before taking $\Lambda \to \infty$. Note that the conformal anomaly can be expressed in terms of the Ricci tensor and the scalar curvature only since the Weyl tensor $C_{\mu\nu\rho\sigma}$ is identically zero in a conformally flat spacetime [3]. The conformal anomaly obtained here (24) agrees with the result obtained by the other regularization methods [9,22,30,31]. We note that one has to be aware of the sign difference in front of the $\Box R$ term in comparing the result here with those in [9,31], which is simply due to the convention of metric and the curvature tensors. For example, Dowker *et al.* [9] adopted $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, ${}^{\text{D}}R_{\mu\nu\sigma}{}^\rho = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \cdots$ and ${}^{\text{D}}R_{\nu\sigma} = {}^{\text{D}}R_{\rho\sigma\nu}{}^\rho$. Therefore we have $R_{\mu\nu} = {}^{\text{D}}R_{\mu\nu}$, and the sign difference comes from the metric sign convention as $\Box R = -\Box^{\text{D}}R$. The same remark applies to the results (39) and (95).

B. Dirac fermion

Adiabatic expansion for a Dirac fermion has been performed recently in [16,17]. This is noticeably different from the scalar field case that is based on the WKB-type expansion. As we are ultimately interested in the de Sitter space superconformal Yang-Mills theory [29], we will follow their convention and consider the action for a Dirac fermion Ψ in the form

$$S = \int d^4 x\sqrt{-g}\left(-\bar{\Psi}\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_{\mu\rho\sigma}\hat{\gamma}^{\rho\sigma}\right)\Psi + m\bar{\Psi}\Psi\right), \qquad (25)$$

where $\bar{\Psi} = i\Psi^\dagger\hat{\gamma}^0$,

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \qquad (26)$$

Here the gamma matrices γ^μ are those in a general curved space, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, while the "hatted" gamma matrices

are those in the Minkowski space, $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\eta^{\mu\nu}$. Note that the spin connection for a conformally flat spacetime is given by $\omega_{\mu\rho\sigma} = -\frac{1}{2}D(\delta^0_{\ \rho}\eta_{\sigma\mu} - \delta^0_{\ \sigma}\eta_{\rho\mu})$, where $D \equiv C'/C = 2a'/a$. It is convenient to introduce the rescaled field $\psi \equiv a^{\frac{3}{2}}\Psi$, $\bar{\psi} = i\psi^\dagger\hat{\gamma}^0$, the field equation for the rescaled field becomes

$$(\hat{\gamma}^\mu\partial_\mu - ma)\psi = 0, \tag{27}$$

which is simply the free field equation for a Dirac fermion in Minkowski space with a time dependent mass.

The Fourier expansion for ψ is given by

$$\psi = \sum_{s=1,2}\int\frac{d^3k}{(2\pi)^3}(c^s_{\vec{k}}u^s_{\vec{k}}(\tau)e^{i\vec{k}\cdot\vec{x}} + d^{s\dagger}_{\vec{k}}v^s_{\vec{k}}(\tau)e^{-i\vec{k}\cdot\vec{x}}), \tag{28}$$

where s is the spin index and $v^s_{\vec{k}}$ is given by the charge conjugation of $u^s_{\vec{k}}$, $v^s_{\vec{k}} = u^{sC}_{\vec{k}} = \hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^3 u^{s*}_{\vec{k}}$. The canonical anticommutation relations are

$$\{\psi_\alpha(\tau,\vec{x}), \psi^\dagger_\beta(\tau,\vec{y})\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}), \tag{29}$$

where α, β are spinor indices. In addition to the field equation, the spinor mode functions are subject to the orthogonality condition,

$$\sum_\alpha u^{s\dagger}_{\vec{k}\alpha}(\tau)u^{s'}_{\vec{k}\alpha}(\tau) = \delta^{ss'}, \tag{30}$$

which guarantees the correct normalization of the scalar product of Ψ [32]. Following [16], we write the spinor mode function $u^s_{\vec{k}}$ as

$$u^s_{\vec{k}} = \begin{pmatrix} h^I_k(\tau,\lambda_s)\xi^s \\ h^{II}_k(\tau,\lambda_s)\xi^s \end{pmatrix}. \tag{31}$$

Here ξ^s is a two component spinor satisfying

$$\sum_\alpha \xi^{s\dagger}_\alpha\xi^{s'}_\alpha = \delta^{ss'}, \quad \sum_s \xi^s_\alpha\xi^{s\dagger}_\beta = \delta_{\alpha\beta}, \quad \frac{\vec{\sigma}\cdot\vec{k}}{k}\xi^s = \lambda_s\xi^s, \tag{32}$$

with $\lambda_s = \pm 1$ the helicity eigenvalues, and $h^{I,II}_k$ are scalar functions depending on λ_s. Substituting (31) into the field equation (27), we obtain

$$(\partial_0 + i\lambda_s k)h^{II}_k = mah^I_k, \quad (-\partial_0 + i\lambda_s k)h^I_k = mah^{II}_k, \tag{33}$$

and it follows from the above equations that the second order differential equations

$$\left(\partial_0^2 - \frac{D}{2}\partial_0 + k^2 + m^2 C + i\frac{D}{2}\lambda_s k\right)h^I_k = 0,$$

$$\left(\partial_0^2 - \frac{D}{2}\partial_0 + k^2 + m^2 C - i\frac{D}{2}\lambda_s k\right)h^{II}_k = 0. \tag{34}$$

Eliminating the first derivative terms by redefining $h^{I,II}_k = a^{1/2}\tilde{h}^{I,II}_k$ yields

$$\left(\partial_0^2 + \Omega^2_F + i\frac{D}{2}\lambda_s k\right)\tilde{h}^I_k = 0,$$

$$\left(\partial_0^2 + \Omega^2_F - i\frac{D}{2}\lambda_s k\right)\tilde{h}^{II}_k = 0, \tag{35}$$

where

$$\Omega^2_F = \omega^2 + \frac{D'}{4} - \frac{D^2}{16}, \quad \text{and} \quad \omega^2 = k^2 + m^2 C. \tag{36}$$

It is important to note that the orthogonality condition (30) implies the normalization condition for the scalar functions

$$|\tilde{h}^I_k(\tau,\lambda_s)|^2 + |\tilde{h}^{II}_k(\tau,\lambda_s)|^2 = 1/a. \tag{37}$$

It is obvious that a simple form of ansatz (20) like that for the scalar field could not solve the normalization condition (37). As demonstrated in [16], one needs to adopt an ansatz where the amplitudes and the phases of the mode functions are independent. Expanding adiabatically, the correct WKB-type solution for a Dirac fermion is of the form

$$h^I_{k(n)} = \sqrt{\frac{\omega - \lambda_s k}{2\omega}}(1 + F_{(1)} + \cdots$$
$$+ F_{(n)})e^{-i\int^\tau(\omega+\omega_{(1)}+\cdots+\omega_{(n)})d\tau'},$$

$$h^{II}_{k(n)} = i\sqrt{\frac{\omega + \lambda_s k}{2\omega}}(1 + G_{(1)} + \cdots$$
$$+ G_{(n)})e^{-i\int^\tau(\omega+\omega_{(1)}+\cdots+\omega_{(n)})d\tau'}. \tag{38}$$

With these ansatz, one can obtain $\omega_{(n)}$, $F_{(n)}$, and $G_{(n)}$ by solving (33) and (37) iteratively. Note that in the adiabatic expansion for a Dirac fermion, terms of all adiabatic order $(n = 0, 1, 2, ...)$ exist unlike the scalar field case where only terms of even adiabatic order are present. Here we will not repeat the same procedure for the adiabatic expansion and the adiabatic regularization of the energy momentum tensor for a Dirac fermion field as it has been carried out in details in [16,17]. The result for the conformal anomaly agrees with the result obtained by the other regularization methods [9,22]

$$\langle T^\mu{}_\mu \rangle_{\text{ren}} = \frac{1}{2880\pi^2}\left[11\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + 6\Box R\right]. \quad (39)$$

III. ADIABATIC EXPANSION FOR $U(1)$ GAUGE FIELD

As we mentioned above, the adiabatic expansion for the gauge field has not been performed before in the literature. As the theory has gauge symmetry, one needs to fix a gauge in order to perform perturbative calculations. The first thing to be clarified is what kind of gauge fixing term should be used. Since the classical action of $U(1)$ gauge theory on a conformally flat spacetime in four dimensions possesses conformal invariance, one may think that it is useful to adopt a gauge fixing term which preserves the classical conformal invariance. For example, by extending the Lorentz gauge fixing in Minkowski space to a conformally flat spacetime while keeping possession of the conformal invariance, we have

$$\mathcal{L}_{\text{gf}} = -\frac{\sqrt{-g}}{2}(\nabla^\mu A_\mu - C^{-1}(\tau)D(\tau)A_0)^2 = -\frac{\sqrt{-g}}{2}(\partial^\mu A_\mu)^2. \quad (40)$$

The above gauge fixing (40) is also known as the so-called W gauge [33]. Using (40), the gauge fixed action with the ghost kinetic term is conformally invariant and can be written precisely as the same form as that in flat Minkowski space. In this case, the gauge field and the ghost fields are simply described, respectively, by collections of four and two massless conformally coupled scalar modes. As a result of this simple description of the fields, the conformal anomaly in the adiabatic regularization amounts to $(4-2) \times \langle T^\mu{}_\mu \rangle_{\text{ren}}^{\text{scalar}}$. This is wrong. The reason why this gives the wrong result is because the gauge fixing term (40) breaks the general covariance, and this leads to the breaking of the covariant conservation of the energy momentum tensor. In this case it is thus impossible to identify the pure conformal anomalous contribution to the expectation value of the trace of energy momentum tensor. Therefore, in order to evaluate the conformal anomaly correctly, we have to use a gauge fixing term that respects the general covariance even though by itself it breaks the classical conformal invariance of the theory. Taking into account the above consideration, we will take the following covariant gauge fixing term with a parameter ξ:

$$\mathcal{L}_{\text{gf}} = -\frac{\sqrt{-g}}{2\xi}(\nabla^\mu A_\mu)^2. \quad (41)$$

To perform the adiabatic expansion for the mode functions in the $U(1)$ gauge theory, we introduce a mass m for the gauge field and a mass m_χ for the (anti)ghost fields $\chi, \bar{\chi}$, respectively, in such a way that the gauge-fixed massless $U(1)$ gauge theory is recovered in the limit $m, m_\chi \to 0$ [9]. The Lagrangian to be considered is thus

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2\xi}(\nabla^\mu A_\mu)^2\right.$$
$$\left. -\frac{1}{2}m^2 g^{\mu\nu}A_\mu A_\nu - i\bar{\chi}g^{\mu\nu}\nabla_\mu\nabla_\nu\chi + im_\chi^2\bar{\chi}\chi\right), \quad (42)$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field equations derived from (42) are

$$\eta^{\rho\sigma}\partial_\rho\partial_\sigma A_\mu + \left(\frac{1}{\xi} - 1\right)\eta^{\rho\sigma}\partial_\mu\partial_\rho A_\sigma - m^2 C A_\mu$$
$$+ \frac{1}{\xi}[\delta_\mu^0(-D\eta^{\rho\sigma}\partial_\rho A_\sigma + D^2 A_0 - D'A_0) - D\partial_\mu A_0] = 0, \quad (43)$$

and

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\chi - D\partial_0\chi - m_\chi^2 C\chi = 0, \quad \text{same for } \bar{\chi}. \quad (44)$$

Here $D = C'/C$ and $C = a^2$ as before.

Next we quantize the theory (42) in the canonical formalism. First of all, the canonical conjugate momenta are defined by

$$\pi_A^\mu = \frac{\partial\mathcal{L}}{\partial\partial_0 A_\mu} = \eta^{\mu\nu}(\partial_0 A_\nu - \partial_\nu A_0) - \frac{1}{\xi}\eta^{\mu 0}(\eta^{\alpha\beta}\partial_\alpha A_\beta - DA_0), \quad (45)$$

$$\pi_\chi = \frac{\partial\mathcal{L}}{\partial\partial_0\chi} = -iC\partial_0\bar{\chi}, \qquad \pi_{\bar{\chi}} = \frac{\partial\mathcal{L}}{\partial\partial_0\bar{\chi}} = iC\partial_0\chi. \quad (46)$$

In terms of (45), the temporal and spatial components of the field equation for the gauge field are written as

$$-\partial_i\pi_A^i + (\partial_0 - D)\pi_A^0 - m^2 C A_0 = 0, \quad (47)$$

$$-\delta_{ik}\partial_0\pi_A^k + \delta^{jk}\partial_j(\partial_k A_i - \partial_i A_k) + \partial_i\pi_A^0 - m^2 C A_i = 0, \quad (48)$$

respectively. To decouple the field equations (47) and (48), we follow the strategy of [34] and separate the canonical variables into the transverse and the longitudinal parts,

$$A_i = B_i + \partial_i A, \qquad \pi_A^i = \delta^{ij}(w_j + \partial_j\pi_A), \quad (49)$$

with $\partial^i B_i = \partial^i w_i = 0$. Substituting the decompositions (49) into the field equations (47) and (48) and using (45), we arrive at three decoupled equations for B_i, π_A^0, and π_A,

$$(\partial_0^2 - \partial_j^2 + m^2 C)B_i = 0, \quad (50)$$

$$(\partial_0^2 - \partial_j^2 - D\partial_0 + \xi m^2 C - D')\pi_A^0 = 0, \quad (51)$$

$$(\partial_0^2 - \partial_j^2 - D\partial_0 + m^2 C)\pi_A = 0, \tag{52}$$

where $\partial_j^2 := \delta^{jk}\partial_j\partial_k$. w^i turns out to be a dependent variable,

$$w_i = \partial_0 B_i, \tag{53}$$

and A_0 and A can be obtained by using (47) and (48) as

$$A_0 = \frac{1}{m^2 C}((\partial_0 - D)\pi_A^0 - \partial_j^2 \pi_A), \quad A = \frac{1}{m^2 C}(\pi_A^0 - \partial_0 \pi_A). \tag{54}$$

The canonical (anti)commutation relations are

$$[A_\mu(\tau, \vec{x}), \pi_A^\nu(\tau, \vec{x}')] = i\delta^\nu{}_\mu \delta^{(3)}(\vec{x} - \vec{x}'), \tag{55}$$

$$\{\chi(\tau, \vec{x}), \pi_\chi(\tau, \vec{x}')\} = i\delta^{(3)}(\vec{x} - \vec{x}'),$$
$$\{\bar{\chi}(\tau, \vec{x}), \pi_{\bar{\chi}}(\tau, \vec{x}')\} = i\delta^{(3)}(\vec{x} - \vec{x}'), \tag{56}$$

while the other (anti)commutators vanish. The Fourier expansions of the dynamical variables B_i, π_A^0, π_A, χ, and $\bar{\chi}$ are given by

$$B_i(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \sum_{p=1,2} (\epsilon_i^p(\vec{k}) a_{\vec{k}}^{(p)} f^{(p)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \tag{57}$$

$$\pi_A^0(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} (a_{\vec{k}}^{(0)} f^{(0)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \tag{58}$$

$$\pi_A(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} (a_{\vec{k}}^{(3)} f^{(3)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \tag{59}$$

$$\chi(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} (b_{\vec{k}} \chi(\tau, k) e^{i\vec{k}\cdot\vec{x}} + b_{\vec{k}}^\dagger \chi^*(\tau, k) e^{-i\vec{k}\cdot\vec{x}}), \tag{60}$$

$$\bar{\chi}(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} (\bar{b}_{\vec{k}} \bar{\chi}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \bar{b}_{\vec{k}}^\dagger \bar{\chi}^*(\tau, k) e^{-i\vec{k}\cdot\vec{x}}), \tag{61}$$

where $\epsilon_i^p(\vec{k})$ is the polarization tensor of the transverse modes that satisfies

$$\sum_i k^i \epsilon_i^p(\vec{k}) = 0, \qquad \sum_i \epsilon_i^p(\vec{k}) \epsilon_i^{p'}(\vec{k}) = \delta^{pp'},$$

$$\sum_{p=1,2} \epsilon_i^p(\vec{k}) \epsilon_j^p(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

According to (53) and (54), the corresponding Fourier expansions for w_i, A_0, and A are obtained as

$$w_i(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \sum_{p=1,2} (\epsilon_i^p(\vec{k}) a_{\vec{k}}^{(p)} \partial_0 f^{(p)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \tag{62}$$

$$A_0(\tau, \vec{x}) = \frac{1}{m^2 C} \int \frac{d^3 k}{(2\pi)^3} (a_{\vec{k}}^{(0)}(\partial_0 - D) f^{(0)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^{(3)} k^2 f^{(3)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}), \tag{63}$$

$$A(\tau, \vec{x}) = \frac{1}{m^2 C} \int \frac{d^3 k}{(2\pi)^3} (a_{\vec{k}}^{(0)} f^{(0)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} - a_{\vec{k}}^{(3)} \partial_0 f^{(3)}(\tau, k) e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}). \tag{64}$$

Now we substitute (57)–(64) into (55) and (56) to solve for the canonical (anti)commutation relations for the creation and annihilation operators and the normalization condition for the mode functions. We obtain

$$\left[a_{\vec{k}}^{(\mu)}, a_{\vec{k}'}^{(\nu)\dagger}\right] = \eta^{\mu\nu}(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'),$$

$$\{b_{\vec{k}}, \bar{b}_{\vec{k}'}^\dagger\} = -\{\bar{b}_{\vec{k}}, b_{\vec{k}'}^\dagger\} = i(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'), \tag{65}$$

where μ, $\nu = 0$, 1, 2, 3, and the following normalization conditions for the mode functions:

$$f^{(1,2)}(\tau, k)\partial_0 f^{(1,2)*}(\tau, k) - \partial_0 f^{(1,2)}(\tau, k) f^{(1,2)*}(\tau, k) = i,$$
$$f^{(0)}(\tau, k)\partial_0 f^{(0)*}(\tau, k) - \partial_0 f^{(0)}(\tau, k) f^{(0)*}(\tau, k) = im^2 C,$$
$$f^{(3)}(\tau, k)\partial_0 f^{(3)*}(\tau, k) - \partial_0 f^{(3)}(\tau, k) f^{(3)*}(\tau, k) = im^2 C k^{-2},$$
$$\chi(\tau, k)\partial_0 \bar{\chi}^*(\tau, k) - \partial_0 \bar{\chi}(\tau, k) \chi^*(\tau, k) = iC^{-1},$$
$$\bar{\chi}(\tau, k)\partial_0 \chi^*(\tau, k) - \partial_0 \chi(\tau, k) \bar{\chi}^*(\tau, k) = iC^{-1}. \tag{66}$$

In terms of the mode functions, the field equations (44), (50)–(52) read

$$(\partial_0^2 + \omega^2) f^{(1,2)}(\tau, k) = 0, \tag{67}$$

$$(\partial_0^2 - D\partial_0 + \omega_0^2 - D') f^{(0)}(\tau, k) = 0, \tag{68}$$

$$(\partial_0^2 - D\partial_0 + \omega^2) f^{(3)}(\tau, k) = 0, \tag{69}$$

$$(\partial_0^2 - D\partial_0 + \omega_\chi^2 - D')\chi(\tau, k) = 0, \tag{70}$$

where

$$\omega^2 := k^2 + m^2 C, \quad \omega_0^2 := k^2 + \xi m^2 C, \quad \omega_\chi^2 := k^2 + m_\chi^2 C. \tag{71}$$

To perform the adiabatic expansion, we notice that the differential equations for $f^{(0)}(\tau, k)$, $f^{(3)}(\tau, k)$, and $\chi(\tau, k)$ include first time derivative terms that can be eliminated by

rescaling the mode functions by appropriate time dependent functions. Defining

$$f^{(0)}(\tau, k) = (m^2 C)^{\frac{1}{2}} Y_0(\tau, k), \qquad (72)$$

$$f^{(3)}(\tau, k) = \left(\frac{m^2 C}{k^2}\right)^{\frac{1}{2}} Y_L(\tau, k), \qquad (73)$$

$$\chi(\tau, k) = C^{-\frac{1}{2}} Y_\chi(\tau, k), \quad \text{same for } \bar{\chi}(\tau, k), \qquad (74)$$

then the differential equations (67)–(70) simplify to the form of a harmonic oscillator with a time dependent frequency,

$$(\partial_0^2 + \Omega_a^2) Y_a(\tau, k) = 0 \qquad (a = 0, L, T, \chi), \quad (75)$$

where we have defined $Y_T(\tau, k) := f^{(1,2)}(\tau, k)$, and

$$\Omega_a^2 := \omega_a^2 + \alpha_a, \qquad (76)$$

with

$$\omega_a = \begin{cases} \omega_0 & (a=0) \\ \omega & (a=L,T), \\ \omega_\chi & (a=\chi) \end{cases} \quad \alpha_a = \begin{cases} -\frac{1}{6}CR & (a=0,\chi) \\ \frac{1}{6}CR - \frac{1}{2}D^2 & (a=L) \\ 0 & (a=T) \end{cases}, \quad (77)$$

and $R = C^{-1}(3D' + \frac{3}{2}D^2)$ being the scalar curvature. Note that the mode functions of the temporal component of the conjugate momentum π_A^0 and of the ghost field χ satisfy the same differential equation as that of a minimally coupled scalar field. Note also that we have

$$Y_0(\tau, k) = Y_\chi(\tau, k) \qquad (m = m_\chi = 0). \qquad (78)$$

At this point it is pleasing to note that the same rescaling also brings the normalization conditions (66) to the same standard form (12) as that of scalar field

$$Y_a \partial_0 Y_a^* - \partial_0 Y_a Y_a^* = i \quad \text{(no sum over } a). \qquad (79)$$

Therefore one can proceed to quantize the theory adiabatically with the choice of the mode functions

$$Y_a(\tau, k) = \frac{1}{\sqrt{2W_a(\tau)}} e^{-i \int^\tau W_a(\tau') d\tau'}, \qquad (80)$$

where

$$W_a^2 = \Omega_a^2 - \left(\frac{W_a''}{2W_a} - \frac{3(W_a')^2}{4W_a^2}\right), \qquad (81)$$

and with the adiabatic vacuum $|0\rangle_A$ defined by

$$a_{\vec{k}}^{(\mu)} |0\rangle_A = b_{\vec{k}} |0\rangle_A = \bar{b}_{\vec{k}} |0\rangle_A = 0. \qquad (82)$$

The adiabatic expansions are obtained by solving (81) iteratively with the zeroth adiabatic order solutions

$$W_{a(0)} = \omega_a. \qquad (83)$$

Then one obtains the following results up to the fourth adiabatic order:

$$\begin{aligned} W_a = \omega_a &- \frac{m_a^2 C''}{8\omega_a^3} + \frac{5m^4 (C')^2}{32\omega_a^5} + \frac{\alpha_a}{2\omega_a} + \frac{m_a^2 C''''}{32\omega_a^5} \\ &- \frac{m_a^4}{128\omega_a^7} (28 C''' C' + 19(C'')^2) \\ &+ \frac{221 m_a^6 C''(C')^2}{256\omega_a^9} - \frac{1105 m_a^8 (C')^4}{2048\omega_a^{11}} - \frac{\alpha_a'' + \alpha_a^2}{8\omega_a^3} \\ &+ \frac{m_a^2}{16\omega_a^5} (5C'\alpha_a' + 3C''\alpha_a) - \frac{25 m_a^4 (C')^2 \alpha_a}{64\omega_a^7} \end{aligned} \qquad (84)$$

and

$$\begin{aligned} \frac{1}{W_a} = \frac{1}{\omega_a} &+ \frac{m_a^2 C''}{8\omega_a^5} - \frac{5m^4 (C')^2}{32\omega_a^7} - \frac{\alpha_a}{2\omega_a^3} - \frac{m_a^2 C''''}{32\omega_a^7} \\ &- \frac{m_a^4}{128\omega_a^9} (28 C''' C' + 21(C'')^2) \\ &- \frac{231 m_a^6 C''(C')^2}{256\omega_a^{11}} + \frac{1155 m_a^8 (C')^4}{2048\omega_a^{13}} + \frac{\alpha_a'' + 3\alpha_a^2}{8\omega_a^5} \\ &- \frac{5m_a^2}{16\omega_a^7} (C'\alpha_a' + C''\alpha_a) + \frac{35 m_a^4 (C')^2 \alpha_a}{64\omega_a^9}. \end{aligned} \qquad (85)$$

Here we have introduced m_a by $\omega_a^2 = k^2 + m_a^2 C$. Explicitly, it is

$$m_a^2 = \begin{cases} \xi m^2 & (a=0) \\ m^2 & (a=L,T) \\ m_\chi^2 & (a=\chi) \end{cases}. \qquad (86)$$

The expressions (84) and (85) have been expressed in ascending (even) powers of time derivatives. The results obtained here for the adiabatic expansion of the $U(1)$ gauge field is new.

IV. ADIABATIC REGULARIZATION OF THE ENERGY MOMENTUM TENSOR AND THE CONFORMAL ANOMALY

Next let us turn to consider the adiabatic regularization of the energy momentum tensor for the $U(1)$ gauge theory (42). We will focus on the conformal anomaly in this paper. The energy momentum tensor obtained from (42) is

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

$$= -\frac{1}{4} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} F_{\alpha\rho} F_{\beta\sigma} + g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{2\xi} g_{\mu\nu} (g^{\alpha\beta} \nabla_\alpha A_\beta)^2$$

$$+ \frac{1}{\xi} (\nabla_\mu A_\nu + \nabla_\nu A_\mu)(g^{\alpha\beta} \nabla_\alpha A_\beta) - \frac{1}{\xi} [\nabla_\mu (A_\nu g^{\alpha\beta} \nabla_\alpha A_\beta) + \nabla_\nu (A_\mu g^{\alpha\beta} \nabla_\alpha A_\beta)]$$

$$+ \frac{1}{\xi} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho (A_\sigma g^{\alpha\beta} \nabla_\alpha A_\beta) - \frac{1}{2} g_{\mu\nu} m^2 g^{\alpha\beta} A_\alpha A_\beta + m^2 A_\mu A_\nu$$

$$+ i g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \bar{\chi} \nabla_\sigma \chi - i(\nabla_\mu \bar{\chi} \nabla_\nu \chi + \nabla_\nu \bar{\chi} \nabla_\mu \chi) + i g_{\mu\nu} m_\chi^2 \bar{\chi} \chi. \tag{87}$$

In the adiabatic regularization scheme, the renormalized energy momentum tensor is given by

$$\langle T_{\mu\nu}(x) \rangle_{\text{ren}} = \langle T_{\mu\nu}^{(m=0)}(x) \rangle - \lim_{m,m_\chi \to 0} {}_A\langle 0|T_{\mu\nu}(x)|0 \rangle_A, \tag{88}$$

where the first term on the right-hand side is evaluated in the vacuum defined by the mode functions in the massless theory. As we have explained before, in order to remove all the divergences and obtain the finite result for the renormalized energy momentum tensor, one should expand the subtraction term $_A\langle 0|T_{\mu\nu}|0 \rangle_A$ up to the fourth adiabatic order.

Now we evaluate the conformal anomaly. Taking the trace of (87), we obtain

$$T^\mu{}_\mu = \frac{2}{\xi} g^{\mu\nu} \nabla_\mu (A_\nu g^{\alpha\beta} \nabla_\alpha A_\beta) - m^2 g^{\mu\nu} A_\mu A_\nu$$

$$+ 2ig^{\mu\nu} \partial_\mu \bar{\chi} \partial_\nu \chi + 4im_\chi^2 \bar{\chi} \chi. \tag{89}$$

One may worry that (89) does not vanish even in the massless limit since the covariant gauge fixing term and the ghost kinetic term breaks the conformal symmetry individually. However, it is easy to check that they indeed cancel each other and give zero contribution to the trace of the energy momentum tensor when we take the expectation value with respect to the vacuum defined in the massless theory where (78) holds,

$$\langle T^\mu{}_\mu^{(m=0)} \rangle = 0. \tag{90}$$

As a result, the conformal anomaly is determined entirely by the adiabatic subtraction term

$$\langle T^\mu{}_\mu \rangle_{\text{ren}} = - \lim_{m,m_\chi \to 0} {}_A\langle 0|T^\mu{}_\mu|0 \rangle_A. \tag{91}$$

Let us start with the contribution from the mass term of the gauge field in (89). The corresponding adiabatic subtraction term that contributes to the conformal anomaly is given by

$$\langle T^\mu{}_\mu \rangle_{\text{ren}}^{\text{mass}} = - \lim_{m \to 0} {}_A\langle 0|(-m^2 g^{\mu\nu} A_\mu A_\nu)|0 \rangle_A$$

$$= \lim_{m \to 0} \frac{m^2}{C} [-{}_A\langle 0|A_0^2|0 \rangle_A + {}_A\langle 0|\delta^{ij} A_i A_j|0 \rangle_A]$$

$$= \lim_{m \to 0} \frac{1}{C^2} \int \frac{d^3 k}{(2\pi)^3} \left(\left| \left(\partial_0 - \frac{D}{2} \right) Y_0(\tau, k) \right|^2 - k^2 |Y_0(\tau, k)|^2 + \left| \left(\partial_0 + \frac{D}{2} \right) Y_L(\tau, k) \right|^2 - k^2 |Y_L(\tau, k)|^2 + 2m^2 C |Y_T(\tau, k)|^2 \right). \tag{92}$$

Expanding this expression up to the fourth adiabatic order, we find that (92) is UV finite and gives a finite contribution to the conformal anomaly. However, we found that the contribution from the mass term alone cannot be expressed in terms of $R_{\mu\nu}^2$, R^2, and $\Box R$ only. The contribution from the other terms is thus important to obtain the correct result. Next we evaluate the contribution from the term proportional to ξ^{-1} in (89),

$$\langle T^\mu{}_\mu \rangle_{\text{ren}}^{\xi} = - \lim_{m \to 0} {}_A\langle 0| \frac{2}{\xi} g^{\mu\nu} \nabla_\mu (A_\nu g^{\alpha\beta} \nabla_\alpha A_\beta)|0 \rangle_A$$

$$= - \lim_{m \to 0} \frac{2}{C^2} [{}_A\langle 0|\xi(\pi_A^0)^2|0 \rangle_A - {}_A\langle 0|A_0(\partial_0 - D)\pi_A^0|0 \rangle_A + {}_A\langle 0|\delta^{ij} \partial_i A \partial_j \pi_A^0|0 \rangle_A]$$

$$= \lim_{m \to 0} \frac{2}{C^2} \int \frac{d^3 k}{(2\pi)^3} \left(- \left| \left(\partial_0 - \frac{D}{2} \right) Y_0(\tau, k) \right|^2 + \omega_0^2 |Y_0(\tau, k)|^2 \right). \tag{93}$$

The fourth adiabatic order contribution from (93) is found to be UV divergent. This UV divergence is canceled by the ghost contribution as we will see below. Finally the ghost contribution is obtained as

$$
\langle T^{\mu}{}_{\mu}\rangle_{\text{ren}}^{\text{ghost}} = -2i \lim_{m_{\chi}\to 0}{}_{A}\langle 0|(g^{\mu\nu}\partial_{\mu}\bar{\chi}\partial_{\nu}\chi + 2m_{\chi}^{2}\bar{\chi}\chi)|0\rangle_{A}
$$

$$
= \lim_{m_{\chi}\to 0}\frac{2}{C^{2}}\int\frac{d^{3}k}{(2\pi)^{3}}\left(\left|\left(\partial_{0}-\frac{D}{2}\right)Y_{\chi}(\tau,k)\right|^{2} - (k^{2}+2m_{\chi}^{2}C)|Y_{\chi}(\tau,k)|^{2}\right). \tag{94}
$$

For large k, taking into account (78), we observe that the two contributions from (93) and (94) have the same form but opposite signs, and so the respective UV divergences cancel each other to give a finite result in the conformal anomaly. We remark that the expressions for the momentum integrations in (92)–(94) are indeed valid for a general vacuum state until we substitute the adiabatic expansions. As a result, the contributions (93) and (94) exactly cancel each other in the massless theory and we obtain (90).

Putting (92), (93), and (94) together, the conformal anomaly for the $U(1)$ gauge theory in the adiabatic regularization is given by

$$
\langle T^{\mu}{}_{\mu}\rangle_{\text{ren}} = \lim_{m,m_{\chi}\to 0}\frac{1}{4\pi^{2}C^{2}}\int_{0}^{\infty}dk k^{2}\left[\frac{2}{W_{0}}\left(\omega_{0}^{2}-\left(\frac{W_{0}'}{2W_{0}}+\frac{D}{2}\right)^{2}-W_{0}^{2}\right)-\frac{1}{W_{0}}\left(k^{2}-\left(\frac{W_{0}'}{2W_{0}}+\frac{D}{2}\right)^{2}-W_{0}^{2}\right)\right.
$$

$$
\left.-\frac{1}{W_{L}}\left(k^{2}-\left(\frac{W_{L}'}{2W_{L}}-\frac{D}{2}\right)^{2}-W_{L}^{2}\right)+2\frac{m^{2}C}{W_{T}}-\frac{2}{W_{\chi}}\left(k^{2}+2m_{\chi}^{2}C-\left(\frac{W_{\chi}'}{2W_{\chi}}+\frac{D}{2}\right)^{2}-W_{\chi}^{2}\right)\right]_{(4)}
$$

$$
= \frac{1}{2880\pi^{2}}\left[-150\frac{(C')^{4}}{C^{6}}+474\frac{(C')^{2}C''}{C^{5}}-162\frac{(C'')^{2}}{C^{4}}-216\frac{C'''C'}{C^{4}}+54\frac{C''''}{C^{3}}\right.
$$

$$
\left.-\log\xi\left(\frac{405}{2}\frac{(C')^{4}}{C^{6}}-\frac{945}{2}\frac{(C')^{2}C''}{C^{5}}+135\frac{(C'')^{2}}{C^{4}}+180\frac{C'''C'}{C^{4}}-45\frac{C''''}{C^{3}}\right)\right]
$$

$$
= \frac{1}{2880\pi^{2}}\left[62\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^{2}\right)-(18+15\log\xi)\Box R\right], \tag{95}
$$

where the subscript (4) denotes the term up to the fourth adiabatic order. In obtaining this result, we have used (84) and (85) in the second equality and (A2) in the third equality. Note that the ξ dependence came entirely from W_{0} and ω_{0} as the other quantities W_{L}, W_{T}, W_{χ} are independent of ξ. The regularization independent term of our result (95) agrees precisely with that obtained [first term of (1)] using other regularization schemes. As for the regularization dependent $\Box R$ term, *a priori* there is no need for our result to agree with any of the previously obtained results. However, to our surprise, our value of d agrees with the results of [9] for $\xi = 1$ obtained using zeta function regularization and of [24] for a general gauge fixing parameter ξ obtained using the DeWitt-Schwinger expansion.

V. SUMMARY

In this article, we have investigated and constructed the adiabatic expansion and regularization for a $U(1)$ gauge field in a conformally flat spacetime. This has never been considered before and our results are new. We argued the necessity of the use of a covariant gauge fixing term for the

sake of covariant conservation of the energy momentum tensor. As in the scalar field case, the adiabatic expansion of the gauge field mode functions are carried out by the WKB-type solutions that preserve the Wronskian-type normalization conditions. It is clear that the adiabatic expansion and the computation of the conformal anomaly for a $U(1)$ gauge field performed here can easily be extended to that for Yang-Mills gauge fields.

Based on the adiabatic expansion, we evaluated the conformal anomaly for the $U(1)$ gauge field in a conformally flat spacetime as an application, and we found that the result exactly agrees with that obtained from ζ function regularization [9,24] in the Dewitt-Schwinger (or local momentum expansion [35]) formalism [1] and from the Hadamard renormalization [36]. Consequently, our construction of the adiabatic expansion and the adiabatic regularization for gauge field in a conformally flat spacetime is confirmed by reproducing the standard result of the conformal anomaly. In addition, we have observed the same gauge dependence in the coefficient of the $\Box R$ term of the conformal anomaly as Eq. (5.1) of [24]. However, the result is different from that obtained using the dimensional regularization with $\xi = 1$ [21,22]. Our

result clearly confirms the regularization dependency of the $\Box R$ term of the conformal anomaly.

While we have focused on the conformal anomaly in this article, evaluation of the renormalized energy momentum tensor (and more general correlation functions) in a specific conformally flat spacetime, e.g., in de Sitter space or in the inflationary universe, is an important application of our adiabatic regularization procedure. Note that the application of the adiabatic regularization scheme that we have constructed here is not restricted to the energy momentum tensor and the conformal anomaly. It will be useful to evaluate various renormalized physical quantities of theories coupled to (Yang-Mills) gauge fields in conformally flat spacetime. Since the adiabatic regularization allows one to compute the particle number density, one can also discuss gauge field particle production in an expanding universe. Another important application is the study of the renormalizability of the $\mathcal{N} = 4$ superconformal Yang-Mills theory on de Sitter space [37].

APPENDIX: SOME GEOMETRICAL TENSORS IN CONFORMALLY FLAT SPACETIME

For a conformally flat spacetime (4) in four dimensions, the Ricci tensor and Ricci scalar are

$$R_{\mu\nu} = \frac{3}{2}\delta^0{}_\mu \delta^0{}_\nu \left(\frac{C'}{C}\right)^2 + \frac{1}{2}(-2\delta^0{}_\mu \delta^0{}_\nu + \eta_{\mu\nu})\frac{C''}{C},$$

$$R = C^{-1}\left[-\frac{3}{2}\left(\frac{C'}{C}\right)^2 + 3\frac{C''}{C}\right]. \tag{A1}$$

Quantities that appear at the fourth adiabatic order in a conformally flat spacetime are

$$R_{\mu\nu}R^{\mu\nu} = \frac{9}{4}\frac{(C')^4}{C^6} - \frac{9}{2}\frac{(C')^2 C''}{C^5} + 3\frac{(C'')^2}{C^4},$$

$$R^2 = \frac{9}{4}\frac{(C')^4}{C^6} - 9\frac{(C')^2 C''}{C^5} + 9\frac{(C'')^2}{C^4},$$

$$\Box R = \frac{27}{2}\frac{(C')^4}{C^6} - \frac{63}{2}\frac{(C')^2 C''}{C^5} + 9\frac{(C'')^2}{C^4}$$
$$+ 12\frac{C'''C'}{C^4} - 3\frac{C''''}{C^3}. \tag{A2}$$

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