Casimir effect for parallel plates in a Friedmann-Robertson-Walker universe

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We evaluate the Hadamard function, the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor for a massive scalar field with a general curvature coupling parameter in the geometry of two parallel plates on a spatially flat Friedmann-Robertson-Walker background with a general scale factor. On the plates, the field operator obeys the Robin boundary conditions with the coefficients depending on the scale factor. In all the spatial regions, the VEVs are decomposed into the boundary-free and boundary-induced contributions. Unlike the problem with the Minkowski bulk, in the region between the plates, the normal stress is not homogeneous and does not vanish in the geometry of a single plate. Near the plates, it has different signs for accelerated and decelerated expansions of the Universe. The VEV of the energy-momentum tensor, in addition to the diagonal components, has a nonzero off-diagonal component describing an energy flux along the direction normal to the boundaries. Expressions are derived for the Casimir forces acting on the plates. Depending on the Robin coefficients and on the vacuum state, these forces can be either attractive or repulsive. An important difference from the corresponding result in the Minkowski bulk is that the forces on the separate plates, in general, are different if the corresponding Robin coefficients differ. We give the applications of general results for the class of α vacua in the de Sitter bulk. It is shown that, compared with the Bunch-Davies vacuum state, the Casimir forces for a given α vacuum may change the sign.

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I. INTRODUCTION

The investigation of quantum effects in cosmological backgrounds is among the most important topics of quantum field theory in curved spacetime (see [1]). There are several reasons for that. Due to the high symmetry of the background geometry, a relatively large number of problems are exactly solvable, and the corresponding results may shed light on the influence of the gravitational field on quantum fields for more complicated geometries. The expectation value of the energy-momentum tensor for quantum fields may break the energy conditions appearing in the formulations of the Hawking-Penrose singularity theorems. This expectation value appears as a source for the gravitational field in the right-hand side of the Einstein equations and, consequently, the quantum effects of nongravitational fields may provide a way to solve the cosmological singularity problem. In the inflationary phase, the quantum fluctuations of fields are responsible for the generation of density perturbations serving as seeds for the large scale structure formation in the Universe. Currently, this mechanism is the most popular one for the generation of cosmological structures. From the cosmological point of view, another interesting quantum field theoretical effect is the isotropization of the cosmological expansion as a result of particle creation.

In a number of cosmological problems, additional boundary conditions are imposed on the operators of quantum fields. These conditions may have different physical origins. For example, they can be induced by nontrivial spatial topology, by the presence of coexisting phases, or by branes in the scenarios of the braneworld type. The boundary conditions modify the spectrum of quantum fluctuations of fields and, as a consequence of that, the expectation values of physical observables are changed. This is the well-known Casimir effect first predicted by Casimir in 1948 (for reviews, see [2]). In the present paper, we consider the influence of the cosmological expansion on the local characteristics of the scalar vacuum in the geometry of two parallel plates. This type of problem for various special cases has been considered previously. In particular, the vacuum expectation values (VEVs) for parallel plates in a background of de Sitter spacetime were investigated in [3,4] and [5] for scalar and electromagnetic fields, respectively.

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The problems with spherical and cylindrical boundaries have been discussed in [6,7] (for the Casimir densities in the anti–de Sitter bulk, see [8] and references therein). All these investigations have been done for the de Sitter invariant Bunch-Davies vacuum state. By using the conformal relation between the Friedmann-Roberston-Walker (FRW) and Rindler spacetimes, the VEVs of the energy-momentum tensor and the Casimir forces for a conformally coupled massless scalar field and for the electromagnetic field, in the geometry of curved boundaries on a background of FRW spacetime with negative spatial curvature, were evaluated in [9] (for a special case of the static background, see also [10]). The electromagnetic Casimir effect in FRW cosmologies with an arbitrary number of spatial dimensions and with power-law scale factors has been considered in [11] (for the topological Casimir densities in the corresponding models with compact dimensions, see [12]).

The present paper generalizes the previous investigations in two directions. First, we consider a spatially flat FRW spacetime with general scale factor and, second, the Casimir effect will be investigated without specifying the vacuum state for a scalar field. The boundary geometry consists of two parallel plates on which the scalar field operator obeys the Robin boundary conditions with, in general, different coefficients on the separate plates. We consider the case when these coefficients are proportional to the scale factor. With this assumption, closed analytical expressions are obtained for the Hadamard function and for the VEVs of the field squared and the energy-momentum tensor without specifying the time dependence of the scale factor.

The paper is organized as follows. In the next section, we describe the bulk and boundary geometries under consideration and the field content. In Sec. III, the Hadamard function is evaluated for a massive scalar field with general curvature coupling parameter and obeying the Robin boundary conditions on two parallel plates. The boundary-induced contributions are explicitly separated for both the single-plate and two-plates geometries. By using the Hadamard function, in Sec. IV, we evaluate the VEVs of the field squared and of the energy-momentum tensor. Expressions are derived for the Casimir forces acting on the plates. Two special cases of the general results are discussed in Sec. V. They include a conformally coupled massless scalar field for a general scale factor and the de Sitter bulk with a massive scalar field for the general case of the curvature coupling. Finally, we leave for Sec. VI the most relevant discussion of the results obtained.

II. PROBLEM FORMULATION AND THE SCALAR MODES

As a background geometry, we take a spatially flat $(1 + D)$-dimensional FRW spacetime described by the line element

$$ds^2 = dt^2 - a^2(t)\sum_{i=1}^{D}(dx^i)^2, \tag{2.1}$$

with the scale factor $a(t)$. Defining the conformal time η in terms of the cosmic time t by $\eta = \int dt/a(t)$, the metric tensor is presented in a conformally flat form $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ with the flat spacetime metric $\eta_{\mu\nu}$. In addition to the Hubble function $H = \dot{a}/a$, we will use the corresponding function for the conformal time:

$$\tilde{H} = a'(\eta)/a(\eta) = a(\eta)H. \tag{2.2}$$

Here and in what follows, the dot specifies the derivative with respect to the cosmic time and the prime denotes the derivative with respect to the conformal time.

Consider a massive scalar field $\phi(x)$ nonminimally coupled to the background. The corresponding action functional has the form

$$S = \frac{1}{2}\int d^{D+1}x\sqrt{|g|}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2 - \xi R\phi^2), \tag{2.3}$$

where ∇_μ stands for the covariant derivative and ξ is the coupling parameter to the curvature scalar R. For the background geometry under consideration, one has

$$R = \frac{D}{a^2}[2\tilde{H}' + (D-1)\tilde{H}^2]. \tag{2.4}$$

By varying the action with respect to the field, one obtains the equation of motion

$$(\nabla_\mu\nabla^\mu + m^2 + \xi R)\phi = 0. \tag{2.5}$$

Additionally, we assume the presence of two flat boundaries located at $z \equiv x^D = z_1$ and $z = z_2$, $z_2 > z_1$, on which the field satisfies the Robin boundary conditions

$$(1 + \beta'_j n^\mu_j \nabla_\mu)\phi = 0, \qquad z = z_j, \qquad j = 1, 2, \tag{2.6}$$

where n^μ_j is the normal to the boundary $z = z_j$, $n_{j\mu}n^\mu_j = -1$. For the region between the plates, $z_1 \le z \le z_2$, one has $n^\mu_j = (-1)^{j-1}\delta^\mu_D/a(\eta)$. The Robin condition is an extension of the Dirichlet and Neumann boundary conditions and is useful for modeling the finite penetration of the field into the boundary with the skin-depth parameter related to the coefficient β'_j [13]. This type of boundary condition naturally arises for bulk fields in braneworld models. In the discussion below, we will consider a class of boundary conditions for which $\beta'_j = \beta_j a(\eta)$, with β_j, $j = 1, 2$, being constants. This corresponds to the physical situation when, for an expanding bulk, the penetration length to the boundary is increasing as well. In this special case, for the region between the plates, the boundary conditions are rewritten as

$$(1+(-1)^{j-1}\beta_j\partial_z)\phi = 0, \qquad z = z_j. \tag{2.7}$$

As will be shown below, the corresponding Casimir problem is exactly solvable for a general case of the scale factor $a(\eta)$.

We are interested in the changes of the VEVs of the field squared and the energy-momentum tensor induced by the imposition of the boundary conditions (2.7). The VEVs of physical observables, quadratic in the field operator, are expressed in terms of the sums over a complete set of solutions to the field equation (2.5) obeying the boundary conditions. In accordance with the geometry of the problem, for the corresponding mode functions, we will use the following ansatz:

$$\phi(x) = f(\eta)e^{i\mathbf{k}\cdot\mathbf{x}_\parallel}h(z), \qquad \mathbf{k} = (k^1, k^2, \ldots, k^{D-1}),$$
$$\mathbf{x}_\parallel = (x^1, x^2, \ldots, x^{D-1}). \tag{2.8}$$

Substituting into the field equation (2.5), we obtain two differential equations:

$$h''(z) = -\lambda^2 h(z) \tag{2.9}$$

and

$$f''(\eta) + (D-1)\tilde{H}f'(\eta) + [\gamma^2 + a^2(m^2+\xi R)]f(\eta) = 0, \tag{2.10}$$

with $\gamma = \sqrt{\lambda^2 + k^2}$ and $k = |\mathbf{k}|$. In particular, from Eq. (2.10), it follows that

$$\{a^{D-1}[f^*(\eta)f'(\eta) - f(\eta)f^{*\prime}(\eta)]\}' = 0, \tag{2.11}$$

where the star stands for the complex conjugate. Note that, introducing the function $g(\eta) = a^{(D-1)/2}f(\eta)$, Eq. (2.10) is written in the form

$$g''(\eta) + \{\gamma^2 + m^2a^2 + D(\xi - \xi_D)[2\tilde{H}' + (D-1)\tilde{H}^2]\}g(\eta) = 0, \tag{2.12}$$

where $\xi_D = (D-1)/(4D)$ is the curvature coupling parameter for a conformally coupled scalar field.

The solution for (2.9) that obeys the boundary condition on the plate $z = z_j$ reads

$$h(z) = \cos[\lambda(z - z_j) + \alpha_j(\lambda)], \tag{2.13}$$

with the function $\alpha_j(\lambda)$ defined by the relation

$$e^{2i\alpha_j(\lambda)} = \frac{i\lambda\beta_j(-1)^j + 1}{i\lambda\beta_j(-1)^j - 1}. \tag{2.14}$$

From the boundary condition on the second plate, it follows that the quantum number λ obeys the restriction condition

$$(1 - b_1b_2u^2)\sin u - (b_1+b_2)u\cos u = 0, \tag{2.15}$$

where

$$u = \lambda z_0, \qquad b_j = \beta_j/z_0, \qquad z_0 = z_2 - z_1. \tag{2.16}$$

Note that the eigenvalue equation (2.15) coincides with the corresponding equation for parallel plates in the Minkowski bulk [14]. We will denote the solutions of the transcendental equation (2.15) by $u = u_n$, $n = 1, 2, \ldots$. For the eigenvalues of the quantum number λ, one has $\lambda = \lambda_n = u_n/z_0$.

So, for the complete set of solutions, one has $\{\phi_\sigma^{(+)}(x), \phi_\sigma^{(-)}(x)\}$, where

$$\phi_\sigma^{(+)}(x) = C_\sigma f(\eta,\gamma)e^{i\mathbf{k}\cdot\mathbf{x}_\parallel}\cos[\lambda_n(z - z_j) + \alpha_j(\lambda_n)], \tag{2.17}$$

$\phi_\sigma^{(-)}(x) = \phi_\sigma^{(+)*}(x)$, with C_σ being a normalization constant and $\sigma = (n, \mathbf{k})$ representing the set of quantum numbers specifying the modes. In (2.17), the dependence of the function f on γ is explicitly displayed.

In accordance with (2.11), we will normalize the function $f(\eta,\gamma)$ by the condition

$$f(\eta,\gamma)\partial_\eta f^*(\eta,\gamma) - f^*(\eta,\gamma)\partial_\eta f(\eta,\gamma) = ia^{1-D}. \tag{2.18}$$

With this normalization, the constant C_σ is determined from the standard orthonormalization condition for the Klein-Gordon equation:

$$\int d^Dx\sqrt{|g|}g^{00}[\phi_\sigma(x)\partial_\eta\phi^*_{\sigma'}(x) - \phi^*_{\sigma'}(x)\partial_\eta\phi_\sigma(x)] = i\delta(\mathbf{k}-\mathbf{k}')\delta_{nn'}. \tag{2.19}$$

By taking into account (2.18), one gets

$$|C_\sigma|^2 = \frac{2}{(2\pi)^{D-1}z_0}\left\{1 + \frac{\sin u_n}{u_n}\cos[u_n + 2\tilde{\alpha}_j(u_n)]\right\}^{-1}, \tag{2.20}$$

where the function $\tilde{\alpha}_j(u)$ is defined in accordance with

$$e^{2i\tilde{\alpha}_j(u)} = \frac{iub_j - 1}{iub_j + 1}, \tag{2.21}$$

for $j = 1, 2$.

Note that the mode functions (2.17) are not yet completely fixed. The function $f(\eta,\gamma)$ is a linear combination of two linearly independent solutions of Eq. (2.10). One of the coefficients is fixed (up to a phase) by the condition (2.18). Among the most important steps in the construction of a quantum field theory in a fixed classical gravitational background is the choice of the vacuum state $|0\rangle$. Different choices of the second coefficient in the linear combination for the function $f(\eta,\gamma)$ correspond to different choices of the vacuum state. An additional condition could be the requirement of the smooth transition to the standard Minkowskian vacuum in the limit of slow expansion. This point will be discussed below for an example of de Sitter bulk.

In the limit of small wavelengths, $\gamma \gg ma, \sqrt{|\tilde{H}'|}, \tilde{H}$, the general solution of Eq. (2.12) is a linear combination of the functions $e^{i\gamma\eta}$ and $e^{-i\gamma\eta}$. For the modes which satisfy the adiabatic condition (for the adiabatic condition see [1]), one takes $g(\eta) \sim e^{-i\gamma\eta}$ and the function $f(\eta,\gamma)$, normalized by the condition (2.18) has the small wavelength asymptotic behavior:

$$f(\eta,\gamma) \approx a^{(1-D)/2}\frac{e^{-i\gamma\eta}}{\sqrt{2\gamma}}. \qquad (2.22)$$

In the limit of slow expansion, these modes approach the positive energy solutions for a scalar field in Minkowski spacetime. The condition on the wavelength, written in terms of the cosmic time t, is in the form $\gamma/a \gg m, \sqrt{|\dot{H}|}, H$. Note that the condition (2.22) does not specify the vacuum state uniquely (for the discussion of related uncertainties in the inflationary predictions of the curvature perturbations, see, for instance, [15]).

III. HADAMARD FUNCTION

Given the complete set of modes, we can evaluate the two-point functions. We consider a free field theory (the only interaction is with the background gravitational field) and all the information about the vacuum state is encoded in two-point functions. As such we take the Hadamard function $G(x,x') = \langle 0|\phi(x)\phi(x') + \phi(x')\phi(x)|0\rangle$ with the mode sum formula

$$G(x,x') = \int d\mathbf{k}\sum_{n=1}^{\infty}\sum_{s=\pm}\phi_{\sigma}^{(s)}(x)\phi_{\sigma}^{(s)*}(x'). \qquad (3.1)$$

Substituting the mode functions (2.17), one gets the representation

$$G(x,x') = \frac{2}{z_0}\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}_{\parallel}}}{(2\pi)^{D-1}}\sum_{n=1}^{\infty}u_n w(\eta,\eta',\gamma_n) \times\frac{\cos[\lambda_n(z-z_j)+\alpha_j(\lambda_n)]\cos[\lambda_n(z'-z_j)+\alpha_j(\lambda_n)]}{u_n+\sin u_n\cos[u_n+2\tilde{\alpha}_j(\lambda_n)]}, \qquad (3.2)$$

with $\Delta\mathbf{x}_{\parallel} = \mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}$, $\gamma_n = \sqrt{\lambda_n^2 + k^2}$, and

$$w(\eta,\eta',\gamma) = f(\eta,\gamma)f^*(\eta',\gamma) + f^*(\eta,\gamma)f(\eta',\gamma). \qquad (3.3)$$

In (3.2), $\lambda_n = u_n/z_0$ and the eigenvalues u_n are given implicitly, as solutions of (2.15). Related to this, the representation (3.2) is not convenient for the evaluation of the VEVs. For the further transformation, we apply to the series over n the summation formula [14,16],

$$\sum_{n=1}^{\infty}\frac{\pi u_n s(u_n)}{u_n+\sin u_n\cos[u_n+2\tilde{\alpha}_j(\lambda_n)]} = -\frac{\pi s(0)/2}{1-b_2-b_1} + \int_0^{\infty}du\, s(u) + i\int_0^{\infty}du\frac{s(iu)-s(-iu)}{c_1(u)c_2(u)e^{2u}-1}, \qquad (3.4)$$

where the notation

$$c_j(u) = \frac{b_j u - 1}{b_j u + 1} \qquad (3.5)$$

is introduced. In (3.4), it is assumed that the function $s(u)$ obeys the condition $|s(u)| < \epsilon(x)e^{c|y|}$ for $|u| \to \infty$, where $u = x + iy$, $c < 2$, and $\epsilon(x) \to 0$ for $x \to \infty$. As the function $s(u)$, we take

$$s(u) = \{\cos(uz_-/z_0) + \cos[u(z_+ - 2z_j)/z_0 + 2\alpha_j(u/z_0)]\}w(\eta,\eta',\sqrt{u^2/z_0^2+k^2}), \qquad (3.6)$$

with

$$z_{\pm} = z \pm z'. \qquad (3.7)$$

After the application of (3.4), the Hadamard function (3.2) is decomposed as

$$G(x,x') = G_j(x,x') + \frac{1}{z_0}\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}_{\parallel}}}{(2\pi)^D}\int_0^{\infty}du\frac{W(\eta,\eta',u,k)}{c_1(u)c_2(u)e^{2u}-1}\left[2\cosh(uz_-/z_0) + c_j(u)e^{u|z_+-2z_j|/z_0} + \frac{e^{-u|z_+-2z_j|/z_0}}{c_j(u)}\right], \qquad (3.8)$$

where

$$W(\eta,\eta',u,k) = i\left[w(\eta,\eta',\sqrt{(iu)^2/z_0^2+k^2}) - w(\eta,\eta',\sqrt{(-iu)^2/z_0^2+k^2})\right]. \qquad (3.9)$$

The part

$$G_j(x,x') = G_0(x,x') + 2\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}_\parallel}}{(2\pi)^D}\int_0^\infty dy\cos[y(z_+ - 2z_j) + 2\alpha_j(y)]w\left(\eta,\eta',\sqrt{y^2+k^2}\right), \tag{3.10}$$

comes from the first integral in the right-hand side of (3.4), and it presents the Hadamard function for the geometry of a single plate at $z = z_j$ when the second plate is absent. In (3.10), the contribution

$$G_0(x,x') = \int d\mathbf{k}_D\frac{e^{i\mathbf{k}_D\cdot\Delta\mathbf{x}}}{(2\pi)^D}w(\eta,\eta',|\mathbf{k}_D|), \tag{3.11}$$

with $\mathbf{x} = (x^1, x^2, \ldots, x^D)$, $\mathbf{k}_D = (k^1, k^2, \ldots, k^D)$, is the Hadamard function in the boundary-free geometry. The second term in the right-hand side of (3.10) is induced by the boundary at $z = z_j$. Consequently, the last term in (3.8) is interpreted as the contribution when one adds the second boundary in the problem with a single boundary at $z = z_j$.

For the further transformation of the boundary-induced contribution in (3.10), we present the cosine function in terms of the exponentials and rotate the integration contour over y by the angles $\pi/2$ and $-\pi/2$ for the parts with the functions $e^{iy|z_+-2z_j|}$ and $e^{-iy|z_+-2z_j|}$, respectively. As a result, for the Hadamard function in the geometry of a single plate at $z = z_j$, we get

$$G_j(x,x') = G_0(x,x') + \int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}_\parallel}}{(2\pi)^D} \times\int_0^\infty dy\frac{\beta_j y+1}{\beta_j y-1}e^{-y|z_+-2z_j|}W(\eta,\eta',yz_0,k). \tag{3.12}$$

Substituting this representation into (3.8), the Hadamard function in the region between two plates is presented in the form

$$G(x,x') = G_0(x,x') + \frac{1}{z_0}\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}_\parallel}}{(2\pi)^D} \times\int_0^\infty du\frac{W(\eta,\eta',u,k)}{c_1(u)c_2(u)e^{2u}-1} \times\left[2\cosh(uz_-/z_0) + \sum_{j=1,2}c_j(u)e^{u|z_+-2z_j|/z_0}\right]. \tag{3.13}$$

This expression can be further simplified by integrating over the angular part of $\mathbf{k}$. The corresponding integral is expressed in terms of the Bessel function. In the regions $z < z_1$ and $z > z_2$, the Hadamard function is given by (3.12) with $j = 1$ and $j = 2$, respectively. Note that the dependence on the mass of the field appears in (3.13) through the function $f(\eta,\gamma)$. Equation (2.10) for the latter contains the mass as a parameter.

IV. VEVs AND THE CASIMIR FORCE

Having the two point function we can evaluate the VEVs of local physical observables bilinear in the field operator.

A. Field squared

We start with the VEV of the field squared. The latter is obtained from the Hadamard function in the coincidence limit of the arguments. Of course, this limit is divergent and a renormalization procedure is required. An important point is that we have separated the part of the Hadamard function corresponding to the boundary-free geometry. For points away from boundaries, the divergences are contained in this part only and the remaining boundary-induced contribution is finite in the coincidence limit. As a consequence, the renormalization is reduced to that for the VEVs in the boundary-free geometry. These VEVs are well investigated in the literature and in the following we will focus on the boundary-induced effects.

Taking the limit $x' \to x$ in (3.13), for the VEV of the field squared, $\langle 0|\phi^2|0\rangle \equiv \langle\phi^2\rangle$, in the region between the plates we get:

$$\langle\phi^2\rangle = \langle\phi^2\rangle_0 + \frac{A_D}{z_0}\int_0^\infty dk k^{D-2}\int_0^\infty du\frac{W(\eta,\eta,u,k)}{c_1(u)c_2(u)e^{2u}-1} \times\left[2 + \sum_{j=1,2}c_j(u)e^{2u|z-z_j|/z_0}\right], \tag{4.1}$$

where

$$A_D = \frac{2^{-D}\pi^{-(D+1)/2}}{\Gamma((D-1)/2)}, \tag{4.2}$$

and $\langle\phi^2\rangle_0$ is the renormalized VEV in the boundary-free geometry. The latter does not depend on the spatial point. In the regions $z < z_1$ and $z > z_2$, the VEVs are obtained from (3.12):

$$\langle\phi^2\rangle_j = \langle\phi^2\rangle_0 + A_D\int_0^\infty dk k^{D-2} \times\int_0^\infty dy\frac{\beta_j y+1}{\beta_j y-1}e^{-2y|z-z_j|}W(\eta,\eta,yz_0,k), \tag{4.3}$$

with $j = 1$ and $j = 2$, respectively.

Alternative expressions are obtained by taking into account that $W(\eta,\eta,u,k) = 0$ for $u < z_0 k$ and

$$W(\eta,\eta,u,k)=U(\eta,\sqrt{u^2-z_0^2k^2}), \tag{4.4}$$

for $u>z_0k$, where

$$U(\eta,z_0x)=i[w(\eta,\eta,ix)-w(\eta,\eta,-ix)]. \tag{4.5}$$

By using the relation

$$\int_0^\infty dxx^{n-1}\int_x^\infty duf_1(u)f_2(\sqrt{u^2-x^2}) = \int_0^\infty duu^nf_1(u)\int_0^1 dss(1-s^2)^{n/2-1}f_2(us), \tag{4.6}$$

the VEV of the field squared in the region between the plates is presented as

$$\langle\phi^2\rangle=\langle\phi^2\rangle_0+\frac{A_D}{z_0^D}\int_0^\infty duu^{D-1}Z(\eta,u) \times\frac{2+\sum_{j=1,2}c_j(u)e^{2u|z-z_j|/z_0}}{c_1(u)c_2(u)e^{2u}-1}, \tag{4.7}$$

with the notation

$$Z(\eta,u)=\int_0^1 dss(1-s^2)^{(D-3)/2}U(\eta,us). \tag{4.8}$$

In a similar way, for the regions $z<z_1$ and $z>z_2$ from (4.3) we get

$$\langle\phi^2\rangle_j=\langle\phi^2\rangle_0+A_D\int_0^\infty dyy^{D-1}Z(\eta,yz_0)\frac{\beta_jy+1}{\beta_jy-1}e^{-2y|z-z_j|}. \tag{4.9}$$

The information on the background geometry is encoded in the function $Z(\eta,u)$.

For the modes which satisfy the adiabatic condition in the limit of small wavelengths, one has the asymptotic condition (2.22). In this case we can obtain simple asymptotic expressions for the VEV of the field squared near the boundaries. From (2.22) it follows that for large x one has $U(\eta,x)\approx 2z_0a^{1-D}/x$ and, hence, for the function $Z(\eta,u)$ we get

$$Z(\eta,u)\approx z_0\frac{\sqrt{\pi}\Gamma((D-1)/2)}{\Gamma(D/2)a^{D-1}u}, \tag{4.10}$$

for $u\gg 1$. In order to find the asymptotic behavior of the VEV (4.7) near the boundary $z=z_j$, we note that in this region the dominant contribution to the integral comes from large values of u. By using (4.10), to the leading order one gets

$$\langle\phi^2\rangle\approx\frac{(1-2\delta_{0\beta_j})\Gamma((D-1)/2)}{(4\pi)^{(D+1)/2}(a|z-z_j|)^{D-1}}. \tag{4.11}$$

This leading term comes from the single plate part (4.9) and coincides with that for the plate in Minkowski bulk with the distance from the plate $|z-z_j|$ replaced by the proper distance $a(\eta)|z-z_j|$ for a fixed η. The latter property is natural, because, due to the adiabatic condition, the influence of the background gravitational field on the modes with small wavelengths is weak and in the region near the plates the main contribution to the VEVs comes from those modes.

The regularization procedure we have employed for the evaluation of the VEV of the field squared is based on the point-splitting technique with combination with the summation formula (3.4). Instead, we could start directly from the divergent expression $\langle\phi^2\rangle=G(x,x)/2$ with $G(x,x)$, obtained from (3.2) in the coincidence limit. In that expression the integration over the angular part of $\mathbf{k}$ is trivial. For the regularization, we can introduce a cutoff function $F(\alpha,\gamma_n)$ with a regularization parameter α, $F(0,\gamma_n)=1$ (for example, $F(x)=e^{-\alpha x}$, $\alpha>0$), and then apply Eq. (3.4) for the summation over n. For points outside the plates, the boundary-induced contribution in the VEV of the field squared is finite and the limit $\alpha\to 0$ can be put directly. The corresponding result for the boundary-induced part will coincide with the last term in Eq. (4.1). Another regularization procedure for the VEVs is the local zeta function technique (see, for instance, [17] and references therein). In the formula for the VEV $\langle\phi^2\rangle$ we can introduce the factor γ_n^{-s}. For sufficiently large Res, the corresponding expression is finite. For the analytic continuation to the physical value $s=0$, we can again use Eq. (3.4). Now, in the generalized Abel-Plana formula the singular points $\pm ik$ should be excluded by small semicircles in the right-half plane. For points away from the plates, the additional contributions to the boundary-induced parts coming from the corresponding integrals vanish in the limit $s\to 0$. The boundary-induced parts are finite for $s=0$ and this value can be directly substituted in the integrand with the results in agreement with those we have displayed before.

B. Energy-momentum tensor

Another important characteristic of the vacuum state is the VEV of the energy-momentum tensor, $\langle 0|T_{\mu\nu}|0\rangle\equiv\langle T_{\mu\nu}\rangle$. Given the Hadamard function and the VEV of the field squared, it is evaluated by using the formula

$$\langle T_{\mu\nu}\rangle=\frac{1}{2}\lim_{x'\to x}\partial_\mu\partial'_\nu G(x,x') +[(\xi-1/4)g_{\mu\nu}\nabla_l\nabla^l-\xi\nabla_\mu\nabla_\nu-\xi R_{\mu\nu}]\langle\phi^2\rangle, \tag{4.12}$$

where for the Ricci tensor one has

$$R_{00}=D\tilde{H}',\qquad R_{ii}=-\tilde{H}'-(D-1)\tilde{H}^2, \tag{4.13}$$

with $i=1,2,\ldots,D$, and the off-diagonal components vanish.

By taking into account the expression (3.13) for the Hadamard function, the diagonal components of the vacuum energy-momentum tensor are presented as (no summation over ν)

$$\langle T_\nu^\nu\rangle = \langle T_\nu^\nu\rangle_0 + \frac{A_D}{z_0 a^2}\int_0^\infty dk k^{D-2}\int_0^\infty du \times \frac{2F_\nu(\eta,u,k) + G_\nu(\eta,u,k)\sum_{j=1,2}c_j(u)e^{2u|z-z_j|/z_0}}{c_1(u)c_2(u)e^{2u}-1}, \tag{4.14}$$

where $\langle T_\nu^\nu\rangle_0$ is the corresponding VEV in the boundary-free geometry. In (4.14), we have defined the functions

$$\begin{aligned} F_0(\eta,u,k) &= W_0(\eta,u,k) - \hat{P}W(\eta,\eta,u,k),\\ F_l(\eta,u,k) &= [-\hat{P}_1 - k^2/(D-1)]W(\eta,\eta,u,k),\\ F_D(\eta,u,k) &= [-\hat{P}_1 + (u/z_0)^2]W(\eta,\eta,u,k), \end{aligned} \tag{4.15}$$

for $l = 1,\ldots,D-1$, and

$$G_\nu(\eta,u,k) = F_\nu(\eta,u,k) + b_\nu(u/z_0)^2 W(\eta,\eta,u,k), \tag{4.16}$$

where $b_\nu = 1-4\xi$ for $\nu \neq D$, $b_D = -1$. For the operators in (4.15), one has

$$\begin{aligned} \hat{P} &= (1/4)\partial_\eta^2 - D(\xi-\xi_D)\tilde{H}\partial_\eta + D\xi\tilde{H}',\\ \hat{P}_1 &= \left(\frac{1}{4}-\xi\right)\partial_\eta^2 + \left[\frac{D-1}{4}-(D-2)\xi\right]\tilde{H}\partial_\eta\\ &\quad + \xi[\tilde{H}' + (D-1)\tilde{H}^2], \end{aligned} \tag{4.17}$$

and

$$W_0(\eta,u,k) = \lim_{\eta'\to\eta}\partial_\eta\partial_{\eta'}W(\eta,\eta',u,k). \tag{4.18}$$

Due to the homogeneity of the background spacetime, the boundary-free contribution $\langle T_\nu^\nu\rangle_0$ to (4.14) does not depend on the spatial point (for the VEV of the energy-momentum tensor in boundary-free FRW cosmologies see, for instance, [1] and Refs. [18] for more recent discussions).

By using Eq. (2.10), it can be seen that

$$W_0(\eta,u,k) = \left[\frac{1}{2}\partial_\eta^2 + \frac{D-1}{2}\tilde{H}\partial_\eta + k^2 - \frac{u^2}{z_0^2} + a^2(m^2+\xi R)\right]W(\eta,\eta,u,k). \tag{4.19}$$

Substituting this into (4.15), we get an alternative expression for the function $F_0(\eta,u,k)$:

$$F_0(\eta,u,k) = (\hat{P}_0 + k^2 - u^2/z_0^2)W(\eta,\eta,u,k), \tag{4.20}$$

with the operator

$$\hat{P}_0 = \frac{1}{4}\partial_\eta^2 + D(\xi+\xi_D)\tilde{H}\partial_\eta + a^2m^2 + D\xi[\tilde{H}' + (D-1)\tilde{H}^2]. \tag{4.21}$$

Note that one has $G_D(\eta,u,k) = -\hat{P}_1 W(\eta,\eta,u,k)$ and this function vanishes for the Minkowski bulk. Hence, in the latter geometry the normal stress is homogeneous. In general, this is not the case for the FRW background.

The problem under consideration is inhomogeneous along the t and z directions. As a consequence of that, in addition to the diagonal components, the vacuum energy-momentum tensor has a nonzero off-diagonal component

$$\langle T_0^D\rangle = -\frac{A_D}{z_0 a^2}\int_0^\infty dk k^{D-2}\int_0^\infty du u \times \frac{\sum_{j=1,2}c_j(u)(-1)^{j-1}e^{2u|z-z_j|/z_0}}{c_1(u)c_2(u)e^{2u}-1}G_{0D}(\eta,u,k), \tag{4.22}$$

with the notation

$$G_{0D}(\eta,u,k) = [(1/2-2\xi)\partial_\eta + 2\xi\tilde{H}]W(\eta,\eta,u,k). \tag{4.23}$$

This corresponds to the energy flux along the direction perpendicular to the plates. If the Robin coefficients for the boundaries are the same, one has $c_1(u) = c_2(u)$. In this special case, the energy flux $\langle T_0^D\rangle$ vanishes at $z=(z_1+z_2)/2$ and has opposite signs in the regions $z < (z_1+z_2)/2$ and $z > (z_1+z_2)/2$. Note that we have the relation

$$\partial_\eta(a^{D-1}G_{0D}(\eta,u,k)) = -a^{D+1}G_D(\eta,u,k), \tag{4.24}$$

between the functions in the expressions for the normal stress and the energy flux.

In the regions $z < z_1$ and $z > z_2$, for the VEV of the energy-momentum tensor one has (no summation over ν)

$$\begin{aligned} \langle T_\nu^\nu\rangle_j &= \langle T_\nu^\nu\rangle_0 + \frac{A_D}{a^2}\int_0^\infty dk k^{D-2} \times \int_0^\infty dy \frac{\beta_j y+1}{\beta_j y-1}\frac{G_\nu(\eta,yz_0,k)}{e^{2y|z-z_j|}},\\ \langle T_0^D\rangle_j &= -\frac{(-1)^j A_D}{a^2}\int_0^\infty dk k^{D-2} \times \int_0^\infty dy y \frac{\beta_j y+1}{\beta_j y-1}\frac{G_{0D}(\eta,yz_0,k)}{e^{2y|z-z_j|}}. \end{aligned} \tag{4.25}$$

with $j = 1$ and $j = 2$, respectively.

By taking into account that

$$\sum_{\nu=0}^{D}F_\nu(\eta,u,k) = \{D(\xi-\xi_D)[\partial_\eta^2 + (D-1)\tilde{H}\partial_\eta] + a^2m^2\}W(\eta,\eta,u,k), \tag{4.26}$$

it can be explicitly checked that the boundary-induced contributions in (4.14) and (4.25), $\langle T_\nu^\nu\rangle_{\rm b} = \langle T_\nu^\nu\rangle - \langle T_\nu^\nu\rangle_0$, obey the trace relation

$$\langle T_\mu^\mu\rangle_{\rm b} = [D(\xi-\xi_D)\nabla_\mu\nabla^\mu + m^2]\langle\phi^2\rangle_{\rm b}, \qquad (4.27)$$

where $\langle\phi^2\rangle_{\rm b} = \langle\phi^2\rangle - \langle\phi^2\rangle_0$ is the boundary-induced part in the VEV of the field squared. For a conformally coupled massless field, the boundary-induced contribution in the VEV of the energy-momentum tensor is traceless. The trace anomaly is contained in the boundary-free part only. As an additional check, we can see that the boundary-induced VEVs satisfy the covariant conservation equation $\nabla_\mu\langle T_\nu^\mu\rangle_{\rm b} = 0$. For the geometry under consideration, it is reduced to the following two equations:

$$\frac{1}{a^{D+1}}\partial_\eta(a^{D+1}\langle T_0^0\rangle_{\rm b}) + \partial_z\langle T_0^D\rangle_{\rm b} - \tilde{H}\langle T_\mu^\mu\rangle_{\rm b} = 0,$$
$$\partial_z\langle T_D^D\rangle_{\rm b} - \frac{1}{a^{D+1}}\partial_\eta(a^{D+1}\langle T_0^D\rangle_{\rm b}) = 0. \qquad (4.28)$$

In particular, the second equation directly follows from the relation (4.24). This equation shows that the inhomogeneity of the normal stress is related to the nonzero energy flux along the direction normal to the plates.

Equivalent representations for the VEVs of the energy-momentum tensor are obtained by using the relation (4.6). In the way similar to that we have used for the VEV of the field squared, for the diagonal components one gets (no summation over ν)

$$\langle T_\nu^\nu\rangle = \langle T_\nu^\nu\rangle_0 + \frac{A_D}{z_0^D a^2}\int_0^\infty du\frac{u^{D-1}}{c_1(u)c_2(u)e^{2u}-1} \times\bigg\{2Z_\nu(\eta,u) + [Z_\nu(\eta,u) + b_\nu(u/z_0)^2 Z(\eta,u)] \times\sum_{j=1,2}c_j(u)e^{2u|z-z_j|/z_0}\bigg\}, \qquad (4.29)$$

with the functions

$$Z_0(\eta,u) = \hat{P}_0 Z(\eta,u) - u^2 Y(\eta,u)/z_0^2,$$
$$Z_l(\eta,u) = \frac{u^2/z_0^2}{D-1}Y(\eta,u) - \left(\hat{P}_1 + \frac{u^2/z_0^2}{D-1}\right)Z(\eta,u),$$
$$Z_D(\eta,u) = (u^2/z_0^2 - \hat{P}_1)Z(\eta,u), \qquad (4.30)$$

and

$$Y(\eta,u) = \int_0^1 ds s^3(1-s^2)^{(D-3)/2}U(\eta,us). \qquad (4.31)$$

For the off-diagonal component, we find

$$\langle T_0^D\rangle = -\frac{A_D}{z_0^D a^2}\int_0^\infty du u^D\frac{\sum_{j=1,2}c_j(u)(-1)^{j-1}e^{2u|z-z_j|/z_0}}{c_1(u)c_2(u)e^{2u}-1} \times[(1/2-2\xi)\partial_\eta + 2\xi\tilde{H}]Z(\eta,u). \qquad (4.32)$$

The dependence of the VEVs on the background geometry enters through the functions $Z(\eta,u)$ and $Y(\eta,u)$.

In the regions $z<z_1$ and $z>z_2$, the alternative expressions for the VEVs are given by

$$\langle T_\nu^\nu\rangle_j = \langle T_\nu^\nu\rangle_0 + \frac{A_D}{a^2}\int_0^\infty du u^{D-1} \times\frac{\beta_j u+1}{\beta_j u-1}\frac{Z_\nu(\eta,uz_0) + b_\nu u^2 Z(\eta,uz_0)}{e^{2u|z-z_j|}},$$
$$\langle T_0^D\rangle_j = -\frac{(-1)^j A_D}{a^2}\int_0^\infty du u^D \times\frac{\beta_j u+1}{\beta_j u-1}\frac{[(1/2-2\xi)\partial_\eta + 2\xi\tilde{H}]Z(\eta,uz_0)}{e^{2u|z-z_j|}}, \qquad (4.33)$$

with $j=1$ and $j=2$, respectively. Note that for the Minkowski bulk the normal stress in the geometry of a single plate vanishes.

Under the adiabatic condition (2.22), we can find simple asymptotic expressions of the VEVs near the boundaries for general case of the scale factor. By taking into account that the dominant contribution to the integral in (4.29) comes from large values of u and using the asymptotic expression (4.10), near the plate at $z=z_j$, to the leading order one finds (no summation over ν)

$$\langle T_\nu^\nu\rangle \approx (2\delta_{0\beta_j}-1)\frac{D\Gamma((D+1)/2)(\xi-\xi_D)}{2^D\pi^{(D+1)/2}(a|z-z_j|)^{D+1}}, \qquad (4.34)$$

for $\nu = 0,1,\ldots,D-1$. For the normal stress, the leading term vanishes and it is needed to keep the next-to-leading term. It is more convenient to find the corresponding asymptotic expression by using the second equation in (4.28) and the asymptotic expression for the energy flux. For the latter from (4.32) and (4.10), we get

$$\langle T_0^D\rangle \approx (2\delta_{0\beta_j}-1)\frac{2(-1)^j D(\xi-\xi_D)H}{(4\pi)^{(D+1)/2}(a|z-z_j|)^D}\Gamma((D+1)/2). \qquad (4.35)$$

Combining this with (4.28), one obtains the asymptotic for the normal stress:

$$\langle T_D^D\rangle \approx (1-2\delta_{0\beta_j})\frac{D(\xi-\xi_D)\Gamma((D-1)/2)}{(4\pi)^{(D+1)/2}(a|z-z_j|)^{D-1}}\frac{\ddot{a}}{a}. \qquad (4.36)$$

The leading terms in the near-plate asymptotic expansions for the diagonal components with $\nu\neq D$, given by (4.34), coincide with the corresponding expressions in the Minkowski bulk, with the distance $|z-z_j|$ replaced by the proper distance $a(\eta)|z-z_j|$. For the Minkowski bulk, the normal stress $\langle T_D^D\rangle$ does not depend on the coordinate z. This property is already seen from the second equation in (4.28), by taking into account that in the Minkowski bulk $\langle T_0^D\rangle = 0$. Hence, we see that the cosmological expansion essentially changes the behavior of the normal stress. In particular, near the plates the normal stress has different signs for accelerated and deccelerated expansions. Eqs. (4.34)–(4.36) present the leading-order terms in the asymptotic expansions of the VEVs over the distance from the plate $z = z_j$. These leading terms do not depend on the field mass and vanish for a conformally coupled field. The next-to-leading order terms, in general, will depend on the mass they do not vanish in the conformally coupled case.

As is seen from Eqs. (4.34)–(4.36), the VEV of the energy-momentum tensor diverges on the boundaries. These types of divergences are well known in quantum field theory with boundaries and they have been investigated for various bulk and boundary geometries. For cosmological backgrounds, an essential difference from the corresponding problem in the Minkowski bulk is that the normal stress diverges on the boundary. For the Minkowski bulk, it remains finite everywhere. Moreover, the corresponding VEV does not depend on z in the region between the plates and vanishes in the regions $z < z_1$ and $z > z_2$. From Eqs. (4.34)–(4.36) it follows that near the plates the VEVs for a field with $\xi \neq \xi_D$ have opposite signs for Dirichlet ($\beta_j = 0$) and non-Dirichlet boundary conditions.

On the base of the results given above, we can investigate the vacuum densities induced by a thick domain wall in the background of FRW spacetime. This is done in the way similar to that used in [19] for a thick brane on the anti–de Sitter bulk. For a thick domain wall with the thickness $2b$, we write the line element for the interior geometry in the form $ds^2 = a^2(\eta)[e^{u(z)}d\eta^2 - e^{v(z)}d\mathbf{x}_\parallel^2 - e^{w(z)}dz^2]$, $|z| < b$. In the regions $|z| > b$, the line element is given by (2.1). The functions $u(z)$, $v(z)$ and $w(z)$ are continuous on the boundaries $z = -b$ and $z = b$. For the symmetric domain wall, these functions are even functions of z. It can be shown that (the details will be presented elsewhere) the VEVs in the region $z > b$ are given by the expressions (4.3) and (4.25) with $z_j = b$ and with the Robin coefficient β_j being a function of the quantum numbers k and λ. This function is determined by the matching conditions for the scalar field modes in the interior and exterior regions.

C. The Casimir force

In the geometry of a single plate the vacuum pressures on the right- and left-hand sides of the plate compensate each other and the corresponding net force is zero. Consequently, for the two plates geometry, the resulting force per unit surface is determined by the part in the normal stress $\langle T_D^D \rangle$ induced by the second plate:

$$P_j = -(\langle T_D^D \rangle - \langle T_D^D \rangle_j)|_{z=z_j}, \tag{4.37}$$

where $\langle T_D^D \rangle$ is the normal stress in the region between the plates. By taking into account the expressions given above, we get

$$P_j = \frac{A_D}{z_0 a^2} \int_0^\infty dk k^{D-2} \times \int_0^\infty du \frac{[2 + c_j(u) + 1/c_j(u)]\hat{P}_1 - 2u^2/z_0^2}{c_1(u)c_2(u)e^{2u} - 1} \times W(\eta, \eta, u, k). \tag{4.38}$$

The force is attractive for $P_j < 0$ and repulsive for $P_j > 0$. In the problem on the Minkowski bulk one has $\hat{P}_1 W(\eta, \eta, u, k) = 0$ and the first term in the numerator of the integrand in (4.38) vanishes. Hence, the Casimir force for the Minkowski bulk is the same for both the plates, regardless of the values of the coefficients in the Robin boundary conditions. This is not the case for general FRW spacetime.

An alternative representation for the Casimir force is obtained by using the expression (4.29) for the normal stress:

$$P_j = \frac{A_D}{z_0^D a^2} \int_0^\infty du u^{D-1} \times \frac{[2 + c_j(u) + 1/c_j(u)]\hat{P}_1 - 2u^2/z_0^2}{c_1(u)c_2(u)e^{2u} - 1} Z(\eta, u), \tag{4.39}$$

with the function $Z(\eta, u)$ defined by (4.8). Depending on the Robin coefficients and on the vacuum state, the forces corresponding to (4.39) can be either attractive or repulsive. In particular, one can have the situation when the forces are repulsive at small separations between the plates and attractive at large separation.

Assuming that the scalar modes satisfy the adiabatic condition with the small wavelength asymptotic (2.22), we can find the asymptotic of the Casimir force at small separation between the plates. Under the assumption $1/(az_0) \gg m, \sqrt{|\dot{H}|}, H$, the dominant contribution in (4.39) comes from the second term in the numerator of the integrand. By using the asymptotic (4.10), to the leading order one gets

$$P_j \approx -\frac{2(4\pi)^{-D/2}}{(z_0 a)^{D+1}\Gamma(D/2)} \int_0^\infty du \frac{u^D}{c_1(u)c_2(u)e^{2u} - 1}. \tag{4.40}$$

The expression in the right-hand side coincides with the Casimir pressure for the plates in the Minkowski spacetime for a massless scalar field.

V. SPECIAL CASES

In this section, we consider some special cases of the general results given above. For the Minkowski bulk $a(t) = 1$ and for the modes realizing the standard Minkwoski vacuum, one has $f(\eta,\gamma) = e^{-i\omega\eta}/\sqrt{2\omega}$ with $\omega = \sqrt{\gamma^2 + m^2}$, and $w(\eta, \eta, \gamma) = 1/\omega$. From here it follows that

$U(\eta, z_0 x) = 0$ for $x < m$ and $U(\eta, z_0 x) = 2/\sqrt{x^2 - m^2}$ for $x > m$. For the function appearing in the expressions (4.1), (4.14) and (4.38), one gets $W(\eta, \eta, u, k) = 0$ for $u < z_0\sqrt{k^2 + m^2}$ and

$$W(\eta, \eta, u, k) = \frac{2}{\sqrt{u^2/z_0^2 - k^2 - m^2}} \tag{5.1}$$

for $u > z_0\sqrt{k^2 + m^2}$. In this special case, the function $Z(\eta, u)$ in the expressions for the VEVs is simplified to

$$Z(\eta, u) = \frac{\sqrt{\pi}\Gamma((D-1)/2)z_0}{\Gamma(D/2)u}[1 - (z_0 m/u)^2]^{D/2-1}, \tag{5.2}$$

for $u \geq z_0 m$ and $Z(\eta, u) = 0$ for $u < z_0 m$. Substituting the expression (5.2) into the general formulas given above, we obtain the VEVs for the Robin plates in Minkowski spacetime (see [14] for the VEVs in the massless case and [20] for a massive scalar field. Note that in [20] the VEVs in the Minkowski bulk are obtained as a limiting case of the corresponding problem with two uniformly accelerated plates moving through the Fulling-Rindler vacuum state).

A. Conformally coupled massless field

For a conformally coupled massless field, one has $\xi = \xi_D$ and $m = 0$. As it follows from (2.12), the general solution for the function $f(\eta, \gamma)$ has the form

$$f(\eta, \gamma) = \frac{a^{(1-D)/2}}{\sqrt{2\gamma}}(c_1 e^{-i\gamma\eta} + c_2 e^{i\gamma\eta}), \tag{5.3}$$

where the factor $1/\sqrt{2\gamma}$ is extracted for the further convenience. One of the coefficients is determined by the normalization condition, whereas the second one is fixed by the choice of the vacuum state. For a vacuum state, we will take the state corresponding to the standard Minkowskian vacuum in the adiabatic limit $a(\eta) = \text{const}$. This corresponds to the choice $c_2 = 0$ and from the normalization condition (2.18) one gets $|c_1|^2 = 1$.

For the function $W(\eta, \eta', u, k)$ in the expressions of the VEVs, we find $W(\eta, \eta', u, k) = 0$ in the region $u < z_0 k$ and

$$W(\eta, \eta', u, k) = 2a^{1-D}\frac{\cosh[(\eta - \eta')\sqrt{u^2/z_0^2 - k^2}]}{\sqrt{u^2/z_0^2 - k^2}}, \tag{5.4}$$

for $u > z_0 k$ and, hence, $U(\eta, x) = 2a^{1-D}z_0/x$. From (4.7), for the VEV of the field squared one finds

$$\langle\phi^2\rangle = \langle\phi^2\rangle_0 + \frac{(az_0)^{1-D}}{(4\pi)^{D/2}\Gamma(D/2)} \times \int_0^\infty du u^{D-2}\frac{2 + \sum_{j=1,2}c_j(u)e^{2u|z-z_j|/z_0}}{c_1(u)c_2(u)e^{2u} - 1}. \tag{5.5}$$

In a similar way, we can see that the VEV of the energy-momentum tensor takes the form

$$\langle T_\nu^\mu\rangle = \langle T_\nu^\mu\rangle_0 - \frac{\text{diag}(1, 1, \ldots, 1, -D)}{(4\pi)^{D/2}\Gamma(D/2+1)(az_0)^{D+1}} \times \int_0^\infty du\frac{u^D}{c_1(u)c_2(u)e^{2u} - 1}. \tag{5.6}$$

In this special case the vacuum energy-momentum tensor is spatially homogeneous and diagonal. Of course, the boundary-induced contributions in (5.5) and (5.6) could be directly obtained from the corresponding expressions in the Minkowski bulk by using the standard result for conformally related problems (see, for instance, [1]).

B. de Sitter bulk

As a next application we consider the case of de Sitter bulk with $a(t) = e^{Ht}$, $H = \text{const}$ (the renormalized expectation value of the energy-momentum tensor for an arbitrary homogeneous and isotropic physical initial state of a scalar field in de Sitter spacetime, in the absence of boundaries, has been investigated in [21]). The corresponding scale factor in conformal time has the form $a(\eta) = -1/(H\eta)$ with $-\infty < \eta \leq 0$. In this case one has $\tilde{H} = -1/\eta$ and $R = D(D+1)H^2$. The general solution of Eq. (2.10) is the linear combination of the functions $|\eta|^{D/2}H_\nu^{(1)}(\gamma|\eta|)$ and $|\eta|^{D/2}H_\nu^{(2)}(\gamma|\eta|)$, with $H_\nu^{(1,2)}(x)$ being the Hankel functions and

$$\nu = \sqrt{D^2 - 4D(D+1)\xi - m^2/H^2}. \tag{5.7}$$

For further convenience, we write the Hankel functions in terms of the Macdonald function $K_\nu(x)$ [22]:

$$f(\eta, \gamma) = \frac{|\eta|^{D/2}}{\sqrt{\pi}\alpha^{(D-1)/2}}[d_1 K_\nu(\gamma|\eta|e^{-\pi i/2}) + d_2 K_\nu(\gamma|\eta|e^{\pi i/2})], \tag{5.8}$$

where the parameter ν is either positive or purely imaginary. From the condition (2.18) we get the relation

$$|d_1|^2 - |d_2|^2 = 1, \tag{5.9}$$

between the coefficients.

By using the relation [22]

$$K_\nu(\gamma|\eta|e^{\pm\pi i}) = e^{\mp\nu\pi i}K_\nu(\gamma|\eta|) \mp \pi i I_\nu(\gamma|\eta|), \tag{5.10}$$

for the function appearing in the expressions of the VEVs we get $W(\eta, \eta', u, k) = 0$ for $u < kz_0$ and

$$W(\eta,\eta',u,k) = 2H^{D-1}|\eta\eta'|^{D/2}\{i\pi(d_1d_2^* - d_1^*d_2)I_\nu(y)I_\nu(y') + (|d_1|^2 + |d_2|^2 + e^{\nu\pi i}d_1d_2^* + e^{-\nu\pi i}d_1^*d_2)[I_{-\nu}(y)K_\nu(y') + I_\nu(y')K_\nu(y)]\}, \tag{5.11}$$

for $u > kz_0$, where

$$y = |\eta|\sqrt{u^2/z_0^2 - k^2}, \qquad y' = |\eta'|\sqrt{u^2/z_0^2 - k^2}. \tag{5.12}$$

For the function in (5.11), one has

$$I_{-\nu}(y)K_\nu(y') + I_\nu(y')K_\nu(y) = -\frac{\pi}{2}\frac{I_\nu(y)I_\nu(y') - I_{-\nu}(y)I_{-\nu}(y')}{\sin(\nu\pi)}, \tag{5.13}$$

which shows that this function is real for both the real and purely imaginary values for ν.

Note that in the expressions of the VEVs only the relative phase of the coefficients d_1 and d_2 is relevant and, hence, by taking into account the relation (5.9), we can take the parametrization

$$d_1 = \cosh\alpha, \qquad d_2 = e^{i\beta}\sinh\alpha, \tag{5.14}$$

in terms of new real parameters α and $0 \le \beta < 2\pi$. With this parametrization, for the function (5.11) one gets

$$W(\eta,\eta',u,k) = 2H^{D-1}|\eta\eta'|^{D/2}\{\pi\sinh(2\alpha)\sin\beta I_\nu(y)I_\nu(y') + [\cosh(2\alpha) + \sinh(2\alpha)\cos(\beta - \nu\pi)] \times [I_{-\nu}(y)K_\nu(y') + I_\nu(y')K_\nu(y)]\}. \tag{5.15}$$

The modes (5.8) correspond to the two-parameter (α,β) family of vacuum states in de Sitter spacetime. As it has been discussed in [23], in the absence of the plates the de Sitter invariant vacuum states correspond to $\beta = 0$. The Bunch-Davies (or Euclidean) vacuum state [24] is a special case of de Sitter invariant vacua and corresponds to $\alpha = 0$. In general, one has a one-parameter family of de Sitter invariant vacuum states specified by the parameter α (α states or α vacua in de Sitter space, for the discussion of the role of these states in inflationary models see, for example, [25]).

The transformations of the boundary-induced contributions in the VEVs, we have described above, are valid for dS invariant vacua only. In this special case the function (5.15) takes the form

$$W(\eta,\eta',u,k) = 2H^{D-1}b(\alpha)|\eta\eta'|^{D/2}[I_{-\nu}(y)K_\nu(y') + I_\nu(y')K_\nu(y)], \tag{5.16}$$

with

$$b(\alpha) = \cosh(2\alpha) + \sinh(2\alpha)\cos(\nu\pi). \tag{5.17}$$

From here it follows that the VEVs of the field squared and of the energy-momentum tensor for a dS invariant vacuum state with a given α are obtained from the corresponding VEVs in the Bunch-Davies vacuum state, investigated in [4], multiplying by the factor $b(\alpha)$. For real values of the parameter ν, this factor is always positive. For purely imaginary ν, the factor $b(\alpha)$ can be negative. In this case, compared with the Bunch-Davies vacuum state, the Casimir forces for the corresponding α vacuum change the sign.

VI. CONCLUSION

We have studied the scalar Casimir effect for the geometry of two parallel plates on the spatially flat FRW background for a general case of the scale factor. On the plates the field obeys the Robin boundary conditions (2.6) with the coefficients proportional to the scale factor. In the model under consideration, all the properties of the vacuum state are encoded in two-point functions and, as the first step in the investigation of the VEVs for physical observables bilinear in the field operator, we have evaluated the Hadamard function. By using the Abel-Plana-type summation formula for the eigenvalues of the quantum number λ, the boundary-induced contribution is explicitly extracted. This contribution in the geometry of a single plate and in the region between two plates is given by the last terms in (3.12) and (3.13), respectively. In the corresponding evaluation we have not fixed the vacuum state. In order to specify the vacuum state, an additional condition should be imposed on the function $f(\eta,\gamma)$ appearing in the expression (2.17) for the scalar modes. In particular, for the modes obeying the adiabatic condition this function has the small wavelength asymptotic (2.22). In the limit of a constant scale factor, these modes approach the positive energy solutions used for the quantization of a scalar field in the Minkowski bulk.

As important local characteristics of the vacuum state, we have considered the VEVs of the field squared and of the energy-momentum tensor. The VEV of the field squared is given by the expression (4.7) in the region between the plates and by (4.9) in the regions $z < z_1$ and $z > z_2$. For points away from the boundaries the renormalization is reduced to that for the boundary free-part $\langle\phi^2\rangle_0$. The information on the background geometry is encoded in the function $Z(\eta,u)$, defined by the relation (4.8). For the vacuum state realized by the modes obeying the adiabatic condition, the leading term in asymptotic expansion of the field squared near the plates is given by the expression (4.11). It is obtained from the corresponding asymptotic in the problem on Minkowski bulk replacing the distance from the plate by the proper distance. Near the plates the dominant contribution to the VEVs come from the modes with small wavelengths, the influence of the gravitational field on which is weak.

The diagonal components of the VEV of the energy-momentum tensor in the region between the plates are given by the formula (4.29), where the functions in the boundary-induced contribution are defined by (4.30). Unlike to the case of the Minkowski bulk, the corresponding normal stress is inhomogeneous. Another feature of the Casimir effect in the expanding bulk is the presence of the nonzero energy flux along the direction normal to the plates. This flux is described by the off-diagonal component of the vacuum energy-momentum tensor, given by the expression (4.32). Depending on the Robin coefficients and on the vacuum state, the flux can be either positive or negative. For boundaries with the same Robin coefficients, the energy flux vanishes on the plane $z = (z_1 + z_2)/2$ and has opposite signs in the right-hand and left-hand regions with respect to this plane. In the regions $z < z_1$ and $z > z_2$ the vacuum energy-momentum tensor coincides with that for the geometry of a single plate and is given by the formulas (4.33). The corresponding normal stress and the energy flux vanish in the Minkowskian limit. Under the adiabatic condition for the scalar modes, the leading term in the near-plate expansion of the diagonal components $\langle T_\nu^\nu \rangle$ for $\nu \neq D$ coincides with that for plates in the Minkowski spacetime. For the energy flux and the normal stress, the corresponding asymptotics are given by the expressions (4.35) and (4.36). In particular, for the normal stress the asymptotic behavior on the FRW bulk is completely different from that for the Minkowski spacetime. In the latter case the normal stress is finite on the plates.

The Casimir force per unit surface of the plate at $z = z_j$ is determined by the expression (4.38). An important difference from the corresponding result in the Minkowski bulk is that for a scalar field with $\beta_1 \neq \beta_2$ the forces acting on the right and left plates, in general, are different. Depending on the Robin coefficients and on the vacuum state under consideration, these forces can be either attractive or repulsive. Assuming that the modes used in the quantization procedure obey the adiabatic condition, for the leading term in the asymptotic expansion of the Casimir force at small distances between the plates one gets the expression (4.40).

In Section V, two special cases of general results are discussed. In the first example we have considered a conformally coupled massless field assuming that the field is prepared in the vacuum state that corresponds to the Minkowskian vacuum in the adiabatic limit. In this case, the boundary-induced contributions to the VEVs of the field squared and of the energy-momentum tensor are obtained from the corresponding VEVs in the Minkowski bulk by using the standard relation for conformally coupled problems. In particular, the vacuum energy-momentum tensor is diagonal. In the second example, the de Sitter spacetime is considered as a background geometry. For this geometry, one has a one-parameter family of the de Sitter invariant vacuum states specified by the real parameter α. The corresponding function $W(\eta, \eta', u, k)$, appearing in the expressions for the VEVs, is given by the expression (5.16) with the coefficient $b(\alpha)$ defined by (5.17). In the special case $\alpha = 0$, we obtain the results for the Bunch-Davies vacuum state previously discussed in the literature. For imaginary values of the parameter ν, depending on the parameter α, the Casimir forces for the α vacua may have opposite signs compared with the Bunch-Davies vacuum.

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[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982); A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, Vacuum Quantum Effects in Strong Fields (Friedmann Laboratory Publishing, St. Petersburg, 1994); S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, England, 1996); L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, England, 2009).

[2] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994); V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997); K. A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002); V. A. Parsegian, *Van der Waals*

Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists (Cambridge University Press, Cambridge, England, 2005); M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, New York, 2009).

[3] M. R. Setare and R. Mansouri, Classical Quantum Gravity **18**, 2659 (2001); M. R. Setare, Phys. Lett. B **637**, 1 (2006).

[4] A. A. Saharian and T. A. Vardanyan, Classical Quantum Gravity **26**, 195004 (2009); E. Elizalde, A. A. Saharian, and T. A. Vardanyan, Phys. Rev. D **81**, 124003 (2010); A. A. Saharian, Int. J. Mod. Phys. A **26**, 3833 (2011); P. Burda, JETP Lett. **93**, 632 (2011).

[5] A. A. Saharian, A. S. Kotanjyan, and H. A. Nersisyan, Phys. Lett. B **728**, 141 (2014); A. S. Kotanjyan, A. A. Saharian, and H. A. Nersisyan, Phys. Scr. **90**, 065304 (2015).

[6] K. Milton and A. A. Saharian, Phys. Rev. D **85**, 064005 (2012).

[7] A. A. Saharian and V. F. Manukyan, Classical Quantum Gravity **32**, 025009 (2015); A. A. Saharian, V. F. Manukyan, and N. A. Saharyan, Int. J. Mod. Phys. A **31**, 1650183 (2016).

[8] E. Elizalde, S. D. Odintsov, and A. A. Saharian, Phys. Rev. D **87**, 084003 (2013); E. R. Bezerra de Mello, A. A. Saharian, and M. R. Setare, Phys. Rev. D **92**, 104005 (2015); S. Bellucci, A. A. Saharian, and V. Vardanyan, Phys. Rev. D **93**, 084011 (2016).

[9] A. A. Saharian and M. R. Setare, Phys. Lett. B **687**, 253 (2010); Classical Quantum Gravity **27**, 225009 (2010).

[10] M. R. Setare and J. Sadeghi, Phys. Lett. B **680**, 94 (2009); M. R. Setare and M. Dehghani, Int. J. Mod. Phys. D **20**, 161 (2011).

[11] S. Bellucci and A. A. Saharian, Phys. Rev. D **88**, 064034 (2013).

[12] A. A. Saharian and A. L. Mkhitaryan, Eur. Phys. J. C **66**, 295 (2010).

[13] V. M. Mostepanenko and N. N. Trunov, Sov. J. Nucl. Phys. **42**, 818 (1985).

[14] A. Romeo and A. A. Saharian, J. Phys. A **35**, 1297 (2002).

[15] D. J. H. Chung, A. Notari, and A. Riotto, J. Cosmol. Astropart. Phys. 10 (2003) 012.

[16] A. A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); Report No. ICTP/2007/082; arXiv:0708.1187.

[17] V. Moretti, Phys. Rev. D **56**, 7797 (1997); Commun. Math. Phys. **201**, 327 (1999); A. A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti, and S. Zerbini, *Analytic Aspects of Quantum Fields* (World Scientific, Singapore, 2003); D. Fermi and L. Pizzocchero, arXiv:1505.00711.

[18] P. R. Anderson and W. Eaker, Phys. Rev. D **61**, 024003 (1999); A. Kaya and M. Tarman, J. Cosmol. Astropart. Phys. 04 (2011) 040; J. D. Bates and P. R. Anderson, Phys. Rev. D **82**, 024018 (2010).

[19] A. A. Saharian and A. L. Mkhitaryan, J. High Energy Phys. 08 (2007) 063.

[20] A. A. Saharian, R. M. Avagyan, and R. S. Davtyan, Int. J. Mod. Phys. A **21**, 2353 (2006).

[21] P. R. Anderson, W. Eaker, S. Habib, C. Molina-París, and E. Mottola, Phys. Rev. D **62**, 124019 (2000).

[22] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).

[23] B. Allen, Phys. Rev. D **32**, 3136 (1985); B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987).

[24] T. S. Bunch and P. C. W. Davies, J. Phys. A **11**, 1315 (1978).

[25] N. Kaloper, M. Kleban, A. Lawrence, S. Shenker, and L. Susskind, J. High Energy Phys. 11 (2002) 037; T. Banks and L. Mannelli, Phys. Rev. D **67**, 065009 (2003); J. de Boer, V. Jejjala, and D. Minic, Phys. Rev. D **71**, 044013 (2005).