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Quantum Hall effect on odd spheres

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We solve the Landau problem for charged particles on odd dimensional spheres S^{2k-1} in the background of constant $SO(2k-1)$ gauge fields carrying the irreducible representation $(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2})$. We determine the spectrum of the Hamiltonian, the degeneracy of the Landau levels and give the eigenstates in terms of the Wigner $\mathcal{D}$-functions, and for odd values of I, the explicit local form of the wave functions in the lowest Landau level (LLL). The spectrum of the Dirac operator on S^{2k-1} in the same gauge field background together with its degeneracies is also determined, and in particular, its number of zero modes is found. We show how the essential differential geometric structure of the Landau problem on the equatorial S^{2k-2} is captured by constructing the relevant projective modules. For the Landau problem on S^5, we demonstrate an exact correspondence between the union of Hilbert spaces of LLLs, with I ranging from 0 to $I_{\max}=2K$ or $I_{\max}=2K+1$ to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ or that of winding number ±1 line bundles over $\mathbb{C}P^3$ at level K, respectively.

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I. INTRODUCTION

In a recent article [1], the relationship between the A-class topological insulators (TIs) and the quantum Hall effect (QHE) on even dimensional spheres has been explored, and it has been recognized that A-class TIs can be realized as QHE on even dimensional spheres [1,2]. A-class TIs are not time-reversal invariant, appear in even dimensions, and can be characterized via an integer topological invariant; while AIII class are also not time reversal invariant, carry an integer topological invariant, but appear in odd dimensions. In addition, AIII-class TIs have chiral symmetry, whereas the A-class TIs do not [3]. Focusing on these connections between the A-class TIs and AIII-class TIs, in a subsequent article, Hasebe [4] considered the possibility of realizing the latter type in terms of a quantum Hall system in odd dimensions. Elaborating on the formulation of the QHE on the three sphere S^3, given by Nair & Daemi [5], Hasebe found that Nambu 3-algebraic geometry can be employed to realize the chiral symmetry of the TI in this setting and modeled the chiral TI as a superposition of two three spheres embedded in S^4 with the SU(2) background monopole fluxes, i.e., in the four-dimensional QHE of Hu and Zhang [6].

In the past decade or so, formulation of the QHE on higher-dimensional manifolds, and investigations on its several aspects, have been a continually appearing theme in contemporary theoretical physics. After the pioneering work of Hu & Zhang [6] in formulating the QHE problem on S^4, a multitude of articles have explored the formulation of the QHE on various higher-dimensional manifolds, such as $\mathbb{C}P^N$, the even-dimensional spheres S^{2k}, complex Grassmann manifolds $\boldsymbol{Gr}_2(\mathbb{C}^N)$, as well as on a particular flag manifold [2,7–11]. One motivation for their study is to understand the generalization of the massless excitations (chiral bosons), which are known to be present at the edge of the two-dimensional quantum Hall samples (see, for instance, [12]). However, it turns out that not only photons and gravitons, but somewhat undesirably even higher massless spin states occur at the edges, which are effectively described by chiral gauged Wess-Zumino-Witten (WZW) theories, and therefore have interesting physical content in their own right [13,14]. There are also strong motivations emerging from the physics of D-branes and strings, as certain configurations with open strings ending on D-branes have low energy limits, which are effectively described by the QHE on spheres [15,16]. The relationship between the matrix algebras describing fuzzy spaces, such as the fuzzy sphere S_F^2, higher-dimensional fuzzy spheres S_F^{2k}, fuzzy complex projective spaces $\mathbb{C}P_F^N$, and the Hilbert spaces of the lowest Landau level (LLL) of the QHE on aforementioned manifolds, have also been explored in the literature to shed further light into the geometrical structure of the LLL [17], while in the present work we will have the opportunity to present yet another facet of this relationship in a particular example. Thus, expanding upon these concrete developments, and along with the novel motivations emerging from the physics of TIs, in this paper, our aim is to investigate the formulation of the QHE on all odd-dimensional spheres S^{2k-1}.

As we have already noted, the QHE problem on S^3 is solved by Nair & Daemi [5], and a complementary

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treatment is recently given in Hasebe's work [4].[1] The clear path for the construction of the QHE over compact higher-dimensional manifolds appears to be closely linked to the coset space realization of such spaces. Indeed, odd spheres can also be realized as coset manifolds as $S^{2k-1} \equiv \frac{SO(2k)}{SO(2k-1)} \equiv \frac{Spin(2k)}{Spin(2k-1)}$. In their approach, Nair & Daemi took advantage of the fact that S^3 can also be realized as $S^3 \equiv \frac{SU(2)\times SU(2)}{SU(2)_D}$ owing to the isomorphisms $\frac{SU(2)\times SU(2)}{\mathbf{Z}_2} = SO(4)$ and $\frac{SU(2)}{\mathbf{Z}_2} = SO(3)$, and they subsequently constructed the Landau problem for a charged particle on S^3 under the influence of a constant $SU(2)_D$ gauge field background carrying an irreducible representation (IRR) of the latter. This quick approach is not immediately applicable to higher-dimensional odd spheres. Nevertheless, coset space realization of S^{2k-1}, in terms of the $SO(2k)/SO(2k-1)$, can be used to handle this problem.

A brief summary of our results and their organization in the present article is in order. In section II, we set up and solve the Landau problem for charged particles on odd-dimensional spheres S^{2k-1} in the background of constant $SO(2k-1)$ gauge fields carrying the irreducible representation $(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2})$. In particular, we determine the spectrum of the Hamiltonian, the degeneracy of the Landau levels, give the eigenstates in terms of the Wigner $\mathcal{D}$-functions, and for odd values of I, the explicit local form of the wave functions in the lowest Landau level. In this section, we also demonstrate in detail how the essential differential geometric structure of the Landau problem on the equatorial S^{2k-2} is captured by constructing the relevant projective modules and the related $SO(2k-2)$ valued curvature two forms. We illustrate our general results on the examples of S^3 and S^5 for concreteness, and in the latter case, we identify an exact correspondence between the union of Hilbert spaces of LLL's with I ranging from 0 to $I_{\max} = 2K$ or $I_{\max} = 2K+1$ to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ at level K or that of winding number ± 1 line bundles over $\mathbb{C}P^3$ at level K, respectively. In section III we determine the spectrum of the Dirac operator on S^{2k-1} in the same gauge field background together with its degeneracies and also compute the number of its zero modes. Some relevant formulas from the representation theory of groups is given in a short appendix for completeness.

II. LANDAU PROBLEM ON ODD SPHERES S^{2k-1}

A. Basic setup and the solution

In this section we aim to set up and solve the Schrödinger equation for charged particles on odd spheres, S^{2k-1}, under the influence of a constant background gauge field. We will give the spectrum of the appropriate Hamiltonian for the problem and determine the associated wave functions. In order to pose the problem in sufficient detail, we start with laying out some definitions and conventions that are going to be used throughout the paper.

A convenient way of specifying the coordinates on S^{2k-1} is to embed it in $\mathbb{R}^{2k}$. Then, $X_a \in \mathbb{R}^{2n}$, $a = (1, 2, \ldots, 2k)$, satisfying the condition $X_a X_a = R^2$, gives the coordinates of S^{2k-1} with radius R. The splitting of X_a into certain spinorial coordinates is going to be of essential interest in what follows. To see how this comes about, let us first note the well-known fact that the odd-dimensional spheres can be represented as the coset spaces

$$S^{2k-1} = SO(2k)/SO(2k-1), \tag{2.1}$$

and the generators of $SO(2k) \approx Spin(2k)$ may be given by

$$\Xi_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \qquad a, b = 1, 2, \ldots, 2k, \tag{2.2}$$

where Γ_a are the generators of the Clifford algebra in $2k$ dimensions. These are $2^k \times 2^k$ matrices satisfying the anticommutation relations $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. We will use the following representation of Γ_as in the present article:

$$\Gamma_\mu = \begin{pmatrix} 0 & -i\gamma_\mu \\ i\gamma_\mu & 0 \end{pmatrix}, \qquad \mu = 1, \ldots, 2k-1,$$
$$\Gamma_{2k} = \begin{pmatrix} 0 & \mathbb{I}_{2^{k-1}\times 2^{k-1}} \\ \mathbb{I}_{2^{k-1}\times 2^{k-1}} & 0 \end{pmatrix},$$
$$\Gamma_{2k+1} = \begin{pmatrix} -\mathbb{I}_{2^{k-1}\times 2^{k-1}} & 0 \\ 0 & \mathbb{I}_{2^{k-1}\times 2^{k-1}} \end{pmatrix}, \tag{2.3}$$

where γ_μ's are the generators of the Clifford algebra in $(2k-1)$ dimensions.

$SO(2k-1)$ is irreducibly generated by

$$\Sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]. \tag{2.4}$$

In fact, $\Sigma_{\mu\nu}$ specifies the irreducible fundamental spinor representation of $SO(2k-1)$, which is 2^{k-1} dimensional. For completeness, let us also indicate that Ξ_{ab} in (2.2) generates $SO(2k)$ reducibly; Ξ_{ab} has the block diagonal form

$$\Xi_{ab} = \begin{pmatrix} \Xi^+_{ab} & 0 \\ 0 & \Xi^-_{ab} \end{pmatrix}, \tag{2.5}$$

indicating that there are two irreducible fundamental representations, $\Xi^\pm_{ab} = (\Xi^\pm_{\mu\nu}, \Xi^\pm_{2k\mu}) = (\Sigma_{\mu\nu}, \mp\frac{1}{2}\gamma_\mu)$, each of dimension 2^{k-1}, generating $SO(2k)$.

[1] Other recent developments in solving the Landau problem and the Dirac-Landau problem in flat higher-dimensional spaces are reported in [18–20].

Let us introduce the 2^k-component spinor

$$\Psi = \frac{1}{\sqrt{2R(R+X_{2k})}}((R+X_{2k})\mathbb{1}_{2^k} + X_\mu \Gamma^\mu)\phi, \quad \Psi^\dagger \Psi = 1, \tag{2.6}$$

where $\mathbb{1}_{2^k}$ stands for a $2^k \times 2^k$ unit matrix and $\phi = \frac{1}{\sqrt{2}}\binom{\tilde{\phi}}{\tilde{\phi}}$, with $\tilde{\phi}$ being a normalized 2^{k-1}-component spinor. It is straightforward to check that Ψ gives us the desired fractionalization, or the "square root" of X_a, via the Hopf-like projection map

$$\frac{X_a}{R} = \Psi^\dagger \Gamma_a \Psi. \tag{2.7}$$

Using the spinor introduced in (2.6), we can construct the spin connection over S^{2k-1}, i.e., the $SO(2k-1)$ gauge field as

$$A = \Psi^\dagger d\Psi, \tag{2.8}$$

whose components are determined to be

$$A_\mu = -\frac{1}{R(R+X_{2k})}\Sigma_{\mu\nu}X_\nu, \qquad A_{2k} = 0. \tag{2.9}$$

Using the covariant derivatives $D_a = \partial_a + iA_a$ and (2.9), components of the field strength

$$F_{ab} = -i[D_a, D_b] = \partial_a A_b - \partial_b A_a + i[A_a, A_b], \tag{2.10}$$

are given as

$$F_{\mu\nu} = \frac{1}{R^2}(X_\nu A_\mu - X_\mu A_\nu + \Sigma_{\mu\nu}), \quad F_{2k\mu} = -\frac{R+X_{2k}}{R^2}A_\mu. \tag{2.11}$$

We find that

$$R^4 \sum_{a<b} F_{ab}^2 = \sum_{\mu<\nu}\Sigma_{\mu\nu}^2. \tag{2.12}$$

The rhs of (2.12) is the Casimir of $SO(2k-1)$, and thus proportional to the identity in an irreducible representation. Thus, a natural choice for a constant gauge field background is the spinor representation given by the highest weight labels [21]

$$\left(\frac{I}{2}\right) \equiv \underbrace{\left(\frac{I}{2}, \ldots, \frac{I}{2}\right)}_{(k-1)\text{terms}}, \qquad I \in \mathbb{Z}, \tag{2.13}$$

since $SO(2k-1)$ is of rank $k-1$. We observe that $(\frac{I}{2}, \ldots, \frac{I}{2})$ can be obtained from the I-fold symmetric tensor product of the fundamental spinor representation $(\frac{1}{2}, \ldots, \frac{1}{2})$. It should readily be understood from the context, which IRR of $SO(2k-1)$ that $\Sigma_{\mu\nu}$ carries; thus in (2.4), this is the 2^{k-1}-dimensional fundamental spinor representation $(\frac{1}{2}, \ldots, \frac{1}{2})$, while in what follows we are going to take it to be in the IRR $(\frac{I}{2}, \ldots, \frac{I}{2})$ due to the reasons just argued.

We can write down the Hamiltonian for a charged particle on S^{2k-1} under the influence of the constant $SO(2k-1)$ gauge field background introduced in the preceding paragraph as

$$H = \frac{\hbar}{2MR^2}\sum_{a<b}\Lambda_{ab}^2, \tag{2.14}$$

where Λ_{ab} are the operators given as

$$\Lambda_{ab} = -i(X_a D_b - X_b D_a), \tag{2.15}$$

which are parallel to the tangent bundle over S^{2k-1}. Commutators of Λ_{ab} give

$$\begin{aligned}[\Lambda_{ab}, \Lambda_{cd}] = {} & i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}) \\ & - i(X_a X_c F_{bd} + X_b X_d F_{ac} - X_b X_c F_{ad} \\ & - X_a X_d F_{bc}),\end{aligned} \tag{2.16}$$

which are not the $SO(2k)$ commutation relations. The reason for Λ_{ab} failing to satisfy the $SO(2k)$ commutation relations is that they just account for the angular momentum of a charged particle on S^{2k-1}, which is not the total angular momentum in the present problem, since the background gauge field also carries angular momentum. Thus, the total angular momentum operators, generating the $SO(2k)$ rotations can be constructed by supplementing Λ_{ab} with the spin angular momentum of the background gauge field by writing

$$L_{ab} = \Lambda_{ab} + R^2 F_{ab}. \tag{2.17}$$

In component form we find

$$L_{\mu\nu} = L_{\mu\nu}^{(0)} + \Sigma_{\mu\nu}, \qquad L_{2k\mu} = L_{2k\mu}^{(0)} - RA_\mu, \tag{2.18}$$

where $L_{ab}^{(0)} = -i(X_a\partial_b - X_b\partial_a)$ are the generators of $SO(2k)$ over S^{2k-1}. In the absence of a magnetic background, $L_{ab}^{(0)}$ would be the generators of angular momentum for a particle on S^{2k-1} and it would be the total angular momentum in that case. In the present case, a straightforward calculation yields

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc}), \tag{2.19}$$

as expected.

Using (2.17), and the fact that Λ_{ab} and F_{ab} are orthogonal, i.e., $\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab} = 0$, we can write (2.14) as

TABLE I. Branching of $SO(2k)$ under $SO(2k-1)$.

$SO(2k)$	$\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k$
$SO(2k-1)$	$\mu_1, \mu_2, \ldots, \mu_{k-1}$
	$\lambda_1 \geq \mu_1 \geq \lambda_2 \cdots \geq \mu_{k-1} \geq \lvert\lambda_k\rvert$

$$H = \frac{\hbar}{2MR^2}\left(\sum_{a<b} L_{ab}^2 - \sum_{\mu<\nu} \Sigma_{\mu\nu}^2\right). \tag{2.20}$$

In order to obtain the spectrum of this Hamiltonian, we have to determine the general form of the IRR of $SO(2k)$ that L_{ab} could carry, given that $\Sigma_{\mu\nu}$ carries the $(\frac{I}{2})$ of $SO(2k-1)$. This problem can be addressed by looking at the branching of $SO(2k)$ IRRs in terms of those of $SO(2k-1)$ [21]. Consider Table I, where the first row indicates a generic IRR of $SO(2k)$ labeled by integers or half odd integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ corresponding respectively to tensor and spinor representations with $\lambda_1 \geq \lambda_2 \cdots \geq |\lambda_k|$, where the last entry λ_k could be positive, negative, or zero and satisfies $|\lambda_k| \geq 0$ for the former and $|\lambda_k| \geq \frac{1}{2}$ for the latter case. IRRs of $SO(2k)$ with opposite sign of λ_k are conjugate representations. The second row stands for the $(\mu_1, \mu_2, \ldots, \mu_{k-1})$ IRR of $SO(2k-1)$, where μ_i are integers or half odd integers satisfying $\mu_1 \geq \mu_2 \cdots \geq \mu_{k-1}$ and $\mu_{k-1} \geq 0$ or $\mu_{k-1} \geq \frac{1}{2}$, respectively. The third line gives the branching rule [21]. Accordingly, for $(\frac{I}{2})$ of $SO(2k-1)$ to appear in this branching, we must have $\lambda_1 \geq \frac{I}{2}$, $\lambda_2 = \lambda_3 = \cdots = \lambda_{k-1} = \frac{I}{2}$ and $|\lambda_k| \leq \frac{I}{2}$. Thus, we may write $\lambda_1 = n + \frac{I}{2}$ for some integer n, and using the notation $\lambda_k = s$ $(|s| \leq \frac{I}{2})$, we see that $(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s)$ is the general form of the $SO(2k)$ IRR, whose branching under $SO(2k-1)$ includes the $(\frac{I}{2})$ IRR of the latter. In fact the complete branching of the former can be written out as the direct sum of $SO(2k-1)$ IRRs as

$$\left(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s\right) = \bigoplus_{\mu_1 = \frac{I}{2}}^{n+\frac{I}{2}} \bigoplus_{\mu_2 = s}^{\frac{I}{2}} \left(\mu_1, \frac{I}{2}, \ldots, \frac{I}{2}, \mu_2\right). \tag{2.21}$$

The spectrum of the Hamiltonian can be written out using the eigenvalues of the quadratic Casimir operators $C^2_{SO(2k)}$ and $C^2_{SO(2k-1)}$ of $SO(2k)$ and $SO(2k-1)$ in the IRRs $(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s)$, $(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2})$, respectively. Eigenvalues for these Casimir operators in generic IRRs are given in the appendix. Explicitly, we have

$$\begin{aligned} E &= \frac{\hbar}{2MR^2}\left(C^2_{SO(2k)}\left(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s\right) - C^2_{SO(2k-1)}\left(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\ &= \frac{\hbar}{2MR^2}\left(n^2 + s^2 + n(I + 2k - 2) + \frac{I}{2}(k-1)\right). \end{aligned} \tag{2.22}$$

Thus, given a fixed background charge I, the lowest Landau level (LLL) is characterized by setting $n = 0$ and $s = 0$ if I is an even integer and setting $n = 0$ and $s = \pm\frac{1}{2}$ if I is an odd integer. In these cases we get

$$E_{LLL} = \begin{cases} \frac{\hbar}{2MR^2}\frac{I}{2}(k-1) & \text{for even } I, \\ \frac{\hbar}{2MR^2}\left(\frac{I}{2}(k-1) + \frac{1}{4}\right) & \text{for odd } I \end{cases}. \tag{2.23}$$

It is possible to interpret n and s as the quantum numbers labeling the Landau levels. We further see that the degeneracy in each Landau level is given by the dimension of the IRR $(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s)$ of $SO(2k)$, which can be written compactly as

$$d(n, s) = \prod_{i<j}^{k}\left(\frac{m_i - m_j}{g_i - g_j}\right)\prod_{i<j}^{k}\left(\frac{m_i + m_j}{g_i + g_j}\right), \tag{2.24}$$

where $g_i = k - i$ and $m_1 = n + \frac{I}{2} + g_1$, $m_i = \frac{I}{2} + g_i$ $(i = 2, \ldots, k-1)$, and $m_k = s + g_k$. It is easy to estimate from (2.24) that for a large I, $d(0,0) \approx I^{\frac{1}{2}(k-1)(k+2)} \approx d(0, \pm\frac{1}{2})$, and shows us how fast the LLL degeneracy grows for a given magnetic background on S^{2k-1}. We note also that for the LLL with an odd I, the degeneracy is doubled since s takes on the values $\pm\frac{1}{2}$.

In the thermodynamic limit I, $R \to \infty$ with a finite "magnetic length" scale $\ell_M = \frac{R}{\sqrt{I}}$, we immediately find

$$E(n, s) \to \frac{\hbar}{2M\ell_M^2}\left(n + \frac{1}{2}(k-1)\right), \quad E_{\mathrm{LLL}} = \frac{\hbar}{2M\ell_M^2}\frac{k-1}{2}, \tag{2.25}$$

and see that the spacing between LL levels remains finite; the LLL energy has the same form as in the standard integer QHE in two dimensions up to an overall constant.

What about the wave functions corresponding to these Landau levels? Compactly, they can be given in terms of the Wigner $\mathcal{D}$-functions $\mathcal{D}^{(n+\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2},s)}(g)_{[L][R]}$ of $SO(2k)$, carrying the $(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s)$ IRR of the latter. Here $[L][R]$ are two sets of collective labels that give the states in this IRR of $SO(2k)$ with respect to the IRRs of $SO(2k-1)$ that appear in the branching (2.21). Since $[R]$ labels all the states in the IRR $(\frac{I}{2}, \ldots, \frac{I}{2})$, $[L]$ is further subject to certain selection rules that restrict both the IRRs in (2.21) and the states in each of the latter, which we do not attempt to determine here. Nevertheless, the 2^{k-1}-component spinors

$$\begin{aligned} \Psi^{\pm} &= \frac{1}{2}\frac{1}{\sqrt{R(R + X_{2k})}}((R + X_{2k})\mathbb{I}_{2^{k-1}} \mp iX_\mu\gamma^\mu)\tilde{\phi}, \\ \Psi &= \begin{pmatrix}\Psi^+ \\ \Psi^-\end{pmatrix}, \end{aligned} \tag{2.26}$$

obtained from (2.6) are indeed the LLL wave functions for $I=1$, with $\pm$ signs corresponding to $s=\frac{1}{2}$ and $s=-\frac{1}{2}$, respectively and 2^{k-1}-fold degeneracy in each sector. Using the compact notation, $\Psi_\alpha^\pm \coloneqq K_{\alpha\beta}^\pm \tilde{\phi}_\beta$, we see that

$$L_{\mu\nu}\Psi_\alpha^\pm = K_{\alpha\beta}^\pm(\Sigma_{\mu\nu})_{\beta\gamma}\tilde{\phi}_\gamma, \qquad L_{2k\mu}\Psi_\alpha^\pm = K_{\alpha\beta}^\pm\left(\mp\frac{1}{2}\gamma_\mu\right)_{\beta\gamma}\tilde{\phi}_\gamma, \tag{2.27}$$

from which, after several steps of calculation, we find

$$\begin{aligned}\sum_{a<b}L_{ab}^2\Psi^\pm &= \sum_{\mu<\nu}\left(\Sigma_{\mu\nu}^2+\frac{1}{4}\gamma_\mu^2\right)\Psi^\pm,\\ &= \sum_{\mu<\nu}\left(\Sigma_{\mu\nu}^2+\frac{1}{2}\left(k-\frac{1}{2}\right)\right)\Psi^\pm,\end{aligned} \tag{2.28}$$

indicating the claimed result upon using (2.20) and (2.22). Thus the LLL wave functions for the case of the odd I are obtained as the I-fold symmetric product of $\Psi_\alpha^\pm$

$$\Psi^I = \sum_{\alpha_1,\cdots\alpha_I} f_{\alpha_1\cdots\alpha_I}\Psi_{\alpha_1}\cdots\Psi_{\alpha_I}, \tag{2.29}$$

where each α takes on values from 1 to 2^{k-1}, and the coefficients $f_{\alpha_1\cdots,\alpha_I}$ are totally symmetric in its indices. These coefficients also satisfy $\Gamma^a_{\alpha_1\alpha_2}f_{\alpha_1\alpha_2\cdots\alpha_I}=0$, $f_{\alpha\alpha\alpha_3\cdots\alpha_I}=0$ to exclude the nonsymmetric representations that appear in the I-fold tensor product of $(\frac{1}{2})$ IRR of $SO(2k-1)$ in the same manner as encountered in [9].

For N particles, the LLL wave function can be obtained via the Slater determinant of Ψ_I and reads

$$\Psi_N^I = \sum_{\alpha_1,\cdots\alpha_I}\varepsilon_{\alpha_1\cdots\alpha_I}\Psi_{\alpha_1}^I(x_1)\cdots\Psi_{\alpha_I}^I(x_N), \tag{2.30}$$

where $\varepsilon_{\alpha_1\cdots\alpha_I}$ is the usual permutation symbol, which is totally antisymmetric in its indices.

B. The Equatorial S^{2k-2}

It appears possible to probe further the physics at the equatorial spheres S^{2k-2}. To see how the physics match with the known results of the Landau problem on even spheres S^{2k-2}, we proceed as follows. We first note that

$$(K^\pm)^2 = \frac{1}{R}(X_{2k}\mathbb{I}_{2^{k-1}} \mp iX_\mu\gamma^\mu). \tag{2.31}$$

On the equatorial S^{2k-2} we have

$$(K_0^\pm)^2 \coloneqq (K^\pm)^2|_{x_{2k}=0} = \mp i\frac{1}{R}X_\mu\gamma^\mu, \tag{2.32}$$

where now R stands for the radius of S^{2k-2}. We may now define an idempotent on S^{2k-2} as

$$Q = i(K_0^\pm)^2, \qquad Q^\dagger = Q, \qquad Q^2 = \mathbb{I}_{2^{k-1}}, \tag{2.33}$$

which allows us to write down the rank-1 projection operators

$$\mathcal{P}_\pm = \frac{\mathbb{I}_{2^{k-1}} \pm Q}{2}. \tag{2.34}$$

Denoting the algebra of functions on S^{2k-2} as $\mathcal{A}$, we may write the free $\mathcal{A}$-module as $\mathcal{A}^{2^{k-1}} = \mathcal{A}\otimes\mathbb{C}^{2^{k-1}}$ and form the projective modules $\mathcal{P}_\pm\mathcal{A}^{2^{k-1}}$. In other words, we may decompose the free $\mathcal{A}^{2^{k-1}}$-module as

$$\mathcal{A}^{2^{k-1}} = \mathcal{P}_+\mathcal{A}^{2^{k-1}} \oplus \mathcal{P}_-\mathcal{A}^{2^{k-1}}, \tag{2.35}$$

where each summand is of dimension 2^{k-2}.

Projections of rank I are obtained by writing

$$\begin{aligned}\mathcal{P}_\pm^I &= \prod_{i=1}^I \frac{\mathbb{I}\pm\mathcal{Q}_i}{2},\\ \mathcal{Q}_i &= \mathbb{I}_{2^{k-1}}\otimes\mathbb{I}_{2^{k-1}}\otimes\cdots\otimes Q\otimes\cdots\otimes\mathbb{I}_{2^{k-1}},\end{aligned} \tag{2.36}$$

where $\mathcal{Q}_i$ is an I-fold tensor product whose ith entry is Q. $\mathcal{P}_\pm^I$ and $\mathcal{Q}_i$ act on the free module $\mathcal{A}_I^{2^{k-1}} = \mathcal{A}\otimes C_I^{2^{k-1}}$, where $C_I^{2^{k-1}}$ is the I-fold symmetric tensor product of $C^{2^{k-1}}$, whose dimension is that of the $(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2})$ IRR of $SO(2k-1)$.

$SO(2k-1)$ and $SO(2k-2)$ are groups of rank $k-1$, and the branching of the $(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2})$ IRR of the former under the IRRs of the latter reads

$$\left(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2}\right) = \bigoplus_{\mu=-\frac{I}{2}}^{\frac{I}{2}}\left(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2},\mu\right). \tag{2.37}$$

$\mathcal{P}_\pm^I$ are indeed the projections to the $(\frac{I}{2},\frac{I}{2},\ldots,\pm|\frac{I}{2}|)$ IRRs of $SO(2k-2)$ appearing in the rhs of the decomposition given in (2.37). These are the projective modules $\mathcal{P}_\pm^I\mathcal{A}_I^{2^{k-1}}$, whose dimensions are equal and given by the dimension of $(\frac{I}{2},\frac{I}{2},\ldots,\pm|\frac{I}{2}|)$.

We are now in a position to observe that the connection two forms associated with $\mathcal{P}_\pm^I$ are [22]

$$\mathcal{F}_\pm = \mathcal{P}_\pm^I d(\mathcal{P}_\pm^I)d(\mathcal{P}_\pm^I). \tag{2.38}$$

Thus, it follows from the remark ensuing (2.37) that, $\mathcal{F}_\pm$ are nothing but the $SO(2k-2)$ constant background gauge fields on S^{2k-2}, which are characterized by the IRRs $(\frac{I}{2},\frac{I}{2},\ldots,\pm|\frac{I}{2}|)$ of $SO(2k-2)$. Finally, we note that the $(k-1)^{th}$ Chern number is given by

$$c_{k-1}^\pm = \frac{1}{k!(2\pi)^k}\int_{S^{2k-2}}\mathcal{P}_\pm^I(d(\mathcal{P}_\pm^I))^{2k-2}, \tag{2.39}$$

where $c_{k-1}\equiv c_{k-1}^+>0$ and $c_{k-1}^- = -c_{k-1}$. $c_{k-1}(I)$ relates with the degeneracy of the LLL on S^{2k-2} via the relation

$c_{k-1}(I) = d_{\rm LLL}^{S^{2k-2}}(k-1, I-1)$ and it matches exactly with the number of zero modes, i.e., the index of the gauged Dirac operator on S^{2k-2}, as an independent solution of the Landau problem and Dirac-Landau problem on S^{2k-2} given in [4] confirms. Our brief analysis in this subsection clarifies the relationship between the QHE problem over even and odd dimensional spheres.

C. QHE on S^3

This is the case considered first by Nair and Daemi [5] and recently by Hasebe [4]. $S^3 \equiv SO(4)/SO(3)$, which follows by setting $k = 2$ in (2.1). The energy spectrum takes the form

$$E = \frac{\hbar}{2MR^2}\left(n^2 + 2n + In + \frac{I}{2} + s^2\right), \quad (2.40)$$

and the degeneracy of (2.40) is given by the dimension of the $(n + \frac{I}{2}, s)$ IRR of SO(4)

$$d(n,s) = \left(n + \frac{I}{2} + s + 1\right)\left(n + \frac{I}{2} - s + 1\right) = \left(n + \frac{I}{2} + 1\right)^2 - s^2. \quad (2.41)$$

For the LLL we have

$$E_{\rm LLL} = \frac{\hbar}{2MR^2}\frac{I}{2}, \qquad \text{I even},$$
$$E_{\rm LLL} = \frac{\hbar}{2MR^2}\left(\frac{I}{2} + \frac{1}{4}\right), \qquad \text{I odd}, \quad (2.42)$$

with the degeneracies

$$d(n = 0, s = 0) = \left(\frac{I}{2} + 1\right)^2,$$
$$d\left(n = 0, s = \pm\frac{1}{2}\right) = d(0, +1/2) + d(0, -1/2) = \frac{1}{2}(I + 1)(I + 3), \quad (2.43)$$

which are all in agreement with the results of [5] and [4].

On the equatorial sphere S^2, we have $Q = \boldsymbol{\sigma}\cdot\hat{X}$ and $\mathcal{P}_\pm = \frac{\mathbb{I}_2 \pm \boldsymbol{\sigma}\cdot\hat{X}}{2}$, with $\hat{X} = \frac{X}{R}$, yielding the usual Abelian Dirac monopole connection $B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F_{\nu\rho} = \frac{I}{2}\frac{X_i}{R^3}$ via (2.38) [1,8,23]. $c_1(I) = I$ yields the zero modes of the Dirac operator on S^2 in the monopole background [24].

D. QHE on S^5

Our next example is $S^5 \equiv SO(6)/SO(5)$, which follows from setting $k = 3$ in (2.1). In this case, the energy spectrum takes the form

$$E = \frac{\hbar}{2MR^2}(n^2 + 4n + In + I + s^2), \quad (2.44)$$

with the degeneracy of (2.44) given by the dimension of the $(n + \frac{I}{2}, \frac{I}{2}, s)$ IRR of SO(6) as

$$d(n,s) = \frac{1}{12}(n+1)^2(n+I+3)\left(\left(n + \frac{I}{2} + 2\right)^2 - s^2\right) \times \left(\left(\frac{I}{2} + 1\right)^2 - s^2\right). \quad (2.45)$$

Inspecting the LLL, we can write down the energy spectrum and degeneracies as

$$E_{\rm LLL} = \frac{\hbar}{2MR^2}I, \qquad \text{I even},$$
$$E_{\rm LLL} = \frac{\hbar}{2MR^2}\left(I + \frac{1}{4}\right), \qquad \text{I odd} \quad (2.46)$$

$$d(n=0, s=0) = \frac{1}{3\cdot 2^6}(I+2)^2(I+3)(I+4)^2, \quad \text{I even}, \quad (2.47)$$

and

$$d\left(n=0, s=\pm\frac{1}{2}\right) = d(0,+1/2) + d(0,-1/2) = \frac{1}{3\cdot 2^5}(I+1)(I+3)^3(I+5), \quad \text{I odd}. \quad (2.48)$$

In this case, we have $Q = \frac{\gamma_\mu X_\mu}{R}$ and $\mathcal{P}_\pm = \frac{\mathbb{I}_4 \pm Q}{2}$ on the equatorial S^4. It can be shown after some algebra that $F_{ij} = \frac{1}{R^2}(X_j A_i - X_i A_j + \Sigma_{ij}^+)$, $F_{5i} = -\frac{R+X_5}{R^2}A_i$, $i = (1, \ldots, 4)$, where $A_i = -\frac{1}{R(R+X_5)}\Sigma_{ij}^+ X_j$, $A_5 = 0$, and $\Sigma_{ij}^+ = -i\frac{1}{4}[\sigma_i, \sigma_j]$ [2,6]. The number of zero modes of the Dirac operator on S^4 with this SU(2) background is given by the second Chern number $c_2(I) = \frac{1}{6}I(I+1)(I+2)$ [24].

There is an exact correspondence between the union of Hilbert spaces of LLLs with I ranging from 0 to $I_{\rm max} = 2K$ or $I_{\rm max} = 2K + 1$, corresponding respectively to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$, or that of the winding number ± 1 line bundle over $\mathbb{C}P^3$ at level K [22,24]. This interesting relationship essentially follows due to the fact that the isometry group SU(4) for $\mathbb{C}P^3$ is isomorphic to that of S^5, which is $\mathrm{Spin}(6) \approx \mathrm{SO}(6)$. We can demonstrate this relationship very easily.

Let us recall that the fuzzy $\mathbb{C}P^3$ at level K is given in term of the matrix algebra $\mathrm{Mat}(d_K)$, where $d_K = \frac{1}{6}(K+3)(K+2)(K+1)$. It covers all the IRRs of SU(4), which emerge from the tensor product [24]

$$\left(\frac{K}{2}, \frac{K}{2}, \frac{K}{2}\right) \otimes \left(\frac{K}{2}, \frac{K}{2}, -\frac{K}{2}\right) = \bigoplus_{k=0}^{K}(k, k, 0). \quad (2.49)$$

Expansion of an element of $\mathrm{Mat}(d_K)$ in terms of the SU(4) harmonics carries the IRRs of SU(4) appearing in the direct sum decomposition given in the rhs of (2.49). We observe, that each summand in the latter is equal to the $\mathrm{SU}(4) \approx SO(6)$ IRR carried by the LLL for $I = 2k$. This readily implies that, for the even I, ($I = 2k$), the union of all of the LLL Hilbert spaces with $0 \le I \le 2K$ has the same dimensions as the matrix algebra $\mathrm{Mat}(d_K)$ of $\mathbb{C}P_F^3$.

Sections of complex line bundles with a winding number 1 over $\mathbb{C}P_F^3$ are described via the tensor product decomposition

$$\left(\frac{K+1}{2}, \frac{K+1}{2}, \frac{K+1}{2}\right) \otimes \left(\frac{K}{2}, \frac{K}{2}, -\frac{K}{2}\right) = \bigoplus_{k=0}^{K} \left(k + \frac{1}{2}, k + \frac{1}{2}, \frac{1}{2}\right). \tag{2.50}$$

Elements in this nontrivial line bundle are $d_{K+1} \times d_K$ rectangular matrices forming a right module $\mathcal{A}^{(1)}(\mathbb{C}P_F^3)$ under the action of $Mat(d_K)$. We observe that each summand in the rhs of (2.50) corresponds to an SO(6) IRR carried by the LLL for $I = 2k + 1$ and $s = \frac{1}{2}$. Thus, the union of all the LLL Hilbert spaces with $0 \le I \le 2K + 1$ has the same dimension as $\mathcal{A}^{(1)}(\mathbb{C}P_F^3)$ over $\mathbb{C}P_F^3$. In particular, it is straightforward to check that the total number of states in this union of LLL is precisely $d_{K+1} d_K$:

$$\sum_{k=0}^{K} \frac{1}{12}(k+4)(k+3)(k+2)^2(k+1) = d_{K+1} d_K. \tag{2.51}$$

A similar correspondence for the unions of LLLs with $s = -\frac{1}{2}$ and $\mathcal{A}^{-1}(\mathbb{C}P_F^3)$ corresponding to the winding number -1 sector is established starting with the tensor product $(\frac{K}{2}, \frac{K}{2}, \frac{K}{2}) \otimes (\frac{K+1}{2}, \frac{K+1}{2}, -\frac{K+1}{2})$. Thus, we observe a novel interpretation of the quantum number $s = \pm\frac{1}{2}$ for the LLL over S^5 as being related essentially to the winding number ± 1 of the monopole bundles over $\mathbb{C}P_F^3$.

III. DIRAC-LANDAU PROBLEM ON S^{2k-1}

In this section, our aim is to determine the spectrum of the Dirac operator for charged particles on S^{2k-1} under the influence of a constant $\mathrm{SO}(2k-1)$ gauge field background.

Let us briefly recall the situation in the absence of a background gauge field. In this case, Dirac operator for odd-dimensional spheres S^{2k-1} is well-known. It can be expressed in the form [25]

$$\mathcal{D}^{\pm} = \frac{1}{2}(\mathbb{I} \mp \Gamma_{2k+1}) \sum_{a<b} \left(-\Xi_{ab} L_{ab}^{(0)} + k - \frac{1}{2}\right), \tag{3.1}$$

where $L_{ab}^{(0)}$ is given after (2.18) and carriers the $(n, 0, \ldots, 0)$ IRR of $\mathrm{SO}(2k)$, and Ξ_{ab} given in (2.2) carries the reducible representation $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2})$ of $\mathrm{SO}(2k)$. The projectors $\mathcal{P}^{\mp} = \frac{1}{2}(\mathbb{I} \mp \Gamma_{2k+1})$ allows us to pick either of the two inequivalent representations. To obtain the spectrum of $\mathcal{D}^{\pm}$, we simply need to observe that

$$(n, 0, \ldots, 0) \otimes \left(\frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2}\right) = \left(n + \frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2}\right) \oplus \left(n - \frac{1}{2}, \frac{1}{2}, \ldots, \mp\frac{1}{2}\right), \tag{3.2}$$

Since the $(\frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2})$ IRRs of $\mathrm{SO}(2k)$ are conjugates, both representations yield the same spectrum for the Dirac operator $\mathcal{D}^{\pm}$ as expected, which is found to be [25]

$$E_{\uparrow} = n + k - \frac{1}{2}, \qquad E_{\downarrow} = -\left(n + k - \frac{3}{2}\right), \tag{3.3}$$

for the spin up and spin down states, respectively. Using the notation $j_{\uparrow\downarrow} = n \pm \frac{1}{2}$, we can express the spectrum of $\mathcal{D}^{\pm}$ more compactly as $E_{\uparrow\downarrow} = \pm(j_{\uparrow\downarrow} + k - 1)$.

Let us now consider the gauged Dirac operator, which can be written by replacing $L_{ab}^{(0)}$ with $\Lambda_{ab} = L_{ab} - R^2 F_{ab}$ as

$$\mathcal{D}_G^{\pm} = \frac{1}{2}(\mathbb{I} \mp \Gamma_{2k+1}) \sum_{a<b} \left(-\Xi_{ab}(L_{ab} - R^2 F_{ab}) + k - \frac{1}{2}\right). \tag{3.4}$$

It is not possible to obtain the spectrum $\mathcal{D}_G$ in the same manner as that of the zero gauge field background case. There is, however, a well-known formula on symmetric spaces that relates the square of the gauged Dirac operator to the gauged Laplacian, the Ricci scalar of the manifold under consideration, and a Zeeman energy term related to the curvature of the background gauge field [26]. Furthermore, on a symmetric coset space, say $K \equiv G/H$, a particular gauge field background which is compatible with the isometries of K generated by G (in the sense that the Lie derivative of the gauge field strength along a Killing vector of K is a gauge transformation of the field strength) is given by taking the gauge group as the holonomy group H and identifying the gauge connection with the spin connection. Then the square of the Dirac operator can be expressed as [26]

$$(i\mathcal{D}_G^{\pm})^2 = C^2(G) - C^2(H) + \frac{\mathcal{R}}{8}, \tag{3.5}$$

where $\mathcal{R}$ is the Ricci scalar of the manifold K and $C^2(G)$ and $C^2(H)$ are quadratic Casimirs of G, H, respectively. $C^2(H)$ is evaluated in the IRR of H characterizing the background gauge field, while $C^2(G)$ is evaluated in certain IRRs of G containing the fixed combinations of the background isospin of the gauge field and the intrinsic spin of the fermion. These considerations fit perfectly with our problem for odd spheres S^{2k-1} under the fixed $SO(2k-1)$ gauge field backgrounds, since in the present problem we have taken the gauge group as the holonomy group $SO(2k-1)$ of the odd spheres and the gauge connection has already been identified with the spin connection and taken explicitly in the IRR of $SO(2k-1)$, which is the I-fold symmetric tensor product of the fundamental spinor representation $(\frac{1}{2},\cdots\frac{1}{2})$. Therefore, we can write

$$(i\mathcal{D}_G^{\pm})^2 = C^2_{SO(2k)}(n+J,J,\ldots,J,\pm|\tilde{s}|) - C^2_{SO(2k-1)}\left(\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2}\right) + \frac{1}{4}(2k^2-3k+1), \tag{3.6}$$

where $2(2k^2-3k+1)$ is nothing but the Ricci scalar of the sphere S^{2k-1}, and J takes on the values $J=\frac{I}{2}+\frac{1}{2}$ $(I\geq 0)$ and $J=\frac{I}{2}-\frac{1}{2}$ $(I\geq 1)$, corresponding to the spin up and spin down states, respectively, and $|\tilde{s}|\leq J$. We find

$$\mathcal{E}_\uparrow = n(n+2k-1)+I(n+k-1) + k(k-1)+\tilde{s}^2, \qquad I\geq 0, \tag{3.7}$$

$$\mathcal{E}_\downarrow = n(n+I+2k-3)+\tilde{s}^2, \qquad I\geq 1. \tag{3.8}$$

It is readily seen that the spectrum for the conjugate $SO(2k)$ IRRs coincide with $\tilde{s}\leftrightarrow -\tilde{s}$.

Degeneracy of $\mathcal{E}_\uparrow$ and $\mathcal{E}_\downarrow$ are given by the dimensions of the IRRs $(n+J,J,\ldots,J,\pm|\tilde{s}|)$, with $J=\frac{I}{2}+\frac{1}{2}$ and $J=\frac{I}{2}-\frac{1}{2}$, respectively. They can be computed from (2.24) with $g_i=k-i$ and $m_1=n+J+g_1$, $m_i=J+g_i$ $(i=2,\ldots,k-1)$, and $m_k=\tilde{s}+g_k$.

The Hamiltonian for the Dirac-Landau problem may be taken as $H=\frac{1}{2MR^2}(i\mathcal{D}_G^{\pm})^2$. For the even I, we see that the LLL is given by taking $n=0$ and $\tilde{s}=\pm\frac{1}{2}$ in (3.8), yielding $\mathcal{E}_\downarrow^{LLL}=\frac{1}{4}$ with the same degeneracy for both of the operators $(i\mathcal{D}_G^{\pm})^2$ and given as $d(n=0,\tilde{s}=\frac{1}{2})=d(n=0,\tilde{s}=-\frac{1}{2})$, which can be computed from (2.24) using the facts given in the previous paragraph. For the odd I, we see that the LLL is given by taking $n=0$ and $\tilde{s}=0$ in (3.8), yielding $\mathcal{E}_\downarrow^{LLL}=0$. These are the zero modes of the Dirac operators $\mathcal{D}_G^{\pm}$ with the degeneracy $d(n=0,\tilde{s}=0)$.

For S^3, we find that the LLL degeneracy for the even I is given as $\frac{I(I+2)}{4}$ and for the odd I it is $\frac{(I+1)^2}{4}$, which is the number of zero modes of Dirac operators $\mathcal{D}_G^{\pm}$. These match with the results of [5]. Another example is S^5, with the LLL degeneracy for the even I given as $\frac{1}{3\cdot 2^6}I(I+2)^3(I+4)$, and for the odd I it is $\frac{1}{3\cdot 2^6}(I+1)^2(I+2)(I+3)^2$.

We may recall that on even dimensional manifolds, the Atiyah-Singer index theorem relates the number of zero modes, i.e., the index of the Dirac operator to Chern classes, which are integers of topological significance [27]. On odd dimensional manifolds, however, no such general index theorem is known. One possible candidate for a topological number on these manifolds could be conceived as the Chern-Simons forms. Nevertheless, for odd spheres it is not too hard to see that these vanish identically when evaluated for the $SO(2k-1)$ connection given in (2.11). Thus, it remains an open question to find out if and how the zero modes of $\mathcal{D}_G^{\pm}$ are related to a number of topological origin.

Finally, let us also note that setting $I=0$ in (3.7), we have $\tilde{s}=\pm\frac{1}{2}$ and we find $\mathcal{E}_\uparrow=(n+k-\frac{1}{2})^2$, which matches with the known result for $\mathcal{D}^{\pm}$ given in (3.3). Explicitly, we have $E_\uparrow=\sqrt{\mathcal{E}_\uparrow}$, while $E_\downarrow=-\sqrt{\mathcal{E}_\uparrow}$ with $n\to n-1$ and $\tilde{s}\to-\tilde{s}$. The latter are necessary to match the IRR $(n+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\pm\frac{1}{2})$ with the second summand in (3.2).

IV. CONCLUSIONS

In this paper, we have solved the Landau problem and the Dirac-Landau problem for charged particles on odd-dimensional spheres S^{2k-1} in the background of constant $SO(2k-1)$ gauge fields. First, using group theoretical arguments, we have determined the spectrum of the Schrödinger Hamiltonian together with its degeneracies at each Landau level. We gave the corresponding eigenstates in terms of the Wigner $\mathcal{D}$-functions in general, while for the odd values of I, an explicit local form of the LLL eigenstates is also obtained. We have noticed a peculiar relation between the Landau problem on S^{2k-1} and that on the equatorial S^{2k-2}, which allowed us to give the background $SO(2k-2)$ gauge fields over S^{2k-2} by constructing the relevant projective modules. Additionally, for the Landau problem on S^5, we were able to demonstrate an exact correspondence between the union of Hilbert spaces of the LLL's with I ranging from 0 to $I_{max}=2K$ or $I_{max}=2K+1$ to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ or that of winding number ±1 line bundles over $\mathbb{C}P^3$ at level K, respectively. This correspondence also means that the quantum number $s=\pm\frac{1}{2}$ for the LLL over S^5 is actually related to the winding number $\kappa=\pm1$ of the monopole bundles over $\mathbb{C}P^3_F$ via $s=\frac{\kappa}{2}$, which permits us to give, in a sense, a topological meaning to the ±1 values of $2s$. In the last section, we have determined the spectrum of the Dirac operators on S^{2k-1} in the same gauge field background together with their degeneracies and found the number of their zero modes as well. Our results are in agreement with the spectra of the ungauged Dirac operators on S^{2k-1} for $I=0$ and generalizes it to all constant $SO(2k-1)$ spin connection backgrounds.

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APPENDIX: SOME REPRESENTATION THEORY

1. Branching Rules

Irreducible representations of $\mathrm{SO}(\mathcal{N})$ and $\mathrm{SO}(\mathcal{N}-1)$ can be given in terms of the highest weight labels $[\lambda] \equiv (\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{N}-1}, \lambda_{\mathcal{N}})$ and $[\mu] \equiv (\mu_1, \mu_2, \ldots, \mu_{\mathcal{N}-1})$, respectively. Branching of the IRR $[\lambda]$ of $\mathrm{SO}(\mathcal{N})$ under $\mathrm{SO}(\mathcal{N}-1)$ IRRs follows from the rule [21,28]

$$[\lambda] = \bigoplus_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_{k-1} \geq |\lambda_k|} [\mu], \quad \text{for } \mathcal{N} = 2k, \tag{A1}$$

$$[\lambda] = \bigoplus_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_{k-1} \geq \lambda_k \geq |\mu_k|} [\mu], \quad \text{for } \mathcal{N} = 2k+1. \tag{A2}$$

2. Quadratic Casimir operators of SO(2k) and SO(2k − 1) Lie algebras

Eigenvalues for the quadratic Casimir operators of $\mathrm{SO}(2k)$ and $\mathrm{SO}(2k-1)$ in the IRRs $[\lambda] \equiv (\lambda_1, \lambda_2 \cdots \lambda_k)$, $[\mu] \equiv (\mu_1, \mu_2 \cdots \mu_{k-1})$, respectively are given as [21]:

$$C_2^{\mathrm{SO}(2k)}[\lambda] = \sum_{i=1}^{k} \lambda_i (\lambda_i + 2k - 2i), \tag{A3}$$

$$C_2^{\mathrm{SO}(2k-1)}[\mu] = \sum_{i=1}^{k-1} \mu_i (\mu_i + 2k - 1 - 2i). \tag{A4}$$

We list a few particular cases for quick reference,

$$C_2^{\mathrm{SO}(2k-1)}\left(\frac{I}{2}, \ldots, \frac{I}{2}\right) = \frac{I^2}{4}(k-1) + \frac{I}{2}(k-1)^2, \tag{A5}$$

$$C_2^{SO(4)}\left(n + \frac{I}{2}, s\right) = \frac{I^2}{4} + In + I + n^2 + 2n + s^2, \tag{A6}$$

$$C_2^{\mathrm{SO}(3)}\left(\frac{I}{2}\right) = \frac{I^2}{4} + \frac{I}{2} \tag{A7}$$

$$C_2^{\mathrm{SO}(6)}\left(n + \frac{I}{2}, \frac{I}{2}, s\right) = \frac{I^2}{2} + In + 3I + n^2 + 4n + s^2, \tag{A8}$$

$$C_2^{\mathrm{SO}(5)}\left(\frac{I}{2}, \frac{I}{2}\right) = \frac{I^2}{2} + 2I, \tag{A9}$$

3. Relationship between Dynkin and Highest weight labels

Throughout this paper highest weight labels (HW) have been used to label the irreducible representations of Lie algebras. Another common way to label the IRRs is given by the Dykin indices. The relationship between Dykin indices and highest weight labels for the groups SO(4), SO(5) and SO(6) are as follows [21,28]:

For a SO(4) IRR the labels are

$$(p, q)_{\mathrm{Dynkin}} \equiv (\lambda_1, \lambda_2)_{HW}, \tag{A10}$$

where the relation between these labels are given by

$$p = (\lambda_1 + \lambda_2), \qquad q = (\lambda_1 - \lambda_2). \tag{A11}$$

For instance, $(n + \frac{I}{2}, s)_{HW}$ which is the IRR used in section II C to label the LL on S^3 corresponds to $(n + \frac{I}{2} + s, n + \frac{I}{2} - s)_{\mathrm{Dynkin}}$, while the LLL are given by either $(\frac{I}{2}, \frac{I}{2})_{\mathrm{Dynkin}}$ or $(\frac{I}{2} \pm \frac{1}{2}, \frac{I}{2} \mp \frac{1}{2})_{\mathrm{Dynkin}}$.

For SO(5) IRRs, the labels are

$$(p, q)_{\mathrm{Dynkin}} \equiv (\lambda_1, \lambda_2)_{HW}, \tag{A12}$$

and the relation between these labels are given by

$$p = \lambda_1 - \lambda_2, \qquad q = 2\lambda_2, \tag{A13}$$

For instance, I-fold symmetric tensor product of $(\frac{1}{2}, \frac{1}{2})_{HW}$ is $(\frac{I}{2}, \frac{I}{2})_{HW}$ and in terms of Dynkin index labels this corresponds to $(0, I)_{\mathrm{Dynkin}}$.

Finally, for SO(6) IRRs labels are given as

$$(p, q, r)_{\mathrm{Dynkin}} \equiv (\lambda_1, \lambda_2, \lambda_3)_{HW}, \tag{A14}$$

and the relationship between these labels is given by

$$p = \lambda_2 + \lambda_3, \qquad q = \lambda_1 - \lambda_2, \qquad r = \lambda_2 - \lambda_3. \tag{A15}$$

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