# Shielding linearized gravity

Robert Beig* and Piotr T. Chruściel†

*Faculty of Physics, University of Vienna, Boltzmanngasse 5, A1090 Vienna, Austria*

(Received 2 January 2017; published 31 March 2017)

We present an elementary argument that one can shield linearized gravitational fields using linearized gravitational fields. This is done by using third-order potentials for the metric, which avoids the need to solve singular equations in shielding or gluing constructions for the linearized metric.

DOI: 10.1103/PhysRevD.95.064063

## I. INTRODUCTION

A fundamental property of Newtonian gravity is that the gravitational field cannot be localized in a bounded region. This is a simple consequence of the equation

$$\Delta\phi = 4\pi G\rho, \tag{1.1}$$

where $\phi$ is the gravitational potential, $G$ is Newton's constant and $\rho$ is the matter density; the requirement that $\rho \geq 0$ and the asymptotic behavior $-M/r$ of $\phi$, where $M$ is the total mass, implies that $\phi$ vanishes at large distances along a curve extending to infinity if and only if there is no matter whatsoever and $\phi \equiv 0$. It is therefore extremely surprising that in general relativity, gravitational fields can be shielded away by gravitational fields, as proved recently in a remarkable paper by Carlotto and Schoen [1].

Since Newtonian gravity is part of the weak-field limit of general relativity (GR) (indeed, this is weak-field GR with small velocities), one wonders if a similar screening can occur for linearized relativity. As it turns out, the analysis of Carlotto and Schoen can be readily generalized to linearized gravitational fields on conelike sets as considered in Ref. [1] (compare Ref. [2]). This, however, requires sophistical mathematical machinery which imposes restrictions to the sets considered and, as an intermediate step, uses solutions blowing up at the relevant boundaries, which leads to difficulties when trying to implement the method numerically. The object of this paper is to point out an alternative elementary method to perform gluings, or achieve screening of linearized gravity by linearized gravitational fields near a Minkowski background. In particular, we give here a very simple proof that at any given time $t$ and given any open set $\Omega \subset \mathbb{R}^3$, every linearized vacuum gravitational field $h_{\mu\nu}$ on $\{t\} \times \mathbb{R}^3$ can be deformed to a new linearized vacuum field $\tilde{h}_{\mu\nu}$ so that $\tilde{h}_{\mu\nu}$ coincides with $h_{\mu\nu}$ on $\Omega$ and vanishes outside a slightly larger set. In other words, the gravitational field has been screened away outside of $\Omega$, and this by using gravitational fields only; no matter fields, whether with positive or negative density, are needed.

We emphasize that the construction of Carlotto-Schoen switches off the gravitational field in sets which have a conelike structure, whether in the linearized case or in the full treatment. In our approach, no restrictions on the geometry of $\Omega$ occur, so that the screening can be done near any set.

Our construction is likely to be useful for the numerical construction of initial data sets with interesting properties, by providing an efficient way of making gluings in the far-away zone, where nonlinear corrections become inessential. Here, as already pointed out, both the Corvino-Schoen and the Carlotto-Schoen gluings require solving elliptic equations in spaces of functions which are singular at the boundary of the gluing region (see Refs. [3,4] for a review), while our gluings are performed by explicit elementary integrations [see (2.31) below], multiplication by a cutoff function, and applying derivatives, once the metric has been put into transverse and traceless (TT)-gauge.

The above leads one naturally to ask similar questions for electric and magnetic fields. Here, we provide a simple proof that Maxwell fields can be shielded by Maxwell fields. Last but not least, we show how to perform the screening in practice, in that we prove that all solutions of sourceless Maxwell equations in a bounded space-time region can be realized by manipulating charges and currents in an enclosing bounded region.

## II. SHIELDING LINEARIZED GRAVITY

Consider $\mathbb{R}^{3+1}$ with a metric which, in the natural coordinates on $\mathbb{R}^{3+1}$, takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $\eta$ denotes the Minkowski metric. Suppose that there exists a small constant $\epsilon$ such that we have

$$|h_{\mu\nu}|, \qquad |\partial_\sigma h_{\mu\nu}|, \qquad |\partial_\sigma\partial_\rho h_{\mu\nu}| = O(\epsilon). \tag{2.2}$$

If we use the metric $\eta$ to raise and lower indices, one has

*robert.beig@univie.ac.at
†piotr.chrusciel@univie.ac.at, homepage.univie.ac.at/piotr.chrusciel.

 

$$R_{\beta\delta} = \frac{1}{2}[\partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\partial_\beta h^\alpha{}_\alpha] + O(\epsilon^2). \tag{2.3}$$

Coordinate transformations $x^\mu \mapsto x^\mu + \zeta^\mu$, with

$$|\zeta_\mu|, \quad |\partial_\sigma\zeta_\mu|, \quad |\partial_\sigma\partial_\rho\zeta_\mu|, \quad |\partial_\sigma\partial_\rho\partial_\nu\zeta_\mu| = O(\epsilon), \tag{2.4}$$

preserve (2.2) and lead to the gauge freedom

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu. \tag{2.5}$$

Imposing the wave-coordinates condition up to $O(\epsilon^2)$ terms,

$$\Box_g x^\alpha = O(\epsilon^2), \tag{2.6}$$

leads to

$$\partial_\beta h^\beta{}_\alpha = \frac{1}{2}\partial_\alpha h^\beta{}_\beta + O(\epsilon^2) \tag{2.7}$$

as well as

$$R_{\beta\delta} = -\frac{1}{2}\Box_\eta h_{\beta\delta} + O(\epsilon^2). \tag{2.8}$$

### A. Cauchy problem for linearized gravity

In what follows, we ignore all $O(\epsilon^2)$ terms in the equations above and consider the theory of a tensor field $h_{\mu\nu}$ with the gauge freedom (2.5) and satisfying the equations

$$0 = \partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\partial_\beta h^\alpha{}_\alpha. \tag{2.9}$$

Solving the following wave equation,

$$\Box\zeta_\alpha = -\partial_\beta h^\beta{}_\alpha + \frac{1}{2}\partial_\alpha h^\beta{}_\beta,$$

where $\Box \equiv \Box_\eta$ is the wave operator of the Minkowski metric, and performing (2.5) leads to a new tensor $h_{\mu\nu}$, still denoted by the same symbol, such that

$$\partial_\beta h^\beta{}_\alpha = \frac{1}{2}\partial_\alpha h^\beta{}_\beta, \tag{2.10}$$

together with the usual wave equation for $h$:

$$\Box h_{\beta\delta} = 0. \tag{2.11}$$

Solutions of this last equation are in one-to-one correspondence with their Cauchy data at $t = 0$. However, those data are not arbitrary, which can be seen as follows: Eqs. (2.10)–(2.11) imply

$$\Box\left(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta\right) = 0. \tag{2.12}$$

It follows that (2.10) will hold if and only if

$$\left(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta\right)\Big|_{t=0} = 0 = \partial_0\left(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta\right)\Big|_{t=0}. \tag{2.13}$$

Equivalently, taking (2.11) into account,

$$\partial_0(h_{00} + h^i{}_i)|_{t=0} = 2\partial_i h^i{}_0|_{t=0}, \tag{2.14}$$

$$\partial_0 h_{0i}|_{t=0} = \left(\partial_j h^j{}_i + \frac{1}{2}\partial_i(h_{00} - h^j{}_j)\right)\Big|_{t=0}, \tag{2.15}$$

$$\Delta h^i{}_i|_{t=0} = \partial_i\partial_j h^{ij}|_{t=0}, \tag{2.16}$$

$$\partial_j(\partial_0 h^j{}_i - \partial_0 h^k{}_k\delta^j_i)|_{t=0} = (\Delta h_{0i} - \partial_i\partial_j h^j{}_0)|_{t=0}. \tag{2.17}$$

The last two equations are of course the linearizations of the usual scalar and vector constraint equations.

There remains the freedom of choosing $\zeta_\alpha|_{t=0}$ and $\partial_t\zeta_\alpha|_{t=0}$. We choose

$$\begin{aligned}(\partial_0 h^k{}_k - 2\partial_k h^k{}_0 - 2\Delta\zeta_0)|_{t=0} &= 0,\\ (h_{00} + 2\partial_0\zeta_0)|_{t=0} &= 0,\\ (h_{0i} + \partial_i\zeta_0 + \partial_0\zeta_i)|_{t=0} &= 0,\\ D_i\left(h^i{}_j - \frac{1}{3}h^k{}_k\delta^i_j + D^i\zeta_j + D_j\zeta^i - \frac{2}{3}D^k\zeta_k\delta^i_j\right)\Big|_{t=0} &= 0,\end{aligned} \tag{2.18}$$

where $D_i \equiv D^i \equiv \partial_i$ in Cartesian coordinates. Indeed, given any $h_{\mu\nu}$ and $\partial_0 h_{\mu\nu}|_{t=0}$, the first equation can be solved for $\zeta_0|_{t=0}$ under suitable natural conditions on the data; the second defines $\partial_0\zeta_0|_{t=0}$; the third defines $\partial_0\zeta_i|_{t=0}$; finally, the last equation is an elliptic equation for the vector field $\zeta_i|_{t=0}$ which can be solved [5] if one assumes that the field

$$\partial_i\left(h^i{}_j - \frac{1}{3}h^k{}_k\delta^i_j\right)\Big|_{t=0} \tag{2.19}$$

belongs to a suitable weighted Sobolev or Hölder space, the precise requirements being irrelevant for our purposes. We simply note that if some components of $h_{ij}$ behave as $1/r$, then $\zeta$ will behave like $\ln r$ in general, which is likely to introduce $\ln r/r$ terms in the gauge-transformed metric. After performing this gauge transformation, we end up with a tensor field $h_{\mu\nu}$ which satisfies

$$\partial_0 h^k{}_k|_{t=0} = h_{00}|_{t=0} = h_{0i}|_{t=0} = \partial_i\left(h^i{}_j - \frac{1}{3}h^k{}_k\delta^i_j\right)\Big|_{t=0} = 0. \tag{2.20}$$

Inserting this into (2.14)–(2.17), we find

$$\partial_0 h_{00}|_{t=0} = 0, \tag{2.21}$$

$$\partial_0 h_{0i}|_{t=0} = -\frac{1}{6}\partial_i h^j{}_j|_{t=0}, \tag{2.22}$$

$$\Delta h^i{}_i|_{t=0} = 0, \tag{2.23}$$

$$\partial_j(\partial_0 h^j{}_i - \partial_0 h^k{}_k\delta^j_i)|_{t=0} = 0. \tag{2.24}$$

The further requirement that $h^i{}_i$ goes to zero as $r$ tends to infinity together with the maximum principle gives

$$h^i{}_i|_{t=0} = 0. \tag{2.25}$$

We conclude (compare Ref. [6]) that at any given time $t = t_0$, every linearized gravitational initial data set $(h_{\mu\nu}, \partial_t h_{\mu\nu})|_{t=t_0}$ can be gauge transformed to the TT-gauge; writing $k_{ij} = \partial_0 h_{ij}$, we have

$$h^k{}_k|_{t=t_0} = \partial_i h^i{}_j|_{t=t_0} = k^k{}_k = \partial_i k^i{}_j = 0. \tag{2.26}$$

From what has been said and from the uniqueness of solutions of the wave equation, we also see that in this gauge, we will have for all $t$

$$h_{00} = h_{0i} = h^k{}_k = \partial_i h^i{}_j = 0, \tag{2.27}$$

which further implies that (2.26) is preserved by evolution.

It should be pointed out that when the construction is carried out on the complement of a ball, e.g., because sources are present or because we perform the construction at large distances only where the nonlinearities become negligible, then (2.25) will not hold in general, and the trace of $h_{ij}$ will be nontrivial, with the usual expansion in terms of inverse powers of $r$, starting with $1/r$ terms associated with the total mass of the configuration. In such cases, our construction below still applies to the transverse-traceless part of the metric.

### B. Third-order potentials

We will need the following result from Ref. [7], which can be summarized as follows: let $h_{ij}$ be a symmetric TT tensor on $\mathbb{R}^3$,

$$\partial_i h^i{}_j = 0 = h^i{}_i. \tag{2.28}$$

Then, there exists a symmetric traceless "third-order potential" $u_{ij}$ such that

$$h_{m\ell} = P(u)_{m\ell}, \tag{2.29}$$

where (here, $g_{ij}$ denotes the Euclidean metric, and $D^i$ denotes the associated covariant derivative)

$$P(u)_{m\ell} \coloneqq \frac{1}{2}\epsilon_m{}^{ij}\partial_i\left(\Delta u_{j\ell} - 2\partial_{(\ell}D^n u_{j)n} + \frac{1}{2}g_{j\ell}D^n D^k u_{nk}\right), \tag{2.30}$$

and where $u = u_{ij}dx^i dx^j$ can be constructed by the following procedure: letting

$$\sigma_{ijk}(\vec{x}) \coloneqq \int_0^1 \epsilon_{ij}{}^{\ell} h_{\ell k}(\lambda\vec{x})\lambda(1-\lambda)^2 d\lambda, \tag{2.31}$$

we set

$$u_{j\ell} = 2x^m x^n x_{(j}\sigma_{\ell)mn} + r^2 x^m \sigma_{m(j\ell)}. \tag{2.32}$$

(This is clearly symmetric, and tracelessness is not very difficult to check. Other third-order potentials $u$ are possible, differing by an element of the kernel of $P$.) One way to see how (2.30) arises is to note that $P(u)$ is, apart from a numerical factor, the linearization at the flat metric of the Cotton-York tensor in the direction of the trace-free tensor $u$. For (2.32), the formulas follow by successively integrating thrice the 2-forms given in Ref. [7], at each step using the Poincaré formula (3.7) below. We sketch the construction in Appendix A.

The converse is also true: given any symmetric trace-free tensor $u_{ij}$, the tensor field $P(u)$ defined by (2.30) is symmetric, transverse, and traceless (of which only the last property and the vanishing of the divergence on the first index are obvious).

As an example, consider $h_{ij}$ describing a plane gravitational wave in TT-gauge propagating in direction $\vec{k}$,

$$h_{ij}(\vec{k}) = \Re(H_{ij}e^{i\vec{k}\cdot\vec{x}}), \qquad \partial_\ell H_{ij} = 0 = H^i{}_i = H_{ij}k^j, \tag{2.33}$$

with possibly complex coefficients $H_{ij}$, where $\Re$ denotes the real part. Then,

$$\sigma_{ijk} = \Re\left(\epsilon_{ij\ell}H^\ell{}_k \int_0^1 e^{i\lambda\vec{k}\cdot\vec{x}}(\lambda - 2\lambda^2 + \lambda^3)d\lambda\right)$$
$$= \Re(W\epsilon_{ij\ell}H^\ell{}_k), \quad \text{where} \tag{2.34}$$

$$W(\vec{x}) = \frac{2ie^{i\vec{k}\cdot\vec{x}}(\vec{k}\cdot\vec{x} + 3i) - \vec{k}\cdot\vec{x}(\vec{k}\cdot\vec{x} - 4i) + 6}{(\vec{k}\cdot\vec{x})^4} \tag{2.35}$$

(which tends to 1/12 when $\vec{k}\cdot\vec{x}$ tends to zero),

$$u_{j\ell} = \Re(W(2x^m x^i x_{(j}\epsilon_{\ell)mk}H^k{}_i - r^2 x^i \epsilon_{ik(j}H^k{}_{\ell)})). \qquad (2.36)$$

As another example, consider the family of fields

$$u_{ij} = \ln(1+r^2)\left(D_i\lambda_j + D_j\lambda_i - \frac{2}{3}D^k\lambda_k g_{ij}\right). \qquad (2.37)$$

Tensors of the form (2.37) *with the* $\ln(1+r^2)$ *term removed* form the kernel of $P$ for any $\lambda_i$ (cf. Appendix A), which easily implies that if $\lambda \sim O(r^\sigma)$ for large $r$, then $h_{ij} \sim O(r^{\sigma-4})$, for all $\sigma \in \mathbb{R}$.

We also note that if $h_{ij}$ is compactly supported to start with, then $u_{ij}$ can also be chosen to be compactly supported; compare Appendix A.

### C. Shielding gravitational Cauchy data

We are ready to prove now a somewhat more general version of our previous claim, that at any given time $t$ and given any region $\Omega \subset \mathbb{R}^3$, every vacuum initial data set for the gravitational field $(h_{ij}, k_{ij})$ can be deformed to a new vacuum initial data set $(\tilde{h}_{ij}, \tilde{k}_{ij})$ which coincides with $(h_{ij}, k_{ij})$ on $\Omega$ and vanishes outside of a slightly larger set.

Indeed, consider any linearized gravitational field in the gauge (2.27). Denote by $(h_{ij}, k_{ij})$ the associated Cauchy data at $t$, and let $(u_{ij}, v_{ij})$ denote the corresponding potentials discussed in Sec. II B; thus,

$$(h_{ij}, k_{ij}) = (P(u)_{ij}, P(v)_{ij}), \qquad (2.38)$$

where $P$ is the third-order differential operator of (2.30). Let $\Omega$ be any open subset of $\mathbb{R}^3$, and let $\tilde{\Omega}$ be any open set containing $\bar{\Omega}$. Let $\chi_\Omega$ be any smooth function which is identically equal to 1 on $\Omega$ and which vanishes outside of $\tilde{\Omega}$. Then, the initial data set

$$(\tilde{h}_{ij}, \tilde{k}_{ij}) = (P(\chi_\Omega u)_{ij}, P(\chi_\Omega v)_{ij}) \qquad (2.39)$$

satisfies the vacuum constraint equations everywhere, coincides with $(h_{ij}, k_{ij})$ in $\Omega$, and vanishes outside of $\tilde{\Omega}$.

When $\Omega$ is bounded, the new fields $(\tilde{h}_{ij}, \tilde{k}_{ij})$ can clearly be chosen to vanish outside of a bounded set. For example, consider a plane-wave solution as in (2.33). Multiplying the potentials (2.36) by a cutoff function $\chi_{B(R_1)}$ which equals 1 on $B(R_1)$ and vanishes outside of $B(R_2)$ provides compactly supported gravitational data which coincide with the plane-wave ones in $B(R_1)$. [Alternatively, one can replace $\vec{k}\cdot\vec{x}$ in the first line of (2.34), or in (2.35) and (2.36), by $\vec{k}\cdot\vec{x}\chi_{B(R_1)}$.] In the limit $\vec{k}=0$, so that $h_{ij}$ is constant and, e.g., $k_{ij}=0$, one obtains data which are Minkowskian in $B(R_1)$, and outside of $B(R_2)$, and describe a burst of radiation localized in a spherical shell. Note that the Minkowskian coordinates for the interior region are distinct from the ones for the outside region. The closest full-theory configuration to this would be Bartnik's time symmetric initial data set [8] which is flat inside a ball of radius $R_1$ and which can be Corvino-Schoen deformed to be Schwarzschildean outside of the ball of radius $R_2$; here, $R_2$ will be much larger than $R_1$ in general, but can be made as close to $R_1$ as desired by making the free data available in Bartnik's construction sufficiently small.

For $\Omega$'s which are not bounded, it is interesting to enquire about falloff properties of the shielded field. This will depend upon the geometry of $\Omega$ and the falloff of the initial field.

For conelike geometries, as considered in Refs. [1,2], and with $h_{\mu\nu} = O(1/r)$, the gravitational field in the screening region will fall off again as $O(1/r)$. This is rather surprising, as the gluing approach of Ref. [1] leads to a loss of decay even for the linear problem. One should, however, keep in mind that the transition to the TT-gauge for a metric which falls off as $1/r$ is likely to introduce $\ln r/r$ terms in the transformed metric, which will then propagate to the gluing region.

As another example, consider the set $\Omega = (a,b)\times\mathbb{R}^2$, which is not covered by the methods of Ref. [1]. Our procedure in this case applies, but if $h_{\mu\nu} = O(1/r)$ and if the cutoff function is taken to depend only upon the first variable of the product $\Omega = (a,b)\times\mathbb{R}^2$, one obtains a gravitational field $\tilde{h}_{\mu\nu}$ vanishing outside a slab $\tilde{\Omega} = (c,d)\times\mathbb{R}^2$, with $[a,b]\subset(c,d)$, which might grow as $r^2\ln r$ when receding to infinity within the slab.

So far, we have been concentrating on "shielding." But of course, the above can be used to glue a linearized field across a gluing region, by interpolating the respective $u$'s to each other in the gluing zone. Equivalently, screen each of the fields which are glued to zero across the gluing region, and add the resulting new fields.

## III. SHIELDING MAXWELL FIELDS

Maxwell equations in Minkowski space-time share at least two features with linearized gravity: the existence of constraint equations and the existence of gauge transformations. It might therefore be unsurprising that there exists a version of the Carlotto-Schoen construction which applies to the Maxwell equations; compare Refs. [4,9] for a discussion of the Maxwell equivalent of the Corvino-Schoen construction, which generalizes without further ado to the Carlotto-Schoen setting. We wish to show here how to carry out the shielding of Maxwell fields with Maxwell fields in an elementary way.

Recall that solutions of the source-free Maxwell equations are in one-to-one correspondence with their initial data at time $t$; these are simply the electric field $\vec{E}$ and the magnetic field $\vec{B}$ at $t$. These fields are not arbitrary but satisfy the constraints

$$\mathrm{div}\vec{E} = \mathrm{div}\vec{B} = 0. \tag{3.1}$$

On $\mathbb{R}^3$, these imply the existence of vector potentials $\vec{\omega}$ and $\vec{A}$ such that

$$\vec{E} = \mathrm{curl}\,\vec{\omega}, \qquad \vec{B} = \mathrm{curl}\,\vec{A}. \tag{3.2}$$

In fact, there is an explicit formula for $\vec{\omega}$,

$$\omega_i = \epsilon_{jik} x^j \int_0^1 E^k(\lambda x)\lambda d\lambda, \tag{3.3}$$

similarly for $\vec{A}$. Using (3.2), it is straightforward to show that at any given time $t$ and given any region $\Omega \subset \mathbb{R}^3$, every sourceless Maxwell field $(\vec{E}, \vec{B})$ can be deformed to new sourceless Maxwell fields which coincide with $(\vec{E}, \vec{B})$ on $\Omega$ and vanish outside a slightly larger set. Indeed, letting $\tilde{\Omega}$ and $\chi_\Omega$ be as in the paragraph following (2.38), the new Maxwell fields at $t$,

$$\vec{E} = \mathrm{curl}(\chi_\Omega \vec{\omega}) \quad \text{and} \quad \vec{B} = \mathrm{curl}(\chi_\Omega \vec{A}), \tag{3.4}$$

are divergence free, coincide with the original fields on $\Omega$, and vanish outside of $\tilde{\Omega}$.

One can solve the Cauchy problem for the Maxwell equations with the new initial data (3.4) to obtain the associated space-time fields, if desired.

The question then arises[1] if every such configuration can be realized by an experimentalist in the lab. Here, "an experimentalist" is defined as someone whose laboratory equipment can produce any desired electric charges $\rho$ and currents $\vec{j}$ subject to the conservation law

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{j} = 0. \tag{3.5}$$

These, in turn, will produce Maxwell fields as dictated by the Maxwell equations written in their tensorial special-relativistic form:

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu, \quad \text{where } (j^\mu) = (\rho, \vec{j}). \tag{3.6}$$

More specifically, let us describe the lab as the following "world volume":

$$\tilde{\mathcal{U}} \coloneqq [t_0, t_3] \times \tilde{\Omega} \subset \mathbb{R}^4.$$

The region within the lab where the desired Maxwell fields need to be produced will be the set

$$\mathcal{U} \coloneqq [t_1, t_2] \times \Omega \subset \tilde{\mathcal{U}},$$

with $t_0 < t_1 \le t_2 < t_3$ and $\bar{\Omega} \subset \tilde{\Omega}$. Let $F_{\mu\nu}$ be a source-free solution of the Maxwell equations in $\mathcal{U}$, as needed to carry out the desired experiments.

The following prescription tells us what the charges and currents outside of $\mathcal{U}$ are which will produce a Maxwell field $\tilde{F}_{\mu\nu}$ coinciding with $F_{\mu\nu}$ in $\mathcal{U}$, out of a vacuum configuration at $t \le t_0$: let $\chi_\mathcal{U}$ be a smooth function which is identically 1 on $\mathcal{U}$ and which vanishes outside of $\tilde{\mathcal{U}}$. Let $A_\mu$ be any 4-vector potential associated with $F_{\mu\nu}$, e.g.,

$$A_\mu(x^\alpha) = x^\nu \int_0^1 F_{\nu\mu}(\lambda x^\alpha)\lambda d\lambda. \tag{3.7}$$

Set $\tilde{A}_\mu = \chi_\mathcal{U} A_\mu$, $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$ and

$$j^\mu \coloneqq \frac{1}{4\pi} \partial_\nu \tilde{F}^{\mu\nu}. \tag{3.8}$$

Then, $\tilde{F}_{\alpha\beta}$ vanishes outside of the lab world volume $\tilde{\mathcal{U}}$ and coincides with the desired field $F_{\alpha\beta}$ in the world volume $\mathcal{U}$ of the experiment. If the experimenter can produce the 4-current (3.8) with her apparatus, she will be able to create the desired Maxwell field in the region where the experiment will take place.

It would be of interest to devise an analogous procedure for the gravitational field, keeping in mind the supplementary difficulty of maintaining positivity of the energy density.

## IV. WEYL TENSOR FORMULATION

As is well known, the vacuum Einstein equations imply a system of equations for the metric and the Weyl tensor [10],

$$\nabla_\mu C^\mu{}_{\alpha\beta\gamma} = 0, \tag{4.1}$$

which implies a symmetrizable-hyperbolic system of equations in dimension $1+3$ (cf., e.g., Ref. [11]). In the linearized case, the equations for the metric and the Weyl tensor decouple, so that one can consider the Weyl tensor equations linearized on Minkowski space-time on their own. We show in Appendix C the equivalence of this approach to the metric one, in the sense that a linearized Weyl tensor is always accompanied by a linearized metric (the reverse property being obvious).

In space-time dimension 4, Eq. (C1) can be rewritten in Maxwell-like form. In this approach (compare Ref. [10]), the evolution equations for two symmetric trace-free tensors $E_{ij}$ and $B_{ij}$,

$$\partial_t E_{ij} = -\epsilon_i{}^{k\ell} \partial_k B_{\ell j}, \qquad \partial_t B_{ij} = \epsilon_i{}^{k\ell} \partial_k E_{\ell j}, \tag{4.2}$$

are complemented by the constraint equations

[1]We are grateful to Peter Aichelburg for pointing out the issue to us.

$$D^i E_{ij} = 0 = D^i B_{ij}. \tag{4.3}$$

Here, $E_{ij}$ is the electric part, and $B_{ij}$ is the magnetic part of the Weyl tensor:

$$E_{ij} = C_{0i0j}, \qquad B_{ij} = \star C_{0i0j}, \tag{4.4}$$

with

$$\star C_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu} C_{\mu\nu\gamma\delta}.$$

The symmetry and tracelessness of $E_{ij}$ as well as tracelessness of $B_{ij}$ are obvious from the symmetries of the Weyl tensor. The symmetry of $B_{ij}$ follows from the less-obvious double-dual symmetry of the Weyl tensor (cf., e.g., Ref. [12], Proposition 4.1),

$$\epsilon_{\alpha\beta}{}^{\mu\nu} C_{\mu\nu\gamma\delta} = \epsilon_{\gamma\delta}{}^{\mu\nu} C_{\mu\nu\alpha\beta}.$$

We show in Appendix C how the vanishing of the divergence of $E_{ij}$ relates to the linearized scalar constraint equation and how the symmetry of $B_{ij}$ relates to the vector constraint equation.

Since both $E_{ij}$ and $B_{ij}$ are transverse and traceless, each of them comes with its own third-order potential $u_{ij}$ as described in Sec. II B, so that shieldings and gluings can be performed on each of them directly, without having to invoke the metric tensor.

## ACKNOWLEDGMENTS

This work was supported in part by the Austrian Science Fund (FWF) Project No. P29517-N16. Useful discussions with Peter Aichelburg and Jérémie Joudioux are gratefully acknowledged.

## APPENDIX A: INTEGRATING 2-FORMS ON $\mathbb{R}^3$

In this Appendix, we address the question of asymptotic behavior of potentials for closed 2-forms. The analysis below has obvious generalizations to $p$-forms on $\mathbb{R}^n$ $(n > 3)$ with $1 < p < n$.

Lemma A.1 Let $\omega_{ij}(x) = \omega_{[ij]}(x)$ be a closed 2-form on $\mathbb{R}^3$ with $\omega_{ij} = O(r^\sigma)$, $\alpha \in \mathbb{R}$. Then, there exists a 1-form $\omega_i(x)$ with $\partial_{[i}\omega_{j]} = \omega_{ij}$ satisfying $\omega_i(x) = O(r^{1+\sigma})$ if $\sigma \neq -2$, $\omega_i(x) = O(r^{-1}\ln r)$ otherwise.

Proof: Consider first the case $\sigma \geq -2$. Then,

$$\omega_i(x) = 2x^j \int_0^1 \omega_{ji}(\lambda x)\lambda d\lambda = O(r^{1+\alpha}) \quad \text{when } \sigma > -2, \tag{A1}$$

and $\omega_i(x) = O(r^{-1}\ln r)$ when $\sigma = -2$. To see this, use spherical coordinates $(r, \theta, \varphi)$ in the argument of $\omega_{ij}$, and substitute $s/r$ for $\lambda$. When $\sigma < -2$, consider

$$\mu_i(x) = -2x^j \int_1^\infty \omega_{ji}(\lambda x)\lambda d\lambda, \tag{A2}$$

which converges and has the right decay at infinity but blows up at the origin. The previous expression $\omega_i$ is still defined and, in the shell $\overline{B(2,0)}\backslash B(1,0)$, differs from $\mu_i$ by a closed 1-form. Since this shell is simply connected, this difference $\Delta_i \coloneqq \omega_i - \mu_i$ satisfies $\Delta_i = \partial_i f$ for some function $f$. Now, extend $f$ smoothly to a function $F$ on all of $B(2,0)$. Then, the 1-form given by $\omega_i + \partial_i F$ in the interior and by $\mu_i$ in the exterior satisfies our requirements. □

An essentially identical argument shows that if $\omega_{ij}$ has compact support, then $\omega_i$ can also be chosen with compact support (which also follows from standard results in algebraic topology (Ref. [13], Corollary 4.7.1).

## APPENDIX B: CONSTRUCTION OF THE POTENTIAL $u$

For the convenience of the reader, we review the construction in Ref. [7] and take this opportunity to correct a minor mistake in the presentation there, namely the second sentence after (3.12) there. Let us define

$$\tau_{ijk} \coloneqq \epsilon_{ij}{}^l h_{lk}. \tag{B1}$$

Since $D_{[i}\tau_{jk]l} = \frac{1}{3}\epsilon_{ijk} D_m h^m{}_l = 0$, there exists a tensor field $U_{ij}$ such that

$$\tau_{ijk} \coloneqq D_{[i} U_{j]k}. \tag{B2}$$

Symmetry of $h_{ij}$ implies that all traces of $\tau_{ijk}$ vanish, which implies in turn that

$$D_{[l} U_i{}^{[k}\delta_{j]}{}^{l]} = 0. \tag{B3}$$

Hence, there exists a tensor field $U_{ijk}$, which can be chosen to be antisymmetric in $jk$, so that

$$D_{[i} U_{j]}{}^{kl} + U_{[i}{}^{[k}\delta_{j]}{}^{l]} = 0. \tag{B4}$$

From the tracelessness of $h_{ij}$, one finds $\tau_{[ijk]} = 0$, which shows that there exists a vector field $V_i$ such that

$$-\frac{1}{3} U_{[jk]} + D_{[j} V_{k]} = 0. \tag{B5}$$

Equations (B4) and (B5) together with some algebra give

$$D_{[l}(2U_{ij]}{}^k - 3V_i \delta_{j]}{}^k) = 0, \tag{B6}$$

which implies the existence of a potential $V_{ij}$:

$$\frac{2}{3} U_{[ij]}{}^k - V_{[i}\delta_{j]}{}^k + D_{[i} V_{j]}{}^k = 0. \tag{B7}$$

Setting

$$u_{ij} := -3V_{(ij)} + \delta_{ij}V_k{}^k,$$

a lengthy calculation shows that

$$U_{ij} = 3D_iV_j + \frac{1}{2}g_{ij}D^kD^l u_{kl} + \Delta u_{ij} - 2D^kD_{(i}u_{j)k} - D_iD_jV_d{}^d. \tag{B8}$$

Thus, neither $V_{[ij]}$ nor $V_i{}^i$ nor $V_i$ contributes to $D_{[i}U_{j]k}$, and we finally obtain (2.30). For (2.31), we have to successively write down expressions for (i) $U_{ij}$, (ii) $(U_{ijk}, V_i)$, and (iii) $V_{ij}$, at each step using formula (A.1), and take the symmetric, trace-free part of $-3V_{ij}$ at the end. In going from i to ii and ii to iii, one uses the identities

$$\int_0^1\int_0^1 F(\lambda\lambda' x)\lambda\lambda'^2 d\lambda d\lambda' = \int_0^1 F(\lambda x)\lambda(1-\lambda)d\lambda \tag{B9}$$

and

$$\int_0^1\int_0^1 F(\lambda\lambda' x)\lambda(1-\lambda)\lambda'^3 d\lambda d\lambda' = \int_0^1 F(\lambda x)\lambda\frac{(1-\lambda)^2}{2}d\lambda, \tag{B10}$$

respectively. The rest is index gymnastics.

If $h_{ij} = O(r^\sigma)$ for large $r$, from what has been said here and in Appendix A, or by analyzing (2.31) for $\sigma \geq -4$, we find that $u_{ij}$ can be chosen to be of $O(r^{\sigma+3})$ when $\sigma \notin \{-4, -3, -2\}$ and $u_{ij}$ of $O(r^{\sigma+3}\ln r)$ otherwise. Furthermore, if $h_{ij}$ is compactly supported, then $u_{ij}$ can also be chosen to be compactly supported.

We end this Appendix with an analysis of the kernel of $P$ on a simply connected region. For this, we follow through the steps starting from (B2) with $\tau_{ijk} = 0$, which implies the existence of a potential $M_i$ such that

$$U_{ij} = D_iM_j. \tag{B11}$$

Next, from (B4), there exists an antisymmetric tensor field $M^{kl} = M^{[kl]}$ such that

$$U_j{}^{kl} + M^{[k}\delta_j{}^{l]} = D_jM^{kl}. \tag{B12}$$

Equation (B5) implies the existence of a function $\phi$ such that

$$V_i - \frac{1}{3}M_i = D_i\phi. \tag{B13}$$

Inserting into (B7), we find that the terms involving $M_i$ cancel so that

$$D_{[i}\left(\frac{2}{3}M_{j]}{}^k + V_{j]}{}^k - \phi\delta_{j]}{}^k\right) = 0. \tag{B14}$$

Consequently,

$$V_{ij} = -\frac{2}{3}M_{ij} + \phi\delta_{ij} + D_iN_j, \tag{B15}$$

so that

$$V_{(ij)} - \frac{1}{3}V_k{}^k = D_{(i}N_{j)} - \frac{1}{3}D_kN^k. \tag{B16}$$

Setting $\lambda_i = -3N_i/2$, we conclude that any tensor field satisfying $P(u) = 0$ on a simply connected region can be written as

$$u_{ij} = D_i\lambda_j + D_j\lambda_i - \frac{2}{3}D^k\lambda_k g_{ij}. \tag{B17}$$

## APPENDIX C: POTENTIAL FOR THE LINEARIZED RIEMANN TENSOR

In this Appendix, we show that every linearized Riemann tensor on a star-shaped subset of $\mathbb{R}^d$ arises from a linearized metric $h_{\mu\nu}$, in arbitrary dimension $> 2$, where $h_{\mu\nu}$ is defined uniquely up to the usual gauge transformations. We leave it as an exercise to the reader to obtain an explicit formula for $h_{\mu\nu}$ by following the steps of our calculation below.

Suppose, thus, that $R_{\mu\nu\rho\sigma}$ is a field on Minkowski spacetime having the algebraic symmetries of the Riemann tensor and satisfying the Bianchi identity $\partial_{[\mu}R_{\nu\rho]\sigma\tau} = 0$. Then,

$$R_{\mu\nu\rho\sigma} = \partial_{[\mu}F_{\nu]\rho\sigma} \tag{C1}$$

with $F_{\mu\nu\rho} = F_{\mu[\nu\rho]}$. But, since $R_{[\mu\nu\rho]\sigma} = 0$,

$$F_{[\mu\nu]\rho} = \partial_{[\mu}H_{\nu]\rho}. \tag{C2}$$

Inserting the identity

$$F_{\nu\rho\sigma} = F_{[\sigma\nu]\rho} + F_{[\sigma\rho]\nu} - F_{[\rho\nu]\sigma} \tag{C3}$$

into (C2) and the resulting equation into (C1), we find the identity,

$$R_{\mu\nu\rho\sigma} = 2\partial_{[\mu}h_{\nu][\rho,\sigma]}, \tag{C4}$$

where

$$h_{\mu\nu} = H_{(\mu\nu)}.$$

The right-hand side of (C4) multiplied by $\epsilon$ is, up to $O(\epsilon^2)$ terms, the Riemann tensor of the metric $\eta_{\mu\nu} + \epsilon h_{\mu\nu}$. Equivalently, $R_{\mu\nu\rho\sigma}$ is the linearized Riemann tensor associated with $h_{\mu\nu}$.

The addition of a pure-trace tensor to $h$ does not change the trace-free part of $R_{\mu\nu\rho\sigma}$. So, for a tensor $C_{\mu\nu\rho\sigma}$ with Weyl symmetries satisfying $\partial_{[\mu}C_{\nu\rho]\sigma\tau}=0$, there exists a second-order potential $h_{\mu\nu}$ as in (C4), which is trace free.

It is instructive to show the equivalence of (C1) to the metric formulation of the theory. For this, we note that, in space-time dimension 4, Eq. (C1) is equivalent to (Ref. [12], Proposition 4.3)

$$\partial_{[\alpha}C_{\beta\gamma]\mu\nu}=0. \tag{C5}$$

As already pointed out, this implies the existence of a symmetric tensor field $h_{\mu\nu}$ such that

$$C_{\mu\nu\rho\sigma}=2\partial_{[\mu}h_{\nu][\rho,\sigma]}. \tag{C6}$$

But the right-hand side of (C6) is the linearized Riemann tensor associated with the linearized metric perturbation $h_{\mu\nu}$. Since the left-hand side of (C6) has vanishing traces, we conclude that the linearized Ricci tensor associated with $h_{\mu\nu}$ vanishes. Equivalently, $h_{\mu\nu}$ satisfies the linearized Einstein equations.

To understand the nature of the divergence constraint $D^iE_{ij}=0$, let us denote by $r_{ijkl}$ the linearized Riemann tensor of the three-dimensional metric $\delta_{ij}+h_{ij}$, with the associated linearized Ricci tensor $r_{ij}=r^k{}_{ikj}$. We have just seen that $C_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}$ for solutions of $\partial_\alpha C^\alpha{}_{\beta\gamma\delta}=0$, which gives for such solutions

$$0=R_{ij}=R^\alpha{}_{i\alpha j}=C^\alpha{}_{i\alpha j}=-C_{0i0j}+r_{ij}=-E_{ij}+r_{ij}. \tag{C7}$$

Here, we have used the fact that the three-dimensional Riemann tensor differs from the four-dimensional one by quadratic terms in the extrinsic curvature; hence, both tensors coincide when linearized at Minkowski space-time. The vanishing of the divergence of the Einstein tensor implies

$$D^ir_{ij}=\frac{1}{2}D_jr,$$

which together with (C7) shows that the constraint equation $D^iE_{ij}=0$ is, for asymptotically flat solutions, equivalent to the linearized scalar constraint $r=0$.

Let us show that symmetry of $B_{ij}$ is equivalent to the vector constraint equation. For this, let

$$k_{ij}=\frac{1}{2}(\partial_0h_{ij}-\partial_ih_{0j}-\partial_jh_{0i})$$

denote the linearized extrinsic curvature tensor of the slices $t=\text{const}$. By a direct calculation, or by linearizing the relevant embedding equations, we find

$$R_{0ij\ell}=\partial_\ell k_{ij}-\partial_jk_{i\ell}. \tag{C8}$$

Again, for solutions of $\partial_\alpha C^\alpha{}_{\beta\gamma\delta}=0$, it holds that

$$\begin{aligned}\epsilon^{n\ell m}B_{\ell m}&=\frac{1}{2}\epsilon^{n\ell m}\epsilon_{mrs}C_{0\ell}{}^{rs}=\frac{1}{2}\epsilon^{n\ell m}\epsilon_{mrs}R_{0\ell}{}^{rs}\\&=2\delta_r^{[n}\delta_s^{\ell]}D^sk_\ell{}^r\\&=D^\ell(k_\ell{}^n-k^m{}_m\delta^n_\ell),\end{aligned} \tag{C9}$$

as claimed.

Let us finally consider the kernel of the map sending $h_{\mu\nu}$ into $R_{\mu\nu\rho\sigma}$. Namely, when $R_{\mu\nu\rho\sigma}=0$, from (C4), we infer

$$h_{\mu[\nu,\rho]}=\partial_\mu A_{\nu\rho}, \tag{C10}$$

where $A_{\nu\rho}=A_{[\nu\rho]}$. But, since $\partial_{[\mu}A_{\nu\rho]}=0$,

$$A_{\mu\nu}=\partial_{[\mu}B_{\nu]}. \tag{C11}$$

Now, defining $k_{\mu\nu}=h_{\mu\nu}+\partial_\mu B_\nu$, we have the result

$$k_{\mu[\nu,\rho]}=h_{\mu[\nu,\rho]}+\partial_\mu\partial_{[\rho}B_{\nu]}=0, \tag{C12}$$

so that $k_{\mu\nu}=\partial_\mu D_\nu$, whence $h_{\mu\nu}=\partial_\mu(D_\nu-B_\nu)$. Finally, using the symmetry of $h_{\mu\nu}$, it follows that

$$h_{\mu\nu}=\partial_{(\mu}\Lambda_{\nu)} \tag{C13}$$

with $\Lambda_\mu=D_\mu-B_\mu$.


[1] A. Carlotto and R. Schoen, Localizing solutions of the Einstein constraint equations, Inventiones Mathematicae **205**, 559 (2016).

[2] P. T. Chruściel and E. Delay, On Carlotto-Schoen-type scalar-curvature gluings, arXiv:1611.00893.

[3] P. T. Chruściel, Anti-gravity à la Carlotto-Schoen, Séminaire Bourbaki **1120**, 1 (2016), arXiv:1611.01808.

[4] J. Corvino and D. Pollack, *Scalar Curvature and the Einstein Constraint Equations*, Surveys in Geometric Analysis and Relativity, Advanced Lectures in Mathematics (ALM) (International Press, Somerville, MA, 2011), Vol. 20, p. 145.

[5] D. Christodoulou and N. Ó. Murchadha, The boost problem in general relativity, Commun. Math. Phys. **80**, 271 (1981).

[6] R. L. Arnowitt, S. Deser, and C. W. Misner, The Dynamics of general relativity, Gen. Relativ. Gravit. **40**, 1997 (2008).

[7] R. Beig, *Mathematics of Gravitation, Part I* (Banach Center Publ., Polish Acad. Sci., Warsaw, 1997), Vol. 41, p. 109.

[8] R. Bartnik, Quasi-spherical metrics and prescribed scalar curvature, J. Diff. Geom. **37**, 31 (1993).

[9] E. Delay, Smooth compactly supported solutions of some underdetermined elliptic PDE, with gluing applications, Commun. Partial Differ. Equ. **37**, 1689 (2012).

[10] R. Geroch, in *Proceedings of the Symposium on Asymptotic Structure of Space-Time, University of Cincinnati, Cincinnati, OH, 1976* (Plenum, New York, 1977), p. 1.

[11] H. Friedrich, On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations, Proc. R. Soc. A **375**, 169 (1981).

[12] D. Christodoulou and S. Klainerman, Asymptotic properties of linear field equations in Minkowski space, Commun. Pure Appl. Math. **43**, 137 (1990).

[13] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics (Springer-Verlag, Berlin, 1982), Vol. 82.