

**Maximum magnetic moment to angular momentum conjecture**John D. Barrow<sup>1</sup> and G. W. Gibbons<sup>1,2,3</sup><sup>1</sup>*DAMTP, Centre for Mathematical Sciences, University of Cambridge,  
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Conjectures play a central role in theoretical physics, especially those that assert an upper bound to some dimensionless ratio of physical quantities. In this paper we introduce a new such conjecture bounding the ratio of the magnetic moment to angular momentum in nature. We also discuss the current status of some old bounds on dimensionless and dimensional quantities in arbitrary spatial dimension. Our new conjecture is that the dimensionless Schuster-Wilson-Blackett number,  $c\mu/JG^{\frac{1}{2}}$ , where  $\mu$  is the magnetic moment and  $J$  is the angular momentum, is bounded above by a number of order unity. We verify that such a bound holds for charged rotating black holes in those theories for which exact solutions are available, including the Einstein-Maxwell theory, Kaluza-Klein theory, the Kerr-Sen black hole, and the so-called STU family of charged rotating supergravity black holes. We also discuss the current status of the maximum tension conjecture, the Dyson luminosity bound, and Thorne's hoop conjecture.

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**I. INTRODUCTION**

Regardless of what one thinks of the debate concerning the relative merits of the traditional Baconian or inductionist versus Bayesian or Popperian viewpoints about the nature of science, few would disagree that making precisely stated conjectures or exhibiting counterexamples has an important place in theoretical physics. In making such conjectures it is important to bare in mind that although it is frequently convenient to adopt units well suited to practical aspects of the subject being discussed, any physically meaningful statement must be independent of an arbitrary choice of units. In fact, adopting an appropriate set of “natural units” can afford insights which may be otherwise obscured. In this paper we are led in Sec. II, by our consideration of natural units for physical quantities which are independent of Planck's constant, to conjecture new fundamental bounds on dimensionless quantities in classical gravitation, in particular that there is an upper bound on the magnetic moment to angular momentum ratio. In Sec. III, we verify that such a bound holds for charged rotating black holes in those theories for which exact solutions are available, including the Einstein-Maxwell theory, Kaluza-Klein theory, the Kerr-Sen black hole, and the so-called STU family of charged rotating supergravity black holes. We discuss the current status of the maximum tension conjecture in Sec. IV, the Dyson luminosity bound in Sec. V, and new approaches to Thorne's hoop conjecture in Sec. VI.

**II. UNITS AND DIMENSIONAL ANALYSES**

Natural units were first introduced into physics and metrology by George Johnstone Stoney at the British

Association Meeting in 1874, in an attempt to cut through the proliferation of parochial units of measurement spawned by the industrial revolution and the expansion of Victorian engineering and commerce [1,2]. He sought to devise units that, unlike feet and horsepower, avoided any anthropomorphic benchmark, and made no use of changing parochial standards, like days or standard weights. A similar universal approach had also been advocated by Maxwell in 1870, who suggested that constants be founded on atomic or optical standards [2,3]. He also saw a new opportunity to promote his prediction of a new elementary particle, which he first dubbed the “electrolion” in 1881 and then renamed the “electron” in 1894, carrying a basic unit of electric charge,  $e$ , whose numerical value he predicted using Faraday's law and Avogadro's number. The electron was subsequently discovered by Thomson in 1897, and Stoney remains the only person to have successfully predicted the numerical value of a new fundamental constant of physics.

**A. Stoney units**

In response to a challenge from the British Association to reduce or organize the plethora of special units that had sprung up to service the industrial revolution and Britain's trading empire, in 1874 Johnstone Stoney first introduced a system of “natural units” of mass, length and time using the speed of light  $c$ , the Newtonian gravitational constant  $G$ , and his proposed electron charge  $e$  [4–6]. Stoney's natural units were

$$M_S = \left(\frac{e^2}{G}\right)^{1/2}, \quad L_S = \left(\frac{Ge^2}{c^4}\right)^{1/2}, \quad T_S = \left(\frac{Ge^2}{c^6}\right)^{1/2}. \quad (1)$$

These were the first natural units. However, we should note that in those days before the theory of special relativity, the speed of light,  $c$ , did not possess the absolute status that it would later assume and  $e$  was still just a hobby horse of Stoney's (for some context see the history in Ref. [7]).

### B. Planck units

In 1899, a similar idea was introduced by Planck [8] to create another set of natural units based on  $c$ ,  $G$ , and  $h$ , the quantum constant of action that bears his name. They differ from Stoney's units by a factor  $\frac{1}{2\pi} \left(\frac{e^2}{\hbar c}\right)^{1/2}$ , the square root of the fine structure constant divided by  $2\pi$ . These units are now commonplace in physics and cosmology and they define units of mass, length and time that combine relativistic, gravitational and quantum aspects of physics:

$$M_{\text{Pl}} = \left(\frac{\hbar c}{G}\right)^{1/2}, \quad L_{\text{Pl}} = \left(\frac{G\hbar}{c^3}\right)^{1/2}, \quad T_{\text{Pl}} = \left(\frac{G\hbar}{c^5}\right)^{1/2}. \quad (2)$$

However, innumerable related Planck units may be constructed for other physical quantities in any number of space dimensions by dimensional analysis. Those involving thermal physics can be included by adding the Boltzmann constant,  $k_B$ , to  $G$ ,  $c$  and  $h$ . Some of the Planck units are especially interesting for classical physics if they do not contain Planck's constant. This signals that they are purely classical in origin and may highlight a limiting physical principle. This is trivially so for the Planck unit of velocity,  $V_{\text{Pl}} = c$ , but less obvious for the Planck units of force  $F_{\text{Pl}} = c^4/G$  and power  $P_{\text{Pl}} = c^5/G$  which are strongly suspected to be maximal quantities in classical physics. It has been conjectured<sup>1</sup> [10–14] that in general relativity (with and without a cosmological constant) there should be a maximum value to any physically attainable force (or tension) given by

$$F_{\text{max}} = \frac{c^4}{4G}, \quad (3)$$

where  $c$  is the velocity of light and  $G$  is the Newtonian gravitational constant. For possible relations to the holographic principle and to quantum clocks, see [15–17].

### C. De Sitter units

If one believes that the observed acceleration of the scale factor of the Universe [18] is due to a classical cosmological constant  $\Lambda$  rather than some form of slowly evolving “dark energy,” with time-dependent density, then a set of absolute de Sitter units of mass, length, and time can be introduced:

<sup>1</sup>For an earlier anticipation of this idea but based on a different physical motivation see [9].

$$M_{dS} = c^2 G^{-1} \Lambda^{-1/2}, \quad L_{dS} = \Lambda^{-1/2}, \quad T_{dS} = c^{-1} \Lambda^{-1/2}. \quad (4)$$

In these units  $c^4/G$  is still the unit of force and the upper bound (3) still appears to hold [13].

### D. Fundamental principles and dimensions

We referred above to “limiting principles,” or what are sometimes called “impotence principles.” In [14] the phrase “maximum tension principle” was used in the usual sense of “fundamental principles,” that is, general statements expected to be true of all viable theories and which may follow as valid consequences of a precisely formulated mathematical statement within any well-defined mathematical theory. Such principles may have heuristic value in motivating and formulating a theory, but cannot be used in themselves to define a theory. For example Heisenberg's uncertainty principle is an elementary theorem in wave mechanics but is insufficient in itself to define wave mechanics. Moreover, it not only rests heavily on translation invariance, but may not hold in more general quantum mechanical theories, such as relativistic quantum field theory, in which the notion of a position observable is problematic. Other examples in general relativity include Mach's principle,<sup>2</sup> equivalence principles, and Thorne's hoop conjecture (to which we return below). Other principles, like the cosmological principle, may be simplifying symmetry assumptions, or approximations, that cannot be precisely true in reality; straightforward methodological principles, like the weak anthropic principle; or various variational principles.

The maximum force conjecture gives rise to the closely related conjecture [20] that there is a maximum power defined by

$$P_{\text{max}} = c F_{\text{max}} = \frac{c^5}{4G}, \quad (5)$$

the so-called Dyson luminosity [21], or some multiple of it [to account for geometrical factors that are  $O(1)$ ]. This will be treated in detail in Sec. IV.

We note that some of the nonquantum Planck units, like the velocity,  $V_{\text{Pl}} = c$ , are independent of the dimension of space but others, like  $F_{\text{Pl}}$ , are not, because in  $N$ -dimensional space the dimensions of  $G$  are  $M^{-1}L^N T^{-2}$ . Thus, in  $N$  dimensions the nonquantum Planck unit is mass  $\times$  (acceleration) $^{N-2}$ , which is only a force when  $N = 3$ , as shown in Ref. [13].

In this paper, we display another physically interesting nonquantum Planck unit formed by the ratio of the magnetic moment of a body,  $\mu$ , to its total angular momentum,  $J$ , and conjecture that classically all bodies satisfy an inequality

<sup>2</sup>For an incisive account of the many inequivalent formulations this can be given see [19].

$$\frac{\mu}{J} < \beta \frac{G^{1/2}}{c}, \quad (6)$$

where  $\beta$  is a numerical factor  $O(1)$ , and we explore the evidence for this maximum bound. Unlike the Planck units of force and power, the Planck unit for the ratio  $\mu/J$  is independent of spatial dimension.

To show this, if we use unrationalized units the dimensions  $[\cdot]$  of magnetic  $Q$  and electric charge  $\bar{Q}$  are the same and are given by the inverse-square laws discovered by Michell and Priestley [22,23], respectively, with

$$[Q] = [\bar{Q}] = M^{1/2} L^{3/2} T^{-1}. \quad (7)$$

The dimensions of a magnetic moment  $\mu$  are therefore

$$[\mu] = M^{1/2} L^{5/2} T^{-1}. \quad (8)$$

Thus, the ratio of magnetic moment to angular momentum  $J$  has dimensions

$$\left[ \frac{\mu}{J} \right] = \left[ \frac{G^{1/2}}{c} \right] = \left[ \frac{Q}{Mc} \right], \quad (9)$$

which is independent of Planck's constant  $\hbar$ . This property continues to hold in  $N$ -dimensional space because there we have  $[Q] = M^{1/2} L^{N/2} T^{-1}$ ,  $[\mu] = M^{1/2} L^{1+N/2} T^{-1}$  and  $[J] = ML^2 T^{-1}$ .

The ratio

$$Z \equiv \frac{Q^2}{GM^2} \quad (10)$$

may be regarded as the separation-independent ratio of the electrostatic repulsion to the gravitational attraction between two identical bodies of mass  $M$  and charge  $Q$ . It has been claimed [24] that Zöllner was the first person to recognize its significance and so one might call it the Zöllner number. A famous, but now discredited, theory of Dirac's predicting the time variation of the gravitation "constant"  $G \propto 1/t$ , with the age of the Universe  $t$  [25], was partly motivated by the very small value of  $Z$  when the mass  $M = m_e$  and charge  $Q = e$  of the electron (or even the proton mass  $m_{pr}$ ) are substituted. This gives

$$N = \frac{e^2}{Gm_e^2} \approx 3 \times 10^{42}, \quad (11)$$

which suggested to Dirac its possible equality (in some yet-to-be-found theory) with the square root of the total number of protons or electrons in the visible universe,  $c^3 t / Gm_e \sim 10^{83}$ , up to a factor  $O(1)$ . In fact, the value of  $N$  and its numerical proximity to the ratio of the classical electron radius to the Hubble radius was first noticed by Weyl in 1919 [5,26] and the numerical "coincidence" is

anthropic because it is equivalent to the statement that the present age of the Universe is of order the main sequence lifetime of a star [5,27].

Classically, we have the Larmor relation

$$\frac{\mu}{J} = \frac{Q}{2Mc} \quad (12)$$

where  $M$  is the mass of a system with charge  $Q$ . More generally, we have

$$\frac{\mu}{J} = g \frac{Q}{2Mc} \quad (13)$$

where  $g$  is the gyromagnetic ratio. Famously, Dirac showed that for electrons  $g = 2$ , at least at lowest order in the fine structure constant  $e^2/\hbar c$  [28], and this value has some significance in supersymmetric theories [29].

After earlier suggestions made by Schuster [30] and Wilson [31], Blackett [32] conjectured that all rotating bodies should acquire a magnetic moment given by

$$\frac{\mu}{J} = \beta \frac{G^{1/2}}{c}, \quad (14)$$

where the dimensionless Schuster-Wilson-Blackett number has  $\beta \approx O(1)$ , and was once regarded as a possible universal constant. Although  $\beta$  is found to be of order unity for a variety of rotating astronomical bodies ranging from the Earth to the Sun and to a variety of stars, as a general statement for macroscopic bodies, the Schuster-Wilson-Blackett conjecture has fallen foul of astronomical data. Yet it remains of interest to inquire whether it provides a natural upper bound for bodies with significant gravitational self-energy.

Since, for electrons

$$\beta = N^{1/2}, \quad (15)$$

no interesting bound holds for the elementary particles. However, it is of interest to ask what is known about  $\beta$  in Einstein-Maxwell and supergravity theories, since for black holes there is typically an upper bound to  $|Q|/G^{1/2}M$  of order unity. For Planck mass particles with charges of order  $e$ , we find  $\beta$  is not far from unity. Such objects can arise in string theory, whose low-energy limit is supergravity theory, so this further motivates the investigation that follows.

### III. THE SCHUSTER-WILSON-BLACKETT NUMBER FOR ELECTRICALLY CHARGED ROTATING BLACK HOLES

Carter first discovered [33] that Kerr-Newman black holes in Einstein-Maxwell theory have a gyromagnetic ratio equal to 2:

$$\frac{\mu}{J} = \frac{Q}{Mc}. \quad (16)$$

Now, to avoid naked singularities, we require (if we assume the black hole has no magnetic charge)

$$GM^2 \geq Q^2 + \frac{J^2}{M^2}. \quad (17)$$

Thus,

$$1 \leq \frac{c^2 \mu^2}{GJ^2} + \frac{J^2}{GM^4}, \quad (18)$$

and so we have the required bound,

$$\left| \frac{\mu}{J} \right| \leq \frac{G^{\frac{1}{2}}}{c}. \quad (19)$$

Hence, we have  $\beta < 1$  for Kerr-Newman black holes. The literature on extensions of Carter's result is quite large. A notable example [34] is a detailed analysis of a current loop surrounding a static black hole. As the loop moves towards the horizon the gyromagnetic ratio smoothly interpolates between the classical value  $g = 1$  and the Carter-Dirac value  $g = 2$ .

It was shown by Reina and Treves [35] that any asymptotically flat solution of the Einstein-Maxwell equations obtained by performing a Harrison transformation on a neutral solution must also have  $g = 2$ . Furthermore, it has been shown [36,37] that, provided any sources obey the constraint that  $G$  times the energy density bounds the charge density, then all asymptotically flat solutions of the Einstein Maxwell equations, possibly with sources of the kind specified which are regular outside a regular event horizon, obey the following Bogomol'nyi bound on the Zöllner number:

$$Z = \frac{Q^2}{GM^2} \leq 1. \quad (20)$$

Combining this with Reina and Treves's result implies

$$\beta < 1. \quad (21)$$

### A. Kerr-Newman AdS black holes

Using the notation of [38], and temporarily setting  $G = c = 1$ , we must distinguish the parameters  $M$ ,  $Q$ ,  $J$  in the spacetime metric from the physical quantities. The latter are denoted by primes. From [39], we introduce

$$M' \equiv \frac{M}{\Xi^2}, \quad J' \equiv \frac{aM}{\Xi^2} \quad (22)$$

where  $\Xi \equiv 1 - \frac{a^2}{l^2}$ .

Aliev gives the physical charge as

$$Q' = \frac{Q}{\Xi}. \quad (23)$$

He finds

$$\mu' = \frac{Qa}{\Xi}, \quad (24)$$

so we have

$$\frac{|\mu'|}{|J'|} = \frac{|Q|}{M} \left( 1 - \frac{a^2}{l^2} \right). \quad (25)$$

Now, for a horizon to exist, we require

$$\Delta_r = \left( 1 + \frac{a^2}{l^2} \right) \left( r^2 - \frac{2Mr}{1 + \frac{a^2}{l^2}} + \frac{Q^2 + a^2}{1 + \frac{a^2}{l^2}} \right) + \frac{r^4}{a^2} \quad (26)$$

to have at least one real root. A *necessary* condition for this is that the quadratic in the first term be negative. This requires

$$\frac{|Q|}{M} < \frac{1}{\sqrt{1 + \frac{a^2}{l^2}}}. \quad (27)$$

Thus, we also require

$$\frac{|\mu'|}{|J'|} < \frac{1 - \frac{a^2}{l^2}}{\sqrt{1 + \frac{a^2}{l^2}}}. \quad (28)$$

Now,

$$\begin{aligned} (1-x)(1+x) &= 1-x^2 \leq 1, \\ \Rightarrow \frac{1-x}{\sqrt{1+x}} &\leq \frac{1}{(1+x)^{\frac{3}{2}}} \leq 1, \end{aligned} \quad (29)$$

so

$$\frac{|\mu'|}{|J'|} < 1. \quad (30)$$

Therefore, we have shown that  $\beta < 1$  for Kerr-Newman-AdS black holes.<sup>3</sup>

<sup>3</sup>Note that since Harrison transformations are not available when the cosmological constant is nonvanishing, there is no analogue of the Reina-Treves result with which to combine the Bogomol'nyi bound of [40] in this case.

### B. Einstein-Maxwell-Dilaton black holes

These have only been discussed for a general dilaton-photon coupling constant  $\alpha$  for the case of slow rotation [41,42]. One has a uniqueness theorem for general  $\alpha$ , angular momentum, electric and magnetic charges [43] provided that  $\alpha^2 \leq 3$ .

In the general slow-rotation case one finds that [41]

$$J = \frac{a}{2} \left( r_+ + \frac{3 - \alpha^2}{3(1 + \alpha^2)} r_- \right), \quad \mu = aQ. \quad (31)$$

If  $a$  is small, then the mass  $M$  and charge  $Q$  are given by

$$M = \frac{1}{2} \left( r_+ + \frac{1 - \alpha^2}{1 + \alpha^2} r_- \right), \quad |Q| = \sqrt{\frac{r_+ r_-}{1 + \alpha^2}}. \quad (32)$$

Since  $r_+ \geq r_- \geq 0$ , we have

$$\frac{|Q|}{M} \leq \sqrt{1 + \alpha^2}, \quad (33)$$

so that in accordance with the Bogomol'nyi bound of [44], this gives

$$M \geq \frac{|Q|}{\sqrt{1 + \alpha^2}}. \quad (34)$$

We have

$$\frac{|J|}{|\mu|} = \frac{1}{2} \sqrt{1 + \alpha^2} \left( \sqrt{\frac{r_+}{r_-}} + \frac{3 - \alpha^2}{3(1 + \alpha^2)} \sqrt{\frac{r_-}{r_+}} \right), \quad (35)$$

so, provided  $\alpha^2 \leq 3$ , this gives

$$\frac{|\mu|}{|J|} \leq \frac{1}{2} \sqrt{1 + \alpha^2} \leq 1. \quad (36)$$

As pointed out in [41], we can then obtain a gyromagnetic ratio:

$$g = 2 - \frac{4\alpha^2 r_-}{(3 - \alpha^2)r_- + 3(1 + \alpha^2)r_+}. \quad (37)$$

### C. Kerr-Kaluza-Klein black holes

The observational and theoretical failures of the old Schuster-Blackett conjecture (14) led some to resort to Kaluza-Klein theory (see [45]). Rotating charged black holes in this theory may be obtained by boosting the neutral Kerr solution (sometimes referred to in this context as a rotating "black string") along the fifth dimension. If  $v$  is the velocity parametrizing the boost, and  $a$  and  $M_s$  the parameters of the original Kerr solution, then in units in which  $G = c = 1$  [46,47], we have

$$M = M_s \left( 1 + \frac{1}{2} \frac{v^2}{1 - v^2} \right), \quad J = \frac{M_s a}{\sqrt{1 - v^2}} \\ Q = M_s \frac{v}{1 - v^2}, \quad \mu = \frac{M_s a v}{\sqrt{1 - v^2}} \quad (38)$$

and the gyromagnetic ratio is  $g = 2 - v^2$ . Restoring units, we have

$$\frac{|\mu|}{|J|} = \frac{G^{\frac{1}{2}}}{c} v, \quad (39)$$

and, remarkably, we see that  $\beta = v/c \leq 1$ .

We may also regard Kaluza-Klein black holes as Einstein-Maxwell-dilaton black holes with  $\alpha = \sqrt{3}$ . For the gyromagnetic ratios of elementary particles in Kaluza-Klein theory and their comparison with black holes, the reader may consult [45,46,48].

### D. Kerr-Sen electrically charged black holes

These black holes satisfy the low-energy equations of motion of heterotic string theory [49] and may be regarded as an Einstein-Maxwell-dilaton black hole with coupling constant  $\alpha = 1$ . According to [49], the mass  $M$ , charge  $Q$ , angular momentum  $J$  and magnetic dipole moment  $\mu$  are given by

$$M = \frac{m}{2} (1 + \cosh \theta), \quad J = Ma \\ Q = \frac{m}{\sqrt{2}} \sinh \theta, \quad \mu = Qa, \quad (40)$$

where  $m$ ,  $a$ ,  $\theta$  are parameters.<sup>4</sup> Thus, we find

$$\beta = \frac{|\mu|}{|J|} = \sqrt{2} \frac{\sinh \theta}{1 + \cosh \theta} = \sqrt{2} \tanh \frac{\theta}{2} \leq \sqrt{2} \quad (41)$$

and

$$g = 2. \quad (42)$$

We also find a Bogomol'nyi inequality (34) where  $\alpha^2 = 1$  is satisfied, that is,

$$\frac{|Q|}{M} \leq \sqrt{2}. \quad (43)$$

### E. STU electrically charged black holes

The electromagnetic properties of a more general family of 4-charged black holes, which are solutions of the so-called STU supergravity theory (characterized by  $S$ ,  $T$ , and  $U$  dualities) are reviewed in [50]. These solutions depend upon four boost parameters,  $\delta_i$ . If  $c_i = \cosh \delta_i$ ,  $s_i = \sinh \delta_i$ ,

<sup>4</sup>Our  $\theta$  is Sen's  $\alpha$ .

$\Pi_c = c_1 c_2 c_3 c_4$ ,  $\Pi_s = s_1 s_2 s_3 s_4$ ,  $\Pi_c^1 = c_2 c_3 c_4$  etc.,  $\Pi_s^1 = s_2 s_3 s_4$  etc., then according to [50]

$$M = \frac{m}{4} \sum_i (c_i^2 + s_i^2), \quad J = ma(\Pi_c - \Pi_s) \quad (44)$$

$$Q_i = 2ms_i c_i, \quad \mu_i = 2ma(s_i \Pi_c^i - c_i \Pi_s^i). \quad (45)$$

Evidently

$$4M \geq \sum_i |Q_i|. \quad (46)$$

We also have

$$\frac{1}{2} \frac{\mu_i}{J} = \frac{s_i \Pi_c^i - c_i \Pi_s^i}{\Pi_c - \Pi_s}. \quad (47)$$

If we assume that  $s_i > 0$ ,  $\forall i$ , then

$$\frac{1}{2} \frac{\mu_i}{J} \leq \frac{s_i(\Pi_c^i - \Pi_s^i)}{c_i(\Pi_c^i - \Pi_s^i)} \leq \tanh \delta_i \leq 1. \quad (48)$$

There are some special cases which coincide with solutions of the Einstein-Maxwell-dilaton theory.

(i) *Einstein-Maxwell black holes*:  $\delta_1 = \delta_2 = \delta_3 = \delta_4$ ,  $Q_i = Q$ ,  $\mu_1 = \mu$ .

Thus, from (45), we have

$$M = m \cosh 2\delta, \quad J = ma \cosh 2\delta, \quad (49)$$

$$Q_i = m \sinh 2\delta, \quad \mu_i = ma \sinh 2\delta, \quad (50)$$

and if we set  $Q = Q_i$  and  $\mu = \mu_i$  so that

$$Q^2 = \frac{1}{4}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2), \quad (51)$$

then we find that  $g = 2$  and

$$\frac{|\mu|}{|J|} = \tanh 2\delta = \frac{|Q|}{M} \leq 1. \quad (52)$$

Evidently both the charge inequality (46) and the dipole inequality (48) are satisfied, the latter by a comfortable margin since for  $x > 0$ ,

$$\tanh 2x \leq 2 \tanh x. \quad (53)$$

(ii) *Kerr-Kaluza-Klein electrically charged black holes*:  $\delta_2 = \delta_3 = \delta_4 = 0$ ,  $\delta_1 = \delta$ .

$$M = \frac{m}{4}(3 + \cosh 2\delta), \quad J = ma \cosh \delta \quad (54)$$

$$Q_1 = m \sinh 2\delta, \quad \mu_1 = 2ma \sinh \delta. \quad (55)$$

Now, if  $v = \tanh \delta$  then from (40) we have

$$M = \frac{M_s}{4}(3 + \cosh 2\delta), \quad J = M_s a \cosh \delta$$

$$Q = \frac{M_s}{2} \sinh 2\delta, \quad \mu = M_s a \sinh \delta. \quad (56)$$

Thus,  $M_s = m$ ,  $Q = \frac{1}{2} Q_1$  and  $\mu = \frac{1}{2} \mu_1$ , so that we have

$$Q^2 = \frac{1}{4} Q_1^2, \quad (57)$$

and we find that

$$\beta = \frac{|\mu|}{|J|} = \tanh \delta \leq 1. \quad (58)$$

We also have

$$M \geq \frac{1}{2} |Q|, \quad (59)$$

which is consistent with (34) provided that  $\alpha^2 = 3$ .

(iii) *String theory*:  $\delta_1 = \delta_2 = \delta$ ,  $\delta_3 = \delta_4 = 0$ .

Now, we have

$$M = \frac{1}{2} m(1 + \cosh 2\delta),$$

$$J = \frac{1}{2} ma(1 + \cosh 2\delta), \quad (60)$$

$$Q_1 = Q_2 = m \sinh 2\delta,$$

$$\mu_1 = \mu_2 = 2ma \sinh \delta \cosh \delta, \quad (61)$$

and if we set  $Q_1 = Q_2 = \sqrt{2}Q$  and  $\mu_1 = \mu_2 = \sqrt{2}\mu$ , so that

$$Q^2 = \frac{1}{4}(Q_1^2 + Q_2^2), \quad (62)$$

we obtain

$$\frac{|\mu|}{|J|} = \sqrt{2} \tanh \delta \quad (63)$$

and

$$\frac{|Q|}{M} = \sqrt{2} \tanh \delta \leq \sqrt{2}. \quad (64)$$

We also find that  $g = 2$  and obtain consistency with (34) and agreement with (41) provided  $\theta = 2\delta$ .

Note that for all these special cases, the conversion from the conventions of [50] and standard (Gaussian) units is

$$Q^2 = \frac{1}{4} \sum_i Q_i^2. \quad (65)$$

#### IV. THE DYSON BOUND

The importance of some multiple of  $c^5/G$  in studies of gravitational radiation appears to have first been noticed in a paper of Dyson [21]. He observed, by a scaling argument, that the luminosity in gravitational radiation of an orbiting binary star system according to Einstein's linearized theory of gravitational radiation must be a dimensionless multiple of  $c^5/G$  (see below for a more precise statement). Subsequently, Thorne [51] introduced it into modern studies of possible sources of gravitational radiation, linear or nonlinear, detectable on Earth using current technology. Thorne's paper seems to have introduced the idea of a Dyson bound [52,53]: a maximum possible luminosity in gravitational waves.<sup>5</sup>

Six years after [21], Dyson wrote a short note [54] posing a question, the answer to which was supplied by Hawking's famous area theorem [55]. It seems reasonable therefore to suggest (see footnote 9 of [56]) that  $c^5/G$ , or some multiple of it, be called "one Dyson." If one accepts this, the maximum luminosity of GW150914 (or the orbital merger of any equal-mass nonspinning black hole) is about 1 milli-Dyson.

#### V. THE MAXIMUM TENSION PRINCIPLE

Independent of these considerations, in an article written for the Festschrift celebrating the 60th birthday of the late Jacob Bekenstein [14], it was conjectured that  $c^4/4G$  was the maximum possible tension or force in classical general relativity. Dimensionally, this makes sense. The Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (66)$$

The Ricci tensor  $R_{\mu\nu}$  has dimensions  $L^{-2}$  and every component of the energy-momentum tensor,  $T_{\mu\nu}$ , has dimensions force per unit area and the Einstein constant  $8\pi G/c^4$  has units of an inverse force.

Some heuristic arguments in favor of this maximum tension conjecture were given in [14] and the factor of  $\frac{1}{4}$

<sup>5</sup>In reply to an inquiry by Christoph Schiller, Dyson replied, "It is not true that I proposed the formula  $c^5/G$  as a luminosity limit for anything. I make no such claim. Perhaps this notion arose from a paper that I wrote in 1962 with the title, 'Gravitational Machines,' published as Chapter 12 in the book, 'Interstellar Communication' edited by Alastair Cameron, [New York, Benjamin, 1963]. Equation (11) in that paper is the well-known formula  $128V^{10}/5Gc^5$  for the power in gravitational waves emitted by a binary star with two equal masses moving in a circular orbit with velocity  $V$ . As  $V$  approaches its upper limit  $c$ , this gravitational power approaches the upper limit  $128c^5/5G$ . The remarkable thing about this upper limit is that it is independent of the masses of the stars. It may be of some relevance to the theory of gamma-ray bursts."

justified by reference to conical singularities and the requirement that the deficit angles of cosmic strings do not exceed  $2\pi$  radians. In fact, the deficit angle is subject to a so-called Bogomol'nyi bound [57] in this case. The extensions in the presence of a cosmological constant were given in [13] but as yet there exists no formal proof, or indeed precise mathematical formulation. Further work on the maximum tension (or force) principle may be found in [10–12,58]. Earlier suggestions regarding a maximum force then came to light. In [9], the authors claimed that  $c^4/4G$  is the maximal force allowed in general relativity and in [20] made the obvious maximal power hypothesis that  $c^5/4G$  is the maximum power allowed in nature. Neither paper makes any reference to [21] or [51]. There are also earlier (unseen) papers on this subject [59,60], whose titles clearly indicate that the author had the same order of magnitude for the maximal force and maximal power in mind [61].

#### VI. THORNE'S HOOP CONJECTURE

The proposed Dyson bound and the maximum tension principle resemble another, as yet unresolved but possibly related issue: how does one formulate in a precise way Thorne's hoop conjecture [62]? Recently, there has been some progress in this direction.

In [63], a precise candidate was proposed for the hoop radius of an apparent horizon in terms of its Birkhoff invariant  $\beta_b$ . The conjecture was that every apparent horizon should satisfy

$$\beta_b \leq 4\pi GM_{\text{ADM}}/c^2, \quad (67)$$

where  $M_{\text{ADM}}$  is the Arnowitt-Deser-Misner (ADM) mass of the system. In [64], considerable support was marshalled for (67) using known exact solutions of supergravity theories. However, more recently, a counterexample was constructed using time-symmetric vacuum initial data in Ref. [65]. Following suggestions in [66,67], one may then reformulate the hoop conjecture as

$$\beta_b \leq 2\pi GM_{\text{BY}}/c^2, \quad (68)$$

where  $M_{\text{BY}}$  is the Brown-York quasilocal mass [68,69] of the apparent horizon. Note that the Brown-York quasilocal mass is only defined for time-symmetric data. Using a result of Paiva [70],<sup>6</sup> one may check that (68) holds for all initial data sets constructed in Ref. [65]. For a proof in Robinson-Trautmann metrics, see Ref. [72].

The Brown-York quasilocal mass [68,69] of the apparent horizon, which is assumed to have positive Gaussian

<sup>6</sup>There is an earlier and weaker result due to Croke [71] which may possibly prove to be of use in the present context.

curvature, therefore admits a unique (up to rigid motions) isometric embedding into Euclidean space as a convex body. The definition of the Brown-York mass inside any 2-surface,  $S$ , lying in a Cauchy surface  $\{\Sigma, \hat{g}\}$  is

$$M_{BY} = \frac{1}{8\pi} \int_S (k_0 - k) dA(S, \hat{g}|_S), \quad (69)$$

where  $k$  is the trace of the fundamental form of  $S$  considered as embedded in  $\{\Sigma, \hat{g}\}$  and  $k_0$  is the trace of the fundamental form of  $\{S, g\}$  when isometrically embedded in Euclidean space  $\{\mathbb{E}^3, \delta_{ij}\}$ , for which we have simply  $dA(S, \hat{g}|_S) = dA(S, \delta_{ij}|_S) = dA$ . From a spacetime point of view, the Brown-York mass depends on both how the spacelike surface  $S$  sits in spacetime  $\{M, g_{\mu\nu}\}$  (it has two fundamental forms) and also the spacelike hypersurface  $\Sigma$  passing through it (which picks out a linear combination of its two second fundamental forms). The Brown-York mass is believed to be a ‘‘quasilocal’’ measure of the amount of ‘‘energy’’ on  $\Sigma$  inside  $S$  [73]. The York-Brown mass suffers from a number of shortcomings but in the present context has been shown that it is positive [74].

Among the shortcomings of the Brown-York mass is that it requires that the surface  $S$  admit an isometric embedding into three-dimensional Euclidean space. This is not possible for the horizon of all Kerr black holes. Embeddings into four-dimensional Euclidean space are known but are believed not to be unique. There exists a unique isometric embedding into hyperbolic three-space [75] and hence a (presumably not unique) embedding into four-dimensional Minkowski spacetime.

The converse of the hoop conjecture remains to be considered; that is, the question that if some surface  $S$  satisfies

$$\beta_b \leq \frac{2\pi GM_{BY}}{c^2}, \quad (70)$$

then must  $S$  be, or lie inside, an apparent horizon? The various papers of Shi and Tam [74,76–82] contain some relevant results here.

### A. Relation to work of Tod

Tod [83] has looked at the hoop conjecture from the point of view of a collapsing shell construction for which an isometric embedding is possible, and seeks to define the hoop radius in terms of a *maximum shadow circumference*  $C_m$ . This is defined as the supremum of the circumference of all orthogonal projections of the surface. He finds that

$$\frac{\pi}{2} C \leq \frac{1}{2} \int k_0 dA \leq 2C_m, \quad (71)$$

where the upper bound is attained for any surface of constant breadth.

Thus, in the context of the time-symmetric initial value problem, rather than the collapsing shell calculation,<sup>7</sup> an apparent horizon must satisfy

$$\frac{C_m}{8} \leq \frac{GM_{BY}}{c^2} \leq \frac{C_m}{2\pi}. \quad (73)$$

Since

$$\beta_b \leq C_m, \quad (74)$$

the lower bound yields

$$\frac{\beta_b}{8} \leq \frac{GM_{BY}}{c^2} \quad (75)$$

which, since  $8 > 2\pi$ , is a weaker statement than (68).

## VII. CONCLUSIONS

We have explored the nature of a number of upper bounds on fundamental quantities in nature. Some of this involves further elaboration and generalization to higher dimensions of earlier upper bounds on forces and power in general relativity, but our discussion has focused on a detailed analysis of our conjecture that the ratio of the magnetic moment to angular momentum is bounded above in nature. Suspicion falls on this combination for a maximum principle because it has a natural Stoney-Planck unit that is independent of the quantum of action,  $h$ , and so it is entirely classical. We find evidence for our conjecture that the ratio  $c\mu/JG^{\frac{1}{2}}$  is bounded by a quantity of order unity by investigating a wide range of testing theoretical situations. In particular, we verified that such a conjecture holds for charged rotating black holes in those theories for which exact solutions are available, including the Einstein-Maxwell and dilaton theories, Kaluza-Klein theory, the Kerr-Sen black hole, and the so-called STU family of charged rotating supergravity black holes. We also discussed the current status of the maximum tension conjecture, the Dyson luminosity bound, and Thorne’s hoop conjecture and saw the possible points of contact between them and our conjecture bounding  $\mu/J$ .

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<sup>7</sup>For which if the apparent horizon lies in a spacelike hyperplane, one does have [63]

$$\frac{\beta_b}{4\pi} \leq \frac{GM_{ADM}}{c^2}. \quad (72)$$



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