

# Noether symmetries and stability of ideal gas solutions in Galileon cosmology

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A class of generalized Galileon cosmological models, which can be described by a pointlike Lagrangian, is considered in order to utilize Noether's theorem to determine conservation laws for the field equations. In the Friedmann-Lemaître-Robertson-Walker universe, the existence of a nontrivial conservation law indicates the integrability of the field equations. Because of the complexity of the latter, we apply the differential invariants approach in order to construct special power-law solutions and study their stability.

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## I. INTRODUCTION

The accuracy of the new cosmological observations [1–5] leads us to the necessity of extending the general theory of relativity. The introduction of a scalar field, which attributes the effects of the so-called dark energy, in the Einstein-Hilbert action is proposed in order to provide new mechanisms in order to explain the various phases of the Universe [6–13]. Canonical scalar fields lead to second-order gravitational theories while at the same time, they can describe the dynamics of higher-order theories of gravity or higher-order terms of noncanonical fields; see [14–16] and references therein. However, it is possible for a noncanonical scalar-field Lagrangian to provide up to a second-order system of differential equations when the Lagrangian is that of Horndeski theories [17]. A special family of noncanonical fields which have been proposed are the so-called Galileons [18,19]. The cosmological theory which includes the action integral of the Galileons is called Galileon cosmology—various cosmological applications of the theory can be found in [20–32]. However, while Galileon cosmology is a second-order gravitational theory and has been well studied in the literature, the existence of actual solutions for the field equations is still an open subject, and it is the reason that motivates this work.

Indeed, in Galileon cosmology, the gravitational field equations are at most of second order while in an isotropic and homogeneous universe, that is, in a Friedmann-Lemaître-Robertson-Walker (FLRW) background geometry, there are 2 degrees of freedom which describe the

scale factor  $a(t)$  and the Galileon field  $\phi(t)$ . The derivation of a solution for a system of differential equations that can be expressed in terms of elementary functions is usually linked to the existence of a sufficient number of independent first integrals, that is, conservation laws or, equivalently, to the existence of a sufficient number of transformations which reduce the differential equations to a system of algebraic equations. When this is true we say that the dynamical system is integrable. These properties are related and in a large extent equivalent to the existence of symmetries. In this work and in order to study the integrability of the Galileon cosmological model under consideration, we search for conservation laws of the action.

We find that the gravitational field equations can follow from the variation of a pointlike Lagrangian and by applying Noether's first theorem we construct the conservation laws of the system. The class of transformations we consider to leave invariant the action and, hence, the field equations are the so-called point transformations. The latter have been applied in various cosmological models and modified theories of gravity for the determination of new cosmological solutions; see for instance [33–42] and references therein. It is important to mention that except for the point transformations, other classes of mappings or different methods for the study of integrability have been applied in gravitational theories; some of these are presented in [43–50]. Furthermore, because the spatial field equations are differential equations of second order bound by a constraint, while the degrees of freedom are 2, we need only determine the unknown functions/parameters of the model whereby the system admits a single nontrivial Noether symmetry, so as to yield it fully integrable. Finally, in order to study the physical properties of the models that we determine from the application of Noether's

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theorem, we perform a study of the stability of power-law solutions.

The plan of the paper is the following: In Sec. II we present the gravitational field equations for the model that we consider in this work. Specifically for the Galileon's Lagrangian, we select the functions in order to describe the so-called cubic Galileon field. The field equations of that model have cubic power of the first derivative of the field  $\phi(t)$ , while there are two unknown functions:  $V(\phi)$ , which corresponds to the potential that forces the field and  $g(\phi)$ , which couples the higher-power terms of the field equations. However, in the limit where  $g(\phi) \rightarrow 0$ , the Galileon field reduces to that of a scalar field minimally coupled to gravity. The application of Noether's theorem for the model of our consideration is given in Sec. III, while power-law solutions (ideal gas solutions) are presented as special solutions for the field equations in Sec. IV. Moreover, the stability of those special solutions is studied. Finally, we discuss our results in Sec. V.

## II. GALILEON COSMOLOGY

The cosmological models with Galileon fields belong to the four-dimensional scalar-tensor theories in which the spatial gravitational field equations are of second order and consequently are free of Ostrogradski instability [51].

The Lagrangian of the generalized Galileon field is given to be [52]

$$\begin{aligned} \mathcal{L}_G = & K - G_3 \square \phi + G_4 R + G_{4,X} [(\square \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi)] \\ & + G_5 G_{\mu\nu} (\nabla^\mu \nabla^\nu \phi) - \frac{1}{6} G_{5,X} [(\square \phi)^3 \\ & - 3(\square \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \\ & + 2(\nabla^\mu \nabla_\alpha \phi)(\nabla^\alpha \nabla_\beta \phi)(\nabla^\beta \nabla_\mu \phi)], \end{aligned} \quad (1)$$

in which the functions  $K$  and  $G_i$  ( $i = 3, 4, 5$ ) depend on the scalar field  $\phi$  and its kinetic energy  $X = -\partial^\mu \phi \partial_\mu \phi / 2$ , while  $R$  is the Ricci scalar and  $G_{\mu\nu}$  is the Einstein tensor. The above Lagrangian can be seen to be equivalent to that of the Horndeski theory [53].

Among the infinite number of models which can be derived from (1), let us assume the special case which leads to a generalized Galileon model with a cubic derivative interaction term, previously studied in [54] for the matter case, and in [55] for the Galileon vacuum, where the free functions are

$$\begin{aligned} K &= X - V(\phi), & G_3 &= -g(\phi)X, \\ G_4 &= \frac{1}{2}, & G_5 &= 0. \end{aligned} \quad (2)$$

Then the gravitational action integral of Galileon cosmology

$$\begin{aligned} S = & \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right. \\ & \left. - \frac{1}{2} g(\phi) \partial^\mu \phi \partial_\mu \phi \square \phi \right) + \int d^4x \sqrt{-g} L_m \end{aligned} \quad (3)$$

can be written in the form

$$S = S_{GR} + S_{SF} + S_{EGT} + S_m, \quad (4)$$

where  $L_m$  denotes the Lagrangian of the matter source,  $S_{GR}$  is the Einstein-Hilbert action integral,  $S_{SF}$  is the action term which corresponds to a minimally coupled scalar field, and  $S_{EGT}$  is the new term which is introduced by the Galileon and includes the cubic derivative term.

Indeed, other definitions of the free functions of (1) exist and provide us with different types of models. However, the selection of  $K$  and  $G_3$  to be linear on  $X$  is the first extension of scalar-field cosmology, and actually in the limit  $g(\phi) \rightarrow 0$ , the (canonical) scalar-field cosmology is recovered.

### A. FLRW cosmology

We assume the cosmological scenario that our Universe is described by the spatially flat FLRW spacetime metric with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (5)$$

for which the Ricci scalar is

$$R = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right]. \quad (6)$$

Consequently, if we assume that the isometries of (5) are also inherited by the matter fields, then it follows that  $\phi \equiv \phi(t)$ . This means that the Galileon field possesses the symmetries of the spacetime and the gravitational field equations are ordinary differential equations. Let us present the gravitational field equations of the action (3).

Variation with respect to the metric tensor and the scalar field of the action integral (3) provides the Friedmann equation

$$3H^2 = \frac{\dot{\phi}^2}{2} (1 - 6g(\phi)H\dot{\phi} + g'(\phi)\dot{\phi}^2) + V(\phi), \quad (7)$$

the acceleration equation

$$2\dot{H} + \dot{\phi}^2 (1 + g'(\phi)\dot{\phi}^2 - 3g(\phi)H\dot{\phi} + g(\phi)\ddot{\phi}) = 0, \quad (8)$$

and the Klein-Gordon-like equation

$$\begin{aligned} & \ddot{\phi}(2\dot{\phi}^2 g'(\phi) - 6Hg(\phi)\dot{\phi} + 1) \\ & + \dot{\phi}^2 \left( \frac{1}{2} \dot{\phi}^2 g''(\phi) - 3g(\phi)\dot{H} - 9H^2 g(\phi) \right) \\ & + 3H\dot{\phi} + V'(\phi) = 0 \end{aligned} \quad (9)$$

where we have assumed that there is not any extra matter source.

A substitution of (6) into (3) followed by integration by parts derives the Lagrange function

$$\begin{aligned} \mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) &= 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi}^2 + a^3V(\phi) \\ &+ g(\phi)a^2\dot{a}\dot{\phi}^3 - \frac{g'(\phi)}{6}a^3\dot{\phi}^4, \end{aligned} \quad (10)$$

from which the two spatial gravitational field equations (8) and (9) can be derived with the action of the Euler operator. Here, it is important to remark that the first Friedmann equation is a constraint equation and can be derived from (10) if we reinstate the gauge invariance in (5) by introducing an arbitrary lapse function  $N(t)$ . At this point, a comment is in order. Although the gauge fixing process is trivial at the level of the equations of motion, this is not true when it takes place in the Lagrangian [56]. That is why, by considering (10), we have to additionally simulate the constraint equation of motion with a first integral which necessarily needs to be zero. The existence of such a conserved quantity is guaranteed due to fact that the system is autonomous. We also have to keep in mind that the nongauged fixed version of a system of this type admits in general different groups of symmetries [57]. However, for the model under consideration, it can be easily checked that the exact same conserved quantity emerges through Noether's first theorem in both considerations. In that respect, we can proceed by fixing  $N = 1$  at the Lagrangian level.

As we did for the action before, we can write Lagrangian (10) in the form

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) = \mathcal{L}_{GR} + \mathcal{L}_{SF} + \mathcal{L}_{EGT} \quad (11)$$

where

$$\mathcal{L}_{EGT} = g(\phi)a^2\dot{a}\dot{\phi}^3 - \frac{g'(\phi)}{6}a^3\dot{\phi}^4 \quad (12)$$

and the rest of the terms of (11) give the Lagrangian of a minimally coupled scalar field in a spatially flat FLRW spacetime. Function (10) is a pointlike Lagrangian, however it differs from that of scalar-tensor theories because of the term  $\mathcal{L}_{EGT}$  which introduces a cubic first-derivative dependent force in the evolution of motion. Therefore the momenta are calculated to be

$$p_a = 6a\dot{a} + g(\phi)a^2\dot{\phi}^3, \quad (13)$$

$$p_\phi = \frac{a^2\dot{\phi}}{3}(9g(\phi)\dot{a}\dot{\phi} - 2ag'(\phi)\dot{\phi}^2 - 3a). \quad (14)$$

We continue our analysis with the determination of the unknown functions  $g(\phi)$  and  $V(\phi)$ , in which the Lagrangian (10) admits Noether (point) symmetries and the corresponding group of transformations under which the action remains form invariant.

### III. POINT TRANSFORMATIONS AND NOETHER SYMMETRIES

In this work we are interested in point transformations, and for convenience of the reader we discuss some preliminary material important for the analysis which follows. Consider a system of second-order ordinary differential equations

$$\ddot{x}^i = \omega^i(t, x^j, \dot{x}^j), \quad (15)$$

where  $t$  is the independent variable and  $x^i$  denotes the dependent variables.

A one-parameter point transformation in the space  $\{t, x^j\}$ , such that  $\{t, x^j\} \rightarrow \{\bar{t}(t, x^j, \varepsilon), \bar{x}^j(t, x^j, \varepsilon)\}$  has the property of mapping solutions of (15) to themselves, satisfies in infinitesimal form the following criterion of invariance:

$$X^{[2]}(\ddot{x}^i - \omega^i) = 0 \quad \text{mod} \quad \ddot{x}^i - \omega^i = 0,$$

where its generator  $X$  is defined as

$$X = \frac{\partial \bar{t}}{\partial \varepsilon} \Big|_{\varepsilon=0} \partial_t + \frac{\partial \bar{x}^i}{\partial \varepsilon} \Big|_{\varepsilon=0} \partial_i \quad (16)$$

and  $X^{[N]}$  indicates the  $N$ th extension of  $X$  in the jet space  $\{t, x^j, x^{(1)j}, \dots, x^{(N)j}\}$  [58]. The generator of the point transformation  $X$  is called a Lie symmetry for the system of differential equations. Symmetries are important in any physical system and play a significant role in every physical theory, especially after the famous work of Emmy Noether published in 1918 [59].

What is more, let us consider that the system (15) follows from the variation of the action integral  $S = \int \mathcal{L} dt$ . The first of Noether's theorems states that when a (finite) group of transformations leaves the action form invariant, i.e., when (for the monparametric transformation considered here)

$$S(t, x^j, \dots) = S(\bar{t}(t, x^j, \varepsilon), \bar{x}^j(t, x^j, \varepsilon), \dots), \quad (17)$$

then a conserved quantity exists which can be constructed with the help of the symmetry generator. In the case of a

Lagrangian containing up to first-order derivatives of the  $x^j$ 's like in our case, condition (17) is expressed in infinitesimal form as

$$X^{[1]}\mathcal{L} + \mathcal{L} \frac{d}{dt} \left( \frac{\partial \bar{t}}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) = \dot{f} \quad (18)$$

and the relative integral of motion is given by

$$I = \left( \dot{x}^j \frac{\partial \mathcal{L}}{\partial \dot{x}^j} - \mathcal{L} \right) \frac{\partial \bar{t}}{\partial \varepsilon} \Big|_{\varepsilon=0} - \frac{\partial \mathcal{L}}{\partial \dot{x}^j} \frac{\partial \bar{x}^j}{\partial \varepsilon} \Big|_{\varepsilon=0} + f. \quad (19)$$

The generator  $X$  is called a Noether symmetry and it is straightforward to show that it is also a Lie symmetry of the equations of motion; however, the inverse is not always true.

Of course conservation laws can be derived without the use of Noether's theorems by using other methods. Nevertheless, the simplicity and the globalization of the applications of Noether's work make them unique in all areas of science and not only in physics.

From the Lagrangian (11) of the Galileon cosmological models, we read that the dependent variables are the scale factor  $a(t)$  and the field  $\phi(t)$ , in which  $t$  is the independent variable. Hence, we assume the generator of the infinitesimal transformation to be

$$X = \xi(t, a, \phi) \partial_t + \eta_a(t, a, \phi) \partial_a + \eta_\phi(t, a, \phi) \partial_\phi. \quad (20)$$

Moreover, as we discussed, we have to consider the first Friedmann equation, i.e., the constraint, as a zero valued integral of motion  $(\dot{x}^j \frac{\partial \mathcal{L}}{\partial \dot{x}^j} - \mathcal{L}) = 0$ , that is, the general form of the Noetherian conservation law for our problem reads

$$I = f - \eta_a \frac{\partial \mathcal{L}}{\partial \dot{a}} - \eta_\phi \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad (21)$$

We continue with the determination of the unknown functions of the Lagrangian (11) in which the field equations are invariant under the action of point transformations.

### A. Symmetry analysis

From the symmetry condition (18) for the Lagrangian (11) of the minimally coupled Galileon field we derive the condition

$$(X^{[1]} + \dot{\xi})(\mathcal{L}_{GR} + \mathcal{L}_{SF}) + (X^{[1]} + \dot{\xi})\mathcal{L}_{EGT} = \dot{f} \quad (22)$$

from where we define a set of constraint equations in order for the coefficient terms of the derivatives of  $\dot{a}$  and  $\dot{\phi}$  in (22) to vanish. The ensuing system is presented in the Appendix. However, since  $\mathcal{L}_{EGT}$  has higher-polynomial derivatives from the other two terms of the Lagrangian (11), it is expected that the potential which is to be defined, should

assume the form of the one in which a minimally scalar field admits an extra Noether symmetry. In the following we assume that  $g(\phi)$  is not constant or zero. It is important to mention that the analysis we perform here is different from the one that is presented in [42]. The main reason is that in this work we see the Lagrangian as a regular system in contrary to [42], which we saw it as a singular system. The two different approaches are complementary; for details see the discussion in [57,60].

As we discussed above, the field equations constitute an autonomous system and admit the Noether symmetry  $\partial_t$  for arbitrary functions  $V(\phi)$  and  $g(\phi)$ . However, in the specific case in which

$$V(\phi) = V_0 e^{-\lambda \phi} \quad \text{and} \quad g(\phi) = g_0 e^{\lambda \phi}, \quad (23)$$

an extra Noetherian symmetry exists, viz.

$$X = t \partial_t + \frac{a}{3} \partial_a + \frac{2}{\lambda} \partial_\phi, \quad (24)$$

with the corresponding conservation law

$$I_1 = - \left( 2a^2 \dot{a} - \frac{2}{\lambda} a^3 \dot{\phi} \right) + g_0 e^{\lambda \phi} a^3 \dot{\phi}^3 - \frac{6}{\lambda} g_0 a^2 e^{\lambda \phi} \dot{a} \dot{\phi}^2. \quad (25)$$

Recall that the same conservation law exists in the limit in which  $V_0 = 0$ . An important observation is that when the Universe is dominated by the potential of the scalar field, then  $g(\phi) \rightarrow 0$ , and the model reduces to that of a minimally coupled scalar field.

As we can see, the linear term of the conservation law is that of a minimally coupled scalar field while the nonlinear terms follow from the  $\mathcal{L}_{EGT}$  of the gravitational Lagrangian. Therefore, with the use of the conservation law (25) and the constraint equation (7) the gravitational field equations are reduced to a system of two first-order ordinary differential equations which are autonomous.

## IV. IDEAL GAS SOLUTION AS GROUP INVARIANT SOLUTION

Above, we made it clear that a Noether symmetry is always a Lie symmetry, where the latter means that there exists a set of invariants in which the equations are independent. However it is easier to construct invariant solutions.

In this respect, for the gravitational field equations (7)–(9) we find the following set of solutions:

the power-law solution

$$a_1(t) = a_0 t^p, \quad \phi(t) = \frac{2}{\lambda} \ln(\phi_0 t),$$

$$g_0 = \frac{\lambda(2 - \lambda^2 p)}{4(3p - 1)\phi_0^2}, \quad V_0 = \phi_0^2 \left( \frac{2}{\lambda^2} + p(3p - 2) \right) \quad (26)$$

and the cubic root solutions

$$a_{2,3}(t) = a_0 t^{\frac{1}{3}}, \quad \phi_{2,3}(t) = \pm \frac{\sqrt{6}}{3} \ln(\phi_0 t),$$

$$V_0 = 0, \quad \lambda_{2,3} = \pm \sqrt{6}. \quad (27)$$

The perfect fluid solution  $a_1(t)$  is similar to the special solution for the exponential potential of a minimally coupled scalar field. We mentioned that the terms  $\mathcal{L}_{EGT}$  in the first Friedmann equation becomes zero in the cubic root solutions. Indeed, the solutions exist for arbitrary coupling ( $g_0$ ), but zero potential ( $V_0 = 0$ ).

Note that these are special solutions of the full system (they are singular solutions). However, what is important from the solution  $a_1(t)$  is that power  $p$  is not related with  $\lambda$ , as in the case of a minimally coupled scalar field. However, the solution of the latter is recovered when we set  $g_0 = 0$  and  $p = \frac{2}{\lambda^2}$ .

### A. Stability of the ideal gas solution

Now, let us follow the approach of [61,62] to investigate the stability of the singular solution

$$a_1(t) = a_0 t^p, \quad \phi(t) = \frac{2}{\lambda} \ln(\phi_0 t),$$

$$g_0 = -\frac{\lambda(\lambda^2 p - 2)}{4(3p - 1)\phi_0^2}, \quad V_0 = \phi_0^2 \left( \frac{2}{\lambda^2} + p(3p - 2) \right).$$

Let us assume  $\phi_0 > 0$ . Introducing the new variables

$$\epsilon = \frac{\lambda\phi}{2 \ln(t\phi_0)} - 1, \quad v = t \left( \frac{\lambda\dot{\phi}}{2 \ln(t\phi_0)} - \frac{\lambda\phi}{2t \ln^2(t\phi_0)} \right),$$

$$t\phi_0 = e^\tau, \quad (28)$$

where  $\phi(t)$  is a general solution of (9). That is, the scaling solution  $\phi_s(t) = \frac{2}{\lambda} \ln(\phi_0 t)$  corresponds to the critical point  $\epsilon = 0$ . Furthermore, by definition we have  $v = \epsilon'$ , where now the prime denotes the derivative with respect to  $\tau$ . Using the variables (28), Eq. (9) recasts as the nonautonomous system

$$\epsilon' = v, \quad (29a)$$

$$v' = -\frac{(\tau v + \epsilon + 1)(6(1 - 3p)p - (\lambda^2 p - 2)e^{2\tau\epsilon}(\tau v + \epsilon + 1)(9p^2 - 3p - 2(\tau v + \epsilon + 1)^2))}{2\tau(-(\lambda^2 p - 2)e^{2\tau\epsilon}(\tau v + \epsilon + 1)(3p - 2(\tau v + \epsilon + 1)) - 3p + 1)}$$

$$- \frac{(3p - 1)(\lambda^2 p(3p - 2) + 2)e^{4\tau - 2\tau(\epsilon + 2)}}{2\tau(-(\lambda^2 p - 2)e^{2\tau\epsilon}(\tau v + \epsilon + 1)(3p - 2(\tau v + \epsilon + 1)) - 3p + 1)} + \frac{(\tau - 2)v + \epsilon + 1}{\tau} \quad (29b)$$

which admits the exact (singular) solution  $(\epsilon, v) = (0, 0)$ . Assuming  $\epsilon \ll 1$ ,  $v \ll 1$ , we obtain the linearized nonautonomous system

$$\begin{pmatrix} \epsilon' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(3p-1)(2\tau+1)}{\tau} & -3p - \frac{2}{\tau} + 1 \end{pmatrix} \begin{pmatrix} \epsilon \\ v \end{pmatrix}. \quad (30)$$

The exact solution of (30) is given by

$$\epsilon(\tau) = e^{\frac{1}{2}(1-3p+\sqrt{9p^2-30p+9})\tau} \epsilon_1(\tau) + e^{\frac{1}{2}(1-3p-\sqrt{9p^2-30p+9})\tau} \epsilon_2(\tau) \quad (31a)$$

$$v(\tau) = e^{\frac{1}{2}(1-3p+\sqrt{9p^2-30p+9})\tau} v_1(\tau) + e^{\frac{1}{2}(1-3p-\sqrt{9p^2-30p+9})\tau} v_2(\tau), \quad (31b)$$

where

$$\epsilon_1(\tau) = \frac{c_1}{\tau}, \quad \epsilon_2(\tau) = \frac{c_2}{\tau}, \quad (32)$$

$$v_1(\tau) = c_2 \left( \frac{1 - 3p + \sqrt{9p^2 - 30p + 9}}{2\tau} - \frac{1}{\tau^2} \right),$$

$$v_2(\tau) = c_1 \left( \frac{1 - 3p - \sqrt{9p^2 - 30p + 9}}{2\tau} - \frac{1}{\tau^2} \right). \quad (33)$$

The two modes of  $\epsilon$  and  $v$  are decaying for  $p > 1/3$ , which are exponentially depressed for  $p > 3$  and shows damped oscillations for  $\frac{1}{3} < p < 3$ . In both cases, the perturbations decrease. For  $p < 1/3$ , one mode is decaying and the other grows as  $\tau \rightarrow \infty$  (the origin is a saddle), so that the perturbation increases with time.

For large  $\tau$ , the system (30) can be approximated by

$$\begin{pmatrix} \epsilon' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2(3p - 1) & 1 - 3p \end{pmatrix} \begin{pmatrix} \epsilon \\ v \end{pmatrix}. \quad (34)$$

The origin has the eigenvalues

$$\left\{ \frac{1}{2} \left( -\sqrt{9p^2 - 30p + 9} - 3p + 1 \right), \right. \\ \left. \frac{1}{2} \left( \sqrt{9p^2 - 30p + 9} - 3p + 1 \right) \right\}.$$

The origin of (34) is a stable node for  $p \geq 3$  or a stable spiral for  $\frac{1}{3} < p < 3$ , and this result seems to be independent on the parameter  $\lambda$ .

For the numerical integration we use the Poincaré projection:

$$\epsilon = \frac{r}{1-r} \sin \theta, \quad v = \frac{r}{1-r} \cos \theta, \quad (35)$$

rescaling the time derivative by  $f' \rightarrow (1-2r)^2 r f'$  and plotting the solutions in the plane  $(\epsilon_r, v_r) = (r \sin \theta, r \cos \theta)$ . This projection shrinks all the trajectories in the phase plane to the unit disk. Furthermore, the points  $N \equiv (0, 1)$ ,  $E \equiv (-1, 0)$ ,  $W \equiv (1, 0)$  correspond respectively to  $(\epsilon = 0, \epsilon' = \infty)$ ,  $(\epsilon = -\infty, \epsilon' = 0)$ , and  $(\epsilon = \infty, \epsilon' = 0)$ . In Fig. 1 it is presented with some orbits of (34) on the Poincaré plane for different choices of  $p$ . In the first case,  $p = 1/6$ , the origin is a saddle and appear with four nontrivial centers. For  $p = 1/3$ , the origin is a saddle and the centers reduce to 2 and the line  $v_r = 0$  is invariant. For  $p = 1/2$  the origin is the only stationary structure of the phase space and it is a stable spiral. For  $p = 2/3$ ,  $p = 1$  the origin is a stable spiral, but two centers appear, in addition

to two saddle points. For  $p = 3$  the centers remain, but the origin now becomes a stable node. There appears two saddle points at the interior of the phase space and two nonhyperbolic points on the Poincaré circle which are saddle points.

Since the system (34) is an approximation of (30), we may expect that the origin of (30) could be a stable node for  $p \geq 3$  or a stable spiral for  $\frac{1}{3} < p < 3$ , and the result should be independent on the parameter  $\lambda$ . Indeed, this result can be confirmed from the exact solution (29) of the linear nonautonomous system. Then, since the system (30) is an approximation of (29) for  $\epsilon \ll 1, \epsilon' \ll 1$ , we might expect analogous results for the full system, at least near the origin. However, since the system (29) is nonautonomous and nonlinear, this heuristic reasoning is not a complete proof, and we have to test the validity of our results, which can be done numerically by integrating the full system (29). Furthermore, since the system is nonautonomous, the orbits cross in the Poincaré representation, but not on the 3D representation of the integral curves  $(\epsilon, \epsilon', \tau)$ .

Notice that the full system (29) can be written as an autonomous system by introducing the new variables

$$\mathcal{E} = e^{-\epsilon\tau} - 1, \quad \mathcal{V} = \epsilon + \tau v, \quad (36)$$

such that  $(\epsilon, v) = (0, 0)$  is mapped onto  $(\mathcal{E}, \mathcal{V}) = (0, 0)$  for all  $\tau$ . And we define the new derivative  $f' = e^{\epsilon\tau} \frac{df}{d\tau}$ . This leads to the dynamical system

$$\mathcal{E}' = -\mathcal{V}, \quad (37a)$$

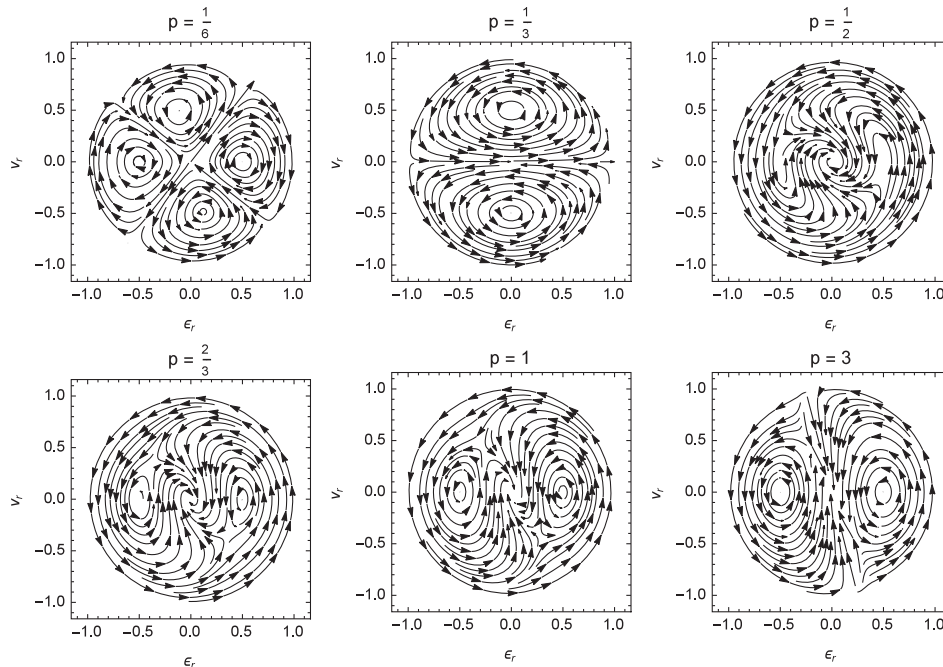


FIG. 1. In the figures are presented some orbits of (34) on the Poincaré plane for  $p = 1/6$ ,  $p = 1/3$ , for radiation ( $p = 1/2$ ) and matter ( $p = 2/3$ ) dominated universes, for zero acceleration solution ( $p = 1$ ), and for accelerated solution ( $p = 3$ ).

$$\mathcal{V}' = \frac{\mathcal{V}^2(\lambda^2 p - 2)(2\mathcal{V}(\mathcal{V} + 2) - 9(p - 1)p)}{2(\mathcal{E} + 1)(3\lambda^2 p^2(\mathcal{V} + 1) + 3p(\mathcal{E}(\mathcal{E} + 2) - 2\mathcal{V} - 1) - 2\lambda^2 p(\mathcal{V} + 1)^2 + (-\mathcal{E} + 2\mathcal{V} + 1)(\mathcal{E} + 2\mathcal{V} + 3))} + \frac{(3p - 1)(\mathcal{V}((\mathcal{E} - 1)(\mathcal{E} + 3) - 3\lambda^2 p^2 + p(-3\mathcal{E}(\mathcal{E} + 2) + 2\lambda^2 + 3)) + \mathcal{E}(\mathcal{E} + 2)(\mathcal{E}(\mathcal{E} + 2) - 3p + 3))}{(\mathcal{E} + 1)(3\lambda^2 p^2(\mathcal{V} + 1) + 3p(\mathcal{E}(\mathcal{E} + 2) - 2\mathcal{V} - 1) - 2\lambda^2 p(\mathcal{V} + 1)^2 + (-\mathcal{E} + 2\mathcal{V} + 1)(\mathcal{E} + 2\mathcal{V} + 3))} + \frac{\mathcal{E}(\mathcal{E} + 2)(\mathcal{E}(\mathcal{E} + 2) + 2)\lambda^2 p(3p - 2)(3p - 1)}{2(\mathcal{E} + 1)(3\lambda^2 p^2(\mathcal{V} + 1) + 3p(\mathcal{E}(\mathcal{E} + 2) - 2\mathcal{V} - 1) - 2\lambda^2 p(\mathcal{V} + 1)^2 + (-\mathcal{E} + 2\mathcal{V} + 1)(\mathcal{E} + 2\mathcal{V} + 3))}. \quad (37b)$$

For the numerical integration we use the Poincaré projection:

$$\mathcal{E} = \frac{r}{1-r} \sin \theta, \quad \mathcal{V} = \frac{r}{1-r} \cos \theta, \quad (38)$$

and plot the solutions in the plane  $(\mathcal{E}_r, \mathcal{V}_r) = (r \sin \theta, r \cos \theta)$ . This projection shrinks all the trajectories in the phase plane to the unit disk. Furthermore, the points  $N \equiv (0, 1)$ ,  $E \equiv (-1, 0)$ ,  $W \equiv (1, 0)$  correspond, respectively, to  $(\mathcal{E} = 0, \mathcal{V} = \infty)$ ,  $(\mathcal{E} = -\infty, \mathcal{V} = 0)$ , and  $(\mathcal{E} = \infty, \mathcal{V} = 0)$ .

In Fig. 2 are plotted some solutions in the plane  $(\mathcal{E}_r, \mathcal{V}_r) = (r \sin \theta, r \cos \theta)$  for the system (37). These numerics support the claim that the phase spaces are topologically equivalent for different choices of  $\lambda$ . Henceforth, the stability results are independent of  $\lambda$ . For  $p = 1/2$  it is confirmed that the origin is a saddle point as it is anticipated from the analysis of the linearized nonautonomous system (and this result is valid for all  $p < 1/3$ ). For  $p = 1/3$  the line  $\mathcal{V} = 0$  is invariant and it is not stable. For the values  $p = 1/2, 2/3, 1$  the origin is a stable spiral as it is anticipated from the analysis of the linearized nonautonomous system; these results are true for  $1/3 < p < 3$ . Finally, for  $p = 3$  (and greater values of  $p$ ) the origin is a stable node. Comparing the results of Figs. 1 and 2 we see that for the same values of  $p$  (and independently of  $\lambda$ ), the dynamics near the origin is topologically equivalent. However, the global features of the phase spaces are rather different; for example, it seems from the diagrams that the system (37) has no center points as the system (34).

### B. Stability of the cubic root solution

For analyzing the stability of the cubic root solution (we choose just the solution on the branch “+”; the analysis for  $a_3(t)$  is quite similar),

$$a_2(t) = a_0 t^{\frac{1}{3}}, \quad \phi(t) = \frac{\sqrt{6}}{3} \ln \phi_0 t, \quad V_0 = 0, \quad \lambda = \sqrt{6},$$

we substitute the values of  $a_2(t)$ ,  $V_0$ , and  $\lambda$  in (9) for an arbitrary  $\phi$  to obtain

$$3g_0 e^{\sqrt{6}\phi} \dot{\phi}^4 + \ddot{\phi} \left( 2\sqrt{6}g_0 e^{\sqrt{6}\phi} \dot{\phi}^2 - \frac{2g_0 e^{\sqrt{6}\phi} \dot{\phi}}{t} + 1 \right) + \frac{\dot{\phi}}{t} = 0.$$

Introducing the new variables

$$\epsilon = \frac{\sqrt{6}\phi}{2 \ln(t\phi_0)} - 1, \quad v = t \left( \frac{\sqrt{6}\dot{\phi}}{2 \ln(t\phi_0)} - \frac{\sqrt{6}\phi}{2t \ln^2(t\phi_0)} \right), \quad t\phi_0 = e^\tau, \quad (39)$$

where  $\phi(t)$  is a general solution of (8) and we have assumed  $\phi_0 > 0$ . Without losing generality we can set  $g_0 \phi_0^2 = 1$ , which means that  $g_0$  is given in units of  $\phi_0^{-2}$ . Using the variables (39), Eq. (8) recasts as the nonautonomous system

$$\epsilon' = v, \quad (40a)$$

$$v' = \left( 1 - \frac{2}{\tau} \right) v - \frac{3(\tau v + \epsilon + 1)(2\sqrt{\frac{2}{3}} e^{2\tau\epsilon} (\tau v + \epsilon + 1)^3 + 1)}{\tau(2\sqrt{6} e^{2\tau\epsilon} (\tau v + \epsilon + 1)(2\tau v + 2\epsilon + 1) + 3)} + \frac{\epsilon + 1}{\tau}, \quad (40b)$$

which admits the exact (singular) solution  $(\epsilon, v) = (0, 0)$ . The linearized equations are

$$\epsilon' = v, \quad v' = -\frac{2v}{\tau},$$

whose solution is

$$\epsilon(\tau) = c_2 - \frac{c_1}{\tau}, \quad v(\tau) = \frac{c_1}{\tau^2}.$$

The solutions satisfy  $\epsilon + v\tau = c_2$ , where  $c_2 = \lim_{\tau \rightarrow \infty} \epsilon(\tau)$ . From these expressions we can have that the  $(0, 0)$  solution is stable as  $\tau \rightarrow \infty$  by choosing  $c_2$  small enough. In Fig. 3 we present some integral curves for the full system (29) and the above feature is illustrated. Now, from the full system (40), we can obtain an autonomous system by introducing the new variables

$$\mathcal{E} = e^{-\epsilon\tau} - 1, \quad \mathcal{V} = \epsilon + \tau v, \quad (41)$$

and the new derivative  $f' = e^{\epsilon\tau} \frac{df}{d\tau}$ . This leads to the dynamical system

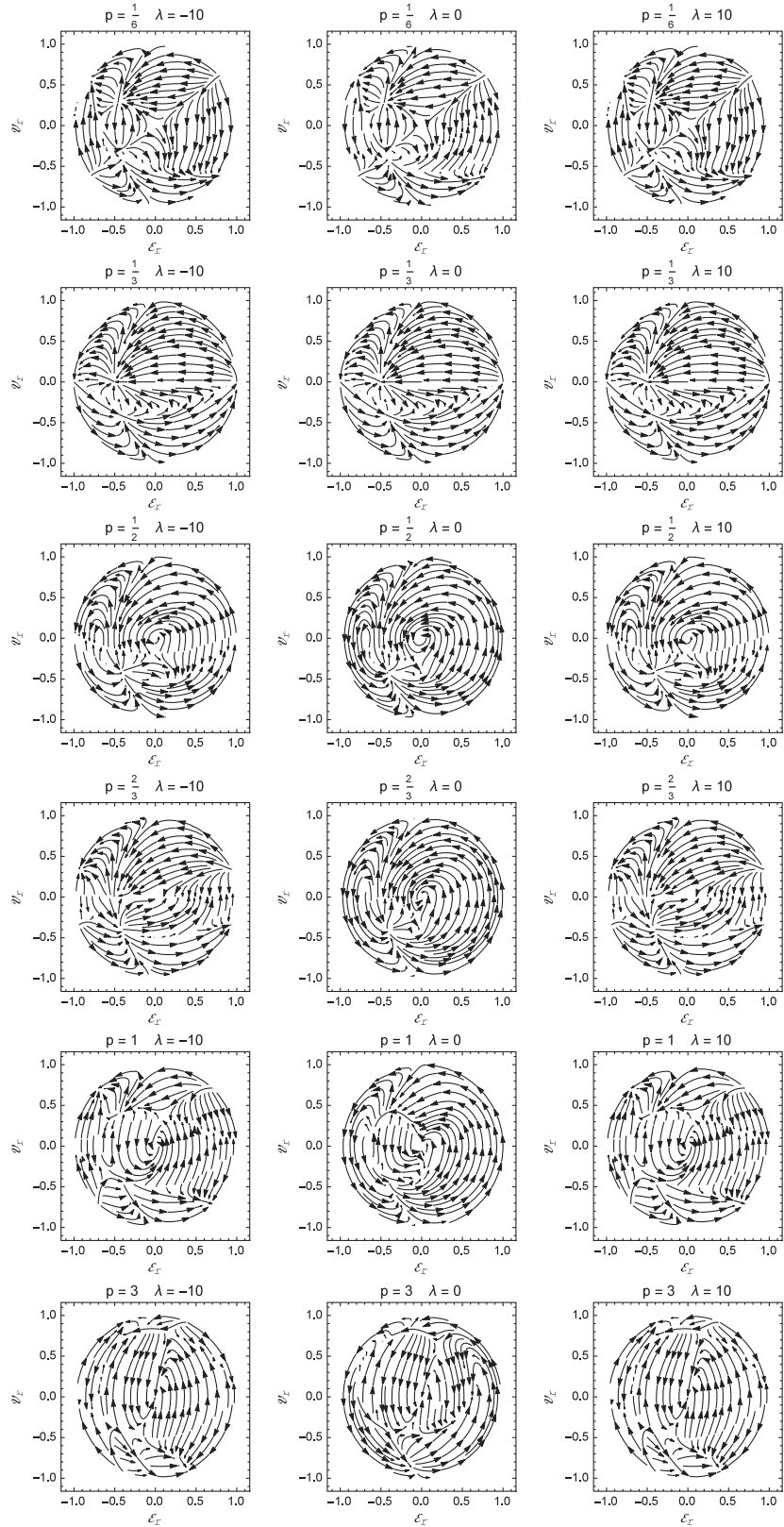


FIG. 2. Plot of the solutions in the plane  $(\mathcal{E}_r, \mathcal{V}_r) = (r \sin \theta, r \cos \theta)$  for the system (37).



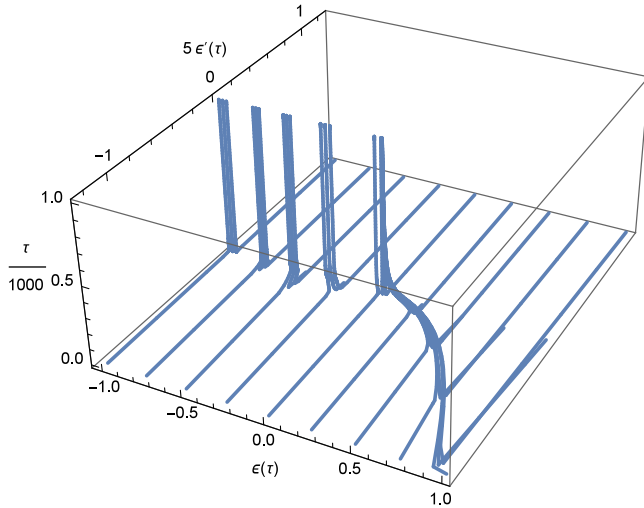


FIG. 3. Some integral curves for the full system (40). It can be shown that the  $(0, 0)$  solution is stable as  $\tau \rightarrow \infty$  by choosing  $c_2$  small enough and for  $\epsilon > 0$ .

$$\mathcal{E}' = -\mathcal{V}, \quad (42a)$$

$$\mathcal{V}' = -\frac{2\sqrt{6}\mathcal{V}^2(\mathcal{V}+1)^2}{3(\mathcal{E}+1)^3 + 2\sqrt{6}(\mathcal{E}+1)(\mathcal{V}+1)(2\mathcal{V}+1)}. \quad (42b)$$

For the numerical integration we use the Poincaré projection:

$$\begin{aligned} \mathcal{E} &= \frac{r}{1-r} \sin \theta, \\ \mathcal{V} &= \frac{r}{1-r} \cos \theta, \end{aligned} \quad (43)$$

and plot the solutions in the plane  $(\mathcal{E}_r, \mathcal{V}_r) = (r \sin \theta, r \cos \theta)$  as shown in Fig. 4.

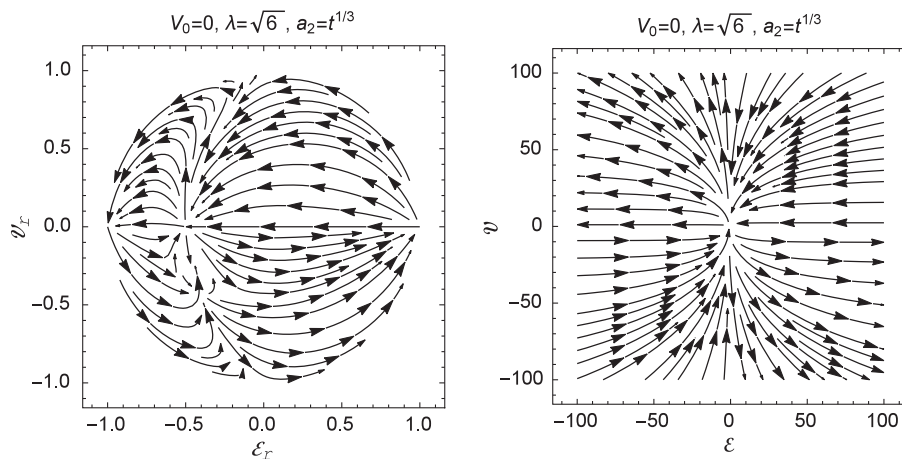


FIG. 4. Poincaré projection of the system (42).

## V. CONCLUSION

There are various ways to study the integrability of a dynamical system. In this work we decided to search for conservation laws given by the application of Noether's symmetries for the pointlike Lagrangian of a Galileon field. For that field we considered a model which can be seen an extension of the minimally coupled canonical scalar field. Specifically the new term which is introduced from the Galileon Lagrangian provides cubic powers of the derivatives in the "Klein-Gordon" equation for the field, while a function is introduced such that, when it goes to zero, the canonical scalar-tensor theories are recovered.

The unknown parameters of the model are two, the potential  $V(\phi)$  and the function  $g(\phi)$  which is introduced from the higher-power derivatives in the gravitational Lagrangian. These two functions drive the evolution of the field equations and provide us with different cosmological models. The demand that the field equations form an integrable system with a conservation law linear in the momentum is sufficient to define the explicit form of these two unknown functions. Specifically, the application of Noether's theorem implies that  $V(\phi)$  and  $g(\phi)$  are exponential such that  $V(\phi) \propto (g(\phi))^{-1}$ . This model is of special interest because, when the Universe is dominated by the scalar-field potential, then the extra terms in the field equations which correspond to the Galileon field do not affect the evolution of the system and the Galileon field behaves like a canonical scalar field.

Because of the complexity of the field equations it was not possible to extract the general solution in a closed form. However, we proved the existence of power-law solutions as special solutions. In order to study the evolution of the system, we followed the method proposed by [61,62]. We found, after linearization, that for the power-law solution there are two modes of  $\epsilon$  and  $\nu$  which are decaying for  $p > 1/3$  at an exponential rate for  $p > 3$ , and the solutions manifest damped oscillations for  $1/3 < p < 3$ . In both cases

the perturbations decrease. For  $p < 1/3$  one mode is decaying and the other grows as  $\tau \rightarrow \infty$  (the origin is a saddle), so that the perturbation increases with time. We have constructed an autonomous dynamical system from the full system. Using numerical integrations, we showed that the stability results are independent of  $\lambda$ ; the dynamics near the origin is topologically equivalent of that of the linearized system. However, the global features of the phase spaces are rather different. For the cubic root solution we found that the origin is stable but not asymptotically stable by choosing proper initial conditions.

The cosmological eras of a canonical scalar field can be recovered from the gravitation action (3) as it has been shown in [54]. However, even in that limit there are various differences between the two models. One of the main can be observed from the analysis, which we performed is in the relation of the exponent  $p$ , in the power-law solution  $a(t) = a_0 t^p$ , with the rate of the exponential potential. Specifically for the canonical scalar-field cosmology, the exponential potential  $V(\phi) = V_0 e^{-\lambda\phi}$  admits a power-law solution in which the power  $p$  and the constant  $\lambda$  are related as follows  $\lambda^2 \simeq p^{-1}$  [63]. On the contrary, this is not necessary for our model where we have shown that  $p$  is independent of the exponential rate of the potential; a fact that is owed to the cubic terms in the Lagrangian.

It is true that power-law solutions describe those of an ideal gas but since  $p$  is now arbitrary, the Galileon field can describe also fluids with a negative equation of state parameter or even phantom fluids with equation of state parameter smaller than minus one. Hence, from our analysis it follows that the integrable model can describe the late-time acceleration of the Universe when the equation of state parameter for the dark energy is different from minus one, or the early inflationary epoch demanding that the power-law solution is unstable so that an exit from the inflationary period does exist. That is in agreement with the various studies of Galileons; for instance, see [64–67] and references therein. Finally, the stability analysis of the power-law solutions differs from that of the canonical field [61] and that is directly related with the existence of the cubic term in the action integral.

To show what the role of the cubic term in the cosmological history is, let us consider now that it dominates in the Lagrangian. Then, the field equations reduce to the following relations:

$$\ddot{\phi} + \frac{1}{2} \lambda \dot{\phi}^2 \simeq 0, \quad \dot{H} - \lambda H \dot{\phi} + 3H^2 + \frac{1}{6} \lambda^2 \dot{\phi}^2 \simeq 0 \quad (44)$$

with solution<sup>1</sup>  $H(t) = \frac{6H_0 - \lambda\dot{\phi}_0}{3((t-t_0)(6H_0 - \lambda\dot{\phi}_0) + 2)} + \frac{\lambda\dot{\phi}_0}{3\lambda\dot{\phi}_0(t-t_0) + 6}$ ,  $\phi(t) = \phi_0 + \frac{2}{\lambda} \ln(1 + \frac{\lambda\dot{\phi}_0}{2}(t-t_0))$ . Hence, for the scale factor follows

<sup>1</sup>Where we have chosen the initial conditions  $H(t_0) = H_0$ ,  $\phi(t_0) = \phi_0$ ,  $\dot{\phi}(t_0) = \dot{\phi}_0$ .

$$a(t) = a_0 \left(1 + \frac{1}{2} \lambda \dot{\phi}_0 (t - t_0)\right)^{\frac{1}{3}} \times \left(1 + \frac{1}{2} (t - t_0) (6H_0 - \lambda \dot{\phi}_0)\right)^{\frac{1}{3}}. \quad (45)$$

By taking initial conditions such that  $\dot{\phi}_0 = 0$  or  $6H_0 - \lambda\dot{\phi}_0 = 0$ , we obtain the cubic root solution that we derived before  $a(t) \propto t^{\frac{1}{3}}$ . On the other hand, if we choose initial conditions such that  $3H_0 - \lambda\dot{\phi}_0 = 0$ , we obtain that  $a(t) \propto t^{\frac{2}{3}}$ , that is, it corresponds to a universe dominated by a dust fluid/dark matter.

In order to understand better the evolution and the dynamics of that model, an analysis of the critical points in the dimensionless variables should be performed. With the latter we will be able to investigate the effects of the various terms of the Lagrangian in the evolution of the Universe and to which eras they provided. The speciality of our model lies in the existence of the second conservation law and it is of special interest to see how the second conservation law fixes the evolution. On the other hand that special case is not explicitly included in the results of [54]. This is still work in progress and will be published elsewhere.

Of course as we have mentioned before, only a special class of the generalized Galileon models was studied in this work. Possible extensions by admitting more terms in the action or more complicated coupling functions is a subject of interest. The reason is that while some can numerically approximate the evolution of the system, the existence of an actual solution of the field equations is an open subject.

## ACKNOWLEDGMENTS

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## APPENDIX: THE SYMMETRY CONDITIONS

In this appendix, for the convenience of the reader, we present the Noether symmetry conditions (18) for the Lagrangian (10) of the Galileon model that we have considered.

$$\begin{aligned}
\xi_{,a} &= 0, & \xi_{,\phi} &= 0, & \eta_{\phi,a} &= 0, & 9g(\phi)a^2\eta_{\phi,t} &= 0, \\
6a\eta_{a,t} - h_{,a} &= 0, & a^3\eta_{\phi,t} + f_{,\phi} &= 0, \\
6a\eta_{a,\phi} - a^3\eta_{\phi,a} &= 0, \\
3a^2g\eta_{a,t} - 2g_{,\phi}a^3\eta_{\phi,t} &= 0, \\
3\eta_a a^2 V(\phi) + \eta_\phi a^3 V_{,\phi} + a^3 V \xi_{,t} - f_{,t} &= 0, \\
-3a\xi_{,t} + 3\eta_a + 6a\eta_{a,a} &= 0, \\
\frac{1}{2}a^3\xi_{,t} - \frac{3}{2}\eta_a a^2 - a^3\eta_{\phi,\phi} &= 0, \\
3g(\phi)a^2\eta_{a,\phi} - 2g_{,\phi}a^3\eta_{\phi,\phi} + \frac{3}{2}g_{,\phi}a^3\xi_{,t} \\
- \frac{3}{2}g_{,\phi}\eta_a a^2 - \frac{1}{2}\eta_\phi a^3 g_{,\phi\phi} &= 0, \\
6\eta_a g(\phi)a + 3\eta_\phi a^2 g_{,\phi} + 3g(\phi)a^2\eta_{a,a} \\
+ 9g(\phi)a^2\eta_{\phi,\phi} - 2a^3 g_{,\phi}\eta_{\phi,a} - 9g(\phi)a^2\xi_{,t} &= 0.
\end{aligned}$$

The solution of the system provides us with the symmetry vectors and the specific forms of the functions  $g(\phi)$  and  $V(\phi)$  that were presented in Sec. III.

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