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Asymptotically flat multiblack lenses

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We present an asymptotically flat and stationary multiblack lens solution with biaxisymmetry of $U(1)\times U(1)$ as a supersymmetric solution in the five-dimensional minimal ungauged supergravity. We show that the spatial cross section of each degenerate Killing horizon admits different lens space topologies of $L(n,1)=S^3/\mathbb{Z}_n$ as well as a sphere S^3. Moreover, we show that, in contrast to the higher-dimensional Majumdar-Papapetrou multiblack hole and multi-Breckenridge-Myers-Peet-Vafa (BMPV) black hole spacetime, the metric is smooth on each horizon even if the horizon topology is spherical.

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I. INTRODUCTION

In recent years, in the string theory and the various contexts of AdS/CFT correspondence, higher-dimensional black holes and other extended black objects have played an important role [1–5]. In particular, the physics of asymptotically flat black holes in the five-dimensional minimal supergravity (Einstein-Maxwell-Chern-Simons theory) has been the subject of increased attention, as it describes a low-energy limit of the string theory. Various types of black hole solutions in the theory have so far been found, with the help of the recent development of solution-generating techniques [6–38].

The topology theorem for stationary black holes generalized to five dimensions [39–42] states that the topology of the spatial cross section of the event horizon is restricted to either a sphere S^3, a ring $S^1\times S^2$, or lens spaces $L(p,q)$ (p, q: coprime integers), if one assumes that the spacetime is asymptotically flat and admits three commuting Killing vector fields, a timelike Killing vector field, and two axial Killing vector fields under the dominant energy condition. As for the first two cases, one knows the corresponding exact solutions in the five-dimensional Einstein theory [2–4] and minimal ungauged supergravity theory [14,16,17]. On the contrary, a regular black hole solution with a lens space topology, at present, has not been found to the five-dimensional vacuum Einstein equation, although a few authors have attempted to construct such a black lens solution by using the combination of the rod diagram and the inverse scattering method [43,44].

Recently, there has been a new development in this field. Asymptotically flat and stationary black lens solutions, whose horizon topology is restricted to $L(2,1)=S^3/\mathbb{Z}_2$, were constructed by Kunduri and Lucietti as supersymmetric solutions to the five-dimensional minimal ungauged supergravity [45] and $U(1)^3$ supergravity [46]. Furthermore, the more general black lenses with the horizon topologies of $L(n,1)=S^3/\mathbb{Z}_n (n\geq 3)$ were also constructed by one of the present authors in the former theory [47]. The basic strategy to get these supersymmetric black lens solutions is to use the well-known method developed by Gauntlett *et al.* in Ref. [6], where the metric of supersymmetric solutions admitting a timelike Killing field is described in an adapted coordinate system by an $\mathbb{R}$ bundle over the four-dimensional hyper-Kähler base space.

It is well known that the Majumdar-Papapetrou solution to the four-dimensional Einstein-Maxwell equation describes an arbitrary number of charged static black holes in an asymptotically flat spacetime by a balance of electromagnetic and gravitational forces [48,49]. Such an asymptotically flat, static multiblack hole solution was previously generalized to higher-dimensional Einstein-Maxwell theories [50]. Furthermore, a multiblack hole solution in a rotational case (this is often called multi-BMPV black hole) was constructed in five-dimensional minimal supergravity [51]. As shown in Refs. [51–53], in contrast to the four-dimensional Majumdar-Papapetrou solution [54], these solutions generalized to higher dimensions do not admit smoothness of the metric at horizons, whereas for the concentric multiblack ring solution [16], the spacetime is known to be analytic at each event horizon. Therefore, it is not entirely clear whether there does exist a regular multiblack lens solution in higher dimensions, simply because a black lens solution with a single horizon exists.

The purpose of this paper is to construct an exact solution describing an arbitrary number of charged rotating black lenses with smooth horizons as an asymptotically flat and stationary supersymmetric solution in five-dimensional minimal supergravity. This work is essentially based on the previous works of Refs. [45,47], where the strategy is to use

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the Gibbons-Hawking space as a hyper-Kähler base space and allow the harmonic functions to have n point sources with appropriate coefficients. We show that, by imposing appropriate boundary conditions on the parameters, the n point sources denote degenerate Killing horizons with the topologies of different lens spaces $L(n_i,1)=S^3/Z_{n_i}$ ($i=1,\ldots,n$). Moreover, introducing an appropriate coordinate system, we also show that the metric and Maxwell's field strength are smooth on each horizon in contrast to the higher-dimensional Majumdar-Papapetrou solutions and multi-BMPV black hole solution.

This paper is organized as follows: In Sec. II, following the work of Gauntlett *et al.* [6], we present the supersymmetric solution on the Gibbons-Hawking base space, which describes multiblack lenses (with n horizons) in the five-dimensional minimal ungauged supergravity. The solution admits three commuting Killing vector fields, i.e., the stationary Killing vector field and two mutually commuting axial Killing vector fields so that the isometry group of the spacetime is $\mathbb{R}\times U(1)\times U(1)$. In Sec. III, we impose the boundary conditions so that the spacetime is asymptotically flat, neither conical nor curvature singularities appear in the domain of outer communications, and no orbifold singularity and no Dirac-Misner string exists on the axis. Furthermore, we require that the spacetime should not admit closed timelike curves (CTCs) in the domain of outer communication. In particular, for $n=2$, we numerically show that there exists no CTC outside the horizons. Section IV is devoted to study some physical properties of such multiple black lenses. In Sec. V, we summarize our result and discuss further generalization.

II. BLACK LENS SOLUTION

We would like to consider supersymmetric solutions in the five-dimensional minimal ungauged supergravity, whose bosonic Lagrangian is described by the Einstein-Maxwell-Chern-Simons theory:

$$\mathcal{L}=R\star 1-2F\wedge\star F-\frac{8}{3\sqrt{3}}A\wedge F\wedge F, \tag{1}$$

where $F=dA$ is the Maxwell field. The metric and gauge potential 1-form are given, respectively, by

$$ds^2=-f^2(dt+\omega)^2+f^{-1}ds_M^2, \tag{2}$$

$$A=\frac{\sqrt{3}}{2}\left[f(dt+\omega)-\frac{K}{H}(d\psi+\chi)-\xi\right], \tag{3}$$

where we choose the hyper-Kähler metric ds_M^2 to be the Gibbons-Hawking metric:

$$ds_M^2=H^{-1}(d\psi+\chi)^2+Hdx^idx^i, \tag{4}$$

$$d\chi=*dH, \tag{5}$$

where $\{x^i\}=(x,y,z)$ ($i=1,\,2,\,3$) are Cartesian coordinates on $\mathbb{E}^3$ and $\partial/\partial\psi$ is a triholomorphic Killing vector. Furthermore,

$$f^{-1}=H^{-1}K^2+L, \tag{6}$$

$$\omega=\omega_\psi(d\psi+\chi)+\hat{\omega}, \tag{7}$$

$$\omega_\psi=H^{-2}K^3+\frac{3}{2}H^{-1}KL+M, \tag{8}$$

$$*d\hat{\omega}=HdM-MdH+\frac{3}{2}(KdL-LdK), \tag{9}$$

$$d\xi=-*dK. \tag{10}$$

Here H, K, L, M are harmonic functions on $\mathbb{E}^3$, where it should be noted that there exists a gauge freedom of redefining harmonic functions [55]:

$$K\to K+aH,\qquad L\to L-2aK-a^2H,$$
$$M\to M-\frac{3}{2}aL+\frac{3}{2}a^2K+\frac{1}{2}a^3H, \tag{11}$$

where a is an arbitrary constant. In fact, under the transformation (11), (f, ω_ψ, χ) remain invariant, whereas the 1-form ξ undergoes a change as $\xi\to\xi-a\chi$. Since this transformation merely amounts to the gauge transformation $A\to A+ad\psi$, the transformation (11) makes the bosonic sector invariant.

Following the papers on supersymmetric black lenses in Refs. [45,47], we consider the next harmonic functions with n point sources:

$$H=\sum_{i=1}^n\frac{h_i}{r_i}\coloneqq\sum_{i=1}^n\frac{n_i}{r_i}, \tag{12}$$

$$M=m_0+\sum_{i=1}^n\frac{m_i}{r_i}, \tag{13}$$

$$K=\sum_{i=1}^n\frac{k_i}{r_i}, \tag{14}$$

$$L=l_0+\sum_{i=1}^n\frac{l_i}{r_i}. \tag{15}$$

Here, each n_i takes not only positive but also negative integers ($n_i=\pm1,\pm2,\ldots$), and $r_i\coloneqq|\boldsymbol{r}-\boldsymbol{r}_i|=\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}$ with constants (x_i,y_i,z_i).

From Eq. (5), (9), and (10), the 1-forms $(\chi,\ \xi,\ \hat{\omega})$ are determined as

$$\chi = \sum_{i=1}^{n} h_i \tilde{\omega}_i, \tag{16}$$

$$\xi = -\sum_{i=1}^{n} k_i \tilde{\omega}_i, \tag{17}$$

$$\hat{\omega} = \sum_{i,j=1(i\neq j)}^{n} \left(h_i m_j + \frac{3}{2} k_i l_j \right) \hat{\omega}_{ij} - \sum_{i=1}^{n} \left(m_0 h_i + \frac{3}{2} l_0 k_i \right) \tilde{\omega}_i + c, \tag{18}$$

where c is a constant and the 1-forms $\tilde{\omega}_i$ and $\hat{\omega}_{ij}$ $(i \neq j)$ on $\mathbb{E}^3$ are, respectively,

$$\tilde{\omega}_i = \frac{z - z_i}{r_i} \frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2}, \tag{19}$$

$$\hat{\omega}_{ij} = -\frac{(\boldsymbol{r} - \boldsymbol{r}_i) \cdot (\boldsymbol{r} - \boldsymbol{r}_j)}{r_i r_j} \frac{[(\boldsymbol{r}_i - \boldsymbol{r}_j) \times (\boldsymbol{r} - \frac{\boldsymbol{r}_i + \boldsymbol{r}_j}{2})]_k dx^k}{|(\boldsymbol{r}_i - \boldsymbol{r}_j) \times (\boldsymbol{r} - \frac{\boldsymbol{r}_i + \boldsymbol{r}_j}{2})|^2}. \tag{20}$$

Throughout this paper, we set $x_i = y_i = 0$ for all i, by which $x\partial/\partial y - y\partial/\partial x$ becomes another Killing field, and assume $z_i < z_j$ for $i < j$. In this case, $\tilde{\omega}_i$ and $\hat{\omega}_{ij}$ are simply written in spherical coordinates $(x, y, z) = (r \sin\theta \cos\phi, \sin\theta \sin\phi, r \cos\theta)$ as

$$\tilde{\omega}_i = \frac{z - z_i}{r_i} d\phi, \tag{21}$$

$$\hat{\omega}_{ij} = \frac{r^2 - (z_i + z_j) r \cos\theta + z_i z_j}{z_{ji} r_i r_j}, \tag{22}$$

where $z_{ji} \coloneqq z_j - z_i$.

III. BOUNDARY CONDITIONS

In the present paper, we would like to obtain a supersymmetric multiple black lens solution such that it describes physically interesting spacetime. We impose suitable boundary conditions at (i) infinity $r \to \infty$, (ii) horizon $\boldsymbol{r} = \boldsymbol{r}_i$ $(i = 1, \ldots, n)$, and (iii) on the z axis $x = y = 0$ in the Gibbons-Hawking space: (i) At infinity $r \to \infty$, the spacetime is asymptotically flat, (ii) each surface $\boldsymbol{r} = \boldsymbol{r}_i$ $(i = 1, \ldots, n)$ should correspond to a smooth degenerate Killing horizon whose spatial cross section has a topology of the lens space $L(n_i, 1) = S^3/\mathbb{Z}_{n_i}$, and (iii) on the z axis $x = y = 0$ in the Gibbons-Hawking space, we require that there should appear no Dirac-Misner string, and orbifold singularity must be eliminated. Moreover, besides these boundaries, in the domain of outer communication, the spacetime allows neither CTCs nor (conical and curvature) singularities.

A. Infinity

First of all, let us consider the boundary condition to satisfy asymptotic flatness. For $r \to \infty$, the metric functions (f, ω_ψ) behave as

$$f^{-1} \simeq l_0 + \left[\left(\sum\nolimits_i k_i \right)^2 + \sum_i l_i \right] r^{-1},$$
$$\omega_\psi \simeq m_0 + \frac{3}{2} l_0 \sum_i k_i. \tag{23}$$

Since the 1-forms $\tilde{\omega}_i$ and $\hat{\omega}_{ij}$ are approximated, respectively, as

$$\tilde{\omega}_i \simeq \cos\theta d\phi, \qquad \hat{\omega}_{ij} \simeq \frac{d\phi}{z_{ji}}, \tag{24}$$

the 1-forms χ and ω behave as, respectively,

$$\chi = \sum_i h_i \hat{\omega}_i \simeq \sum_i n_i \cos\theta d\phi, \tag{25}$$

$$\omega \simeq \left(m_0 + \frac{3}{2} l_0 \sum_i k_i \right) (d\psi + \cos\theta d\phi) - \sum_i \left(m_0 h_i + \frac{3}{2} l_0 k_i \right) \cos\theta d\phi + \left(\sum_{i,j(i\neq j)} \frac{h_i m_j + \frac{3}{2} k_i l_j}{z_{ji}} + c \right) d\phi. \tag{26}$$

The asymptotic flatness demands that the parameters should satisfy

$$l_0 = 1, \tag{27}$$

$$\sum_{i=1}^{n} n_i = 1, \tag{28}$$

$$c = -\sum_{i,j(i\neq j)} \frac{h_i m_j + \frac{3}{2} k_i l_j}{z_{ji}}, \tag{29}$$

$$m_0 = -\frac{3}{2} \sum_i k_i. \tag{30}$$

In terms of the radial coordinate $\rho = 2\sqrt{r}$, the metric asymptotically $(\rho \to \infty)$ behaves as

$$ds^2 \simeq -dt^2 + d\rho^2 + \frac{\rho^2}{4} [(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2]. \tag{31}$$

This coincides with the metric of Minkowski spacetime, where the metric of S^3 is written in terms of Euler angles (ψ, ϕ, θ). The avoidance of conical singularities requires the range of angles to be $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, and $0 \le \psi < 4\pi$ with the identification $\phi \sim \phi + 2\pi$ and $\psi \sim \psi + 4\pi$.

B. Horizon

Next, we show that each point source $\boldsymbol{r} = \boldsymbol{r}_i$ ($i = 1, \ldots, n$) denotes a degenerate Killing horizon whose spatial topology is a lens space $L(n_i, 1) = S^3/\mathbb{Z}_{n_i}$. In terms of the radial coordinate redefined by $r \coloneqq |\boldsymbol{r} - \boldsymbol{r}_i|$, near the ith point source $r = 0$, the four harmonic functions H, K, L, and M behave as

$$H \simeq \frac{n_i}{r} + \sum_{j(\neq i)} \frac{n_j}{|z_{ji}|}, \qquad K \simeq \frac{k_i}{r} + \sum_{j(\neq i)} \frac{k_j}{|z_{ji}|},$$
$$L \simeq \frac{l_i}{r} + l_0 + \sum_{j(\neq i)} \frac{l_j}{|z_{ji}|}, \qquad M \simeq m_0 + \frac{m_i}{r} + \sum_{j(\neq i)} \frac{m_j}{|z_{ji}|}, \tag{32}$$

and the functions f^{-1} and ω_ψ are approximated, respectively, by

$$f^{-1} \simeq \frac{k_i^2/n_i + l_i}{r} + c'_{1(i)},$$
$$\omega_\psi \simeq \frac{k_i^3/n_i^2 + 3k_i l_i/2n_i + m_i}{r} + c'_{2(i)}. \tag{33}$$

Here, the constants $c'_{1(i)}$ and $c'_{2(i)}$ are defined, respectively, by

$$c'_{1(i)} \coloneqq l_0 + \sum_{j(\neq i)} \frac{1}{n_i^2 |z_{ji}|} [2n_i k_i k_j - k_i^2 n_j + n_i^2 l_j], \tag{34}$$

$$c'_{2(i)} \coloneqq m_0 + \frac{3}{2h_i} k_i l_0 + \sum_{j(\neq i)} \frac{1}{2n_i^3 |z_{ji}|} [-(4k_i^3 + 3n_i k_i l_i) n_j + 3n_i(2k_i^2 + n_i l_i) k_j + 3n_i^2 k_i l_j + 2n_i^3 m_j]. \tag{35}$$

The asymptotic behaviors of the 1-forms $\tilde{\omega}_i$ and $\hat{\omega}_{ij}$ are

$$\tilde{\omega}_i \simeq \cos\theta d\phi, \qquad \tilde{\omega}_j \simeq -\frac{z_{ji}}{|z_{ji}|} d\phi \quad (j \neq i), \tag{36}$$

$$\hat{\omega}_{ij} \simeq -\frac{\cos\theta}{|z_{ji}|} d\phi \quad (j \neq i),$$
$$\hat{\omega}_{jk} \simeq \frac{z_{ji} z_{ki}}{|z_{ji} z_{ki}| z_{kj}} d\phi \quad (j, k \neq i, j \neq k), \tag{37}$$

which leads to

$$\hat{\omega} = \Bigg[\sum_{j(\neq i)} \left(n_i m_j - n_j m_i + \frac{3}{2} k_i l_j - \frac{3}{2} k_j l_i\right) \frac{-\cos\theta}{|z_{ji}|} + \sum_{j,k\neq i (j\neq k)} \left(n_j m_k + \frac{3}{2} k_j l_k\right) \frac{z_{ji} z_{ki}}{|z_{ji} z_{ki}| z_{kj}} - \left(m_0 n_i + \frac{3}{2} l_0 k_i\right) \cos\theta - \sum_{j(\neq i)} \left(-m_0 + \frac{3}{2} l_0 k_j\right) \frac{-z_{ji}}{|z_{ji}|} + c\Bigg] d\phi, \tag{38}$$

and

$$\chi = h_i \hat{\omega}_i + \sum_{j(\neq i)} h_j \hat{\omega}_j \simeq \left(n_i \cos\theta - \sum_{j(\neq i)} n_j \frac{z_{ji}}{|z_{ji}|}\right) d\phi. \tag{39}$$

It is obvious that the metric components g_{rr} and $g_{r\psi'}$ apparently diverge at $r = 0$. However, the apparent divergence can be eliminated by introducing new coordinates (v, ψ') given by

$$dv = dt - \left(\frac{A_0}{r^2} + \frac{A_1}{r}\right) dr,$$
$$d\psi' = d\psi - \sum_{j(\neq i)} n_j \frac{z_{ji}}{|z_{ji}|} d\phi - \frac{B_0}{r} dr, \tag{40}$$

where the constants A_0 and B_0 are determined by demanding the $1/r^2$ term in g_{rr} and the $1/r$ term $g_{r\psi'}$ should vanish and the constant A_1 is determined to remove the $1/r$ term in g_{rr}, which results in

$$A_0 = \frac{1}{2} \sqrt{3k_i^2 l_i^2 + 4n_i l_i^3 - 4m_i(2k_i^3 + 3n_i k_i l_i + n_i^2 m_i)}, \tag{41}$$

$$A_0 B_0 = \frac{2k_i^3 + 3k_i l_i n_i + 2m_i n_i^2}{2}, \tag{42}$$

$$4A_0 A_1 = -4k_i^3 m_0 + 3l_i k_i^2 - 6n_i k_i l_i m_0 + 6n_i l_i^2 - (4m_0 n_i^2 + 6k_i n_i) m_i + \sum_{j(\neq i)} \frac{1}{|z_{ji}|} [2(l_i^3 - 3k_i l_i m_i - 2n_i m_i^2) n_j + 3(k_i l_i^2 - 4k_i^2 m_i - 2n_i l_i m_i) k_j + 3(k_i^2 l_i + 2n_i l_i^2 - 2n_i k_i m_i) l_j - 2(2k_i^3 + 3n_i k_i l_i + 2n_i^2 m_i) m_j]. \tag{43}$$

In terms of this coordinate system, we see that the metric is then analytic in r and therefore can be uniquely extended into the $r<0$ region. Moreover, one can easily confirm that the null surface $r=0$ is the Killing horizon for the Killing field $V=\partial/\partial v$.

Taking the limit as $(v,r)\to(v/\epsilon,\epsilon r)$ and $\epsilon\to 0$ [13], after short computations, we obtain the near-horizon geometry as

$$ds^2_{\rm NH}=\frac{R^2_{2(i)}}{4}\left[d\psi'+n_i\cos\theta d\phi-\frac{2k_i(2k_i^2+3n_il_i)-4n_i^2m_i}{R^4_{1(i)}R^2_{2(i)}}rdv\right]^2+R^2_{1(i)}(d\theta^2+\sin^2\theta d\phi^2)-\frac{4r^2}{R^2_{1(i)}R^2_{2(i)}}dv^2-\frac{4}{R_{2(i)}}dvdr, \tag{44}$$

and

$$A=\frac{\sqrt{3}}{2}\left[\frac{n_ir}{k_i^2+n_il_i}dv+\frac{2k_i^3+3n_ik_il_i+2n_i^2m_i}{2n_iR^2_{1(i)}}(d\psi'+n_i\cos\theta d\phi)\right], \tag{45}$$

where we have defined

$$R^2_{1(i)}\coloneqq k_i^2+n_il_i, \tag{46}$$

$$R^2_{2(i)}\coloneqq\frac{3k_i{}^2l_i{}^2+4n_il_i{}^3-4m_i(2k_i^3+3n_ik_il_i+n_i^2m_i)}{R^4_{1,i}}. \tag{47}$$

This is isometric to the near-horizon geometry of the BMPV black hole. In order to remove CTCs near all horizons, we must require the inequalities

$$R^2_{1(i)}>0,\qquad R^2_{2(i)}>0. \tag{48}$$

As will be shown, it can be expected that these are also sufficient conditions for the avoidance of CTCs throughout the outside region of black holes. The cross section of the event horizon can be extracted by $v=$ const and $r=0$ in (44) as

$$ds^2_{\rm H}=\frac{R^2_{2(i)}}{4}(d\psi'+n_i\cos\theta d\phi)^2+R^2_{1(i)}(d\theta^2+\sin^2\theta d\phi^2), \tag{49}$$

which is the squashed metric of the lens space $L(n_i,1)=S^3/\mathbb{Z}_{n_i}$.

C. Axis

We impose $\hat\omega_\phi=0$ on the z axis of $\mathbb{E}^3$ (i.e., $x=y=0$) in the Gibbons-Hawking space, which implies that no Dirac-Misner string pathology happens on the z axis. As is discussed in Ref. [56], there is a danger of CTCs arising from Dirac-Misner strings at $\theta=0,\ \pi$ because the space-space part of the metric $ds_4^2=Pf^2H^{-2}(d\psi+\chi-P^{-1}H^2\omega_\psi\hat\omega_\phi d\phi)^2+f^{-1}H(r^2\sin^2\theta-P^{-1}\hat\omega_\phi{}^2)d\phi^2+f^{-1}H(dr^2+r^2d\theta^2)$ $(P\coloneqq f^{-3}H-\omega_\psi^2H^2)$ cannot become positive there unless $\hat\omega_\phi$ vanishes on the polar axis. For the black lens solution with a single horizon [45,47], $\hat\omega_\phi=0$ is a direct consequence of the bubble equations, whereas for the multiblack lens solution obtained here, this is not the case. This is why we demand $\hat\omega_\phi=0$ on the z axis.

The z axis of $\mathbb{E}^3$ in the Gibbons-Hawking space splits up into the $(n+1)$ intervals: $I_-=\{(x,y,z)|x=y=0,z<z_1\}$, $I_i=\{(x,y,z)|x=y=0,z_i<z<z_{i+1}\}\quad(i=1,\ldots,n-1)$, and $I_+=\{(x,y,z)|x=y=0,z>z_n\}$. On the z axis, the 1-forms $\hat\omega_{ij}$ and $\tilde\omega_i$ take simple forms, respectively,

$$\hat\omega_{ij}=\frac{(z-z_i)(z-z_j)}{z_{ji}|z-z_i||z-z_j|}d\phi,\qquad\tilde\omega_i=\frac{z-z_i}{|z-z_i|}d\phi. \tag{50}$$

In particular, on $I_\pm$, $\hat\omega_{ij}$ and $\tilde\omega_i$ become, respectively,

$$\hat\omega_{ij}=\frac{1}{z_{ji}}d\phi,\qquad\tilde\omega_i=\pm d\phi. \tag{51}$$

Hence, on $I_\pm$, $\hat\omega$ automatically vanishes, since

$$\begin{aligned}\hat\omega&=\sum_{k,j(k\neq j)}\left(h_km_j+\frac{3}{2}k_kl_j\right)\hat\omega_{kj}-\sum_j\left(m_0h_j+\frac{3}{2}k_j\right)\hat\omega_j+cd\phi\\&=\sum_{k,j(k\neq j)}\left(h_km_j+\frac{3}{2}k_kl_j\right)\frac{d\phi}{z_{jk}}\mp\sum_j\left(m_0h_j+\frac{3}{2}k_j\right)d\phi-\sum_{k,j(k\neq j)}\left(h_km_j+\frac{3}{2}k_kl_j\right)\frac{d\phi}{z_{jk}}\\&=\mp\sum_j\left(m_0h_j+\frac{3}{2}k_j\right)d\phi\\&=\mp\left(m_0\sum_jn_j+\frac{3}{2}\sum_jk_j\right)d\phi=0,\end{aligned} \tag{52}$$

where we have used Eqs. (28) and (30) in the last equality.

On the intervals I_i $(i = 1, \ldots, n-1)$, we should impose $\hat{\omega}_\phi = 0$, since it does not automatically vanish there. Let us note that if and only if the constants $\hat{\omega}_\phi[I_1] - \hat{\omega}_\phi[I_-]$, $\hat{\omega}_\phi[I_i] - \hat{\omega}_\phi[I_{i-1}]$ $(i = 2, \ldots, n-1)$, $\hat{\omega}_\phi[I_+] - \hat{\omega}_\phi[I_{n-1}]$ vanish, the 1-form $\hat{\omega}$ vanishes on all the intervals I_i $(i = 1, \ldots, n-1)$.

The constant $\hat{\omega}_\phi[I_1] - \hat{\omega}_\phi[I_-]$ can be written as

$$\begin{aligned}
\hat{\omega}_\phi[I_1] - \hat{\omega}_\phi[I_-] &= -\sum_{j\neq 1}\left[h_1 m_j - h_j m_1 + \frac{3}{2}(k_1 l_j - k_j l_1)\right]\frac{1}{z_{j1}} + \sum_{k,j(k\neq j,k,j\neq 1)}\left(h_k m_j + \frac{3}{2}k_k l_j\right)\frac{1}{z_{jk}} - \left(m_0 h_1 + \frac{3}{2}k_1\right) \\
&\quad + \sum_{j\neq 1}\left(m_0 h_j + \frac{3}{2}k_j\right) + c - \sum_{j\neq 1}\left[h_1 m_j - h_j m_1 + \frac{3}{2}(k_1 l_j - k_j l_1)\right]\frac{1}{z_{j1}} - \sum_{k,j(k\neq j,k,j\neq 1)}\left(h_k m_j + \frac{3}{2}k_k l_j\right)\frac{1}{z_{jk}} \\
&\quad - \sum_{j}\left(m_0 h_1 + \frac{3}{2}k_j\right) - c \\
&= -2\sum_{j\neq 1}\left[h_1 m_j - h_j m_1 + \frac{3}{2}(k_1 l_j - k_j l_1)\right]\frac{1}{z_{j1}} - 2\left(m_0 h_1 + \frac{3}{2}k_1\right).
\end{aligned} \tag{53}$$

Similarly, the constant $\hat{\omega}_\phi[I_i] - \hat{\omega}_\phi[I_{i-1}]$ can be simplified as

$$\begin{aligned}
\hat{\omega}_\phi[I_i] - \hat{\omega}_\phi[I_{i-1}] &= \sum_{k,j\leq i(k\neq j)}\left(h_k m_j + \frac{3}{2}k_k l_j\right)\frac{1}{z_{jk}} - \sum_{k,j(j\leq i<k)}\left(h_k m_j - h_j m_k + \frac{3}{2}(k_k l_j - k_j l_k)\right)\frac{1}{z_{jk}} \\
&\quad + \sum_{k,j(k,j>i,k\neq j)}\left(h_k m_j + \frac{3}{2}k_k l_j\right)\frac{1}{z_{jk}} - \sum_{j(\leq i)}\left(m_0 h_j + \frac{3}{2}k_j\right) + \sum_{j(>i)}\left(m_0 h_j + \frac{3}{2}k_j\right) + c \\
&\quad - \sum_{k,j\leq i-1(k\neq j)}\left(h_k m_j + \frac{3}{2}k_k l_j\right)\frac{1}{z_{jk}} + \sum_{k,j(j\leq i-1<k)}\left(h_k m_j - h_j m_k + \frac{3}{2}(k_k l_j - k_j l_k)\right)\frac{1}{z_{jk}} \\
&\quad - \sum_{k,j(k,j>i-1,k\neq j)}\left(h_k m_j + \frac{3}{2}k_k l_j\right) + \sum_{j(\leq i-1)}\left(m_0 h_j + \frac{3}{2}k_j\right) - \sum_{j(>i-1)}\left(m_0 h_j + \frac{3}{2}k_j\right) - c \\
&= \sum_{j(<i)}\left(h_i m_j - h_j m_i + \frac{3}{2}(k_i l_j - k_j l_i)\right)\frac{1}{z_{ji}} + \sum_{j(<i)}\left(h_i m_j - h_j m_i + \frac{3}{2}(k_i l_j - k_j l_i)\right)\frac{1}{z_{ji}} \\
&\quad - \sum_{k(>i)}\left(h_k m_i - h_i m_k + \frac{3}{2}(k_k l_i - k_i l_k)\right)\frac{1}{z_{ik}} - \sum_{j(>i)}\left(h_i m_j - h_j m_i + \frac{3}{2}(k_i l_j - k_j l_i)\right)\frac{1}{z_{ji}} \\
&\quad - \left(m_0 h_i + \frac{3}{2}k_i\right) - \left(m_0 h_i + \frac{3}{2}k_i\right) \\
&= 2\sum_{j(<i)}\left(h_i m_j - h_j m_i + \frac{3}{2}(k_i l_j - k_j l_i)\right)\frac{1}{z_{ji}} - 2\sum_{j(>i)}\left(h_i m_j - h_j m_i + \frac{3}{2}(k_i l_j - k_j l_i)\right)\frac{1}{z_{ji}} \\
&\quad - 2\left(m_0 h_i + \frac{3}{2}k_i\right).
\end{aligned} \tag{54}$$

The constant $\hat{\omega}_\phi[I_+] - \hat{\omega}_\phi[I_{n-1}]$ becomes

$$\begin{aligned}
\hat{\omega}_\phi[I_+] - \hat{\omega}_\phi[I_{n-1}] &= -2\sum_{j(<n)}\left(h_n m_j - h_j m_n + \frac{3}{2}(k_n l_j - k_j l_n)\right)\frac{1}{z_{nj}} \\
&\quad - 2\left(m_0 h_n + \frac{3}{2}k_n\right),
\end{aligned} \tag{55}$$

where we have used Eqs. (28) and (30) in the last equality.

From Eqs. (53)–(55), it turns out that to assure $\hat{\omega}=0$ on the z axis, for $i=1,\ldots,n-1$, the parameters should be subject to the constraints

$$m_0 n_i+\frac{3}{2}k_i=-\sum_{j(\neq i)}\left[n_i m_j-n_j m_i+\frac{3}{2}(k_i l_j-k_j l_i)\right]\frac{1}{|z_{ji}|}. \tag{56}$$

For the discussion of the issue of orbifold singularities, we consider the rod diagram of the multiblack lenses. On $I_\pm$, we get

$$\chi=\pm d\phi, \tag{57}$$

and on I_i,

$$\begin{aligned}\chi&=\left(\sum_{j\le i}h_j\frac{z-z_j}{|z-z_j|}+\sum_{i+1\le j\le n}h_j\frac{z-z_j}{|z-z_j|}\right)d\phi\\&=\left(\sum_{j\le i}h_j-\sum_{i+1\le j\le n}h_j\right)d\phi\\&=\left(\sum_{j\le i}n_j-\sum_{i+1\le j\le n}n_j\right)d\phi\\&=\left(2\sum_{j\le i}n_j-1\right)d\phi.\end{aligned} \tag{58}$$

Therefore, we can write the two-dimensional (ϕ,ψ) part of the metric on the intervals $I_\pm$ and I_i in the form

$$ds_2^2=(-f^2\omega_\psi^2+f^{-1}H^{-1})(d\psi+\chi_\phi d\phi)^2. \tag{59}$$

Let us now use the coordinate basis vectors $(\partial_{\phi_1},\partial_{\phi_2})$ of 2π periodicity, instead of $(\partial_\phi,\partial_\psi)$, where these coordinates are defined by $\phi_1\coloneqq(\psi+\phi)/2$ and $\phi_2\coloneqq(\psi-\phi)/2$. From (59), one can observe that the Killing vector $v\coloneqq\partial_\phi-\chi_\phi\partial_\psi$ vanishes on each interval. Namely,

(1) on the interval I_+, the Killing vector $v_+\coloneqq\partial_\phi-\partial_\psi=(0,-1)$ vanishes,
(2) on each interval I_i $(i=1,\ldots,n-1)$, the Killing vector $v_i\coloneqq\partial_\phi-(2n-2i+1)\partial_\psi=(1-\sum_{j(\le i)}n_j,-\sum_{j(\le i)}n_j)$ vanishes, and
(3) on the interval I_-, the Killing vector $v_-\coloneqq\partial_\phi+\partial_\psi=(1,0)$ vanishes.

From these, we can observe that the Killing vectors $v_\pm, v_i$ on the intervals satisfy

$$\det(v_+^T,v_{n-1}^T)=n_n,\qquad \det(v_i^T,v_{i-1}^T)=n_i, \tag{60}$$

with

$$\det(v_1^T,v_-^T)=n_1. \tag{61}$$

Equation (60) assures that the metric smoothly joints at the end points $z=z_i$ $(1\le i\le n)$ of the intervals [41], which means that there exist no orbifold singularities at adjacent intervals. Furthermore, Eqs. (60) and (61) show that the spatial cross section of the ith Killing horizon $\boldsymbol{r}=\boldsymbol{r}_i$ is topologically the lens space $L(n_i,1)=S^3/\mathbb{Z}_{n_i}$.

IV. PHYSICAL PROPERTIES

In this section, we study physical properties of the multiblack lenses.

A. Conserved quantities

Let us investigate conserved quantities of the multiblack lens solution. The Arnowitt-Deser-Misner (ADM) mass and two ADM angular momenta can be computed as

$$M=\frac{\sqrt{3}}{2}|Q|=3\pi\left[\left(\sum\nolimits_i k_i\right)^2+\sum_i l_i\right], \tag{62}$$

$$J_\psi=4\pi\left[\left(\sum\nolimits_i k_i\right)^3+\sum_i m_i+\frac{3}{2}\left(\sum_i k_i\right)\left(\sum_i l_i\right)\right], \tag{63}$$

$$J_\phi=6\pi\left[-\left(\sum_i k_i\right)\left(\sum_i z_i\right)+\left(\sum_i k_i z_i\right)\right], \tag{64}$$

where Q is an electric charge, which saturates the Bogomol'nyi bound.

The surface gravity and the angular velocities of the horizon vanish, as expected for supersymmetric black objects in the asymptotically flat spacetime [10]. The area of the ith horizon reads from (44) as

$$\text{area}=8\pi^2R_{1(i)}^2R_{2(i)}. \tag{65}$$

The interval I_i $(i=1,\ldots,n-1)$ represents the bubble between adjacent two horizons which is topologically an annulus $S^1\times[0,1]$. The magnetic flux through I_i is defined as

$$q[I_i]\coloneqq\frac{1}{4\pi}\int_{I_i}F. \tag{66}$$

Since the Maxwell gauge potential 1-form A_μ is smooth at the horizons and bubbles, these fluxes are given by only the contribution from the horizons $q[I_i]=[-A_\psi]_{z=z_i}^{z=z_{i+1}}$, which leads to

$$q[I_i]=\frac{\sqrt{3}}{2}\left[\frac{k_il_i+2n_im_i}{2(k_i^2+n_il_i)}-\frac{k_{i+1}l_{i+1}+2n_{i+1}m_{i+1}}{2(k_{i+1}^2+n_{i+1}l_{i+1})}\right]\quad (i=1,\ldots n-1). \tag{67}$$

Let us see whether there exists a parameter region such that magnetic fluxes vanish. For simplicity, we now consider the two-black-lens solution ($n = 2$). From Eq. (67), the magnetic flux $q[I_1]$ between the two horizons can be written as

$$q[I_1] = \frac{\sqrt{3}}{2}\left(\frac{k_1 l_1 + 2n_1 m_1}{2(k_1^2 + n_1 l_1)} - \frac{k_2 l_2 + 2n_2 m_2}{2(k_2^2 + n_2 l_2)}\right). \quad (68)$$

From Eq. (68), when m_2 is denoted by

$$m_2 = \frac{k_1 l_1 (k_2^2 + n_2 l_2) - k_2 l_2 (k_1^2 + n_1 n_1)}{2n_2 (k_1^2 + n_1 l_1)}, \quad (69)$$

the magnetic flux $q[I_1]$ vanishes. Here, let us recall that, for this case, the condition (56) for the absence of Dirac-Misner string singularities on the z axis is simply written as

$$-\frac{3}{2} n_1 (k_1 + k_2) + \frac{3}{2} k_1 = -\left[(n_1 m_2 - n_2 m_1) + \frac{3}{2}(k_1 l_2 - k_2 l_1)\right]\frac{1}{z_{21}}, \quad (70)$$

which gives

$$z_{21} = \frac{k_1 l_2 - k_2 l_1 + 2n_1 m_2/3}{n_1 k_2 - n_2 k_1}, \quad (71)$$

where we have put $m_1 = 0$ from Eq. (11) without the loss of generality. From our assumption, the constant z_{21} must be positive. As shown in Fig. 1, there exists a parameter region such that the magnetic flux vanishes for $R_1^2 > 0$, $R_2^2 > 0$, and $z_{21} > 0$.

As for the supersymmetric black lens solution in obtained in Refs. [45,47], there exists no limit such that all the magnetic fluxes vanish. Therefore, one can consider that, at least, for the supersymmetric black lens with the single horizon of the topology $L(n, 1) = S^3/\mathbb{Z}_n$ ($n = 2, 3, \ldots$) in Refs. [45,47], the existence of the magnetic fluxes plays an essential role in supporting the horizon of the black lens. On the other hand, for the supersymmetric multiblack lenses obtained in this paper, it seems that the magnetic flux does not necessarily need to exist.

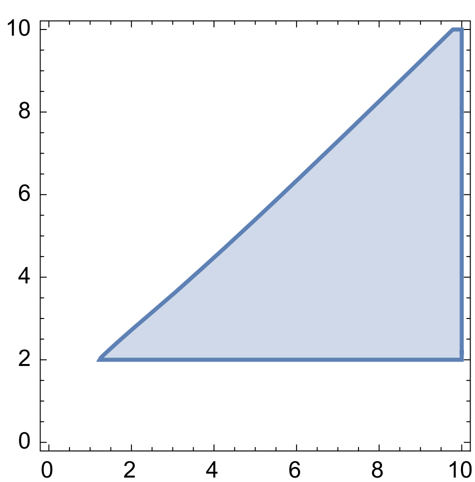

FIG. 1. Region in a (k_1, k_2) plane such that $R_1^2 > 0$, $R_2^2 > 0$, $z_{21} > 0$, and $q[I_1] = 0$ for $n_1 = 5$, $n_2 = -4$, and $l_1 = l_2 = 1$.

B. No CTC

We demand that the domain of outer communication in the five-dimensional spacetime does not admit CTCs. This is achieved if the inequalities

$$g_{\theta\theta} > 0, \qquad g_{\psi\psi} > 0, \qquad g_{\psi\psi} g_{\phi\phi} - g_{\psi\phi}^2 > 0 \quad (72)$$

are satisfied on and outside all horizons. Explicitly, these conditions are replaced by

$$D_1 \coloneqq K^2 + HL > 0, \quad (73)$$

$$D_2 \coloneqq \frac{3}{4} K^2 L^2 - 2K^3 M - 3HKLM + HL^3 - H^2 M^2 > 0, \quad (74)$$

$$D_3 \coloneqq D_2 r^2 \sin^2\theta - \hat{\omega}_\phi^2 > 0. \quad (75)$$

It is a considerably troublesome problem to prove their positivity.

As seen in Fig. 2, for $n = 2$, we have checked the absence of CTCs by seeing numerically the positivity of D_i ($i = 1, 2, 3$) and found that there appear no causal violations in the domain of outer communications. We can expect that, for $n = 2$, the inequalities (48) are sufficient to remove CTCs in the whole domain of outer communication. We also expect that, even for $n > 2$, (48) are sufficient to remove causal pathologies on and outside the horizon.

C. Critical surfaces

One of the physically interesting features is that the harmonic function H becomes negative around $\boldsymbol{r} = \boldsymbol{r}_i$ ($i = 1, \ldots, n$), which leads to the $(-,-,-,-)$ signature of the Gibbons-Hawking base space. However, the signature of the five-dimensional spacetime metric remains Lorentzian, because the function $f^{-1}H$ is positive. In this case, a so-called evanescent ergosurface appears [37] at the places which $f = 0$ corresponding to $H = 0$.

At this surface, one must impose the regularity condition $K \neq 0$ when $H = 0$, since, if this does not hold, the spacetime could not become regular there. In other words, if there exist points $z = z_c$ on the axis such that

$$H = \sum_i \frac{n_i}{|z_c - z_i|} = 0, \qquad K = \sum_i \frac{k_i}{|z_c - z_i|} = 0, \quad (76)$$

these critical surfaces are singular, which leads to

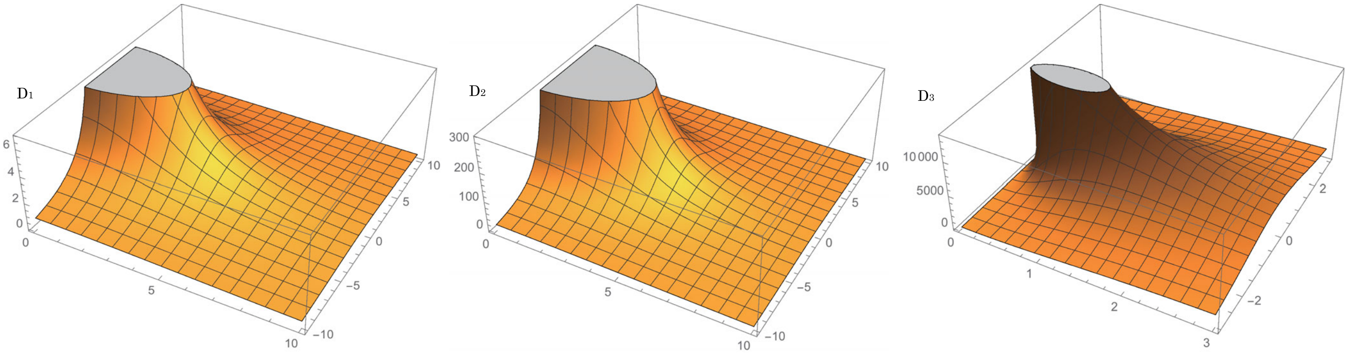

FIG. 2. Plots of D_i's against ($\rho = \sqrt{x^2+y^2}$, z) for $n_1 = 2$, $n_2 = -1$, $z_1 = 0$, $z_2 \simeq 0.0011$, $k_1 = k_2 = 4$, $l_1 = l_2 = 1$, $m_1 = 0$, and $m_2 = 0.001$. No naked CTC appears.

$$\sum_{i\geq 2} \frac{n_1 k_i - n_i k_1}{|z_c - z_i|} = 0. \tag{77}$$

Hence, for instance, one of the sufficient conditions to avoid the singularities at critical surfaces is that for any i ($i = 2, \ldots, n$), $n_1 k_i - n_i k_1$ must have the same signs:

$$n_1 k_i - n_i k_1 > 0 \quad \text{or} \quad n_1 k_i - n_i k_1 < 0. \tag{78}$$

For $n = 2$, one of the evanescent ergosurfaces exists at

$$z = \frac{n_1 z_2 + n_2 z_1}{n_1 + n_2} \tag{79}$$

for $z \in I_+$ when $n_1 > 0$ and $z \in I_-$ when $n_1 < 0$, whereas the other exists at

$$z = \frac{n_1 z_2 - n_2 z_1}{n_1 - n_2} \tag{80}$$

for $z \in I_1$.

V. SUMMARY

In this work, we have constructed an asymptotically flat and stationary multiblack lens solution as a supersymmetric solution in the bosonic sector of the five-dimensional minimal supergravity. We have shown that this solution describes a mechanical equilibrium state of an arbitrary number of charged black lenses and the degenerate Killing horizons admit different lens space topologies $L(n_i, 1) = S^3/\mathbb{Z}_{n_i}$ ($i = 1, \ldots, n$), where each n_i takes nonzero different integers but must satisfy the constraint equation $\sum_{i=1}^n n_i = 1$. This multiblack lens spacetime has a spatial symmetry of $U(1) \times U(1)$, because all horizons are aligned on the z axis in the Gibbons-Hawking space. Moreover, we have also computed the conserved charges including the (positive and Bogomol'nyi-Prasad-Sommerfeld (BPS)-saturating) mass, two angular momenta, and the magnetic fluxes on the bubbles.

For the supersymmetric single black lens in Refs. [45,47], the existence of the magnetic fluxes plays an essential role in supporting the horizon of the black lens, whereas for the multiblack lenses obtained in this paper, this is not the case, since the magnetic flux vanishes at least for $n = 2$. This is one of the surprising features for the multiblack lenses. This can be interpreted as follows: Even if the magnetic fluxes do not exist, the electric force acting on the black lenses (black holes) can support the horizons. Furthermore, this fact leads to the following natural question. Does a nonsupersymmetric black lens exist, for instance, in pure Einstein gravity? If such a vacuum black lens solution exists, the horizon must be supported by the centrifugal force of a rotating black lens the same as the black ring. This issue deserves further study.

In this work, we have considered the supersymmetric solution subject to the constraint (28), which comes from the requirement that the topology of spatial infinity $r \to \infty$ should be S^3, when the spacetime asymptotically becomes flat. This constraint seems to impose a considerably strong restriction on the topologies of horizons. However, if one replaces the harmonic function H in the Gibbons-Hawking base space with, for instance, another one

$$\sum_{i=1}^{n} \frac{n_i}{r_i} - \sum_{i=n+1}^{N} \frac{1}{r_i} \left(N := \sum_{i=1}^{n} n_i - 1 \right),$$

and, moreover, if at each point $\boldsymbol{r} = \boldsymbol{r}_i$ ($i = n+1, \ldots, N$) where the harmonic function diverges, one demands regularity (this corresponds to the conditions $c_2 = 0$ in Ref. [47]), one no longer may need to impose the constraint (28). In this case, each horizon $\boldsymbol{r} = \boldsymbol{r}_i$ ($i = 1, \ldots, n$) can have an independent lens space topology of $L(n_i, 1)$.

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